# The Telephone Coordination Game Revisited: From Random to Deterministic Algorithms

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**Abstract**—Two players wishing to communicate are placed each in a room with $N$ telephones connecting the two rooms. The players do not know how the telephones are interconnected. In each round, each player picks up a phone and says "hello" until when they hear each other. The problem is to devise an algorithm minimising the delay to establish communication. The above problem, called the *Telephone Coordination Game*, also termed as the *Telephone Problem*, is of fundamental importance in distributed algorithm design. In this paper, we investigate a generalised version where among $N$ telephones, only a subset can establish communication between the two players. We are interested in *devising the deterministic strategy achieving **bounded** rendezvous delay and minimising the worst-case rendezvous delay*. Specifically, we first establish the lower-bound of worst-case rendezvous delay. We then characterise the structure of the phone pick sequences that can guarantee rendezvous without any prior coordination. Assuming each player has a globally unique ID, we further devise a deterministic strategy that (1) guarantees rendezvous between the players regardless of their telephone labeling functions and their relative time difference and (2) approaches the performance bound within a constant factor proportional to the ID length.

**Index Terms**—Telephone coordination game, rendezvous search game, distributed algorithm design

## 1 Introduction

### 1.1 The Telephone Problem

The *Telephone Coordination Game*, also referred to as the *Telephone Problem*, was first formally introduced by Alpern in 1976 [1] as follows: Two players wishing to communicate with each other are placed each in a distinct room with $N$ telephone lines connecting the two rooms. The players do not know how the telephones are interconnected. The game is played in discrete steps termed as rounds. In each round $t = 0, 1, 2, \ldots$, each player picks up a phone and says "hello" until when they hear each other. The common aim of the two players is to minimise the time until they can hear each other. The major difficulty that makes the telephone problem non-trivial is the *lack of any form of coordination* between the two players, as summarised in the following:

- *No common phone labeling*. The $N$ telephones are identical and not labeled or ordered in any way. In other words, if each player puts a label on each phone for the purpose of identification, by no means they can have any common labeling of the phones.
- *No time synchronisation*. Each player is unaware of the moment when the other starts the game. That is, there does not exist any external signal coordinating the searching process such as starting or stopping signals.
- *No pre-assigned roles*. The players do not have pre-assigned roles as a caller or a callee because such role assignment requires some form of coordination. Even if such coordination is possible, a player may wish to be a caller to initiate a communication session and a callee at the same time to accept incoming communication sessions from other players. In this case, it is impossible to assign a role to each player.

The Telephone Problem reflects a typical paradigm of distributed algorithm design without any prior coordination among agents. Despite (or thanks to) its simple and generic formulation, a number of engineering problems can be cast into the Telephone Problem, ranging from communication link establishment, meeting scheduling to web-crawling. Taking the first one as an example, two communication nodes wishing to communicate in a multi-channel wireless network need to meet each other on a common channel via a "rendezvous" process [2].

Mathematically, the Telephone Problem belongs to the field of *Rendezvous Search Games* [3], [4]. Due to its application in many engineering problems, the Telephone Problem and its variants have attracted extensive research attention (cf. Section 2 for a brief overview of related work and the comparison to our work), but the original Telephone Problem still remains open, even though it is simple to state and intuitively understandable. Until today, very few results have been reported on the structure of the optimal strategy, among which the most important progress toward characterising the structure of the optimal probabilistic strategy was made by Anderson and Weber [5].

### 1.2 Anderson and Weber Strategy

Anderson and Weber studied the Telephone Problem using another formulation as a symmetrical rendezvous search

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*Manuscript received 24 Apr. 2014; revised 29 Dec. 2014; accepted 5 Jan. 2015. Date of publication 8 Jan. 2015; date of current version 10 Sept. 2015. Recommended for acceptance by J. Cao.*
*For information on obtaining reprints of this article, please send e-mail to: reprints@ieee.org, and reference the Digital Object Identifier below.*
*Digital Object Identifier no. 10.1109/TC.2015.2389799*

game on $N$ locations between two players [5]. Each player can switch across different locations freely from one to any other location and the delay required to pass from one location to another is negligible. The objective is to find the optimal strategy of visiting a location each round to minimise the expected rendezvous delay. By regarding locations as telephones, the rendezvous search game considered in [5] can be cast into the Telephone Problem.

Aiming at minimising the expect rendezvous delay, Anderson and Weber developed a probabilistic strategy, referred to as *Anderson and Weber strategy*, in which each player waits where it is for $N-1$ periods with probability $\theta$ and switch to a random permutation of the remaining $N-1$ locations with probability $1-\theta$. The strategy is repeated each $N-1$ periods until rendezvous. By calculating the value of $\theta$, they have demonstrated the optimality of the proposed strategy in minimising the expected rendezvous delay for the cases $N=2$ and 3. However, for other values of $N$, even small values, the Telephone Problem remains unsolved.

### 1.3 Our Main Results

In this paper, we revisit the classical Telephone Problem in its generic form. *Specifically, we investigate a generalised version of the Telephone Problem by studying the situation where among the N telephone lines, only a subset of them, unknown to the players, can establish the communication between the two players.* Others are connected to telephones in other rooms than the rooms of the players. This generalisation of the original Telephone Problem, termed as *Generalised Telephone Problem*, can capture a number of engineering and system constraints in practical settings, e.g., in the channel establishment problem, a channel may be temporally occupied by other users and thus cannot be accessed; in the problem of robot rendezvous, some places may be inaccessible for some robots due to resource constraint or security reasons. Compared to the original Telephone Problem, the Generalised Telephone Problem we consider has one more difficulty, recaptured as follows:

- *Partial telephone connection*. Only a subset of the $N$ telephones can establish the communication between the two players. The players do not know the pairwise connectivity of any telephone in their rooms.

In the Generalised Telephone Problem, we are interested in finding an optimal *deterministic* strategy that is able to achieve *bounded* rendezvous delay and that minimises the *worst-case* rendezvous delay. Our work in this paper consists of a complementary research thrust compared to that pioneered by Anderson and Weber in [5] seeking *probabilistic* strategies minimising the *expected* meeting delay. Our focus on the deterministic strategy is motivated by the long-tail effect of the probabilistic strategies in which two players may experience extremely long and unbounded delay before they can meet each other and consequently the need of rendezvous strategies that can satisfy engineering applications requiring bounded rendezvous delay.

Particularly, we investigate the following natural while fundamental questions:

- What is the structure of a deterministic strategy that can achieve bounded rendezvous delay?
- What is the worst-case rendezvous delay bound for any deterministic strategy?
- How to design a deterministic strategy that approaches the worst-case rendezvous delay bound?

By the analysis presented in this paper, we give the answers to the above questions.

- *D-bounded phone pick sequence*. We characterise the structure of the phone pick sequences of the deterministic strategies (termed as $D$-bounded phone pick sequences) that can guarantee rendezvous without any coordination;
- *Worst-case rendezvous delay bound*. We prove that the lower-bound of the worst-case rendezvous delay of any deterministic strategy is $O(N^2)$;
- *Zero-knowledge Rendezvous*. Assuming each player has a globally unique ID, we devise a deterministic strategy, called Zero-knowledge Rendezvous, that (1) guarantees rendezvous between the players regardless of their telephone labeling functions and their relative time difference and (2) approaches the performance bound within a constant factor proportional to the ID length.

Our results and the method employed in the analysis form a theoretical basis of devising rendezvous strategies in a variety of engineering applications requiring bounded rendezvous delay. The rest of the paper is organised as follows. Section 2 gives a brief overview of related work and compares our results with existing results. Section 3 formulates the Generalised Telephone Problem and the deterministic rendezvous strategy. Sections 4 and 6 establish the theoretic performance bound for deterministic and probabilistic rendezvous strategies, respectively. Section 5 presents the design of an order-optimal deterministic strategy, Zero-knowledge Rendezvous, and mathematically establishes its performance. Section 7 concludes the paper by discussing a number of related searching and rendezvous problems and strategies.

## 2 Related Work

The Telephone Problem and the related discovery and rendezvous problems belong to the field of Rendezvous Search Games, which is extensively surveyed in [3]. In the following, we briefly summarize the related work.

*Original telephone problem*. Despite significant research efforts, the original Telephone Problem still remains open today, even though it is simple to state and intuitively understandable. The optimal probabilistic strategy minimising the expected rendezvous delay has only been derived for the cases $N=2$ and 3 [5], [6]. For other values of $N$ (even small values), characterising the optimal probabilistic strategy remains unsolved. Our work in this paper consists of a complementary research strand to the existing work on probabilistic strategies pioneered by Anderson and Weber, by establishing theoretical bounds for deterministic rendezvous strategies and developing an order-optimal deterministic rendezvous strategy that approaches the performance bound.

*Rendezvous games on graphs*. Recently, motivated by the rendezvous problems between robots, the rendezvous games on graphs and their different variants have attracted significant research attention, both from probabilistic and deterministic perspectives (cf. [7], [8], [9], [10] and the references therein). Particularly, concerning the deterministic strategies which are more related to our work, although a number of solutions have been proposed to achieve rendezvous on graphs, the majority of them are focused on specific graphs and develop strategies with polynomial complexity, leaving the optimality of the propositions unaddressed. Motivated by this observation, we focus on establishing the theoretical performance bound for any deterministic strategy and devising strategies that can approach the performance bound. The problem we address in this paper can be regarded as a specific version of rendezvous game on graphs where players can switch freely from one vertex to any other vertex.

