Conflicts and Incentives in Wireless Cooperative Relaying: A Distributed Market Pricing Framework

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Abstract—Extensive research in recent years has shown the benefits of *cooperative relaying* in wireless networks, where nodes overhear and cooperatively forward packets transmitted between their neighbors. Most existing studies focus on physical-layer optimization of the effective channel capacity for a given transmitter-receiver link; however, the interaction among simultaneous flows between different endpoint pairs, and the conflicts arising from their competition for a shared pool of relay nodes, are not yet well understood. In this paper, we study a distributed pricing framework, where sources pay relay nodes to forward their packets, and the payment is *shared* equally whenever a packet is successfully relayed by several nodes at once. We formulate this scenario as a Stackelberg (leader-follower) game, in which sources set the payment rates they offer, and relay nodes respond by choosing the flows to cooperate with. We provide a systematic analysis of the fundamental structural properties of this generic model. We show that multiple follower equilibria exist in general due to the nonconcave nature of their game, yet only one equilibrium possesses certain continuity properties that further lead to a unique system equilibrium among the leaders. We further demonstrate that the resulting equilibria are reasonably efficient in several typical scenarios.

Index Terms—Cooperative communication, relay selection, Stackelberg game, pricing.

1 Introduction

Cooperative relaying has emerged in recent years as an important technique in wireless networks with unstable links. Cooperative relaying takes advantage of the broadcast nature of the medium and provides additional diversity against link outages (caused, e.g., by dynamic fading or shadowing effects) by allowing nearby nodes that overhear the transmitted signal to make additional transmissions to assist in delivering the data to its destination. The extensive research on the topic has resulted in a wide variety of proposed cooperation methods. For a single relay, these range from simple *decode-and-forward* of the data packet itself [1], [2] to *coded cooperation* where the relay transmits additional error-correcting code bits rather than retransmitting the original data [3]. Similar ideas have been extended to multiple-relay cooperation, where the receiver decodes the data by combining the relayed signals received either over separate multiplexed subchannels (e.g., CDMA [4] or TDMA [5]), or over the same subchannel with multiple receiving antennas using space-time codes [6], [7].

The vast majority of studies in the area have tackled the issue from a pure optimization perspective, focusing on strategies to maximize the performance (measured in terms of capacity or effective error rate) of the communication between a given source/destination pair with the aid of nearby relay nodes. However, when there are several pairs of nodes communicating in each other's vicinity, the use of cooperative relaying techniques creates a conflict if the relay nodes are within range of several endpoint pairs simultaneously. This conflict arises since, if a relay node takes a cooperative action (transmission) to assist a packet on one link, then it cannot do the same on another link at the same time. Moreover, the additional transmissions by the relay node(s) increase the interference for other links and thus adversely impact their capacities, even if the other links do not employ cooperative relaying directly. Such interactions among flows between different source/destination pairs in the presence of cooperative relaying, and distributed mechanisms for efficient allocation of cooperative relay nodes among the different flows (which may have different requirements and utilities), remain not yet well understood.

Motivated by the above observation, this paper proposes a pricing framework, based on the idea of "pay for cooperation," to encourage efficient use of the relay nodes. Under this framework, each flow (i.e., a source-destination pair) offers a payment per successfully received packet, which is shared equally among all relay nodes that participated in the delivery of that packet. Hence, the utility of a relay node is defined as its share of received payment minus its own cost of cooperation (e.g., due to energy spent for relaying). For a flow, the utility is defined as a generic concave function of the packet delivery rate minus the cost paid to the relay nodes. We model the

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Manuscript received 12 Jan. 2010; revised 7 May 2010; accepted 13 May 2010; published online 23 Aug. 2010.
Recommended for acceptance by C. Pinotti.
For information on obtaining reprints of this article, please send e-mail to: tpds@computer.org, and reference IEEECS Log Number TPDS-2010-01-0037.
Digital Object Identifier no. 10.1109/TPDS.2010.152.

resulting scenario as a Stackelberg game [8], in which the relay nodes are the *followers* that respond to payment rates set by the flows (i.e., each relay chooses which flow or mix of flows to serve so as to maximize its utility, given the payment rates offered by the flows and the actions of its competing peers); and the flows are the *leaders* that set the payment rates to maximize their own utility in anticipation of the Nash equilibrium (NE) response of the followers. We point out that, notwithstanding our interest in the cooperative relaying context, this model is very generic and can be applied in any scenario where a set of "jobs" compete for the services of a pool of "workers," such that the jobs set their payment rates, workers are free to choose the job(s) they will attempt, and the payment from each job is eventually shared equally among all the workers that completed it successfully.

The idea of pricing as a distributed control mechanism, to encourage autonomous and independent network users to make rational decisions that result in a social benefit for the entire network, has a long history. Its best-known application is the Network Utility Maximization (NUM) framework, originally proposed by Kelly et al. [9] and widely used since in the contexts of congestion control in the Internet (cf. [10] and references therein), as well as power and rate control in wireless networks [11], [12]. In the NUM approach, the network sets *shadow prices* for using its resources, and the goal is to set them so that the optimization of individual user responses will coincide with the distributed optimization of the social network utility. However, the NUM framework has limited applicability in the wireless cooperative relaying context, where the interaction among network "users" and "resources" can be complicated. Since nodes can play the dual role of sources of their own flows and relays for other flows at the same time, the distinction between the network "resources" and "users" is blurred, and there may not be an independent entity that can objectively set optimal prices in the network.

The research on alternative frameworks, suitable for networks of autonomous nodes, has ranged from incentive mechanisms such as reputation or mutual credits (see [13] and references therein) to, more recently, market-based schemes where nodes directly negotiate rates to be paid in exchange for forwarding each other's packets [14], [15], [16]. The operation of such market-based schemes is markedly different from the NUM approach, in that the flow sources take an active part in setting the prices they are willing to pay for forwarding their traffic. However, the results in [14], [15], [16] apply for a network model where the market is decoupled from the routing protocol. In other words, the routes of flows are assumed to be determined by a low-layer protocol and the nodes can only decide whether or not to route their peers' traffic along those routes; in particular, it is implicit that every packet can only be forwarded along a single route. Several mechanisms are proposed in [17] to encourage selfish nodes to declare their true costs and actions in order for the routing algorithm to choose the optimal paths, and a distributed protocol that uses cryptographic techniques to enforce the forwarding is described in [18].

Our work is inspired by the market-based pricing methodology, and can be viewed as its adaptation to the cooperative relaying context with multiple flows and relay nodes. However, the opportunistic nature of cooperative relaying with multiple relays—where any given packet may end up being relayed along a number of routes and it is not known in advance which one(s) will be successful—leads to several important differences between our model and those in past studies of packet forwarding in ad hoc networks. First, unlike most pricing methods in existing literature that only involve one kind of selfish players, our framework features two types of players (the relay nodes and the flow sources), each of which is not only in competition with its peers but also with players of the other type. In particular, this fact sets it apart from the scenario considered in [19], which studied the properties of a pricing game involving an access point (AP) and a single source node aided by a single (predetermined) cooperative relay node using an Amplify-and-Forward or Decode-and-Forward scheme; and the scenario in [20], which used a buyer/seller game model for relay selection to stimulate cooperation and improve the system performance, but only for a single-flow case. Furthermore, in any scheme where prices are applied to control forwarding along a single predetermined route, the expected utility of a player depends only on its own strategy once the prices are set, whereas in our case the payment from a flow is *shared* among all relay nodes participating in the relaying of that flow. As a result, a node's utility depends on the strategies of its peers, which leads to a competition scenario with much more complex interactions among the players. It turns out that the sharing of payment leads to utility functions that are nonconcave, requiring an original study of the game equilibrium properties that cannot draw on existing well-known results.

This paper presents a detailed study of the Stackelberg game among the flows and relay nodes and its equilibrium properties. Specifically, we establish that the followers' game admits two kinds of Nash equilibria, including a unique symmetrical NE where all relay nodes play the same mixed strategy, and a boundary NE where each relay node dedicates itself to a single flow. We further show that, from the leaders' perspective, a Stackelberg equilibrium may not exist if the followers play the boundary NE, yet it always exists and exhibits desirable uniqueness and convergence properties if the followers respond with the symmetrical NE. We emphasize that these novel results are substantially different from any other pricing method studied in the past. We also present several numerical examples to demonstrate the prices and utilities achieved in the game, and show that the equilibrium in general is reasonably efficient (i.e., has a low "price of anarchy" [21]).

The rest of this paper is structured as follows: Section 2 presents our system model and pricing framework and formulates the Stackelberg cooperative relaying game. Section 3 analyzes the properties of the followers' game and the corresponding equilibria, while Section 4 investigates equilibrium properties in the leaders' game, including existence, uniqueness, and dynamic convergence thereto. Section 5 includes some numerical examples demonstrating the efficiency of the equilibria. Finally, the paper is concluded in Section 6.

