Anti-\(k\)-labeling of graphs

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\begin{tabular}{ll}
\textbf{Keywords:} & \textbf{A B S T R A C T} \\
Anti-\(k\)-labeling problem & It is well known that the labeling problems of graphs arise in many (but not limited to) networking and telecommunication contexts. In this paper we introduce the anti-\(k\)-labeling problem of graphs which we seek to minimize the similarity (or distance) of neighboring nodes. For example, in the fundamental frequency assignment problem in wireless networks where each node is assigned a frequency, it is usually desirable to limit or minimize the frequency gap between neighboring nodes so as to limit interference.

No-hole anti-\(k\)-labeling number & Let \(k \geq 1\) be an integer and \(\psi\) is a labeling function (anti-\(k\)-labeling) from \(V(G)\) to \([1, 2, \ldots, k]\) for a graph \(G\). A no-hole anti-\(k\)-labeling is an anti-\(k\)-labeling using all labels between 1 and \(k\). We define \(w_{\psi}(e) = |\psi(u) - \psi(v)|\) for an edge \(e = uv\) and \(w_{\psi}(G) = \min\{w_{\psi}(e) : e \in E(G)\}\) for an anti-\(k\)-labeling \(\psi\) of the graph \(G\). The anti-\(k\)-labeling number of a graph \(G\), \(\lambda_k(G)\), is \(\max\{w_{\psi}(G) : \psi\}\). In this paper, we first show that \(\lambda_k(G) = \lfloor \frac{k}{k-1} \rfloor\) and the problem that determines the anti-\(k\)-labeling number of graphs is NP-hard. We mainly obtain the lower bounds on no-hole anti-\(n\)-labeling number for trees, grids and \(n\)-cubes.

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Channel assignment problem, &
\end{tabular}

1. Problems

All graphs considered here are simple and finite. Definitions which are not given here may be found in [1]. Let \(k \geq 1\) be an integer. An anti-\(k\)-labeling \(\psi\) of a graph \(G\) is a mapping from \(V(G)\) to \([1, 2, \ldots, k]\). An anti-\(k\)-labeling \(\psi\) of \(G\) is called a no-hole anti-\(k\)-labeling if it uses all labels between 1 and \(k\). We define \(w_{\psi}(e) = |\psi(u) - \psi(v)|\) for an edge \(e = uv\) and \(w_{\psi}(G) = \min\{w_{\psi}(e) : e \in E(G)\}\) for an anti-\(k\)-labeling \(\psi\) of the graph \(G\). The anti-\(k\)-labeling number (the no-hole anti-\(k\)-labeling number) of a graph \(G\), \(\lambda_k(G)\), is \(\max\{w_{\psi}(G) : \psi\}\). We refer to a labeling \(\psi\) with \(w_{\psi}(G) = \lambda_k(G)\) as an optimal anti-\(k\)-labeling (an optimal no-hole anti-\(k\)-labeling) for a graph \(G\). Such (no-hole) anti-\(k\)-labeling number problem is our focus in this paper.

The above labeling problem represents a generic class of labeling problems arising in many (but not limited to) networking and telecommunication contexts, in which we seek to minimize the similarity (or distance) of neighboring nodes. For example, in the fundamental frequency assignment problem in wireless networks where each node is assigned a frequency, it is usually desirable to limit or minimize the frequency gap between neighboring nodes so as to limit interference. Another example relates to the content sharing systems such as peer-to-peer file sharing systems, where resources (e.g., files) are

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replicated at network nodes to reduce resource retrieval time and increase system robustness. In these systems, to maximize performance gain, we usually want to place different items in the vicinity of each node or to place the same items far from each other.

These problems can be cast to the labeling problem where we seek a node labeling maximizing the minimum labeling distance among neighboring nodes. Surprisingly, this labeling problem has not yet been analyzed (not even formulated in a mathematical sense).

Let \( T \) be a set of nonnegative integers. Find a function \( f : V(G) \to \mathbb{Z}^+ \) such that \( |f(x) - f(y)| \notin T \) for \( xy \in E(G) \). This function \( f \) is called a \( T \)-coloring of \( G \). The span under \( f \) is \( \max\{|f(x) - f(y)| : x, y \in V(G)\} \). We denote the minimum span over all \( T \)-colorings by \( \text{sp}_T(G) \). If \( T = \{0, 1, \ldots, m - 1\} \), then this \( T \)-coloring is called an \( m \)-distant coloring. Moreover, if all colors are used, then this \( m \)-distant coloring is called a no-hole \( m \)-distant coloring. When \( m = 1 \), then an \( m \)-distant coloring is an ordinary graph coloring. Hence, \( m \)-distant coloring is a generalization of ordinary graph coloring.

In some sense, our focus problem is also \( m \)-distant coloring. In fact, \( \lambda_k(G) > 0 \) if and only if \( k \geq \chi(G) \) for a graph \( G \), where \( \chi(G) \) is the chromatic number of the graph \( G \). Hence, \( \chi(G) \) is the minimum number of \( k \) such that \( \lambda_k(G) > 0 \) for a graph \( G \). Since determining the chromatic number of graphs is \( NP \)-hard, the anti-\( k \)-labeling problem is also \( NP \)-hard.