*Channel rendezvous problem.* Our work is also related and applicable to the channel rendezvous problem in multichannel networks, in which a number of distributed channel rendezvous solutions have been proposed in the literature recently [2], [11], [12], [13], [14], [15], [16]. However, none of them addressed all the three challenges due to the lack of coordination among players, i.e., common telephone labeling function, synchronisation, and preassigned player roles. We would like to point out that it is the holistic combination of the three challenges that makes the design of rendezvous strategies far from trivial. Particularly, most, if not all, of the existing work implicitly assumes identical channel labeling for the rendezvous pair. However, establishing reliable channel labeling requires coordination between the rendezvous pair, which cannot be satisfied before the rendezvous is achieved. Some rendezvous schemes use the quality of a channel as its label. This approach is not reliable, either because the perceived quality of the same channel may vary significantly at different nodes due to their locations. Moreover, due to the application-oriented approach, none of existing work has a complete study from the theoretical perspective as that in this paper on the channel rendezvous problem which can be cast into the Telephone Problem.

## 3 Problem Formulation

In this section we formulate the Generalised Telephone Problem and the deterministic rendezvous strategy.

### 3.1 The Generalised Telephone Problem

Consider two players, Alice ($a$) and Bob ($b$), who wish to communicate with each other, each placed in a distinct room. In the room of Alice (Bob), there are $N_a$ ($N_b$) telephones among which $N_c \leq \min\{N_a, N_b\}$ telephones can be used to connect the two rooms. Other telephones may be connected to those in places other than the rooms of Alice and Bob. In our analysis, we focus on the extreme case by setting $N_c = 1$, i.e., there is only one telephone connecting the two rooms. Our focus on the extreme case allows us to concentrate on the essential properties of the problem and the resulting strategy. The two players do not know how the telephones are interconnected.

Time is divided into rounds. In each round $t = 0, 1, 2, \ldots$, each player picks up a phone and says "hello" until when they hear each other, termed as a pairwise *rendezvous*. The common aim of the two players is to minimise the worst-case delay until they can hear each other. As in the original Telephone Problem specified in the beginning of the paper, no prior coordination is possible between Alice and Bob, meaning that they do not have (1) common phone labeling, (2) time synchronisation and (3) pre-assigned roles.

### 3.2 Telephone Labeling Function

In this section, we introduce the *telephone labeling function* to formalise the first constraint.

**Definition 1 (Telephone Labeling Function).** *Denote $\mathcal{C}$ the index set of all telephone lines where each index $c \in \mathcal{C}$ denotes the pair of telephones connected by connection $c$. For each player $i$ ($i = a, b$) with a set $\mathcal{N}_i \subseteq \mathcal{C}$ of telephones in its room, we define the channel label function $f_i$ as an bijective mapping*

$$f_i : \mathcal{N}_i \to \{0, 1, \ldots, N_i - 1\},$$

*where $\forall c_1, c_2 \in \mathcal{N}_i$, $c_1 \neq c_2$ implies $f_i(c_1) \neq f_i(c_2)$. We define $f_i^{-1}$: $\{0, 1, \ldots, N_i - 1\} \to \mathcal{N}_i$ as the inverse mapping of $f_i$.*

**Remark 1.** Definition 1 basically states that each player $i$ has its own labeling of the $N_i$ telephones in his room that may differ from the global label set of the $N_i$ telephones $\mathcal{N}_i$. Using Definition 1, we can express formally the constraint that there is only one telephone connecting the two rooms ($|\mathcal{N}_a \bigcap \mathcal{N}_b| = 1$) as follows: There exists a unique telephone line $c^* \in \mathcal{C}$ such that there exist $0 \leq h_a \leq N_a - 1$ and $0 \leq h_b \leq N_b - 1$ such that $f_a(c^*) = h_a$ and $f_b(c^*) = h_b$, or equivalently, $f_a^{-1}(h_a) = f_b^{-1}(h_b) = c^*$.

The following example further illustrates Definition 1.

**Example 1.** Consider a system with $\mathcal{C} = \{c_0, c_1, c_2, c_3\}$. There are two and three telephones in the room of Alice and Bob, respectively, with $\mathcal{N}_a = \{c_0, c_1\}$, $\mathcal{N}_b = \{c_1, c_2, c_3\}$. The telephone labeling functions of Alice and Bob are: $f_a(c_0) = 0$, $f_a(c_1) = 1$ and $f_b(c_1) = 2$, $f_b(c_2) = 1$, $f_b(c_3) = 0$. It can be noted that Alice and Bob can communicate only via telephone line $c_1$. Mathematically, $f_a^{-1}(1) = f_b^{-1}(2) = c_1$.

### 3.3 Phone Pick Sequence

As analysed in the Introduction, any probabilistic strategy cannot achieve bounded rendezvous delay and suffers from the long-tail rendezvous latency problem in which Alice and Bob may experience extremely long delay before they can rendezvous. Motivated by this observation, we consider deterministic rendezvous strategies in which each player picks up a phone each round based on a specific sequence so as to rendezvous with its peer. We call such a sequence the *phone pick sequence* and give its formal definition in the following.

**Definition 2 (Phone Pick Sequence).** *The phone pick sequence of a player is defined a sequence $\mathbf{u} \triangleq \{u_t\}_{0 \leq t \leq T_u - 1}$ where $u_t$ is the index of the telephone picked by the player in round $t$ based on its own labeling, $T_u$ is the period of the sequence.*[1]

1. A probabilistic rendezvous strategy can be regarded as a special case where $T_u \to \infty$.

| Round index | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| **Alice:** | 0 ($c_0$) | 1 ($c_1$) | 0 ($c_0$) | 1 ($c_1$) | 0 ($c_0$) | 1 ($c_1$) | ... |
| **Bob:** | 0 ($c_3$) | 1 ($c_2$) | 2 ($c_1$) | 0 ($c_3$) | 1 ($c_2$) | 2 ($c_1$) | ... |

| Round index | 0 | 1 | 2 | 3 | ... |
|---|---|---|---|---|---|
| **Alice:** | 0 ($c_0$) | 1 ($c_1$) | 0 ($c_0$) | 1 ($c_1$) | ... |
| **Bob:** | 2 ($c_1$) | 1 ($c_2$) | 0 ($c_3$) | 1 ($c_2$) | ... |

Fig. 1. Example of phone pick sequences: left: $\mathbf{u} = \{0,1\}$, $\mathbf{v} = \{0,1,2\}$; right: $\mathbf{u} = \{0,1\}$, $\mathbf{v} = \{2,1,0,1\}$.

Given the phone pick sequences of Alice and Bob denoted as $\mathbf{u}$ and $\mathbf{v}$, whose periods are denoted as $T_a$ and $T_b$, if there exists $t \in [0, T_aT_b - 1]$ and $c \in \mathcal{N}_a \bigcap \mathcal{N}_b$ such that $f_a^{-1}(u_t) = f_b^{-1}(v_t) = c$, Alice and Bob can rendezvous in round $t$ on telephone line $c$. $t$ is called the rendezvous round and $c$ the rendezvous telephone.

**Example 2.** Consider the setting of Example 1 with the following phone pick sequences for Alice and Bob: $\mathbf{u} = \{0,1\}$ with $T_a = 2$ and $\mathbf{v} = \{0,1,2\}$ with $T_b = 3$. It can be noted that they can rendezvous in round 5 on telephone $c_1$. However, if Alice and Bob operate on the following phone pick sequences: $\mathbf{u} = \{0,1\}$ with $T_a = 2$ and $\mathbf{v} = \{2,1,0,1\}$ with $T_b = 4$, they can never rendezvous. The phone pick sequences and the rendezvous process are illustrated in Fig. 1.

To model the situation where the players are not synchronised such that they may start their search in different time instances, we apply the concept of *cyclic rotation* to the phone pick sequences. Specifically, given a phone pick sequence $\mathbf{w}$, we denote $\mathbf{w}(k)$ a cyclic rotation of $\mathbf{w}$ by $k$ rounds and $k$ is referred to as the *cyclic rotation phase*. Consider an example where $\mathbf{u} = \{0,1,2\}$ with $T_u = 3$, we have $\mathbf{u}(2) = \{2,0,1\}$. The situation where $k$ is fractional, corresponding to the case where the rounds of Alice and Bob are not aligned, is analysed in Section 5.3.

### 3.4 $D$-Bounded Phone Pick Sequence

In this section, we define the $D$-bounded rendezvous system consisting of the $D$-bounded phone pick sequences, any pair of which can rendezvous within $D$ rounds regardless of the cyclic rotation phases and the telephone labeling functions of the players.

**Definition 3 ($D$-bounded Rendezvous System).** *A D-bounded rendezvous system is defined as a set of phone pick sequences such that any two distinct sequences* $\mathbf{u}$ *and* $\mathbf{v}$ *satisfy the following property:*

$$\exists 0 \leq t < D \text{ such that } f^{-1}[u_t(t_0)] = f'^{-1}[v_t(t'_0)], \forall f, f' \in \mathcal{F}, t_0, t'_0,$$

*where $\mathcal{F}$ denotes the set of telephone labeling functions.*

**Definition 4 ($D$-bounded Phone Pick Sequence).** *The phone pick sequences in a D-bounded rendezvous system are called D-bounded phone pick sequences.*

Armed with the above definitions and the mathematic notations introduced in this section, we can formalise the Generalised Telephone Problem with deterministic strategy as follows.