2 System Model and Pricing Framework

For the sake of concreteness, we present the game model and its analysis in the context of cooperative relaying in wireless networks. Nevertheless, as explained in the Introduction, the model can be readily applied in a variety of distributed scenarios where generic "jobs" compete for a common pool of "workers." Therefore, the following description and the use of terms such as "links," "flows," "transmission," etc., should be understood generically.

2.1 Wireless Network Model

We consider a set $\mathcal{F}$ of flows in a synchronized slotted wireless network. We use S_f and D_f to denote the source and destination of flow $f \in \mathcal{F}$, respectively. Each flow transmits a continuous stream of packets, where each packet from every flow takes an identical transmission time (a slot). A set $\mathcal{R}$ of potential *relay nodes*, with $|\mathcal{R}| = R \geq 2$, may retransmit packets to assist them reach their destinations. We assume that the different flows coexist on different "channels" in the network, such as CDMA or FDMA. Thus, if relay node $i \in \mathcal{R}$ decides to cooperate with flow f, it must tune itself to receive the packets from S_f; a node cannot overhear multiple flows simultaneously. All nodes cooperating with flow f relay every packet from that flow to D_f immediately after receiving it, i.e., simultaneously to each other. We assume that the different relay signals do not mutually interfere; e.g., they can be multiplexed on separate subchannels (as in [4], [5]) or based on space-time coding with multiple-antenna receivers [7]. A packet is considered successful if it is received error-free from at least one of the relay retransmissions.[1]

Accordingly, we consider a simple wireless link model in which any link is either "good" (i.e., error-free), or "bad" otherwise. We assume that links between different pairs of nodes are independent, which generally holds, in practice, for any realistic spacing between nodes (of at least a wavelength of the carrier frequency). For the sake of simplicity, the analysis, in this paper, assumes links that are memoryless on a packet time scale; that is, the probability of being in the "good" state is fixed and independent between subsequent slots. We denote this probability by P^f_{sn} for the link between S_f and any relay node, and P^f_{nd} for the link between any relay node and D_f. Thus, we assume a symmetrical setting where link probabilities are identical a priori among all relays (though not necessarily among all flows). The extension of our analysis to asymmetric and/or nonmemoryless (e.g., on-off) links is left for future work.

2.2 Pricing Framework

We denote the cost (e.g., in terms of energy) for a relay node to cooperate with a packet from flow f by e^f. For a selfish relay node to make a cooperative transmission, it must expect to receive a payment in return that is greater than its energy cost. Each flow f offers a payment of C^f per successful packet, where C^f is decided by the flow itself (i.e., C^f is the *strategy* of f). Hence, the utility of flow $f \in \mathcal{F}$ is defined as the net payoff that f gets per slot:

$$U_f \triangleq u_f\left(P^f_{suc}\right) - C^f P^f_{suc}, \tag{1}$$

where P^f_{suc} is the probability of a packet of f to be successfully received at the destination (which depends on the relay node strategies, as described below). The function $u_f(P^f_{suc})$ characterizes the application payoff (e.g., satisfaction level) of f from a delivery probability of P^f_{suc}. We assume $u_f(P^f_{suc})$ is continuously differentiable, strictly increasing and weakly concave in P^f_{suc} (i.e., $u''_f(P^f_{suc}) \leq 0$), with $u_f(0) = 0$.

We turn to consider the utility of the relay nodes. If a packet is successful, the payment of C^f is shared equally among all nodes that successfully relayed it to D_f. We denote by r^f_i the probability of node i to relay a packet from flow f; thus, the vector $\mathbf{r_i} = \{r^f_i, f \in \mathcal{F}\}$, where $\sum_f r^f_i \leq 1$, is the *strategy* of relay node i. For brevity, we henceforth denote $K^f \triangleq P^f_{sn} P^f_{nd}$. Thus, the expected payoff per slot for node i is

$$V_i \triangleq \sum_{f \in \mathcal{F}} \left[C^f K^f r^f_i \sum_{l=0}^{R-1} \frac{P^f(l)}{l+1} - e^f r^f_i \right], \tag{2}$$

where

$$P^f(l) \triangleq \sum_{\substack{\mathcal{T} \subseteq \mathcal{R} \setminus \{i\} \\ |\mathcal{T}| = l}} \prod_{j_1 \in \mathcal{T}} K^f r^f_{j_1} \prod_{\substack{j_2 \notin \mathcal{T} \\ j_2 \neq i}} \left(1 - K^f r^f_{j_2}\right) \tag{3}$$

is the probability that there are l additional nodes beside i that successfully relay the packet of f to its destination as well. Note that, from the flow's perspective, the total success probability is $P^f_{suc} = 1 - \prod_{i \in \mathcal{R}} (1 - K^f r^f_i)$.

2.3 Stackelberg Game Formulation

We model the cooperative transmission with pricing as a *Stackelberg game*, in which the *leaders* choose their strategy first, and the *followers* respond by choosing their strategies accordingly, knowing the leaders' strategies [8]. In our setting, the game is defined as follows:

Follower's problem. Each follower (relay node i) chooses its strategy $\mathbf{r_i}$ to maximize its utility V_i in response to the leaders' strategies $\mathbf{C} \triangleq \{C^f, f \in \mathcal{F}\}$ and the strategies of its peers $\mathbf{r_{-i}} \triangleq \{\mathbf{r_j}, j \neq i\}$. Thus, each node i solves the following problem:

$$\mathbf{r^*_i}(\mathbf{r_{-i}}, \mathbf{C}) = \text{argmax} V_i(\mathbf{r_i}, \mathbf{r_{-i}}, \mathbf{C}), \tag{4}$$

and a vector of followers' strategies is an NE if it corresponds to a fixed point of (4).

Leader's problem. Each leader (i.e., a flow f or its source-destination pair) chooses its strategy C^f to maximize its utility function U_f, given the strategies of its peers $\mathbf{C^{-f}} \triangleq \{C^{f'}, f' \neq f\}$ and anticipating that the followers will eventually respond with a collection of strategies that constitutes an NE according to (4). Thus, the leader's problem is described as

$$C^{f^*} = \text{argmax} U_f(C^f, \mathbf{C^{-f}}, \mathbf{r^*_i}(\langle C^f, \mathbf{C^{-f}} \rangle)). \tag{5}$$

The solution of the game is characterized by a *Stackelberg-Nash equilibrium (SNE)*, a strategy profile from which no player (leader or follower) has incentive to deviate unilaterally.

To conclude this section, we point out that the main goal in this work is to establish and demonstrate some fundamental structural properties of the resulting equilibria in the

1. We do not consider coded cooperation in this paper, where relay transmissions consist of error-correcting code bits rather than the packet itself, so that the packet can be recovered from a combination of relay signals even if no individual one is error-free; this extension is left to future work.

above generic market-based framework. Accordingly, our model contains several implicit simplifying assumptions, which we highlight below. First, we assume that the flow endpoints are honest, and indeed make the payments in equal shares to all successful relay nodes, as detected by the destination. We do not consider in detail the question of *payment enforcement* or other design issues that would arise in a practical mechanism implementing our scheme, other than mentioning that existing solutions (e.g., the cryptographic approach of [18]) could generally be used for our case as well.[2] Second, in order to allow the discussion to focus on the fundamental properties and insights, we assume, for simplicity, that the flow sources and relay nodes are distinct, and that all flows and nodes operate at the same rate (thus, the notion of "time slot" is identical for all players, and every flow has an endless supply of new packets to transmit in every slot). These assumptions are made mainly for the purpose of presentation clarity; they are not essentially critical to the analysis or the resulting equilibrium properties, and can be alleviated in a straightforward manner (so as, e.g., to cover the case where nodes may simultaneously play the role of sources and relay nodes for different flows). Finally, we point out that, even though our analysis focuses only on the case of symmetrical relay nodes, the numerical evaluation part (Section 5) includes an illustration of a nonsymmetrical case as well, and demonstrates that, indeed, certain system properties from our analysis apply in the general case to some extent. Space constraints do not allow us to consider the above extensions in greater detail within this paper, and therefore, they are left for future work.