Another related labeling problem (namely, \( L(2, 1) \)-labeling) will be mentioned in Section 4.

2. \( \lambda_k(G) \) and \( \chi(G) \) of graphs

Observation 1. If \( G \) is a subgraph of \( G \), then \( \lambda_k(G) \geq \lambda_k(G) \).

Proof. Clearly, for an arbitrary anti-\( k \)-labeling \( \psi \), \( w_\psi(H) \geq w_\psi(G) \) holds. Suppose \( \psi \) is an optimal anti-\( k \)-labeling of \( G \) (i.e., \( w_\psi(G) = \lambda_k(G) \)). Then \( w_\psi(H) \geq w_\psi(G) \leq \lambda_k(G) \). Hence, \( \lambda_k(H) \geq \lambda_k(G) \) by the definition of anti-\( k \)-labeling number. \( \square \)

Suppose that \( G_1 \) and \( G_2 \) are two graphs with \( V(G_1) \cap V(G_2) = \emptyset \). The union \( G \) of \( G_1 \) and \( G_2 \), denoted by \( G = G_1 \cup G_2 \), is the graph whose vertex set is \( V(G_1) \cup V(G_2) \), and edge set is \( E(G_1) \cup E(G_2) \).

Observation 2. If \( G = G_1 \cup G_2 \), then \( \lambda_k(G) = \min\{\lambda_k(G_1), \lambda_k(G_2)\} \).

Proof. \( \lambda_k(G) \leq \min\{\lambda_k(G_1), \lambda_k(G_2)\} \) following from Observation 1 and the fact that \( G_1 \) and \( G_2 \) are subgraphs of \( G \). On the other hand, an anti-\( k \)-labeling of \( G_1 \) together with an anti-\( k \)-labeling of \( G_2 \) makes an anti-\( k \)-labeling \( \psi \) of \( G_1 \cup G_2 \) so that \( w_\psi(G) \geq \min\{\lambda_k(G_1), \lambda_k(G_2)\} \). Hence \( \lambda_k(G) \geq \min\{\lambda_k(G_1), \lambda_k(G_2)\} \). \( \square \)

Theorem 3. Let \( G \) be a graph with chromatic number \( \chi \geq 2 \). Then \( \lambda_k(G) = \lceil \frac{k-1}{\chi-1} \rceil \) for all \( k \).

Proof. We first show that \( \lambda_k(G) \geq \lceil \frac{k-1}{\chi-1} \rceil \). It suffices to show that there exists an anti-\( k \)-labeling \( \psi \) such that \( w_\psi(G) = \lceil \frac{k-1}{\chi-1} \rceil \) for a graph \( G \). Let \( V_1, V_2, \ldots, V_\chi \) be a proper \( \chi \)-coloring of \( G \). Then we consider the following labeling \( \psi \): label the vertices of \( V_1 \) by \( 1 \) and \( i - 1 \) \( \frac{k-1}{\chi-1} \) for \( i = 1, 2, \ldots, \chi \), Note that \( \chi - 1 \frac{k-1}{\chi-1} \leq k \) and \( V_i \) \( i = 1, 2, \ldots, \chi \) is an independent set. We have \( w_\psi(G) = \min\{w_\psi(e) : e \in E(G)\} = \frac{k-1}{\chi-1} \). Hence, \( \lambda_k(G) \geq \lceil \frac{k-1}{\chi-1} \rceil \).

We next show that \( \lambda_k(G) \leq \lceil \frac{k-1}{\chi-1} \rceil \). Let \( \psi \) be an optimal anti-\( k \)-labeling of \( G \) and \( V_1, V_2, \ldots, V_\chi \) is a partition of \( V(G) \) under \( \psi \), where the vertices in \( V_i \) have label \( i \), \( i = 1, 2, \ldots, k \). Assume \( \lambda_k(G) \leq \lceil \frac{k-1}{\chi-1} \rceil + 1 \). We colour the vertices of \( V_1 \) with color \( c_i \) \( i = 1, 2, \ldots, \chi \), and \( \chi - 1 \) -colour the vertices of \( V_1 \) with color \( c_\chi - 1 \). Note that \( k \leq (\chi - 1) \left( \frac{k-1}{\chi-1} \right) + \chi - 1 \). And the vertices of \( V_1 \) are not adjacent to the vertices of \( V_j \) \( 1 \leq j \leq k \), \( j \in \{ i - \frac{k-1}{\chi-1}, i - \frac{k-1}{\chi-1} + 1, \ldots, i + \frac{k-1}{\chi-1} \} \) by the assumption \( \lambda_k(G) \geq \lceil \frac{k-1}{\chi-1} \rceil + 1 \). Thus, the vertices of coloring \( c_i \) \( i = 1, 2, \ldots, \chi \) are not adjacent. This implies a proper \( (\chi - 1) \)-coloring of \( G \), a contradiction. Therefore \( \lambda_k(G) = \lceil \frac{k-1}{\chi-1} \rceil \). \( \square \)

By Theorem 3, \( \lambda_k(G) \geq \lambda_k(G) \) holds for \( k' \geq k \). And for some integer \( k \), if \( \lambda_k(G) = m \), then \( \frac{k-1}{m+1} + 1 < \chi(G) \leq \frac{k-1}{m} + 1 \). In particular, if \( k \) is the minimum number with \( \lambda_k(G) = m \), then \( \chi = \frac{k-1}{m} + 1 \). This is line with the following Theorem.