*Generalised telephone problem with deterministic strategy.* The Generalised Telephone Problem with deterministic strategy consists of devising $D$-bounded phone pick sequences $\mathbf{u}$ and $\mathbf{v}$ for Alice and Bob to minimise the worst-case rendezvous delay bound $D$. In other words, we seek an algorithm to construct phone pick sequences to achieve bounded and minimal worst-case rendezvous delay among all possible telephone labeling functions and all cyclic rotation phases of the two players in the setting depicted in the beginning of this section.

## 4 Theoretic Performance Bound for Deterministic Strategies

In this section, we establish the worst-case rendezvous delay bound for any deterministic strategy. We also analyse the structure of the phone pick sequence to guarantee rendezvous between Alice and Bob regardless of their telephone labeling functions and cyclic rotation phases. The results derived in this section, as recaptured in the following, serve as design guidelines for the order-optimal deterministic strategy devised later in Section 5 that approaches the performance bound.

- *D-bounded phone pick sequence structure (Lemma 1).* Given two $D$-bounded phone pick sequences $\mathbf{u}$ and $\mathbf{v}$, for any cyclic rotation phases $t_0^a$ and $t_0^b$, the pair $(u_t(t_0^a), v_t(t_0^b))$ $(0 \leq t < D)$ must cover all the possible telephone label couples in $[0, N_a - 1] \times [0, N_b - 1]$;
- *Rendezvous delay bound (Theorem 1).* The worst-case rendezvous delay is lower-bounded by $N_aN_b$, or $O(N^2)$ if $N_a \simeq N_b \simeq N$;
- *Phone pick sequence period (Theorem 2).* To ensure rendezvous regardless of the players' phone labeling functions and cyclic rotation phases, the phone pick sequence period of player $i$ $(i \in \{a, b\})$ cannot be shorter than $N_i^2$.

We start by showing a structural property of the $D$-bounded phone pick sequences.

**Lemma 1.** *If Alice and Bob can rendezvous in D rounds by using the D-bounded phone pick sequences* $\mathbf{u}$ *and* $\mathbf{v}$*, then for any cyclic rotation phases $t_0^a$ and $t_0^b$ and any telephone label pair $(h_a, h_b)$ where $0 \leq h_a \leq N_a - 1$ and $0 \leq h_b \leq N_b - 1$, there exists $t < D$ such that $u_t(t_a^0) = h_a$ and $v_t(t_b^0) = h_b$.*

**Proof.** We prove the lemma by contradiction. Assume that there exits a telephone label pair $(h_a^0, h_b^0)$ such that there does not exist $t < D$ such that $u_t(t_a^0) = h_a^0$ and $v_t(t_b^0) = h_b^0$. Let $c^*$ denote the unique telephone line via which Alice and Bob can communicate with each other. We consider a pair of random telephone label functions $f_a$ and $f_b$ and proceed our proof in the following cases.

- If $f_a(c^*) = h_a^0$ and $f_b(c^*) = h_b^0$, rendezvous cannot be achieved within $D$ rounds following the assumption.

- Otherwise, we construct the following telephone label functions for Alice and Bob, denoted as $f'_a$ and $f'_b$:

$$f'_a(c) = \begin{cases} f_a(c) & c \neq c^*, f_a^{-1}(h_a^0), \\ h_a^0 & c = c^*, \\ f_a(c^*) & c = f_a^{-1}(h_a^0); \end{cases}$$

$$f'_b(c) = \begin{cases} f_b(c) & c \neq c^*, f_b^{-1}(h_b^0), \\ h_b^0 & c = c^*, \\ f_b(c^*) & c = f_b^{-1}(h_b^0). \end{cases}$$

  Recall the assumption that there does not exist $t < D$ such that $u_t(t_a^0) = h_a^0$ and $v_t(t_b^0) = h_b^0$, it holds that Alice and Bob cannot rendezvous in $D$ rounds under the constructed telephone label functions $f'_a$ and $f'_b$.

Consequently, in both cases, we can construct telephone label functions for Alice and Bob under which rendezvous cannot be achieved, which contradicts with the fact that Alice and Bob can rendezvous within $D$ rounds. □

**Remark 2.** Lemma 1 shows that given any cyclic rotation phases $t_0^a$ and $t_0^b$, to ensure rendezvous within $D$ rounds, the pair $(u_t(t_0^a), v_t(t_0^b))$ $(0 \leq t < D)$ must cover all the possible telephone label couples $(h_a, h_b)$ where $0 \leq h_a \leq N_a - 1$ and $0 \leq h_b \leq N_b - 1$. In other words, the two $D$-bounded phone pick sequences of Alice and Bob should cover all couples in $[0, N_a - 1] \times [0, N_b - 1]$.

We now proceed to establish the worst-case rendezvous delay bound for any deterministic strategy by showing that for any pair of $D$-bounded phone pick sequences of Alice and Bob, the worst-case rendezvous delay of any deterministic rendezvous strategy is at least $N_aN_b$.

**Theorem 1 (Worst-Case Rendezvous Delay Lower Bound).** *For any pair of $D$-bounded phone pick sequences, the worst-case rendezvous delay among all possible telephone labeling functions and all cyclic rotation phases cannot be lower than $N_aN_b$, i.e., $D \geq N_aN_b$.*

**Proof.** We prove the theorem by contradiction. Assume that there exist a pair of $D$-bounded phone pick sequences, $\mathbf{u}$ for Alice and $\mathbf{v}$ for Bob, with which the worst-case rendezvous delay is less than $N_aN_b$. It follows from Lemma 1 that for any cyclic rotation phases $t_0^a$ and $t_0^b$ and any telephone label pair $(h_a, h_b) \in [0, N_a - 1] \times [0, N_b - 1]$, there exists $t < D$ such that $u_t(t_a^0) = h_a$ and $v_t(t_b^0) = h_b$. That is, the pair $(u_t(t_0^a), v_t(t_0^b))$ $(0 \leq t < D)$ must cover all the possible telephone label couples $(h_a, h_b)$, which is impossible with $D < N_aN_b$. □

Having established the worst-case rendezvous delay bound of any deterministic rendezvous strategy, we turn to investigate the structure of the $D$-bounded phone pick sequences that can guarantee rendezvous between Alice and Bob regardless of their telephone labeling functions and their cyclic rotation phases. Without loss of generality, we investigate the structure of the phone pick sequence of Alice $\mathbf{u}$ by focusing on its period $T_u$. The following results (Theorem 2 hold symmetrically for Bob, whose phone pick sequence is $\mathbf{v}$ of period $T_v$.

**Theorem 2 (Lower-bound of $T_u$).** *If the rendezvous can be guaranteed between Alice and Bob regardless of their cyclic rotation phases and $N_a$, $N_b$, then it holds that $T_u \geq N_a^2$.*

**Proof.** Assume, by contradiction, that $T_u < N_a^2$, we consider a specific symmetrical setting where $N_b = N_a$, which leads to $T_u = T_v$ where $T_v$ is the period of the phone pick sequence of Bob $\mathbf{v}$.

Let $n_{u,h}$ $(n_{v,g})$ denote the number of rounds in sequence $\mathbf{u}$ $(\mathbf{v})$ in which Alice (Bob) picks telephone $h \in [0, N_a - 1]$ $(g \in [0, N_b - 1])$. Recall that $N_b = N_a$, we can express the period length of $\mathbf{u}$ and $\mathbf{v}$ as follows:

$$T_u = T_v = \sum_{h=0}^{N_a-1} n_{u,h} = \sum_{g=0}^{N_a-1} n_{v,g} = \sum_{h=0}^{N_a-1} \sum_{g=0}^{N_a-1} \frac{n_{u,g} + n_{v,h}}{2N_a}. \tag{1}$$

Moreover, since both players use the phone pick sequences of the same period $T_u$, if they can rendezvous, they must rendezvous within $T_u$ rounds.

We now fix $\mathbf{u}$ and cyclically rotate $\mathbf{v}$ by $l$ rounds where $l = 0, 1, \ldots, T_u - 1$. Recall Lemma 1, for any $l$, there exists at least one round $t$ such that $u_t = g$ and $v_t(l) = h$ for any pair of $(g, h) \in [0, N_a - 1] \times [0, N_a - 1]$. It follows that the total accumulated number of rounds in which $u_t = g$ and $v_t(l) = h$, as $l$ is incremented from $0$ to $T_u - 1$, is at least $T_u$. On the other hand, we can count the total accumulated number of rounds in which $u_t = g$ and $v_t(l) = h$, as $l$ is incremented from $0$ to $T_u - 1$, as $n_{u,g} \cdot n_{v,h}$. Hence we have $n_{u,g} \cdot n_{v,h} \geq T_u$.

Recall (1), we have $T_u = \sum_{h=0}^{N_a-1} \sum_{g=0}^{N_a-1} \frac{n_{u,g}+n_{v,h}}{2N_a} \geq \sum_{h=0}^{N_a-1} \sum_{g=0}^{N_a-1} \frac{\sqrt{n_{u,g}n_{v,h}}}{N_a} \geq N_a\sqrt{T_u}$. It then follows that $T_u \geq N_a^2$, which contradicts with the assumption $T_u < N_a^2$ and completes the proof. □

### 4.1 Summary

To conclude this section, we summarise the established theoretic bounds on deterministic rendezvous strategies:

- *Rendezvous delay bound (Theorem 1)*: The worst-case rendezvous delay is lower-bounded by $N_aN_b$, or $O(N^2)$ if $N_a \simeq N_b \simeq N$;
- *$D$-bounded phone pick sequence structure (Lemma 1)*: Given two $D$-bounded phone pick sequences $\mathbf{u}$ and $\mathbf{v}$, for any cyclic rotation phases $t_0^a$ and $t_0^b$, the pair $(u_t(t_0^a), v_t(t_0^b))$ $(0 \leq t < D)$ must cover all the possible telephone label couples in $[0, N_a - 1] \times [0, N_b - 1]$;
- *Phone pick sequence period (Theorem 2)*: To ensure rendezvous regardless of the players' phone labeling functions and cyclic rotation phases, the phone pick sequence period of player $i$ $(i \in \{a, b\})$ cannot be shorter than $N_i^2$.