3 Equilibrium Analysis of the Followers' Game

The goal of our subsequent analysis is to find and characterize the properties of the SNE of the above game. To that end, we first study the followers' game and obtain the best-response strategies and equilibrium properties for a given vector of leaders' strategies $\mathbf{C}$. Before proceeding, we emphasize that, in general, the utility functions in this game (V_i) are not concave. To see this, consider the simple example of a single flow served by three nodes over perfectly reliable links ($K^f = 1$) with $C^f = 1$ and $e^f = 0$. Then, the utility function of relay node 1 reduces to

$$V_1 = r_1\left[(1-r_2)(1-r_3) + \frac{r_2 + r_3 - 2r_2r_3}{2} + \frac{r_2r_3}{3}\right],$$

which is, in fact, nonconcave; for example,

$$V_1\begin{pmatrix} r_1 = 1 \\ r_2 = 0.5 \\ r_3 = 0.5 \end{pmatrix} = \frac{7}{12} < \frac{1}{2}\left[V_1\begin{pmatrix} r_1 = 1 \\ r_2 = 0 \\ r_3 = 0 \end{pmatrix} + V_1\begin{pmatrix} r_1 = 1 \\ r_2 = 1 \\ r_3 = 1 \end{pmatrix}\right] = \frac{2}{3}.$$

As a result, many well-known generic properties of games with concave utility functions, such as equilibrium existence and uniqueness, do not hold in our case. This fact is illustrated in the following simple yet insightful example, which also demonstrates some of the equilibrium properties we prove subsequently.

Example 1. Consider a system with two flows offering identical payments of $C^1 = C^2 = 1$, $e^1 = e^2 = 0$, and two relay nodes with perfectly reliable links. In this system, the following equilibria exist in the followers' game:

- $\mathbf{r_1} = \mathbf{r_2} = (\frac{1}{2}, \frac{1}{2})$ (i.e., each node allocates half of its cooperation to each of the flows). To see that this is an NE, note that, for $\mathbf{r_2}$ fixed at $(\frac{1}{2}, \frac{1}{2})$, the utility function of node 1 reduces to $V_1 = (r_1^1 + r_1^2) \cdot (\frac{1}{2} + \frac{1}{4})$, which is maximized by any strategy with $r_1^1 + r_1^2 = 1$ (intuitively, the node maximizes the received payment by increasing its cooperation effort to the maximum, and is indifferent between the two flows). The same logic holds for node 2 with $\mathbf{r_1}$ fixed. We refer to this NE, where all nodes apply an identical strategy, as the *symmetrical* NE.
- $\mathbf{r_1} = (1, 0)$, $\mathbf{r_2} = (0, 1)$ (or vice versa). Indeed, when each node cooperates with one flow, there is no incentive for any of them to deviate by shifting some of the cooperation probability to the other flow, where the expected payment rate is lower due to competition with the other node. We refer to such an NE, where every node cooperates only with one flow, as a *boundary* NE.

It is easily confirmed that the system has no other equilibria.

As we show below, this simple example is indicative of the followers' game properties in general. It can be noted that the existence of an equilibrium per se in the followers' game could be readily shown with well-known generic tools, e.g., the Kakutani fixed point theorem. However, such tools do not provide the more detailed structural properties of the equilibria, which we obtain via direct analysis. Indeed, the main result of this section is that, for any $\mathbf{C}$, the followers' game always admits a unique symmetrical equilibrium, as well as at least one boundary equilibrium. These properties will subsequently turn out to be crucial for the discussion of the leaders' game and the overall system equilibrium in Section 4.

3.1 Symmetrical Equilibria

We commence our discussion of the equilibrium properties by considering the best response of relay node i, with the strategy $\mathbf{r_i} = (r_i^1, \ldots, r_i^{|\mathcal{F}|})$. The corresponding optimization problem from the perspective of node i can be stated as follows:

$$\max_{\mathbf{r_i}} V_i(\mathbf{r_i}, \mathbf{r_{-i}}) \quad s.t. \quad \sum_{f\in\mathcal{F}} r_i^f \leq 1 \quad \text{and} \quad r_i^f \geq 0, \forall f \in \mathcal{F}. \quad (6)$$

Since V_i is continuously differentiable in r_i^f, it follows that the first-order Kuhn-Tucker conditions corresponding to problem (6) are necessary for optimality. On the other hand, we note from (2) that, for a fixed $\mathbf{r_{-i}}$, the function $V_i(\mathbf{r_i}, \mathbf{r_{-i}})$ is linear in $\mathbf{r_i}$ (the coefficient of each r_i^f is constant). This implies that the same Kuhn-Tucker conditions are sufficient for optimality as well. We conclude that a strategy profile is an

2. It is important to emphasize that we do not assume the *relay nodes* to be honest: by definition, there is no upper-layer protocol prescribing to relay nodes how to forward packets, and each relay can freely choose its cooperation probability with every flow and thus seek to maximize its own utility.

equilibrium if and only if there exist $\lambda_i \geq 0$ and $\{\mu_i^f \geq 0, f \in \mathcal{F}\}$ such that the following conditions are met:

$$\frac{\partial V_i}{\partial r_i^f} = \lambda_i - \mu_i^f \quad \forall f \in \mathcal{F}, \tag{7}$$

$$\lambda_i \left(\sum_{f \in \mathcal{F}} r_i^f - 1 \right) = 0, \tag{8}$$

$$\mu_i^f r_i^f = 0. \tag{9}$$

We now focus on *symmetrical* strategy profiles, where all nodes use an identical strategy. To that end, we define the function

$$g^f(x) \triangleq \left. \frac{\partial V_i}{\partial r_i^f} \right|_{r_j^f = x, \forall j \in \mathcal{R}} \tag{10}$$

(note the index i is dropped since the function does not depend on the choice of any specific i). Computing the derivative of (2) and noticing that, in a symmetrical strategy profile, all the summation terms in the right-hand side of (3) are identical, we obtain an explicit expression for $g^f(x)$:

$$\begin{aligned} g^f(x) &= C^f K^f \sum_{l=0}^{R-1} \binom{R-1}{l} \frac{(K^f x)^l (1 - K^f x)^{R-1-l}}{l+1} - e^f \\ &= C^f \frac{1 - (1 - K^f x)^R}{Rx} - e^f; \end{aligned} \tag{11}$$

in particular, $g^f(0) = \lim_{x \to 0} g^f(x) = C^f K^f - e^f$. For convenience, we also define $h^f(x) \triangleq \frac{1-(1-K^f x)^R}{Rx}$; thus $g^f(x) = C^f h^f(x) - e^f$.

At this point, we state some monotonicity properties that will be useful in several subsequent proofs.

Lemma 1. *The following monotonicity properties hold:*

1. *$g^f(x)$ and $h^f(x)$ are strictly decreasing in x;*
2. $$\frac{dh^f(x)}{dx}$$ *is strictly increasing in x;*
3. $$\frac{dh^f(x)/dx}{[h^f(x)]^2}$$ *is strictly decreasing in x.*
4. $$-\frac{h'^f(x)}{h^f(x)} \frac{K^f x}{(1 - K^f x)^{R-1}}$$ *is increasing in x.*

Proof. See Appendix. □

We now state the main result of this section.

Theorem 1. *For any vector of flow prices* $\mathbf{C}$*, there exists a unique set of $\{\rho^f, f \in \mathcal{F}\}$ such that the symmetrical strategy profile $r_j^f = \rho^f, \forall j \in \mathcal{R}$ is a Nash equilibrium. Furthermore, there exist $\lambda \geq 0$,$\{\mu^f \geq 0, f \in \mathcal{F}\}$, such that*

$$g^f(\rho^f) = \lambda - \mu^f \quad \forall f \in \mathcal{F}, \tag{12}$$

$$\lambda \left(\sum_{f \in \mathcal{F}} \rho^f - 1 \right) = 0, \tag{13}$$

$$\mu^f \rho^f = 0 \quad \forall f \in \mathcal{F}. \tag{14}$$

Proof. First, we note that, at the symmetrical strategy profile defined by $\{\rho^f, f \in \mathcal{F}\}$, we have (by definition)

$$\left. \frac{\partial V_i}{\partial r_i^f} \right|_{r_i^f = \rho^f, \mathbf{r_{-i}}} = g^f(\rho^f).$$

Thus, (12)-(14) coincide with the Kuhn-Tucker conditions (7)-(9) in this case. Accordingly, the set $\{\rho^f, f \in \mathcal{F}\}$ corresponds to a symmetrical NE if and only if it satisfies (12)-(14).