Theorem 4 [5]. \( \text{sp}_T(G) = \chi(G) - 1 \) for \( T = \{0, 1, \ldots, m - 1\} \).

3. \( \lambda_n^{nh}(G) \) of graphs

In this section we consider no-hole anti-\( k \)-labeling for \( k = n \).

Observation 5. If \( G' \) is a spanning subgraph of \( G \), then \( \lambda_n^{nh}(G') \geq \lambda_n^{nh}(G) \).

Proof. Suppose \( \lambda_n^{nh}(G) = l \) with an optimal labeling \( \psi \). Let \( w_\psi(G') = w_\psi'(e) \). Then \( \lambda_n^{nh}(G') \geq w_\psi'(G') = w_\psi'(e) \geq w_\psi(G) = l \) by the definitions. Therefore, \( \lambda_n^{nh}(G') \geq l \). \( \square \)

Observation 6. For a graph \( G \) with \( n \) vertices, \( \lambda_n(G) \geq \lambda_n^{nh}(G) \) for all \( n \geq 2 \).
Proof. It is obvious that $\lambda_n(G) \geq \lambda_n^{nh}(G)$. □

We denote by $\delta$ and $\Delta$ the minimum degree and maximum degree of a graph $G$. We have the following.

Observation 7. For a connected graph $G$ with $n$ vertices, $\lambda_n^{nh}(G) \geq 1$ and $\lambda_n^{nh}(G) \leq \min\{n - \Delta, \lfloor \frac{n-1}{2} \rfloor, \lfloor \frac{n-\Delta+1}{2} \rfloor\}$ hold for all $n \geq 2$.

Proof. For each no-hole anti-$n$-labeling $\psi$, $w^{nh}_\psi(G) \geq 1$. Thus, $\lambda_n^{nh}(G) \geq 1$.

Note that the vertex with the maximum degree has $\Delta$ neighbors which have distinct labels for any no-hole anti-$n$-labeling. Then $\lambda_n^{nh}(G) \leq n - \Delta$.

Let $v$ be the vertex having label $\lceil \frac{n}{2} \rceil$ for an optimal no-hole anti-$n$-labeling $\psi$ of $G$, then there is an edge $e$ incident to $v$ so that $w^{nh}_\psi(e) \leq \lfloor \frac{n-\Delta+1}{2} \rfloor$ since there are at least $\delta$ vertices adjacent to $v$ in $G$. Therefore $\lambda_n^{nh}(G) \leq \lfloor \frac{n-\Delta+1}{2} \rfloor$.

It is clear that $\lambda_n^{nh}(G) \leq \lambda_n(G) = \lfloor \frac{n-1}{2} \rfloor$ by Observation 6 and Theorem 3. Thus, the claim holds. □

Theorem 8 [8]. For a graph $G$, $\lambda_n^{nh}(G) \geq n$ if and only if $G$ has no edges.

Let $G$ be a simple graph. The complement graph $G'$ of $G$ is the simple graph with vertex set $V(G)$, two vertices being adjacent in $G'$ if and only if they are not adjacent in $G$. An $m$-path with $m' > m$ vertices is a sequence of $m'$ distinct vertices of $G$, $v_1, v_2, \ldots, v_{m'}$, where $v_i, v_{i+1}, \ldots, v_{i+m'}$ form a clique $(i = 1, 2, \ldots, m' - m)$. An $m$-path with $m' \leq m$ vertices is simply a clique of order $m'$. A Hamilton $m$-path of $G$ is an $m$-path containing all vertices of $G$.

Theorem 9 [8]. For a graph $G$, $\lambda_n^{nh}(G) \geq m + 1$ if and only if there exists a Hamilton $m$-path for $G'$.

By Theorem 9, one can see that the no-hole anti-$n$-labeling number implies some structural properties of graphs.

Corollary 10. For a graph $G$, $\lambda_n^{nh}(G) \geq 2$ if and only if there exists a Hamilton path for the complement graph $G'$ of $G$.

Proof. This is an immediate consequence of $m = 1$ in Theorem 9. □

Corollary 11. For a non-empty graph $G$ (i.e., $G$ has at least an edge), $\lambda_n^{nh}(G) \leq \alpha(G)$, where $\alpha(G)$ is the independence number of $G$.

Proof. Suppose $\lambda_n^{nh}(G) = m$. Then $G'$ contains a Hamilton $(m - 1)$-path by Theorem 9, and $m < n$ by Theorem 8, since $G$ has at least an edge. And so $G'$ contains a clique of order $m$. That is, $G$ has an independent set of order $m$. Hence, $\alpha(G) \geq m$. □

Next, we consider the no-hole anti-$n$-labeling number of some special graphs.