## 5 An Order-Optimal Deterministic Strategy: Zero-Knowledge Rendezvous

In this section, we develop an order-optimal deterministic rendezvous strategy, Zero-knowledge Rendezvous, which

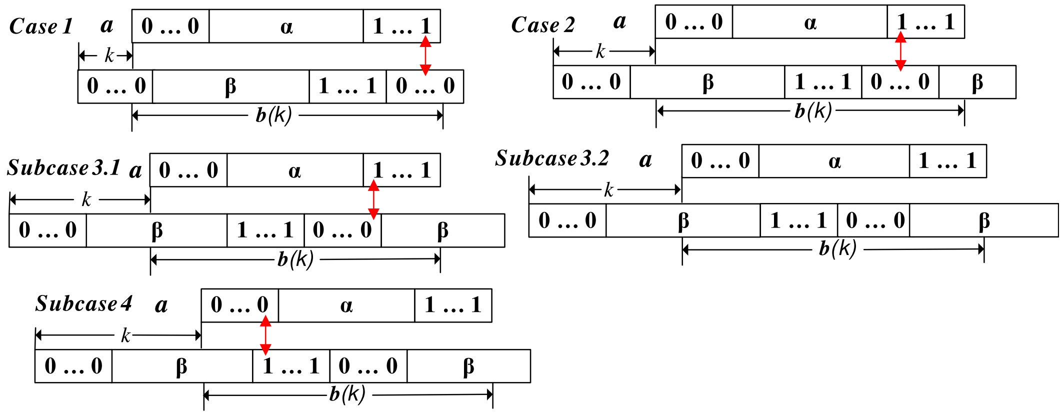


Fig. 2. Illustration of the four cases in the proof of Lemma 2.

(1) guarantees rendezvous between Alice and Bob regardless of their telephone labeling functions and their cyclic rotation phases and (2) approaches the performance bound derived in Section 4 without any prior knowledge or coordination.

### 5.1 Phone Pick Sequence Construction

The phone pick sequence of each player is constructed based on its ID, which is globally unique. Examples of such globally unique IDs includes one's passport number, biometric identities such as DNA sequence and in case of a communication device its address. Mathematically, we define a player's ID as a binary sequence of length $l$ composed of a sequence of bits where each bit takes either a value of $0$ or $1$.

**Remark 3.** Using globally unique IDs in the design of Zero-knowledge Rendezvous is a way of breaking the symmetry between the two players and is realistic in many practical engineering problems. A natural question that arises is whether it is possible to devise a deterministic strategy with bounded rendezvous latency without breaking any form of symmetry between the players. In other words, players are treated as indistinguishable entities. The response is unfortunately no. We next give a counterexample which is simple while sufficient to demonstrate the impossibility of devising a rendezvous strategy without any form of symmetry breaking. Consider the case where Alice and Bob are perfectly synchronised and $N_a = N_b$. Without symmetry breaking, it holds that $\mathbf{u} = \mathbf{v}$. If the unique telephone connecting them $c_0$ is indexed differently by Alice and Bob, i.e., $f_a^{-1}(c_0) \neq f_b^{-1}(c_0)$, the rendezvous can never be achieved.

The phone pick sequence construction process is composed of three steps, summarised as follows.

- *Step 1:* Each player $i$ independently generates a padded binary sequence $\mathbf{e}^i$ based on its ID such that the binary sequences of any two players are *cyclic rotationally distinct* one to the other;
- *Step 2:* Each player $i$ independently generates a sequence $\mathbf{o}^i$ based on $\mathbf{e}^i$ such that for any two players $i$, $j$, there always exists $t_1$ and $t_2$ such that $o^i_{t_1}(t^i_0) = o^j_{t_1}(t^j_0)$ and $o^i_{t_2}(t^i_0) \neq o^j_{t_2}(t^j_0)$ for any cyclic rotation phases $t^i_0$ and $t^j_0$. We denote such sequences $\mathbf{e}^i$ as *regular sequences*;
- *Step 3:* Each player $i$ generates its phone pick sequence based on $\mathbf{o}^i$.

*1) Step 1: Constructing cyclic rotationally distinct padded binary sequence.* As the first step of constructing its phone pick sequence, each player independently generates a binary sequence based on its ID such that the binary sequences of any two players are *cyclic rotationally distinct* one to the other. Note that the sequences resulting from cyclic rotations of a sequence are not considered to be cyclic rotationally distinct with respect to each other and the original sequence. We next show how to construct such cyclic rotationally distinct binary sequences.

Let $\alpha$ denote the ID of a player (say, Alice) and let $\mathbf{1}$ ($\mathbf{0}$) denote a sequence of $1$ ($0$) of length $l' = \lceil \frac{l}{2} \rceil$. We construct the padded ID of Alice as the concatenation of $\mathbf{0}$, $\alpha$ and $\mathbf{1}$, denoted as $\mathbf{0}||\alpha||\mathbf{1}$. By the following lemma, we show that the padded ID sequences generated in such way based on different ID sequences are cyclic rotationally distinct one to another.

**Lemma 2.** *Given any two padded ID sequences* $\mathbf{a}$ *and* $\mathbf{b}$ *generated from two ID sequences* $\alpha$ *and* $\beta$ *in the way that* $\mathbf{a} \triangleq \mathbf{0}||\alpha||\mathbf{1}$ *and* $\mathbf{b} \triangleq \mathbf{0}||\beta||\mathbf{1}$, *it holds that*

$$\alpha \neq \beta \quad \Longrightarrow \quad \mathbf{a} \neq \mathbf{b}(k), \ \forall k \in (0, l + 2l'],$$

*where* $\mathbf{b}(k)$ *is* $\mathbf{b}$ *with a cyclic rotation of* $k$ *bits.*

**Proof.** We prove the lemma by considering four possible scenarios illustrated in Fig. 2, and showing, in each scenario, that a bit in $\mathbf{a}$ and another bit in $\mathbf{b}(k)$ have different values although the two bits are in the same position within the respective padded ID sequences. This is sufficient to prove that the two padded ID sequences $\mathbf{a}$ and $\mathbf{b}$ are cyclic rotationally distinct one to the other.

*Case 1.* $k \in (0, l')$. As indicated by the arrow in Fig. 2, it holds that $a_L = 1$ and $b_L(k) = 0$ where $L = l + 2l'$.

*Case 2.* $k \in [l', l + l')$. As indicated by the arrow in Fig. 2, it holds that $a_{l+l'} = 1$ and $b_{l+l'}(k) = 0$.

*Case 3.* $k = l + l'$. We further distinguish the following two subcases:

- *Subcase 3.1.* $l$ is odd. It holds that $l = 2l' - 1$. As indicated by the arrow in Fig. 2, it holds that $a_{l'} = 0$ and $b_{l'}(k) = 1$.

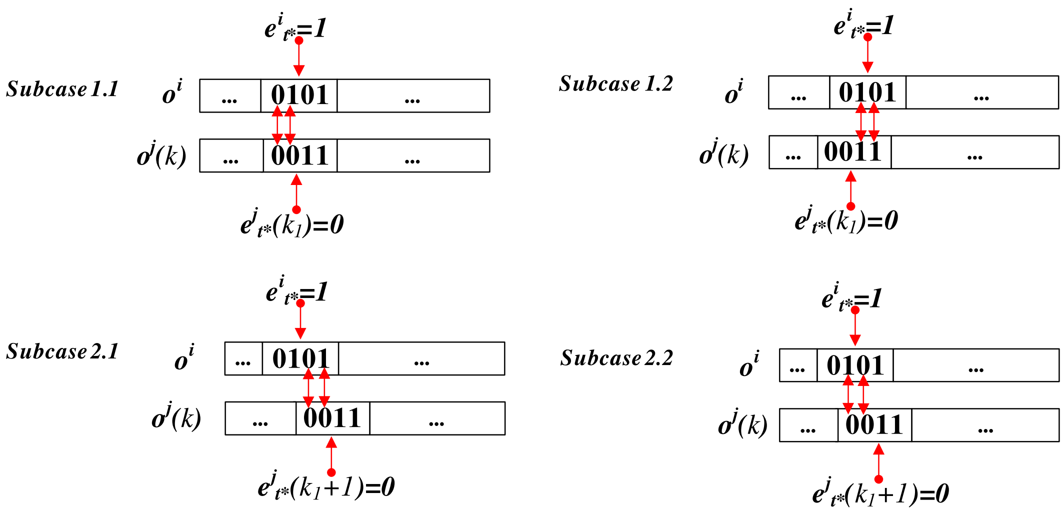


Fig. 3. Illustration of the two cases in the proof of Lemma 3.

- *Subcase 3.2.* $l$ is even. It holds that $l = 2l'$. As indicated by the arrow in Fig. 2, since $\mathbf{a} \neq \mathbf{b}$, $\mathbf{a} = \mathbf{1}||\mathbf{0}$ and $\mathbf{b} = \mathbf{1}||\mathbf{0}$ cannot hold simultaneously; there must exists $l_0$ such that $a_{l_0} \neq b_{l_0}(k)$.