It remains to show that there exists a unique combination of $\{\rho^f\}$, λ, and $\{\mu^f\}$ satisfying (12)-(14). To that end, define the function $V(\mathbf{x})$, where $\mathbf{x} = (x^1, \ldots, x^{|\mathcal{F}|})$, as follows: $V(\mathbf{x}) \triangleq \sum_{f \in \mathcal{F}} \int_0^{x^f} g^f(\xi) d\xi$. Consider the following optimization problem:

$$\max_{\mathbf{x}} V(\mathbf{x}) \quad s.t. \quad \sum_{f \in \mathcal{F}} x^f \leq 1 \quad \text{and} \quad x^f \geq 0, \quad \forall f \in \mathcal{F}. \tag{15}$$

Since $V(\mathbf{x})$ is a sum of integrals of decreasing functions (Lemma 1), it is continuously differentiable and concave, and, therefore, the above constrained optimization problem over a compact region must have a unique solution, which can be denoted by $\{\rho^f, f \in \mathcal{F}\}$. This solution must satisfy the Kuhn-Tucker conditions corresponding to problem (15), which are precisely the conditions listed in (12)-(14). □

3.2 Boundary Equilibria

We now turn our attention to equilibria consisting of *boundary* strategies. We say that node i plays a boundary strategy if r_i^f is either 0 or 1 for all $f \in \mathcal{F}$. Our main result in this section is stated next.

Theorem 2. *For any vector of flow prices* $\mathbf{C}$*, there exists a boundary equilibrium with $r_i^f \in \{0, 1\}$ for all $i \in \mathcal{R}$, $f \in \mathcal{F}$.*

Proof. We construct a simple algorithm that finds a boundary equilibium, as follows: Initially, start with the strategy $r_i^f = 0$ for all $i \in \mathcal{R}$, $f \in \mathcal{F}$. Thereafter, proceed with R iterations to assign the relay nodes to flows, where in each iteration $k = 1, \ldots, R$,

- denote $R^f = \sum_i r_i^f$, $f \in \mathcal{F}$, to be the number of nodes assigned to flow f so far;
- find the flow f^* such that setting $r_k^{f^*} = 1$ maximizes V_k;
- if assigning $r_k^{f^*} = 1$ results in $V_k \leq 0$, stop; otherwise assign $r_k^{f^*} = 1$ and proceed to the next iteration.

We elaborate on the second step in the above. From (2), if node k is assigned to flow f in the iteration where R^f

other nodes already assign a strategy probability of 1 to that flow, then the node's utility will be

$$C^f K^f \sum_{l=0}^{R^f} \frac{1}{l+1} \binom{R^f}{l} (K^f)^l (1-K^f)^{R^f-l} - e^f = C^f \cdot \frac{1-(1-K^f)^{R^f+1}}{R^f+1} - e^f, \tag{16}$$

and the flow f^* to which node k should be assigned is thus the one that maximizes (16) among all flows in the iteration.

We now show by induction that after each iteration k, the strategies so far $\{\mathbf{r_i}\}$ constitute an equilibrium among relay nodes $i = 1, \ldots, k$. This is clearly true for $k = 1$. Assume that it is true for $k = n-1$, and consider iteration $k = n$. Let f_n be the flow chosen by node n. By construction, f_n is the flow offering the maximum payment share to node n, which thus has no incentive to deviate any probability to other flows where the expected payment share is smaller. Since the relay nodes are symmetric, the same holds for all other nodes assigned to flow f_n before iteration n. Now, consider any node j assigned to a flow $f_j \neq f_n$. By the induction assumption, node j had no incentive to deviate before iteration n, and since the only change from iteration $n-1$ to n is that R_{f_n} has increased (i.e., f_n will now offer an even lesser payment share to new nodes than before), it follows that j will have no incentive to deviate after iteration n either. Therefore, the strategies after each iteration are in equilibrium among the nodes assigned so far, which eventually leads to a boundary equilibrium after all R nodes are assigned. □

We emphasize that, unlike the symmetrical equilibrium, the boundary equilibrium is not necessarily unique (even after allowing for permutation of the nodes), since (16) in a particular iteration may be maximized by more than one flow. For example, consider again the system from Example 1, except that now $C^1 = 2$ and $C^2 = 1$. Applying the algorithm in the proof of Theorem 2, the first node is assigned to flow 1; then, the second node becomes indifferent between the two flows (it can either cooperate with flow 2 and receive the full payment of $C^2 = 1$, or with flow 1 and receive half of the payment of $C^1 = 2$). Consequently, $\mathbf{r_1} = (1, 0)$ and either $\mathbf{r_2} = (1, 0)$ or $\mathbf{r_2} = (0, 1)$ are boundary equilibria (in fact, $\mathbf{r_2} = (r_2^1, r_2^2)$ with any $r_2^1 + r_2^2 = 1$ will bring about an equilibrium).

We conclude this section by showing the following property of strictly interior equilibria in the followers' game. To that end, a strictly interior equilibrium is defined as an equilibrium where the cooperation probability of any relay node with every flow is strictly positive.

Theorem 3. *If a strictly interior equilibrium exists in the followers' game, then it is symmetrical.*

Proof. Suppose, to the contrary, that in a strictly interior equilibrium $\mathbf{r}$, there exists a flow f_0 and relay nodes i, j such that $r_i^{f_0} < r_j^{f_0}$. Consider the partial derivative

$$\frac{\partial V_i}{\partial r_i^f} = C^f K^f \sum_{l=0}^{R-1} \frac{P^f(l)}{l+1} - e^f, \tag{17}$$

where $P^f(l)$ is defined by (3). Separating the terms in (17) that depend on r_j^f from those that do not, we obtain

$$\frac{\partial V_i}{\partial r_i^f} = C^f K^f \left[\sum_{l=0}^{R-2} \frac{P_j^f(l)}{l+1} - K^f r_j^f \sum_{l=0}^{R-2} \frac{P_j^f(l)}{(l+2)(l+1)} \right] - e^f, \tag{18}$$

where

$$P_j^f(l) \triangleq \sum_{\substack{\mathcal{T} \subseteq \mathcal{R} \setminus \{i,j\} \\ |\mathcal{T}| = l}} \prod_{j_1 \in \mathcal{T}} K^f r_{j_1}^f \prod_{\substack{j_2 \notin \mathcal{T} \\ j_2 \neq i,j}} \left(1 - K^f r_{j_2}^f\right) \tag{19}$$

does not depend on either r_i^f or r_j^f. A similar expression can be obtained for $\frac{\partial V_j}{\partial r_j^f}$, separating the terms that depend on r_i^f from those that do not. It follows that, for the flow f_0,

$$r_i^{f_0} < r_j^{f_0} \Longrightarrow \frac{\partial V_i}{\partial r_i^{f_0}} > \frac{\partial V_j}{\partial r_j^{f_0}}. \tag{20}$$

On the other hand, consider the Kuhn-Tucker conditions (7)-(9) for the equilibrium $\mathbf{r}$. Since $\mathbf{r}$ is strictly interior, it follows that $\mu_i^f = 0, \forall i \in \mathcal{R}, f \in \mathcal{F}$. Thus, from (7) and (20), we have

$$\lambda_i = \frac{\partial V_i}{\partial r_i^{f_0}} > \frac{\partial V_j}{\partial r_j^{f_0}} = \lambda_j, \tag{21}$$

which therefore leads to $r_i^f < r_j^f$ for *all* $f \in \mathcal{F}$ (not just f_0), and, therefore, $\sum_{f \in \mathcal{F}} r_i^f < \sum_{f \in \mathcal{F}} r_j^f \leq 1$. However, (8) then implies $\lambda_i = 0$, which is obviously a contradiction with (21). □

4 Analysis of the Leaders' Game

In this section, we study the properties of the leaders' game and its equilibrium (which is the overall SNE of the system). Our task is complicated by the findings of section 3, namely, that once the leaders' strategies $\mathbf{C}$ are set, the followers' equilibrium is not unique. To get around this complication, we first analyze the leaders' game under the assumption that the followers always respond by playing in their symmetrical equilibrium (which is always unique). For this case, we establish that an equilibrium in the leaders' game always exists, is unique, and is strongly stable in the sense that the game always converges to it from any initial vector of flow prices under best-response dynamics. We then show that an SNE may not exist at all if the followers may play an equilibrium other than the symmetrical one.

4.1 Followers Play Symmetrical Equilibrium

We explore how the followers' symmetrical equilibrium defined in Theorem 1 depends on the payment rate C^f of a specific flow $f \in \mathcal{F}$, with all other rates $\mathbf{C^{-f}}$ remaining fixed. To streamline the discussion, we view the value of ρ^f in the equilibrium corresponding to a setting of C^f as a function $\rho^f = F(C^f)$ (a scalar function, since we focus only on ρ^f and are not interested in the strategy values for other flows). Also, we define the value of λ that satisfies (13) in the equilibrium as a function $\lambda = \Lambda(C^f)$.

We begin by exploring these functions for extreme values of C^f. Clearly, with $C^f = 0$, the utility of cooperating with flow f for any relay node is nonpositive, implying $\rho^f = F(C^f = 0) = 0$. Denote $\lambda_0 = \Lambda(C^f = 0)$; then, if C^f is gradually increased, the equilibrium remains unchanged as long as $g^f(0) = K^f C^f - e^f \le \lambda_0$, since (12) can still be satisfied with some $\mu^f \ge 0$. We conclude that $F(C^f) = 0$ and $\Lambda(C^f) = \lambda_0$ for all $C^f \le C^f_{\min} \triangleq \frac{\lambda_0 + e^f}{K^f}$.