3.1. $\lambda_n^{nh}(G)$ of complete multipartite graphs

Theorem 12 [8]. If $G$ contains a complete $t$-partite subgraph $H$ and $|V(G)| - |V(H)| < (t - 1)(m - 1)$, then $\lambda_n^{nh}(G) < m$.

Corollary 13. Let $K_{n_1, \ldots, n_t}$ be a complete $t$-partite graph with $n$ vertices. Then $\lambda_n^{nh}(K_{n_1, \ldots, n_t}) = 1$ holds for all $n \geq 2$.

Proof. It is clear according to Observation 7 and $m = 2$ of Theorem 12. □

We next consider an example for graph operations. Suppose $G_1$ and $G_2$ are two graphs with disjoint vertex sets. The join $G$ of $G_1$ and $G_2$, denoted by $G = G_1 + G_2$, is the graph obtained from $G_1 \cup G_2$ by adding all edges between vertices in $V(G_1)$ and vertices in $V(G_2)$.

Corollary 14. If $G = G_1 + G_2$, then $\lambda_n^{nh}(G) = 1$.

Proof. If $G = G_1 + G_2$, then $G'$ is a spanning subgraph of $G$, where $G'$ is a complete bipartite graph with bipartition $(V(G_1), V(G_2))$. Hence $\lambda_n^{nh}(G) = 1$ by Observation 5 and Corollary 13. □

3.2. $\lambda_n^{nh}(G)$ of trees

Theorem 15. Let $P_n$ be a path on $n$ vertices. Then $\lambda_n^{nh}(P_n) = \lceil \frac{n}{2} \rceil$.

Proof. Since $\delta(P_n) = 1$, $\lambda_n^{nh}(P_n) \leq \lceil \frac{n}{2} \rceil$ according to Observation 7.

Let $v_1, v_2, \ldots, v_n$ be vertices of $P_n$ such that $v_i$ is adjacent to $v_{i+1}$, $1 \leq i \leq n - 1$. Now we show that $\lambda_n^{nh}(P_n) \geq \lceil \frac{n}{2} \rceil$. It suffices to show that there is a no-hole anti-$n$-labeling $\psi$ such that $w^{nh}_\psi(P_n) = \lceil \frac{n}{2} \rceil$ for $P_n$. Consider the following labeling:

(i) If $n$ is even, then we define

$$
\psi(v_i) = \begin{cases} 
\frac{2 - i}{2} & \text{if } i \text{ is odd}, \\
\frac{n + 1 - i}{2} & \text{if } i \text{ is even}.
\end{cases}
$$
We denote by $T_n$ a tree with $n$ vertices. Note that a tree is a bipartite graph. A **leaf** in a tree is a vertex of degree 1.

**Lemma 16.** For a tree $T_n$ with bipartition $(X_1, X_2)$, and $|X_1| < |X_2|$, we have $X_2$ contains a leaf of $T_n$.

**Proof.** Given a vertex $v$ of degree 1, we label $v$ with 0. For every vertex $u$ of degree 2, label the vertices adjacent to $u$ with 1 and 2. If $v$ is a leaf, then its label is 0. Otherwise, at least one of the neighbors of $v$ is a leaf and thus its label is 0. This can be continued until all vertices are labeled. This labeling is a hole labeling.

**Theorem 17.** For a tree $T_n$ with bipartition $(X_1, X_2)$, $|X_1| = q_i$. Then we have $\lambda_{n}^{nh}(T_n) = \min(q_1, q_2) + 1$. The labeling of $T_n$ is unique.

**Proof.** We prove the result carefully holds for $n \geq 2$. Without loss of generality, we assume that $q_1 < q_2$ and $q_1 + 1 < q_2$. We show that $\lambda_{n}^{nh}(T_n) \geq q_1$ by giving a no-hole anti-$n$-labeling $\psi_n$ of $T_n$ with $w_{\psi_n}^{nh}(T_n) \geq q_1$. For $v \in X_1$ (resp. $v \in X_2$), we define $\psi_n(v) = 0$ (resp. $\psi_n(v) = 1$) for $v \in X_1$ (resp. $\psi_n(v) = 1$). For $v \in X_2$ (resp. $v \in X_1$), we define $\psi_n(v) = 0$ (resp. $\psi_n(v) = 1$) for $v \in X_2$ (resp. $\psi_n(v) = 1$). Then $\psi_n$ is a no-hole anti-$n$-labeling of $T_n$. This labeling is unique according to the previous result.

We next construct a no-hole anti-$n$-labeling $\psi_n$ of $T_n$ by induction on $n \geq 4$. We assume that $\psi_m$ is a no-hole anti-$k$-labeling of $T_k$ satisfying the requirement for $k < n$. We label $T_n$ based on the labeling $\psi_k$ of $T_k$ as below.