*Case 4.* $k \in (l + l', l + 2l')$. As indicated by the arrow in Fig. 2, it holds that $a_{l'} = 0$ and $b_{l'}(k) = 1$.

Noticing that $\alpha \neq \beta \Longrightarrow \mathbf{a} \neq \mathbf{b}$, we thus conclude that $\mathbf{a} \neq \mathbf{b}(k),\ \forall k \in [0, l + 2l')$. □

*2) Step 2: Generating regular sequence.* Denote the padded ID sequence as $\mathbf{e}^i$ for player $i$, the next step for each player is to generate a sequence $\mathbf{o}^i$ based on $\mathbf{e}^i$ such that for any two players $i$, $j$, there always exists $t_1$ and $t_2$ such that $o^i_{t_1}(t^i_0) = o^j_{t_1}(t^j_0)$ and $o^i_{t_2}(t^i_0) \neq o^j_{t_2}(t^j_0)$ for any cyclic rotation phases $t^i_0$ and $t^j_0$. We denote such sequences $\mathbf{e}^i$ as *regular sequences*. In the following we develop an algorithm that can generate regular sequences.

**Algorithm 1.** Construct a regular sequence $\mathbf{o}^i$

**Input:** Padded ID sequence $\mathbf{e}^i$ of $l + 2l'$ bits
**Output:** Regular sequence $\mathbf{o}^i$

```
for t = 0 to l + 2l' − 1 do
    switch e^i_t
        case 1: expand e^i_t into four bits 0101
        case 0: expand e^i_t into four bits 0011
end for
o^i ← the expanded sequence of e_i
```

**Lemma 3.** *The sequence generated by Algorithm 1 is regular.*

**Proof.** It suffices to show that for any $k \in [0, 4(l + 2l')]$, there exist $t_1$ and $t_2$ such that $o^i_{t_1} = o^j_{t_1}(k)$ and $o^i_{t_2} \neq o^j_{t_2}(k)$. Let $k = 4k_1 + k_2$ where $k_1 \triangleq \lceil \frac{k}{4} \rceil - 1 \in [0, l + 2l' - 1]$ and $k_2 \triangleq k \bmod 4 \in [0, 3]$, we distinguish the following two cases, as illustrated in Fig. 3.

- *Case 1.* $k_2 = 0$ or $1$. Let $\mathbf{e}^i$ and $\mathbf{e}^j$ denote the padded ID sequences of player $i$ and $j$. Recall Lemma 2 and the notations in Section 3.3, there exists $t^*$ such that $e^i_{t^*} \neq e^j_{t^*}(k_1)$. Without loss of generality, assume that $e^i_{t^*} = 1$ and $e^j_{t^*}(k_1) = 0$. It follows from Algorithm 1 that

$$\begin{cases} [o^i_{4t^*},\ o^i_{4t^*+1},\ o^i_{4t^*+2},\ o^i_{4t^*+3}] = [0,1,0,1], \\ [o^j_{4t^*-k_2}(k),\ o^j_{4t^*-k_2+1}(k),\ o^j_{4t^*-k_2+2}(k), \\ \qquad o^j_{4t^*-k_2+3}(k)] = [0,0,1,1]. \end{cases}$$

We further distinguish the following two subcases.
   - *Subcase 1.1.* $k_2 = 0$. In this subcase, we have $o^i_{4t^*} = o^j_{4t^*}(k) = 0$ and $o^i_{4t^*+1} = 1 \neq o^j_{4t^*+1}(k) = 0$.
   - *Subcase 1.2.* $k_2 = 1$. In this subcase, we have $o^i_{4t^*+1} = o^j_{4t^*+1}(k) = 1$ and $o^i_{4t^*+2} = 0 \neq o^j_{4t^*+2}(k) = 1$.

- *Case 2.* $k_2 = 2$ or $3$. Recall Lemma 2, there exists $t^*$ such that $e^i_{t^*} \neq e^j_{t^*}(k_1 + 1)$. Without loss of generality, assume that $e^i_{t^*} = 1$ and $e^j_{t^*}(k_1 + 1) = 0$. It follows from Algorithm 1 that

$$\begin{cases} [o^i_{4t^*},\ o^i_{4t^*+1},\ o^i_{4t^*+2},\ o^i_{4t^*+3}] = [0,1,0,1], \\ [o^j_{4(t^*+1)-k_2}(k),\ o^j_{4(t^*+1)-k_2+1}(k),\ o^j_{4(t^*+1)-k_2+2}(k), \\ \qquad o^j_{4(t^*+1)-k_2+3}(k)] = [0,0,1,1]. \end{cases}$$

We further distinguish the following two subcases.
   - *Subcase 2.1.* $k_2 = 2$. In this subcase, we have $o^i_{4t^*+2} = o^j_{4t^*+2}(k) = 0$ and $o^i_{4t^*+3} = 1 \neq o^j_{4t^*+3}(k) = 0$.
   - *Subcase 2.2.* $k_2 = 3$. In this subcase, we have $o^i_{4t^*+1} = 1 \neq o^j_{4t^*+1}(k) = 0$ and $o^i_{4t^*+2} = o^j_{4t^*+2}(k) = 0$.

The sequence generated by Algorithm 1 is thus regular. □

*3) Step 3: Generating phone pick sequence.* Each player $i$ then constructs his phone pick sequence $\mathbf{u}$ based on the regular sequence $\mathbf{o}^i$ whose length is $L_o \triangleq 4(l + 2l)'$. To that end, let $p^0_i$ and $p^1_i$ denote the two smallest prime numbers not smaller than $N_i$ and co-prime with $L_o$, i.e., $p^1_i \geq p^0_i \geq N_i$. The phone pick sequence of player $i$, $\mathbf{u}$, is

**ID α=01, Padded ID a=0011,**
**$N_a$=3, $p_a^0$=3, $p_a^1$=5, $t_0^a$=0**

| Round index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $o^a$ : | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | ... |
| u : | 0 | 1 | 2 | r | 1 | 2 | 1 | 2 | 2 | r | 1 | 1 | 0 | r | 2 | 0 | ... |

Fig. 4. A phone pick sequence example for Alice ($r$ denotes a randomly chosen telephone).

constructed as follows:

$$u_t = \begin{cases} [t]_{p_i^0} & o_t^i = 0 \text{ and } [t]_{p_i^0} < N_i, \\ [t]_{p_i^1} & o_t^i = 1 \text{ and } [t]_{p_i^1} < N_i, \\ \text{rand}(N_i - 1) & \text{otherwise,} \end{cases} \tag{2}$$

where $[x]_y$ denotes $x$ mod $y$, rand$(N_i - 1)$ denotes a random integer in $[0, N_i - 1]$. It can be noted that the period of the phone pick sequence $\mathbf{u}$ is $L_o p_i^0 p_i^1$ without taking into account the randomly chosen telephones. Fig. 4 provides an example of the phone pick sequence in Zero-knowledge Rendezvous.

**Algorithm 2.** Construct the phone pick sequence for Alice: the overall algorithm

**Input:** ID sequence $\alpha$ of $l$ bits
**Output:** Phone pick sequence $\mathbf{u}$
1: Form the padded ID sequence $\mathbf{a} = \mathbf{0}||\alpha||\mathbf{1}$
2: Construct the regular sequence $\mathbf{o}^a$ (Algorithm 1)
3: Choose two prime numbers $p_a^0$ and $p_a^1$ larger than $N_a$ and coprime to $L_o = 4\left(l + \lfloor \frac{l}{2} \rfloor\right)$
4: Construct the phone pick sequence $\mathbf{u}$ based on (2)

*4) The overall algorithm*. Algorithm 2 summarises the majors steps to construct the phone pick sequence in the proposed Zero-knowledge Rendezvous algorithm by taking Alice as an example (the same steps hold for Bob to construct $\mathbf{v}$).

## 5.2 Correctness and Worst-Case Rendezvous Delay Bound

In the following theorem, we prove the correctness of Zero-knowledge Rendezvous in guaranteeing rendezvous and establish the worst-case rendezvous delay bound.

**Theorem 3 (Correctness and Worst-Case Rendezvous Delay Bound of Zero-Knowledge Rendezvous).** *Zero-knowledge Rendezvous can ensure rendezvous between Alice and Bob regardless of their telephone labeling functions and cyclic rotation phases. Let $p_a^1$ and $p_b^1$ denote the larger prime numbers chosen by Alice and Bob, the worst-case rendezvous delay is upper-bounded by $L_o p_a^1 p_b^1$.*

**Proof.** Given any pair of cyclic rotation phases $t_a^0$ and $t_b^0$, by Lemma 3, there exit $0 \le l_1, l_2 < L_o$ such that $o_{l_1}^a(t_0^a) = o_{l_1}^b(t_0^b)$ and $o_{l_2}^a(t_0^a) \neq o_{l_2}^b(t_0^b)$. Without loss of generality, assume that $o_{l_1}^a(t_0^a) = o_{l_1}^b(t_0^b) = 0$, $o_{l_2}^a(t_0^a) = 0$ and $o_{l_2}^b(t_0^b) = 1$.

Since $p_a^i$ and $p_b^i$ $(i = 0, 1)$ are prime numbers, it holds that $p_a^0$ is co-prime with either $p_b^0$ or $p_b^1$. Without loss of generality, assume that $p_a^0$ is co-prime with $p_b^0$.