On the other hand, if C^f is very large (more precisely, $C^f \ge C^f_{\max} \triangleq [(\max_{f' \ne f} C^{f'} K^{f'} - e^{f'}) + e^f] \frac{R}{1-(1-K^f)^R}$, which implies that $g^f(1) = C^f \frac{1-(1-K^f)^R}{R} - e^f \ge g^{f'}(0) = C^{f'} K^{f'} - e^{f'}$ for any $f' \ne f$), then the equilibrium conditions will be satisfied by $\rho^f = F(C^f) = 1$ and $\lambda = \Lambda(C^f) = g^f(1)$, with $\mu^{f'} > 0$ and therefore $\rho^{f'} = 0$ for all $f' \ne f$. Intuitively, if C^f is so large that even a share of just $\frac{1}{R}$ of the payment from f is larger than the full payment from any other flow, then no node will deviate from cooperating fully with f.

We now proceed to explore the functions $F(C^f)$ and $\Lambda(C^f)$ between these extremes, i.e., in the range $C^f_{\min} \le C^f \le C^f_{\max}$. The following lemmas state the necessary monotonicity properties of these functions; for clarity of presentation, the detailed proofs of these lemmas are deferred to the Appendix.

Lemma 2. *The function $\Lambda(C^f)$ is continuous and nondecreasing in C^f.*

Lemma 3. *The function $F(C^f)$ is continuous, and, in the range $C^f_{\min} \le C^f \le C^f_{\max}$, strictly increasing in C^f.*

Lemma 4. *The function $F(C^f)$ is concave in C^f in the range $C^f_{\min} \le C^f \le C^f_{\max}$.*

Proof. See Appendix. □

We are now equipped to explore how the utility received by flow f depends on its choice of C^f, i.e., we consider (with a slight abuse of notation) the function $U_f(C^f, \mathbf{C^{-f}})$. We also define the *best-response function* of flow f to be $B^f(\mathbf{C^{-f}}) \triangleq \operatorname{argmax}_{C^f} \quad U_f(C^f, \mathbf{C^{-f}})$. The following two lemmas state important properties of these functions:

Lemma 5. *For a fixed $\mathbf{C^{-f}}$, the function $U_f(C^f, \mathbf{C^{-f}})$ is concave in C^f.*

Proof. Consider the derivative of the utility function with respect to C^f, assuming that the followers respond with a symmetrical equilibrium as analyzed above:

$$\frac{\partial U_f}{\partial C^f} = \left[u'_f(P^f_{suc}(\rho^f)) - C^f\right] \frac{\partial P^f_{suc}(\rho^f)}{\partial \rho^f} \frac{\partial \rho^f}{\partial C^f} - P^f_{suc}(\rho^f), \quad (22)$$

where $\rho^f = F(C^f)$. Since $u_f(P^f_{suc})$ is concave by assumption, $P_{suc}(\rho^f) = 1 - (1 - K^f \rho^f)^R$ is increasing and concave in ρ^f, and ρ^f is concave in C_f by Lemma 4, it follows that $\frac{\partial U_f}{\partial C^f}$ is nonincreasing in C_f, i.e., U_f is indeed concave in C^f. □

Lemma 6. *The best-response function of flow f is bounded by $0 \le B^f(\mathbf{C^{-f}}) \le u'_f(0)$.*

Proof. Notice that U_f can be written as

$$U_f = \left[\frac{u_f(P^f_{suc})}{P^f_{suc}} - C^f\right] P^f_{suc}.$$

Obviously, in the best response, the utility is nonnegative (a utility of 0 can always be obtained by $C^f = 0$). Hence, $0 \le B^f(\mathbf{C^{-f}}) \le \max_{P^f_{suc}} \frac{u_f(P^f_{suc})}{P^f_{suc}}$. However, the concavity of $u_f(P^f_{suc})$ and $u_f(0) = 0$ imply that $\frac{u_f(P^f_{suc})}{P^f_{suc}} \le u'_f(0)$ for any $0 \le P^f_{suc} \le 1$, and the lemma follows. □

From Lemma 6, we conclude that if $u'_f(0) \le C^f_{\min} = \frac{\Lambda(C^f=0)+e^f}{K^f}$, then the flow can never achieve a positive utility. The strategy C^f used in this case is immaterial, since no node will cooperate with f under any $0 \le C^f \le u'(0)$, thus the payment of f to the nodes is 0 in any case. For concreteness, we define $B^f(\mathbf{C^{-f}}) = u'_f(0)$ in that case; this ensures the continuity of the best-response function with respect to $\mathbf{C^{-f}}$, since $\Lambda(0)$ is itself a continuous function of $\mathbf{C^{-f}}$.

Otherwise, if $u'(0) > C^f_{\min}$, the optimal C^f is obtained by solving $\frac{\partial U_f}{\partial C^f} = 0$. Due to the concavity of U_f in C^f (Lemma 5), a unique solution is guaranteed; furthermore, we observe that if u_f is continuously differentiable, then the best-response function is continuous as well.

We now state the main result of this section.

Theorem 4. *If the followers always respond by playing in their (unique) symmetrical NE, then an equilibrium of the leaders' game (i.e., an SNE of the overall system) exists and is unique.*[3]

Proof. Define the mapping $\mathbf{B}(\mathbf{C}) = \{B^f(\mathbf{C^{-f}}), f \in \mathcal{F}\}$ to be the collection of best-response functions to the respective strategy vectors of other flows. Since each component of $\mathbf{B}(\mathbf{C})$ is continuous and bounded (Lemma 6), the entire mapping is continuous and bounded. Therefore, it has a fixed point, which is an equilibrium of the leaders' game. This establishes the existence of the SNE.

In order to prove the uniqueness of the fixed point, we note that, in an equilibrium, for every flow $f \in \mathcal{F}$, unless $\rho^f = 0$, the equation $\frac{\partial U_f}{\partial C^f} = 0$ must be satisfied. The proof then proceeds by contradiction, showing that it is impossible to have two different points that solve the equation. For convenience, the details of the (long) proof are presented in the Appendix. □

So far, we have considered only the static properties of existence and uniqueness of the equilibrium in the leaders' game. In the final part of this section, we show that the game always converges to the equilibrium dynamically from any initial vector of strategies under a general assumption on the sequence of best-response strategy updates.

Lemma 7. *The best-response function $B^f(\mathbf{C^{-f}})$ is monotonic and nondecreasing in each component of $\mathbf{C^{-f}}$.*

Proof. Consider two vectors of flow prices $\langle C^f_a, \mathbf{C^{-f}_a} \rangle$, $\langle C^f_b, \mathbf{C^{-f}_b} \rangle$, such that $C^f_a = B^f(\mathbf{C^{-f}_a})$, $C^f_b = B^f(\mathbf{C^{-f}_b})$, and the only difference between $\mathbf{C^{-f}_a}$ and $\mathbf{C^{-f}_b}$ is that one component $C^{f'}$, $f' \ne f$, is changed between $C^{f'}_a$ and $C^{f'}_b$, where $C^{f'}_a < C^{f'}_b$. The lemma then states that $C^f_a \le C^f_b$.

If $C^f_b = u'_f(0)$, then the lemma holds trivially since C^f_b is already the upper bound of possible values of C^f (Lemma 6). Therefore, assume $C^f_b < u'_f(0)$, i.e., C^f_b is the

3. To be precise, the equilibrium is unique subject to the above definition of $B^f(\mathbf{C^{-f}})$ in the case of $u'_f(0) \le C^f_{\min}$; in other words, we assume a flow's best response is always uniquely defined and ignore the degree of freedom that exists if the flow's best-response utility is 0 anyway.

solution of the equation $\frac{\partial U_f}{\partial C^f} = 0$ at $\langle C_b^f, \mathbf{C_b^{-f}} \rangle$, and denote λ_b and ρ_b^f to be the respective quantities of the corresponding followers' equilibrium.

Next, consider the followers' equilibrium at the price vector $\langle C_b^f, \mathbf{C_a^{-f}} \rangle$, and denote the respective quantities by λ_{ba}, ρ_{ba}^f. By applying Lemma 2 from the perspective of flow f', we conclude that $\lambda_{ba} \leq \lambda_b$; consequently, (12) and the monotonicity of $g^f(\rho^f)$ then implies that $\rho_{ba}^f \geq \rho_b^f$.