**Case 1.** $q_1 < q_2$.

By **Lemma 16**, there exists a vertex $u \in X_2$ of $T_n$. Let $T_{n-1} = T_n - u$. Clearly, $|X_1(T_{n-1})| = |X_1(T_n)| - 1 = q_1$. By the induction hypothesis, there exists a no-hole anti-$n$-labeling $\psi_{n-1}$ of $T_{n-1}$ such that $w_{\psi_n}^{nh}(T_{n-1}) \geq q_1$ and each vertex of $X_1(T_{n-1})$ (resp. $X_2(T_{n-1})$) is labeled by $i \leq q_1$ (resp. $i \leq q_1$). We obtain the labeling $\psi_n$ by labeling the vertex $u$ by $q_1$ based on $\psi_{n-1}$. It is obvious that $w_{\psi_n}^{nh}(T_n) \geq q_1$ and each vertex of $X_1(T_{n-1}) \times X_2(T_{n-1})$. resp.) is labeled by $i \leq q_1$ (resp. $i \leq q_1$). The labeling $\psi_n$ is unique according to the previous result.

**Case 2.** $q_1 = q_2 = q = 2$.

Clearly, there is a vertex (say $u$) whose neighbors are all leaves except one vertex for any tree $T_n$. Without loss of generality, we assume that $u \in X_2$ and $u$ has $m$ leaves as its neighbors. We consider the graph $T_{n-1}$ obtained from $T_n$ by removing the vertex $u$ and the $m$ neighbors (the $m$ leaves) of $u$. Note $|X_1(T_{n-1})| = |X_1(T_n)| - m = n - m$ and $|X_2(T_{n-1})| = |X_2(T_n)| - 1 = n - m$. By the induction hypothesis, there exists a no-hole anti-$(n-1)$-labeling $\psi_{n-1}$ so that $w_{\psi_n}^{nh}(T_{n-1}) \geq q_1$ and each vertex of $X_1(T_{n-1}) \times X_2(T_{n-1})$. resp.) is labeled by $i \leq q_1$ (resp. $i \leq q_1$). We now label $T_n$ by the following rules (i.e., $\psi_n$): relabel the vertex with label $i > \frac{n}{2} - m$ in $T_{n-1}$ by $i + m$, label the vertex $u$ by $n$, and label the $m$ neighbors of $u$ by $\frac{n}{2} - m + 1, \frac{n}{2} - m + 2, \ldots, \frac{n}{2}$. Clearly, $\psi_n(v) \leq \frac{n}{2}$ $\psi_n(v) > \frac{n}{2}$ for $v \in X_1(T_n) \times X_2(T_n)$, resp.) in the labeling $\psi_n$ of $T_n$.**
Next we show $w_{\psi_n}^{wh}(T_n) \geq \frac{n}{2}$, i.e., $w_{\psi_n}^{wh}(e) \geq \frac{n}{2}$ for all $e = uv \in E(T_n)$ in $\psi_n$. If $e \in E(T_{n-\lambda})$, then $w_{\psi_n}^{wh}(T_n) \geq \frac{n}{2}$ since $w_{\psi_n}^{wh}(T_{n-\lambda}) \geq \frac{n}{2} - m$ by the induction hypothesis and $w_{\psi_n}^{wh}(e) \geq |\psi_n(u) - \psi_n(v)| = |\psi_{n-\lambda}(u) - \psi_{n-\lambda}(v)| + m$. If $e \notin E(T_{n-\lambda})$, then $e$ is incident to $u$. Note that $u$ is labeled by $n$ and its neighboring vertices are labeled by some integer $i \leq \frac{n}{2}$ in $\psi_n$. We have $w_{\psi_n}^{wh}(e) \geq \frac{n}{2}$. Hence, $w_{\psi_n}^{wh}(T_n) \geq q$ (see Fig. 2(3)). □

**Remark 18.** For an arbitrary bipartition $(X_1, X_2)$, $|X_1| = q_1 \leq |X_2| = q_2$, there is a tree $T_n$ such that $\lambda_n^{wh}(T_n) = q_1$. We consider the tree $T_n$ as following: $T_n$ is obtained by joining $q_1 - 1$ new vertices to leaves in the star graph $K_{1,q_2}$. Since $\Delta(T_n) = q_2$, then $\lambda_n^{wh}(T_n) \leq n - q_2 = q_1$ by Observation 7. Therefore, $\lambda_n^{wh}(T_n) = q_1$ by Theorem 17.

We also pose a conjecture below.

**Conjecture 19.** For a tree $T_n$ with bipartition $(X_1, X_2)$, $|X_i| = q_i$, $i = 1, 2$, we have $\lambda_n^{wh}(T_n) = q$, where $q = \min\{q_1, q_2\}$.

### 3.3. $\lambda_n^{wh}(G)$ of 2-Dimensional grids $P_m \times P_n$

In this subsection, we generalize the result on paths to 2-Dimensional grids.

**Observation 20.** Let $G$ be a 2-Dimensional grid $P_m \times P_n (m \leq n)$. Then $\lambda_n^{wh}(G) \geq \lceil \frac{m-n+1}{2} \rceil$.

**Proof.** Let $P_m \times P_n$ (i.e., $m$ rows and $n$ columns) as a chessboard. Like in the chessboard, we have white and black alternating squares (see Fig. 3).