We examine the rounds $t_k = l_1 + kL_o$ where $k \in \mathbb{N}$. More specifically, we consider the subsequences of $\mathbf{u}(t_0^a)$ and $\mathbf{v}(t_0^b)$ in these rounds, i.e., $\{u_{t_k}(t_0^a)\}$ and $\{v_{t_k}(t_0^b)\}$. Recall (2), we can write $u_{t_k}(t_0^a)$ and $v_{t_k}(t_0^b)$ as follows:

$$\begin{cases} u_{t_k}(t_0^a) = [t_0^a + l_1 + kL_o]_{p_a^0}, \\ v_{t_k}(t_0^b) = [t_0^b + l_1 + kL_o]_{p_b^0}. \end{cases}$$

Recall that (1) $p_a^0$ is co-prime with $p_b^0$, and (2) both $p_a^0$ and $p_b^0$ are co-prime with $L_o$, it follows from the Chinese Remainder Theorem [19] that for any couple $(h_a, h_b) \in [0, N_a - 1] \times [0, N_b - 1]$, there exists $k_0 < p_a^0 p_b^0$ such that $[k_0 L_o]_{p_a^0} = h_a$ and $[k_0 L_o]_{p_b^0} = h_b$. Hence, for any telephone labeling functions of Alice and Bob $f_a$ and $f_b$ where the only telephone connecting them $c^*$ are labeled $h_a^*$ by Alice and $h_b^*$ by Bob, i.e., $f_a(c^*) = h_a^*$, $f_b(c^*) = h_b^*$, there exists $t_{k^*} < p_a^0 p_b^0$ such that $[t_0^a + l_1 + k^* L_o]_{p_a^0} = h_a^*$, and $[t_0^b + l_1 + k^* L_o]_{p_b^0} = h_b^*$. It then follows that $f_a^{-1}[u_{t_{k^*}}(t_0^a)] = f_b^{-1}[v_{t_{k^*}}(t_0^b)]$, meaning that the rendezvous can be achieved at round $t_{k^*}$ with the worst-case rendezvous delay bounded by $L_o p_a^0 p_b^0$.

Similarly, when $p_a^0$ is co-prime with $p_b^1$ ($p_a^1$ is co-prime with $p_b^0$, $p_a^1$ is co-prime with $p_b^1$, respectively), we can prove that the worst-case rendezvous delay is upper-bounded by $L_o p_a^0 p_b^1$ ($L_o p_a^1 p_b^0$, $L_o p_a^1 p_b^1$, respectively). Recall that $p_a^0 < p_a^1$ and $p_b^0 < p_b^1$, it holds that the worst-case rendezvous delay of Zero-Knowledge Rendezvous is upper-bounded by $L_o p_a^1 p_b^1$. □

**Remark 4.** Asymptotically, it follows from Theorem 3 that the worst-case rendezvous delay of Zero-knowledge Rendezvous approaches $L_o N_a N_b$, or $O(N^2)$ if $N_a \simeq N_b \simeq N$ which approaches the established rendezvous delay lower-bound established in Theorem 1 in Section 4.

Zero-knowledge Rendezvous can be adapted and simplified if players have pre-assigned roles, such as in half-duplex communication. Specifically, each player is either a caller or a callee, and a rendezvous is required between a caller and a callee. In such context, we can attribute a one-bit ID $\mathbf{a} = 0$ for any caller and an one-bit ID $\mathbf{b} = 1$ for any callee. Following the same analysis as that in this section, we can show that the worst-case rendezvous delay of Zero-knowledge Rendezvous using the two one-bit IDs is upper-bounded by $p_a^1 p_b^1$, a factor of $L_o$ shorter than the case without pre-assigned roles.

Zero-knowledge Rendezvous can also be adapted and simplified if players are synchronised, i.e., they start the rendezvous search at the same time. In this case, we do not need to pad the ID sequences of the two players to ensure that they are cyclic rotationally distinct one to the other. The sequences $o^i$ can be generated directly using the ID sequence. In terms of rendezvous delay, the worst-case delay is upper-bounded by $4lp_a^1 p_b^1$ in the synchronised case, which is 50 percent shorter than the case without preassigned roles.

To complete our study on the rendezvous delay of Zero-knowledge Rendezvous, we now derive the upper-bound on the average rendezvous delay. Recall the proof of Theorem 3 by using the same notation there, given a random pair of $t_0^a$ and $t_0^b$, the expectation of $k^*$ is bounded by $\frac{p_a^1 p_b^1 - 1}{2}$. Assume that the players' IDs can be regarded as random binary sequences, the expectation of $l_1$ is bounded by $\frac{L_o}{2}$. The average rendezvous delay is thus upper-bounded by $\frac{L_o p_a^1 p_b^1}{2}$, and asymptotically when $p_i^0 \simeq p_i^1 \simeq N_i \simeq N$, it can be bounded by $\frac{L_o N^2}{2}$.

### 5.3 Rendezvous Analysis with Asymmetrical Round Duration

Our previous analysis implicitly assumes identical round duration at the two players. In this section, we relax this assumption to study the effect of round non-alignment and asymmetrical round length due to lack of coordination or relative clock drift between the two players.

We first study the case of non-aligned rounds with identical duration. We regard the local time at Alice as the time reference by normalising its round duration and assume that its rounds start at time $k \in \mathbb{Z}$. The rounds of Bob start at time $k + \delta$ with $k \in \mathbb{Z}$ and $\delta \in (-1/2, 1/2]$. In this case, the previous analysis can be directly applied, the difference being that instead of an entire overlap, a rendezvous in this case is a partial overlap of time $1 - \delta$.

We now investigate a more interesting and practical case with non-identical round durations. Clearly the rounds of Alice and Bob are not necessarily aligned to each other. Mathematically, we assume that each round of Bob lasts $\rho$ time by regarding the round duration of Alice as the unit time. Without loss of generality, assume that $\rho < 1$.

We first establish the following property which is useful in later analysis.

**Lemma 4.** *Given any $\rho \in \mathbb{R}$, let $p(n)$ denote the nth prime number, it holds that $\lim_{n\to\infty} \frac{p(\lfloor \rho n \rfloor)}{p(n)} = \rho$. That is, $\rho$ can be well approximated by $\frac{p(\lfloor \rho n \rfloor)}{p(n)}$.*

**Proof.** Recall the prime number theorem (PNT) [17] that $\lim_{n\to\infty} p(n) = n \ln(n)$, we have

$$\lim_{n\to\infty} \frac{p(\lfloor \rho n \rfloor)}{p(n)} = \frac{\rho n \ln(\rho n)}{n \ln(n)} = \lim_{n\to\infty} \frac{\rho n \ln(\rho) + \rho n \ln(n)}{n \ln(n)} = \rho,$$

which completes the proof. □

The following theorem establishes the rendezvous delay bound of Zero-knowledge Rendezvous.

**Theorem 4 (Rendezvous Delay Bound: Asymmetrical Round Duration).** *Regard the round of Alice as unit time, Alice and Bob are guaranteed to rendezvous with each other. The rendezvous delay is upper-bounded by $\rho L_o p_a^1 p_b^1$.*

**Proof.** We decompose each round of Alice and Bob into mini-rounds of duration $\frac{1}{p(n)}$ where $n$ is sufficiently large. Under the decomposition, each slot of Alice and Bob contain $p(n)$ and $p(\lfloor \rho n \rfloor)$ mini-rounds, respectively. We can express the phone pick sequence of Alice and Bob in mini-rounds

$$u_t^a = \begin{cases} [tp(n) - l_a]_{p_a^0 p(n)} & o_t^a = 0 \text{ and } [t]_{p_a^0 p(n)} < N_a, \\ & 0 \le l_a \le p(n) - 1, \\ [tp(n) - l_a]_{p_a^1 p(n)} & o_t^a = 1 \text{ and } [t]_{p_a^1 p(n)} < N_a, \\ & 0 \le l_a \le p(n) - 1, \\ \text{rand}(N_a - 1) & \text{otherwise,} \end{cases}$$

$$u_t^b = \begin{cases} [tp(\lfloor \rho n \rfloor) - l_b]_{p_b^0 p(\lfloor \rho n \rfloor)} & o_t^b = 0 \text{ and } [t]_{p_b^0 p(\lfloor \rho n \rfloor)} < N_b, \\ & 0 \le l_b \le p(\lfloor \rho n \rfloor) - 1, \\ [tp(\lfloor \rho n \rfloor) - l_b]_{p_b^1 p(\lfloor \rho n \rfloor)} & o_t^b = 1 \text{ and } [t]_{p_b^1 p(\lfloor \rho n \rfloor)} < N_b, \\ & 0 \le l_b \le p(\lfloor \rho n \rfloor) - 1, \\ \text{rand}(N_b - 1) & \text{otherwise,} \end{cases}$$

When $n$ is sufficiently large, specifically, $p(\lfloor \rho n \rfloor) > \max\{L_o, p_a^1, p_b^1\}$, we can use the same analysis as that in the proof of Theorem 3 to show that within at most $O(L_o p_1^a p_1^b p(n) p(\lfloor \rho n \rfloor))$ mini-rounds (i.e., $O(\rho L_o p_1^a p_1^b)$ time by regarding the rounds of Alice as unit time), there exist a mini-round $t_m$ such that Alice and Bob can rendezvous on the telephone $c^*$ with $l_a = l_b = 0$. It then follows that from mini-round $t_m$ to $t_m + p(\lfloor \rho n \rfloor)$ (i.e., one round for Bob), both Alice and Bob stick to $c^*$, and can thus rendezvous with each other. □

The results obtained in this section, particularly Theorem 4, demonstrate that the rendezvous delay bound of Zero-knowledge Rendezvous established in previous analysis holds even when the rounds of Alice and Bob have asymmetrical duration and drift away from each other for an arbitrary amount of time.