Finally, consider (22). Since each of the terms on the right-hand side of (22) is decreasing in ρ^f, it follows that if $\frac{\partial U_f}{\partial C^f} = 0$ for $\rho^f = \rho_b^f$, then $\frac{\partial U_f}{\partial C^f}$ must be nonpositive for $\rho^f = \rho_{ba}^f$ when the value of C^f is the same ($C^f = C_b^f$). Lemma 5 then implies that the solution of $\frac{\partial U_f}{\partial C^f} = 0$ under $\mathbf{C_a^{-f}}$ is obtained for $C^f = C_a^f$ such that $C_a^f \leq C_b^f$. □

Theorem 5. *Assume that the leaders' game follows a best-response dynamics from some initial vector of prices $\mathbf{C^f}(0)$, i.e., from time to time, an asynchronous update step is taken where some flow $f \in \mathcal{F}$ updates its strategy from $C^f(n)$ to $C^f(n+1) = B^f(\mathbf{C^{-f}}(n))$, and the sequence of flows doing the update steps can be arbitrary as long as the number of steps between consequent updates of every individual flow is bounded. Then, $\lim_{n\to\infty} \mathbf{C^f}(n) = \mathbf{C}^*$, where $\mathbf{C}^*$ is the (unique) equilibrium of the leaders' game.*

Proof. First, consider an arbitrary sequence of update steps commencing from an initial vector of $\mathbf{C^f}(0) = \langle 0, 0, \ldots, 0 \rangle$, and denote the resulting sequence of flow price vectors by $\mathbf{C^f_{\min}}(n)$. Obviously, for any flow f, the first time the flow updates its strategy will be a nondecreasing update. In light of Lemma 7, it follows by induction that all updates must be nondecreasing, i.e., $\mathbf{C^f_{\min}}(n)$ is a nondecreasing sequence. Since $\mathbf{C^f_{\min}}(n)$ is bounded as well (Lemma 6), it follows that it must converge to a limit. Due to the continuity of the best-response function $\mathbf{B}(\mathbf{C^f})$, this limit must be its (unique) fixed point $\mathbf{C}^*$.

In a similar manner, consider a sequence of best-response updates from an initial vector of $\mathbf{C^f}(0) = \{u_f'(0)\}$ (i.e., the upper bounds of the respective flows' best responses, as per Lemma 6), denoted by $\mathbf{C^f_{\max}}(n)$. By the same token, Lemma 7 implies that all the updates in the sequence must be nonincreasing, and the sequence must therefore converge to $\mathbf{C}^*$.

Finally, consider a sequence of best-response updates $\mathbf{C^f}(n)$ commencing from an arbitrary initial vector of flow prices $\mathbf{C^f}(0)$. Without loss of generality, assume that all the prices are within the bounds set by Lemma 6 (otherwise, consider instead the sequence only after every flow has had at least one opportunity to update its strategy). From Lemma 7, it follows that $\mathbf{C^f_{\min}}(n) \leq \mathbf{C^f}(n) \leq \mathbf{C^f_{\max}}(n)$, provided that for every n, the update step is performed by the same flow in all three sequences. Since, as established above, $\mathbf{C^f_{\min}}(n)$ and $\mathbf{C^f_{\max}}(n)$ converge to $\mathbf{C}^*$, it follows that the same is true for $\mathbf{C^f}(n)$ as well. □

4.2 Followers Play Boundary Equilibrium

In this section, we show that the SNE existence property established in Theorem 4 does not extend in general to the case that the followers' response maybe any other than the symmetrical equilibrium. To that end, we first prove a structural property of any SNE in a system where the followers play in a boundary equilibrium, and then show that in some cases no strategy profile can possibly satisfy that property.

Theorem 6. *If relay nodes always respond to flow price settings by playing a boundary equilibrium, then at any SNE:*

1. *if R^f denotes the number of nodes cooperating with flow f, then either the set $\mathcal{F}' = \{f' | R^{f'} = 0\}$ is nonempty or the utility of every relay node is 0;*
2. *the utility values of all relay nodes are identical and equal to $H_{th} \triangleq \max f' \in \mathcal{F}'[u_{f'}(K^{f'}) - e^{f'}]$ (or 0 if H_{th} is negative).*

Proof. To show the first property, assume to the contrary that the set $\mathcal{F}'$ is empty (i.e., $R^f > 0$ for all $f \in \mathcal{F}$). The utility of each node cooperating with f is given by

$$H^f(R^f) \triangleq \frac{C^f\left[1 - (1 - K^f)^{R^f}\right]}{R^f} - e^f. \tag{23}$$

Consider the flow $f \in \mathcal{F}$ with the highest $H^f(R^f)$, and assume that $H^f(R^f) > 0$. Then, since $H^f(R^f) \geq H^{f'}(R^{f'}) > H^{f'}(R^{f'} + 1)$ for any $f' \in \mathcal{F}$, there exists an $\epsilon > 0$ by which C^f can be reduced such that the new $H^f(R^f)$ is still both positive and higher than $H^{f'}(R^{f'} + 1)$ for any $f' \in \mathcal{F}$, and, therefore, no node will deviate from cooperating with f. Therefore, C^f cannot be the best-response strategy of f.

If $\mathcal{F}'$ is nonempty, consider the flow $f \notin \mathcal{F}'$ with the highest $H^f(R^f)$. We observe that, if $H^f(R^f) > C^{f'}K^{f'} - e^{f'}$ for all $f' \in \mathcal{F}'$, then, again, C^f is not the best-response strategy for f since it can be reduced by some $\epsilon > 0$ without triggering a deviation of any relay node. On the other hand, if there exists any flow $\hat{f} \notin \mathcal{F}'$ with $H^{\hat{f}}(R^{\hat{f}}) < H_{th}$, then it follows that there exists a $f' \in \mathcal{F}$ which can "poach" one of relay nodes currently cooperating with $\hat{f}$ and obtain a positive utility, by setting $C^{f'} = \frac{u_{f'}(K^{f'})}{K^{f'}} - \epsilon$ for some sufficiently small $\epsilon > 0$. Combining the above observations, we conclude that, if $H_{th} \geq 0$, then $H(R^f) = H_{th}$ for any $f \notin \mathcal{F}'$, i.e., all relay nodes receive an identical utility of H_{th}. □

Corollary. *In a system with two symmetrical flows (i.e., with identical $u_f(\cdot)$ and K^f for $f \in \{1, 2\}$) and $e^f = 0, f \in \{1, 2\}$, an SNE does not exist.*

Proof. Consider the options allowed by Theorem 6. If the utility of all nodes is 0, then, with $e^f = 0$, this implies $C^f = 0$ for both flows. Clearly, this is not an SNE since each flow has an incentive to increase its C^f to a small positive value so as to encourage the nodes to cooperate with it and thereby obtain a positive utility.

On the other hand, if all nodes receive a positive utility of $H_{th} > 0$, the first property of the theorem implies that one of the flows (say, flow 2) does not have any nodes cooperating with it, and therefore, the other flow (say, flow 1) is bearing the payment for all the nodes, i.e.,

$$C^1 \frac{[1 - (1 - K^f)^R]}{R} = H_{th} = u_f(K^f).$$

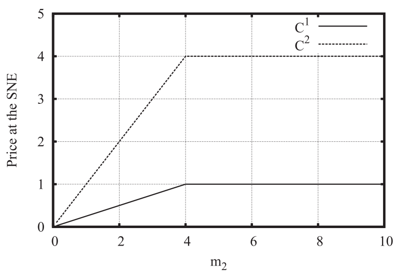

Fig. 1. Prices C^1, C^2 at the SNE (single relay case).

The utility of flow 1 is therefore

$$\begin{aligned} U_1 &= u_f\big(1-(1-K^f)^R\big) - C^1\big[1-(1-K^f)^R\big] \\ &= u_f\big(1-(1-K^f)^R\big) - R\cdot u_f(K^f) \\ &< u_f\big(R\cdot K^f\big) - R\cdot u_f(K^f) \le 0, \end{aligned}$$

where the inequalities follow from the monotonicity and concavity of u_f and the fact that $R \ge 2$. It follows that the first flow cannot be in a best-response strategy. Therefore, no SNE is possible in this system. □

Remark. The fact that $R \ge 2$ is crucial in the proof of the corollary above. If $R = 1$, i.e., there is only one relay node in the network, then the "followers' equilibrium" degenerates simply to that node cooperating with the flow f that provides the highest $H^f(1) = C^f K^f - e^f \ge 0$, resulting in $P_{suc}^f = K^f$ for that flow. It can be seen that a vector $\mathbf{C}$ that satisfies the following conditions is then a system SNE:

- $0 < C^{\hat f} \le u_{\hat f}(K^{\hat f})/K^{\hat f}$ and $C^{\hat f}K^{\hat f} - e^{\hat f} \ge 0$ for one particular $\hat f \in \mathcal{F}$;
- for all other $f' \ne \hat f$, $u_{f'}(K^{f'}) - e^{f'} \le C^{\hat f}K^{\hat f} - e^{\hat f}$;
- for at least one $f' \ne \hat f$, $u_{f'}(K^{f'}) - e^{f'} = C^{\hat f}K^{\hat f} - e^{\hat f}$.