(i) If at least one of $m$ and $n$ is even (i.e., $mn$ is even), we have in the “white” squares the labels from the range $[1, \frac{mn}{2}]$ and in the “black” squares the labels from the range $[\frac{mn}{2} + 1, mn]$. Without loss of generality, we assume that the left upper square is white. Take the following labeling $\psi$: put 1 in the left upper corner (put $\frac{mn}{2} + 1$ in the second square in the first row of grid, resp.) and subsequently put in the white (black, resp.) squares from left to right and row by row the upper range labels: $2,3, \ldots, \frac{mn}{2}, \frac{mn}{2} + 2, \frac{mn}{2} + 3, \ldots, mn$, resp.). Let $v$ be labelled by $i$, $i \leq \frac{mn}{2}$ ($i > \frac{mn}{2}$, resp.). Then the vertices adjacent to $v$ are labelled by $i + \frac{mn}{2}, i + \frac{mn}{2} - 1, i + \frac{mn}{2} - 2, \ldots, i + \frac{mn}{2} - (m - 1)$, $i + \frac{mn}{2} - (m - 1) - 1, i + \frac{mn}{2} - (m - 1) - 2, \ldots, i$, $i + \frac{mn}{2} - (m - 1), i + \frac{mn}{2} - (m - 1) + 1, \ldots, i + \frac{mn}{2} - 1$ (in the first square in the second row of grid, resp.) and subsequently put in the white (black, resp.) squares from left to right and row by row the upper range labels: $2,3, \ldots, \frac{mn}{2}, \frac{mn}{2} + 2, \frac{mn}{2} + 3, \ldots, mn$, resp.). Hence, $\lambda_n^{wh}(G) \geq \lceil \frac{mn}{2} \rceil$ (see Fig. 3(1)).

(ii) If $m$ and $n$ are odd (i.e., $mn$ is odd), we have in the “white” squares the labels from the range $[1, \frac{mn}{2}]$ and in the “black” squares the labels from the range $[\frac{mn}{2} + 1, mn]$. Take the following labeling $\psi$: put 1 in the left upper corner (put $\frac{mn}{2} + 1$ in the second square in the first row of grid, resp.) and subsequently put in the white (black, resp.) squares from left to right and row by row the upper range labels: $2,3, \ldots, \frac{mn}{2}, \frac{mn}{2} + 2, \frac{mn}{2} + 3, \ldots, mn$, resp.). We have $\lambda_n^{wh}(G) \geq \frac{mn}{2} - m$ by the argument of (i) (see Fig. 3(2)). □

**Conjecture 21.** Let $G$ be a 2-Dimensional grid $P_m \times P_n$. Then $\lambda_n^{wh}(G) = \lceil \frac{mn}{2} \rceil$, where $m = \min\{m, n\}$.

### 3.4. $\lambda_n^{wh}(G)$ of $n$-cubes

**Theorem 22.** For a cycle $C_n$ of length $n$, $\lambda_n^{wh}(C_n) = \lceil \frac{n+1}{2} \rceil$ (Fig. 4).

**Proof.** Since $\delta(C_n) = 2$, $\lambda_n^{wh}(C_n) \leq \lceil \frac{n+1}{2} \rceil$ according to Observation 7.

Now we show that $\lambda_n^{wh}(C_n) \geq \lceil \frac{n+1}{2} \rceil$. It suffices to show that there is a labeling $\psi$ such that $w_{\psi}^{wh}(C_n) = \lceil \frac{n+1}{2} \rceil$. Let $C_n$ be $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n \rightarrow v_1$. Consider the following labeling:

(i) If $n$ is odd, then we define

$$\psi(v_i) = \begin{cases} \frac{n+1}{2} - \frac{i}{2} & i \text{ is odd,} \\ n + 1 - \frac{i}{2} & i \text{ is even.} \end{cases}$$

(ii) If $n$ is even, then we define

$$\psi(v_i) = \begin{cases} \frac{n+1}{2} - \frac{i}{2} & i \text{ is odd,} \\ n - \frac{i}{2} & i \text{ is even.} \end{cases}$$
Proof. Theorem 22. Let \( Q_n \) be an \( n \)-cube. Then, for all \( n \geq 2 \), \( \lambda_{2n}^{nh}(Q_n) \geq 2^{n-2} \).

\[ \psi_n(t_i) = \begin{cases} 1 & \text{if } i = 1, \\ n & \text{if } i = 3, \\ \frac{n-1}{2} & \text{if } i \text{ is odd and } i \neq 1, 3, \\ \frac{n}{2} - 1 + \frac{1}{2} & \text{if } i \text{ is even.} \end{cases} \]

It is easy to show that \( w_{\psi_n}^{nh}(e), e \in E(C_n) \), defined above is \( \frac{n}{2} \) or \( \frac{n}{2} - 1 \) for even \( n \), and \( \frac{n-1}{2} \) or \( \frac{n+1}{2} \) for odd \( n \). Hence \( \lambda_{2n}^{nh}(Q_n) \geq \left\lfloor \frac{n-1}{2} \right\rfloor \). □

An \( n \)-cube can be defined inductively as follows. An 1-cube is a \( P_2 \). An \( n \)-cube \( Q_n \) may be constructed from the disjoint union of two \((n-1)\)-cubes \( Q_{n-1} \), by adding an edge from each vertex in one copy of \( Q_{n-1} \) to the corresponding vertex in the other copy. The joining edges form a perfect matching.