## 6 Theoretic Performance Analysis for Probabilistic Strategies

In this section, we investigate probabilistic rendezvous strategies in the Generalised Telephone Problem to provide a comparison reference for the deterministic strategy developed in Section 5 and to make our analysis complete. By playing a probabilistic strategy, a player picks a telephone each round based on certain probability so as to minimise the expected rendezvous delay. Note that the rendezvous delay cannot be bounded under probabilistic strategies.

Specifically, we derive the expected rendezvous delay for three representative probabilistic strategies, based on which we make a conjecture on the minimal expected rendezvous delay of probabilistic strategies.

### 6.1 Purely Random Strategy

We start by investigating the simplest probabilistic strategy, the purely random strategy where each player $i$ picks each telephone $h \in \mathcal{N}_i$ with equal probability $\mu_i(h)$, i.e., $\mu_i(h) = \frac{1}{N_i}$. This process is repeated each round. Let $c^*$ denote the only telephone via which the two players can rendezvous, the probability that both Alice and Bob picks it in a round is $\frac{1}{N_a N_b}$. The expected rendezvous delay, denoted as $d$, is thus

$$d = \sum_{t=1}^{\infty} t\left(1 - \frac{1}{N_a N_b}\right)^{t-1} \frac{1}{N_a N_b} = N_a N_b,$$

Asymptotically when $N_a \simeq N_b \simeq N$, $d \simeq O(N^2)$.

### 6.2 Anderson-Weber Strategy

We next analyse the Anderson-Weber strategy developed in [5]. Under the Anderson-Weber strategy, each player $i$ sticks to a telephone for $N_i$ rounds with probability $\theta$ and hops across a random permutation of the $N_i$ telephones with probability $1-\theta$. The process is repeated each $N_i$ rounds until rendezvous. To make the analysis tractable on the expected rendezvous delay $d$ as in their paper [5], we consider the case where $N_a = N_b = N$ and both players begin the search simultaneously. The reasons why we focus on this simplified case are two-fold:

- Dropping either assumption makes the analytical characterisation of expected rendezvous delay intractable;
- The simplified case is sufficient to provide order-magnitude performance bound on this strategy.

To derive the expected rendezvous delay $d$, we consider the following three cases:

*Case 1: With probability $\theta^2$, both players stick to a random telephone for $N$ rounds.* In this case, with probability $\frac{1}{N}$, a player picks the right telephone $c^*$ that can make them rendezvous. Hence with probability $\frac{1}{N^2}$, rendezvous can be achieved with a delay 1 and with probability $1-\frac{1}{N^2}$, rendezvous cannot be achieved within $N$ rounds, thus resulting an expected rendezvous delay $N+d$. The expected rendezvous delay in this case is thus $\frac{1}{N^2} + (1-\frac{1}{N^2})(N+d)$.

*Case 2: With probability $2\theta(1-\theta)$, one player sticks to a telephone for $N$ rounds and the other player hops across the $N$ telephones in its room.* In this case, with probability $\frac{1}{N}$, the player sticking to a random telephone chooses the telephone $c^*$ that makes them rendezvous, resulting in an expected rendezvous delay $\frac{N}{2}$. With the complementary probability $1-\frac{1}{N}$, rendezvous cannot be achieved within $N$ rounds, resulting in an expected rendezvous delay $N+d$. The expected rendezvous delay in this case is thus $\frac{1}{N}\cdot\frac{N}{2} + (1-\frac{1}{N})(N+d)$.

*Case 3: With probability $(1-\theta)^2$, both players hop across the $N$ telephones in their rooms in a random permutation.* In This case, it can be calculated that the probability that they can rendezvous within $N$ rounds is

$$\frac{\binom{N}{1}[(N-1)!]^2}{(N!)^2} = \frac{1}{N}$$

with an average rendezvous delay $\frac{N}{2}$. With the complementary probability $1-\frac{1}{N}$, rendezvous cannot be achieved within $N$ rounds, thus resulting an expected rendezvous delay $N+d$. The expected rendezvous delay in this case is thus $\frac{1}{N}\cdot\frac{N}{2} + (1-\frac{1}{N})(N+d)$.

Based on the above analysis, we can established the following equation:

$$\begin{aligned} d = {} & \theta^2\left[\frac{1}{N^2} + \left(1-\frac{1}{N^2}\right)(N+d)\right] + 2\theta(1-\theta) \\ & \times\left[\frac{1}{2} + \left(1-\frac{1}{N}\right)(N+d)\right] + (1-\theta)^2 \\ & \times\left[\frac{1}{2} + \left(1-\frac{1}{N}\right)(N+d)\right]. \end{aligned}$$

After some algebraic operations, we have

$$d = \frac{N[N + \frac{(1-\theta)^2}{2}]}{1-\theta^2} - N.$$

The minimal expected rendezvous delay can be derived as $d_{min} = N^2 - \frac{N}{2}$ when $\theta = 0$. The Anderson-Weber strategy achieving minimal expected rendezvous delay is a Markovian strategy where each player repeatedly picks all the telephones sequentially following a random permutation of them. Asymptotically, we have $d \simeq O(N^2)$.

### 6.3 Impatient Markovian Strategy

The results on the delay of the Anderson-Weber strategy in the generalised telephone problem demonstrate that it is better to explore than sticking to one telephone. Based on this observation, we investigate a natural strategy aiming at further decreasing the rendezvous delay by repeatedly exploring a subset of $\alpha N$ $(0 \le \alpha \le 1)$ telephones. We term this strategy as the *impatient Markovian strategy* where $\alpha$ characterises the degree of patience of the players. When $\alpha = 1$, the impatient Markovian strategy becomes a patient strategy and degenerates to the Anderson-Weber strategy with $\theta = 0$.

Specifically, the impatient Markovian strategy works in an epoch-based way in which each epoch is composed of $\alpha N$ rounds. In each epoch, each player randomly picks a subset of $\alpha N$ telephones sequentially following a random permutation of them. Such operation is repeated until when the rendezvous is achieved.

We now derive the expected rendezvous delay for the above impatient Markovian strategy. As in the previous section, we consider the case where $N_a = N_b = N$ and both players begin the search simultaneously. Let $\pi$ denote the probability that the rendezvous can be achieved in an epoch. We can compute $\pi$ as

$$\pi = \frac{\binom{\alpha N}{1}[P(N-1, \alpha N - 1)]^2}{[P(N, \alpha N)]^2} = \frac{\alpha}{N},$$

where $P(n_1, n_2) \triangleq \frac{n_1!}{(n_1-n_2)!}$ denotes the number of permutations of $n_2$ elements from a set of $n_1$. We further consider the following two cases:

- *Case 1*. With probability $\pi$, the rendezvous can be achieved with an expected delay of $\frac{\alpha N}{2}$.
- *Case 2*. With probability $1-\pi$, the rendezvous cannot be achieved within the current epoch, thus resulting an expected rendezvous delay $\alpha N + d$.

It follows that

$$d = \pi \cdot \frac{\alpha N}{2} + (1-\pi)(d + \alpha N).$$

We can thus solve $d$ as $d = N(N - \frac{\alpha}{2})$, which is minimised at $\alpha = 1$ with the minimum $N^2 - \frac{N}{2}$. In this case, patience actually leads to minimal rendezvous latency.

### 6.4 Discussion

It is insightful to compare the rendezvous delay of probabilistic strategies in the Generalised Telephone Problem to the original Telephone Problem.

Globally, we note that the expected rendezvous delay increases from $O(N)$ in the original version to $O(N^2)$ in the generalised version. Such difference in rendezvous delay can be intuitively explained by the problem setting that in the original Telephone Problem, there are $N$ telephones that connecting the players compared to only one such telephone in the Generalised Telephone Problem. Consequently, the expected rendezvous delay in the original problem is one order smaller than that in the generalised one.

Specifically, the Anderson-Weber strategy is shown to be optimal for $N = 2, 3$ in the original Telephone Problem and has a asymptotical delay of $0.83N$, compare to $N$ for the purely random strategy. In the Generalised Telephone Problem, the Anderson-Weber strategy that minimises the expected rendezvous delay degenerates to the Marcovian strategy of repeatedly picking the $N$ telephones sequentially by following a random permutation of them. This difference can be explained by noticing that in the original problem, the best situation for the players is that one follows the *wait-for-mummy* strategy by sticking to a telephone while the other follows the strategy of *exploration* by picking its telephones sequentially. This setting can guarantee rendezvous with an expected delay $\frac{N}{2}$. However, since the players cannot coordinate to achieve such setting, and the situation with both players choosing wait-for-mummy leads to longer expected rendezvous delay than the situation with both of them exploring, the optimal strategy consists of striking a balance with a probability $\theta = 0.25$ in the asymptotic case corresponding to a larger probability of exploration. The situation is different in the generalised problem. For Alice, if Bob chooses the wait-for-mummy strategy, it follows from the analysis in the previous section on the Anderson-Weber strategy that Alice is better off to choose exploration because wait-for-mummy leads to a rendezvous probability $\frac{1}{N^2}$ with an average delay of $1$ round while exploration leads to a rendezvous probability $\frac{1}{N}$ with an average delay of $\frac{N}{2}$. If Bob chooses to explore, regardless of the strategy of Alice (i.e., wait-for-mummy or exploration), the rendezvous is achieved with probability $\frac{1}{N}$ with an average delay of $\frac{N}{2}$. Therefore, in the Generalised Telephone Problem, each player is always better off exploring than waiting for mummy. The optimal Anderson-Weber strategy in this case thus degenerates to the Marcovian strategy of repeatedly exploring the $N$ telephones.