In particular, such a vector always exists for symmetrical flows with $e^f = 0$ for all $f \in \mathcal{F}$, by setting $C^f = \frac{u_f(K^f)}{K^f}$ for all flows. In the corresponding equilibrium, the relay node will then cooperate with one of the flows $\hat f$, yet the utility of all flows is 0 and cannot be improved: flow $\hat f$ cannot reduce its $C^{\hat f}$ by any amount since that will cause the node to switch to a different flow, while any attempt to increase the offered payment by another flow will only result in a negative utility for that flow.

5 Numerical Examples

In this section, we demonstrate some of the theoretical results of the paper and gain further insight on the behavior of the game via a numerical study. We aim to present several scenarios indicative of the typical interactions among the players in the game, beginning with the simple case of two flows competing for the service of one relay, continuing with two flows with multiple relay nodes (up to $R = 5$), and finally considering the asymptotic case where the number of flows and relay nodes is large. In particular, we use these examples to comment on the issue of *equilibrium efficiency*, which was not explicitly addressed in the analytical part of the paper.

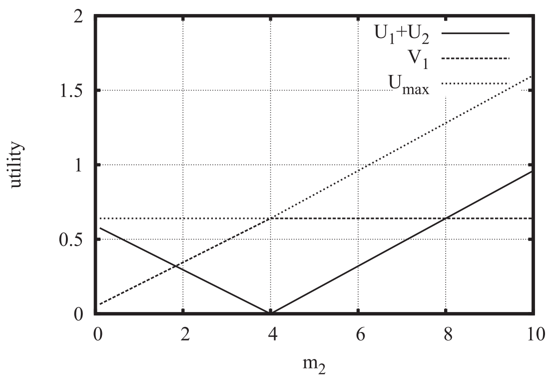

Fig. 2. Utility partition among flows and relay.

5.1 Competition between Heterogeneous Flows: Single Relay

We start with a degenerate scenario consisting of two flows and single relay node ($\mathcal{F} = \{1,2\}, \mathcal{R} = \{1\}$). For the flows, we adopt a linear utility function, as follows: $u_f(P_{suc}^f) = m_f P_{suc}^f, f \in \mathcal{F}$. In the following, we fix $m_1 = 1$ and vary m_2 to study how the game results depend on the heterogeneous flow utilities. We set $P_{sn}^1 = P_{nd}^1 = 0.8$ and $P_{sn}^2 = P_{nd}^2 = 0.4$ (which translates to $K^1 = 0.64$ and $K^2 = 0.16$), reflecting a difference in channel qualities between the endpoint pairs of the flows and the relay. Finally, we assume $e^f = 0$ for both flows. It is easily verified that the socially optimal operating point (i.e., one that maximizes the total utility of all flows) is achieved if the relay node cooperates entirely with the flow with the higher utility, i.e., $\mathbf{r} = (1,0)$ if $m_1K^1 \ge m_2K^2$, and $\mathbf{r} = (0,1)$ otherwise; therefore, the total maximum social utility is $U_{\max} = \max(0.64, 0.16m_2)$.

Fig. 1 plots the flow strategies C^1, C^2 in the resulting SNE as a function of m_2. Clearly, the relay node serves the more profitable flow with probability 1 at the equilibrium. Fig. 2 illustrates the utility for the flow side ($U_1 + U_2$) and the relay side (V_1), as well as the maximum social utility U_{max}. It is evident that $U_{max} = V_1 + U_1 + U_2$, i.e., the SNE always coincides with the socially optimal operating point. In other words, the proposed pricing mechanism and the resulting competition between the flows guides the relay node to operate efficiently without any information on the flows' utilities.

It is interesting to observe how the total system utility gets divided between the flows and the relay node. If m_2 is small, flow 1 can obtain the relay service for a very cheap cost, since flow 2 is limited in the price it can offer due to its own low utility. The full utility is thus retained by flow 1. As m_2 increases, flow 1 must increase its price so as to remain just slightly more attractive to the relay than the maximum offer flow 2 is able to make. Thus, the relay node gets paid more for its service, while the utility retained by the flow decreases. At $m_2 = \frac{K_1}{K_2} = 4$, the price war between the flows is at its peak, and the entire system utility of 0.64 is enjoyed by the relay node. For $m_2 > 4$, the first flow can no longer compete with the price able to be offered by flow 2; therefore, flow 2 can secure the service of the relay node by matching (or offering just slightly above) the maximum of flow 1, i.e., $C_2 = 4$. From that point, the utility retained by

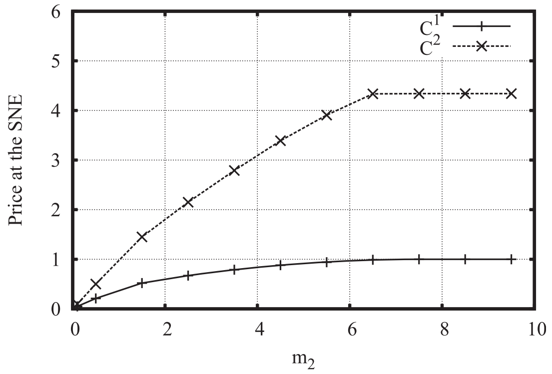

Fig. 3. Prices C^1, C^2 at SNE (two-relay scenario).

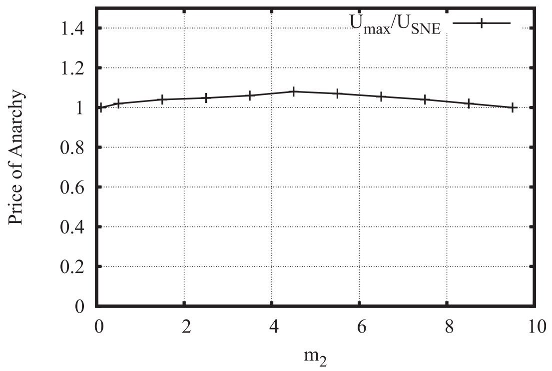

Fig. 4. Price of anarchy for two-relay scenario.

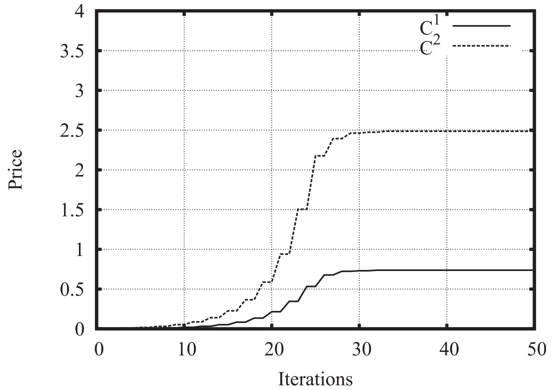

Fig. 5. Convergence of prices to SNE: scenario 2.

the relay node is constant, and all further increases in m_2 are reflected in the utility of flow 2.

5.2 Multiple Flows and Multiple Relay Nodes

We first consider a scenario with two flows and two relay nodes, which is more representative of the interactions among players in a multiple-flow and multiple-relay system. Apart from the second relay, all parameters are set to the same values as before. The results (for the case where the relay nodes play the symmetrical equilibrium) are shown in Figs. 3, 4, and 5. Fig. 3 shows the prices offered by the flows in the SNE. Fig. 4 displays the equilibrium efficiency as the "Price of Anarchy" [21], defined as the ratio between the optimal social utility and the system utility achieved at the equilibrium. Fig. 5 plots the convergence trajectories of the flows' best-response strategies for the case of $m_2 = 3$.

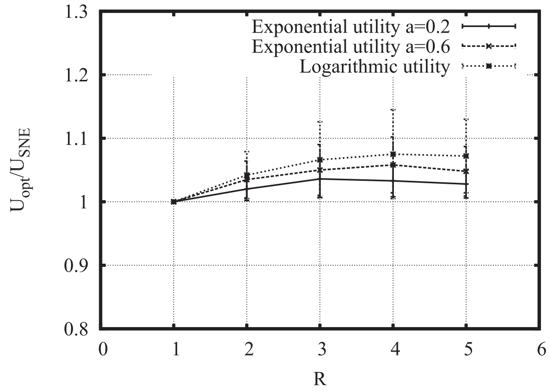

Fig. 6. Average ratio between U_{opt} and U_{SNE}.