**Theorem 23.** Let \( Q_n \) be an \( n \)-cube. Then, for all \( n \geq 2 \), \( \lambda_{2n}^{nh}(Q_n) \geq 2^{n-2} \).

**Proof.** We show \( \lambda_{2n}^{nh}(Q_n) \geq 2^{n-2} \) by constructing a no-hole anti-2\(^n\)-labeling \( \psi_n \) such that \( w_{\psi_n}^{nh}(Q_n) \geq 2^{n-2} \), and one end has label at most \( 2^{n-1} \) and the other end has label greater \( 2^{n-1} \) for each edge in \( Q_n \). If \( n = 2 \), then \( Q_2 = C_4 \). By Theorem 22, \( \lambda_{2n}^{nh}(Q_2) = 1 \geq 2^{2-2} \). Let \( \psi_2 \) be the optimal no-hole anti-2\(^2\)-labeling defined in Theorem 22 of \( Q_2 \). Clearly, for each edge \( e \) of \( Q_2 \), one end of \( e \) has label at most \( 2^{2-1} = 2 \) and the other end of \( e \) has label greater 2 under \( \psi_2 \), see Fig. 5(1). For \( m \leq n \), we assume there exists a labeling \( \psi_m \) such that \( w_{\psi_m}^{nh}(Q_m) \geq 2^{m-2} \), and one end has label at most \( 2^{m-1} \) and the other end has label greater \( 2^{m-1} \) for each edge in \( Q_m \). We next construct the labeling \( \psi_{n+1} \) satisfying the assumption above for \( Q_{n+1} \) from the labeling \( \psi_n \) defined above of \( Q_n \) as follows.
Note that an \((n + 1)\)-cube \(Q_{n+1}\) can be obtained by adding a perfect matching \(PM\) between two copies of an \(n\)-cube, denoted by \(Q_n^1\) and \(Q_n^2\) (Each edge of \(PM\) joins two vertices having the same labels). We relabel the vertices with label \(i > 2^n-1\) in \(Q_n^1\) by \(i + 2^n-1\), and we relabel the vertices with label \(i \leq 2^n-1\) in \(Q_n^2\) by \(i + 2^n + 2^{n-1}\).

We next show that the assumption above holds for \(\psi_{n+1}\) in \(Q_{n+1}\). Let \(e = uv\) be an edge of \(E(Q_{n+1})\). We firstly assume \(e \in E(Q_n^1)\) and \(\psi_n(u) > \psi_n(v)\). By the induction hypothesis, we have \(\psi_n(u) > 2^n-1\), \(\psi_n(v) \leq 2^n-1\) and \(\psi_n(u) - \psi_n(v) \geq 2^{n-2}\). Therefore \(\psi_{n+1}(u) = \psi_n(u) + 2^n-1 \geq 2^n\), \(\psi_{n+1}(v) = \psi_n(v) \leq 2^n-1 + 2^n\), and \(\psi_{n+1}(u) - \psi_{n+1}(v) = \psi_n(u) - \psi_n(v) \geq 2^n-1 + 2^n - 2^{n-2} = 2^n-1\). Finally, we assume \(e \in E(PM)\). Without loss of generality, we assume \(u \in V(Q_n^1)\) and \(v \in V(Q_n^2)\). Then \(\psi_n(u) = \psi_n(v)\). If \(\psi_n(u) \leq 2^n-1\), then \(\psi_{n+1}(u) = \psi_n(u) \geq 2^n\), \(\psi_{n+1}(v) = \psi_n(v) + 2^n-1 > 2^n\), and \(\psi_{n+1}(u) - \psi_{n+1}(v) = \psi_n(u) + 2^n - 2^n - 2^{n-2} = 2^n-1\). We finally complete the proof.}

\[\square\]

**Theorem 24.** Let \(Q_3\) be a 3-cube. Then \(\lambda_{bh}^2(Q_3) = 2\).

**Proof.** We have \(\lambda_{bh}^2(Q_3) \geq 2\) by Theorem 23. We next show \(\lambda_{bh}^2(Q_3) \leq 2\) by contradiction. Suppose \(\lambda_{bh}^2(Q_3) \geq 3\). Let \(\psi\) be an optimal labeling and we denote by \(v_i\) the vertex with label \(i\) under \(\psi\). Then \(v_4\) may only be adjacent to vertices \(v_1, v_7, v_8\). \(v_5\) may only be adjacent to vertices \(v_1, v_2, v_8\), \(v_6\) may only be adjacent to vertices \(v_1, v_2, v_3\) in \(Q_3\) due to \(m_{bh}^2(Q_3) \geq 3\). Note that \(Q_3\) is a bipartite graph. Let the bipartition of \(Q_3\) be \(\{X, Y\}\), and \(|X| = |Y| = 2 = 2^3 - 4 = 4\). Without loss of generality, we assume \(v_4 \in X\). Then \(\psi(v_1) \geq \psi(v_2) \geq \psi(v_6) \geq \psi(v_5) \geq \psi(v_8) \in Y\). Hence \(v_1, v_2, v_3, v_7, v_8 \in Y\), that is, \(|Y| = 5\), a contradiction. 