Furthermore, our results on the impatient Markovian strategy demonstrate that the full exploration minimises the expected rendezvous delay.

We can summarise our theoretical results on probabilistic strategies in the Generalised Telephone Problem:

- Exploration outperforms wait-for-mummy;
- A full exploration outperforms a partial one.

The two findings motivate us to make the following conjecture, the proof or counter-examples of which is far from trivial[2] and consists of an interesting direction for future research.

**Conjecture 1.** *In the Generalised Telephone Problem, the Markovian strategy of picking all the telephones sequentially by following a random permutation of them minimises the expected rendezvous delay.*

## 7 Variants and Extensions

The Generalised Telephone Problem and the rendezvous strategy analysed in this paper can generate many related searching and rendezvous problems that are interesting and applicable in many engineering domains. In this section we review and investigate a number of them.

### 7.1 Trading Off Worst-Case and Average Rendezvous Delay

We have analysed deterministic strategies (Sections 4 and 5). Section 6 provides a quantitative analysis on probabilistic strategies. It is insightful to conduct a quantitative comparison of their performance so as to further combine their respective advantages while avoiding their drawbacks. Specifically, as analysed in Section 6, the purely random probabilistic strategy achieves an average rendezvous delay of $N_a N_b$, with the delay of the other two probabilistic strategies slightly shorter. The worst-case rendezvous delay of a probabilistic strategy cannot be bounded. On the other hand, the rendezvous delay of the deterministic strategy Zero-knowledge Rendezvous has a larger average rendezvous delay of $\frac{L_o p_a^1 p_b^1}{2}$ as analysed in Section 5. The worst-case rendezvous delay of Zero-knowledge Rendezvous is bounded by $L_o p_a^1 p_b^1$. A natural question is *how to improve the average performance of Zero-knowledge Rendezvous while still ensuring a bounded rendezvous delay*.

In this section, we investigate how a desired tradeoff between the worst-case and the average rendezvous delay can be achieved by properly choosing the two prime numbers $p_0^i$ and $p_1^i$. To make our analysis tractable, we focus on a synchronised case where $N_a = N_b = N$ and Alice and Bob choose the same prime numbers. However, the idea presented via this example also holds in the general cases. Recall the phone pick sequence of Zero-knowledge Rendezvous (equation (2)), we note that for rounds $N_i \leq [t]_{p_i^0} \leq p_i^0$ (when $o_t^i = 0$) and $N_i \leq [t]_{p_i^1} \leq p_i^1$ (when $o_t^i = 1$), each player randomly picks a telephone. We can configure the duration of such "random periods" via $p_i^0$ and $p_i^1$ so as to improve the average performance while still ensuring the bounded rendezvous delay by the operations in the remaining "deterministic rounds". Specifically, choosing larger $p_i^0$ and $p_i^1$ results in longer "random periods", thus improving the average performance at the price of increasing the worst-case delay. By choosing proper $p_i^0$ and $p_i^1$, we can trade off the worst-case and the average rendezvous delay.

2. Note that even in the original Telephone Problem, it remains an open problem to find the optimal probabilistic strategy that minimises the expected rendezvous delay.

We next provide an approximative quantitative analysis on the above tradeoff. Consider the case where $N$ is sufficiently large and $p_i^0 \simeq p_i^1 \simeq p$. Approximatively, within each $p$ rounds, there are $p - N$ "random rounds" where player $i$ randomly picks a telephone. We call such $p - N$ "random rounds" a random frame. The probability that a rendezvous can be achieved within one random frame can be calculated as

$$q = 1 - \left(1 - \frac{1}{N^2}\right)^{p-N}.$$

Recall that the worst-case delay is bounded by $L_o p^2$ rounds, i.e., $L_o p$ random frames, we can then calculate the upper-bound of the average rendezvous delay $\overline{d}$ as follows:

$$\overline{d} \le q \cdot p + (1-q)q \cdot 2p + \cdots + (1-q)^{L_o p - 2} q \cdot (L_o p - 1)p^2 + (1-q)^{L_o p - 1} \cdot L_o p^2.$$

Given a target expected delay bound $\bar{d}$, $p$ can be chosen based on the above inequality. To get more insight, we consider the case where we set $p$ sufficiently larger than $N$ but linear to $N$, i.e., $p = (1+\lambda)N$ with sufficiently large $\lambda$. We have $q \simeq \frac{\lambda}{N}$. After some algebraic operations, the average delay is bounded by

$$\overline{d} < \sum_{k=0}^{\infty} (1 - q^k) q \cdot (k+1) p = \left(1 + \frac{1}{\lambda}\right) N^2,$$

with the worst-case rendezvous delay being $L_o(1+\lambda)^2 N^2$. We can now clearly see the possibility of trading off the worst-case and the average rendezvous delay in Zero-knowledge Rendezvous by configuring $\lambda$.

### 7.2 The Case of Picking Multiple Telephones Simultaneously

Throughout our analysis in the paper, we implicitly assume that each player can pick only one telephone at a time. In this section, we relax this constraint by considering the case where each player can pick two telephones simultaneously. The analysis in this section can be generalised to the case where each player can pick an arbitrary number of telephones. In a broader sense of a rendezvous or searching game, relaxing the constraint of picking only one telephone a time allows to study the case where each player can search more than one places simultaneously or where the game is played between two groups of players and the communication is possible among the players in the same group.

We first establish the lower-bound of the worst-case rendezvous delay. To this end, by performing a similar analysis as that in the proof of Theorem 1, we can show that the worst-case rendezvous delay, among all possible telephone labeling functions and all cyclic rotation phases, cannot be lower than $\frac{N_a N_b}{4}$, and more generically $\frac{N_a N_b}{n^2}$ if each player is allowed to pick $2n$ telephones simultaneously.

We then devise a deterministic rendezvous strategy that achieves $O(N_a N_b)$ worst-case rendezvous delay. Specifically, we take Alice as an example to derive the phone pick sequence. Since she can pick two telephones each time, we denote the sequences $\mathbf{u_l} = \{u_{t,l}\}$ and $\mathbf{u_r} = \{u_{t,r}\}$ the sequences of telephones picked by her left and right hands, respectively. We devise $\mathbf{u_l} = \{u_{t,l}\}$ as follows:

$$u_{t,l} = \begin{cases} [t]_{p_a^l} & [t]_{p_a^l} < N_a, \\ \text{rand}(N_a - 1) & \text{otherwise}, \end{cases}$$

where $p_a^l$ is a prime number not smaller than $N_a$. The sequence $\mathbf{u_r} = \{u_{t,r}\}$ can be devised symmetrically using a different prime number $p_a^r$. To establish that the rendezvous is guaranteed between Alice and Bob, it suffices to note that at least one of $p_a^l$ and $p_a^r$ is co-prime to at least one of $p_b^l$ and $p_b^r$, the two prime numbers of the phone pick sequences of the left and right hands of Bob. The worst-case rendezvous delay can be proved to be upper-bounded by $p_a^r p_b^r$ if we specify that the prime number of the right hand is larger than that of the left hand.

When each player can pick two telephones simultaneously, a symmetry breaking technique is no more necessary as that based on user ID because the phone pick sequence of the left hand can be designed in a different way as that of the right hand, thus naturally breaking the symmetry and resulting in a rendezvous between the left-hand and the right-hand sequences. Consequently, a shorter worst-case rendezvous delay, which is a factor $L_o$ shorter than baseline case, can be achieved.

### 7.3 The Case of Partially Active Rounds

Throughout our analysis, we implicitly assume that each player listens to the picked telephone during the hold round. An interesting extension consists of studying the case where each player stays active (i.e., listens to the telephone) only part of each round. This has a practical implication when the players are limited by energy and thus only remain active periodically, such as mobile sensor nodes and robots. Specifically, these devices, usually battery-powered, alternate between active and sleeping modes by turning their radios on only periodically to save energy.[3] In this context, a rendezvous strategy consists not only of a sequence of telephones to pick, but also of an activation schedule so that Alice and Bob are ensured to activate at the same time instant on the same phone, regardless of their telephone labeling functions and their cyclic rotation phases. This new problem, with an additional dimension in the strategy space, requires a dedicated study and is thus left for our future research.

## 8 Conclusion

We have studied the generalised Telephone Problem. We have established theoretical limit of any deterministic strategy with bounded rendezvous delay that can guarantee rendezvous between two players regardless of their telephone labeling functions and cyclic rotation phases. We have also devised a deterministic strategy, termed as Zero-knowledge Rendezvous, that guarantees rendezvous and approaches the performance limit without any prior knowledge or coordination. Our work presented in this paper consists of a

3. The energy-saving technique allowing each node to alternate between active and sleeping modes is called *duty cycling*, where duty cycle refers to the fraction of time a device stays active [18].

complementary research thrust compared to existing work on probabilistic strategies. Our results and the method employed in the analysis form a theoretical basis of devising rendezvous strategies in a variety of engineering applications requiring bounded rendezvous delay.

## ACKNOWLEDGMENTS

The work of L. Chen was supported by the ANR grant Green-Dyspan (ANR-12-IS03). Lin Chen is the corresponding author.

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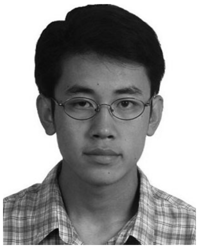

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