Note that the bound in Theorem 23 is sharp for \(n = 2, 3\). \(\lambda_{bh}^2(Q_n) \leq \lceil \frac{2^{n-1}}{n-1} \rceil \) holds by Observation 7. We pose the following problem.

**Conjecture 25.** For all \(n \geq 2\), \(\lambda_{bh}^2(Q_n) = 2^{n-2}\).

4. **Anti-\(L_d(2, 1)\)-labeling of graphs**

Given a simple graph \(G = (V, E)\) and a positive number \(d\), an \(L_d(2, 1)\)-labeling of \(G\) is a function \(f : V(G) \to [0, \infty)\) such that whenever \(x, y \in V\) are adjacent, \(|f(x) - f(y)| \geq 2d\), and whenever the distance between \(x\) and \(y\) is two, if \(|f(x) - f(y)| \geq d\). The \(L_d(2, 1)\)-labeling number \(\lambda_d(G, d)\) is the smallest number \(m\) such that \(G\) has an \(L_d(2, 1)\)-labeling \(f\) with \(\max\{f(v) : v \in V\} = m\). When \(d = 1\), the \(L_d(2, 1)\)-labeling problem is the \(L(2, 1)\)-labeling problem. The \(L(2, 1)\)-labeling problem of graphs has been discussed for many graph families, see [2, 4, 7, 9, 10].

Similarly, we define the anti-\(L_d(2, 1)\)-labeling problem: given a simple graph \(G = (V, E)\) and a positive number \(d\), a labeling of \(G\) is a function \(f : V(G) \to [1, k]\) such that \(|f(x) - f(y)| \geq 2d\) if \(xy \in E(G)\), \(|f(x) - f(y)| \geq d\) if \(d(x, y) = 2\). The anti-\(L_d(2, 1)\)-labeling number of \(G\), denoted by \(\lambda_d^\perp(G)\), is the largest number \(2d\).

By the proofs of Observations 1 and 2, we have the results of Observations 26 and 27 as following.

**Observation 26.** If \(H\) is a subgraph of \(G\), then \(\lambda_d^\perp(H) \geq \lambda_d^\perp(G)\).

**Observation 27.** If \(G = G_1 \cup G_2\), then \(\lambda_d^\perp(G) = \min\{\lambda_d^\perp(G_1), \lambda_d^\perp(G_2)\}\).

**Lemma 28 [7].** \(\lambda_d(G, d) = d \cdot \lambda_d(G, 1)\) for a non-negative integer \(d\).

**Lemma 29 [6].** \(\lambda_d(G, 1) \leq \Delta^2 + \Delta - 2\).

**Theorem 30.** Let \(G\) is a simple graph. Then \(\lambda_d^\perp(G) \geq 2\lceil \frac{k-1}{2d+\Delta-2} \rceil \).

**Proof.** Suppose that \(\lambda(G, d) = m\) for a graph \(G\). Then there exists a labeling \(f : V(G) \to [0, m]\) such that whenever \(x, y \in V\) are adjacent, \(|f(x) - f(y)| \geq 2d\). Therefore, there exists a labeling \(\psi : V(G) \to [1, m+1]\) such that \(w_\psi(G) = 2d\) for \(k = m+1 = \lambda_d(G, d) = 1\). According to Lemma 28, there exists a labeling \(\psi\), such that \(w_\psi(G) = 2\lceil \frac{k-1}{2d+\Delta-1} \rceil\) for all \(k\). Therefore \(\lambda_d^\perp(G) \geq 2\lceil \frac{k-1}{2d+\Delta-1} \rceil\) for all \(k\) according to the definition of the anti-\(L_d(2, 1)\)-labeling number \(\lambda_d^\perp(G)\). Combining with Lemma 29, we have \(\lambda_d^\perp(G) \geq 2\lceil \frac{k-1}{\Delta^2+\Delta-2} \rceil \).

**Theorem 31.** If \(\lambda_d^\perp(G) = 2d\) for a positive number \(k\), then \(\frac{k-1}{2d+\Delta-1} \leq \lambda_d(G, 1) \leq \frac{k-1}{d-1}\).

**Proof.** Suppose \(\lambda_d^\perp(G) = 2d\) for a graph \(G\). Then \(\lambda(G, d) + 1 \leq k < \lambda_d(G, d+1) + 1\). In fact, it is obvious that \(\lambda(G, d) + 1 \leq k\), since \(G\) has an \(L_d(2, 1)\)-labeling. Then \(\lambda(G, d+1) + 1\) is the smallest number \(m\) such that \(G\) has an \(L_d(2, 1)\)-labeling \(f\). Suppose \(k \geq \lambda_d(G, d+1) + 1\). Then there exists a labeling \(\psi\), such that \(w_\psi(G) = 2(d+1)\). Hence \(\lambda_d^\perp(G) \geq 2(d+1)\) according to the definition of \(\lambda_d^\perp(G)\). Combining with Lemma 28, that is \(\frac{k-1}{2d+\Delta-1} \leq \lambda_d(G, 1) \leq \frac{k-1}{d-1}\). \(\square\)
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References