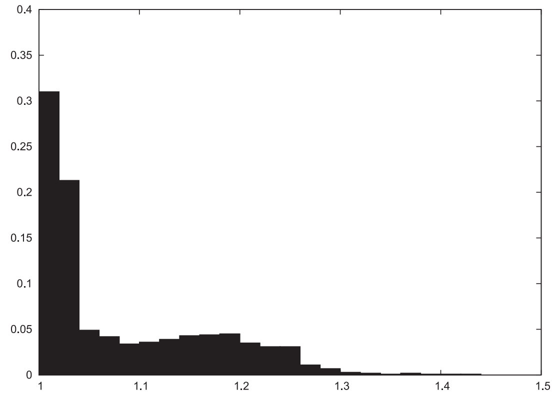

Fig. 7. Histogram of U_{SNE}/U_{opt} for $R = 4$.

From the results, we observe that the price of anarchy tends to 1 when the flows are heterogeneous, i.e., when m_2 is either very small or very large. This is explained by the fact that, in those extremes, both the equilibrium strategy and the global optimum require the relay nodes to cooperate fully with only one of the flows, respectively. Otherwise, for intermediate ranges of m_2, the SNE is less efficient since the symmetrical followers' equilibria tend to assign a nonzero cooperation probability to each of the flows, as neither flow is in a position to offer a price large enough to attract both relays entirely to itself. Nevertheless, we observe that even the worst price of anarchy is only slightly greater than 1.

To further investigate the efficiency of the proposed pricing framework, we conduct a range of simulations with two flows and R ($1 \leq R \leq 5$) relay nodes, using more sophisticated utility functions, namely power-law utility ($u_f(P^f_{suc}) = m_f(P^f_{suc})^a,\ 0 < a \leq 1$) and logarithmic utility ($u_f(P^f_{suc}) = m_f log(1 + P^f_{suc})$). For each R, we run 100 random scenarios with m_f,e^f, and K^f uniformly distributed in the range of $m_f \in [1, 100]$, $e^f \in [0, 10]$, $K^f \in [0, 1]$, respectively. Fig. 6 plots the average ratio between U_{opt}, the total system utility at the global optimum, and U_{SNE}, the total utility at the SNE. Fig. 7 shows a representative histogram of the ratio U_{SNE}/U_{opt} (for $R = 4$ with logarithmic utility; a similar histogram is observed in other cases). These results suggest that the proposed pricing framework can bring about a reasonably efficient equilibrium with only a small system utility loss due to players' selfishness.

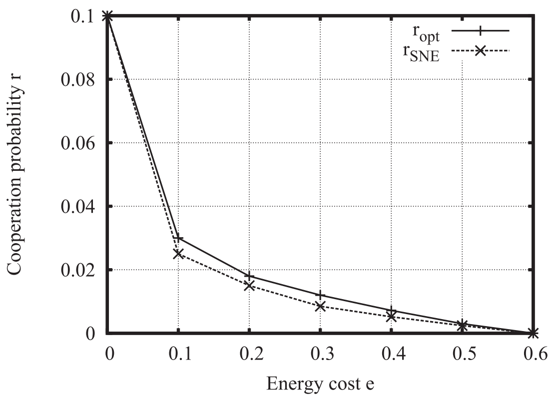

Fig. 8. Cooperation probability as a function of e.

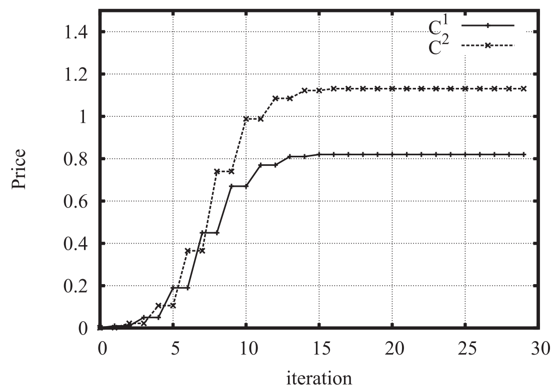

Fig. 10. Trajectory of prices: asymmetrical scenario.

5.3 Large System Scenario: Many Flows and Many Relay Nodes

In this section, we demonstrate the asymptotic properties of the game in a large-scale symmetrical scenario, consisting of 100 relay nodes and 10 flows with identical parameters of $K^f = 0.6$, $u_f(P^f_{suc}) = P^f_{suc}$, and $e^f = e$ for all flows. Clearly, in this case, the cooperation probabilities of all relay nodes with all flows are identical ($r^f_i = r, \forall i \in \mathcal{R}, f \in \mathcal{F}$), whether in the (unique) system equilibrium or in the optimal operating point (i.e., one that maximizes the total system utility). Accordingly, we vary the energy cost e and compare the cooperation probability and the total utility achieved in the unique SNE versus the optimum, as a function of e.

The results are shown in Figs. 8 and 9. When the energy cost is either very high or very low, the strategy of the relay nodes at the SNE tends to the same as in the optimum, i.e., respectively, not to cooperate at all ($r = 0$) or to cooperate completely ($r^f_i = \frac{1}{|\mathcal{F}|}$ for every $f \in \mathcal{F}$). Otherwise, we observe that the relay nodes' strategy at the SNE is more conservative than in the optimum, leading to a lower P^f_{suc} for the flows and consequently a lower total system utility; however, once again we observe that the price of anarchy remains small, not exceeding 1.28 anywhere.

5.4 An Asymmetrical Scenario

Our analysis in this paper has established several fundamental properties of equilibria in systems with symmetrical relay nodes, i.e., where K^f and e^f for any $f \in \mathcal{F}$ are the same across all nodes. In the final part of this section, we aim to illustrate that, to some extent, the equilibrium and convergence properties carry over to asymmetrical scenarios as well, where the success probability K^f_i for flow f is not the same for different $i \in \mathcal{R}$. To that end, we consider a system of two flows and two relay nodes, with $K^1_1 = K^2_1 = 0.8$, $K^1_2 = 0.5$, $K^2_2 = 0.4$, $e^1 = e^2 = 0$, $u_f(P^f_{suc}) = m_f P^f_{suc}$, where $m_1 = 1$ and $m_2 = 2$.

With regard to the followers' game, for any $\mathbf{C} = (C^1, C^2)$, an interior strategy profile $\mathbf{r} = \{r^f_i\}, 0 \le r^f_i \le 1$ is an equilibrium if it satisfies the condition

$$\frac{\partial V_i}{r^f_i} = 0 \quad i, f = 1, 2.$$

Consequently, we explore the dynamics of the game from the leaders' perspective, starting initially from $C^1 = C^2 = 0$ and allowing each flow, in turn, to make a best-response update of its strategy, i.e., at each step n, find $C^f(n)$ that maximizes $u_f(C^f(n), C^{-f}(n-1))$, under the assumption that the followers will play in an interior equilibrium (if it exists). The resulting price trajectories are shown in Fig. 10. We observe that $\mathbf{C(n)}$ converges to $\mathbf{C}^* = (0.780, 1.106)$, which is a fixed point of the best response, or a system SNE; the corresponding followers' equilibrium is $\mathbf{r}_1 = (r^1_1, r^2_1) = (0.335, 0.665)$ and $\mathbf{r}_2 = (r^1_2, r^2_2) = (0.194, 0.806)$. Incidentally, it is interesting to note that, with the same vector of prices $\mathbf{C}^*$, a boundary equilibrium in the followers' game exists as well (namely, $\mathbf{r}_1 = (0, 1)$, $\mathbf{r}_2 = (1, 0)$). This example clearly shows that some of the properties established in the paper, e.g., the existence of interior and boundary equilibria in the followers' game and the convergence to the SNE, apply to some extent beyond the limited symmetrical node model considered in the paper. Establishing the precise conditions for the existence and uniqueness of the system equilibrium, and the dynamic convergence thereto, for the general asymmetric case, remains an important topic for future work.

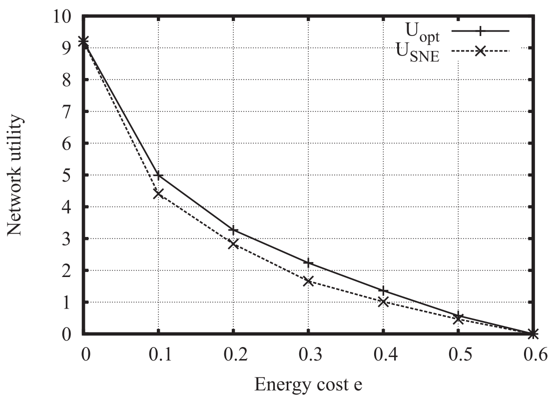

Fig. 9. Network utility at symmetric optimum and SNE.

6 Conclusion

We have proposed a market-based pricing framework for wireless networks with autonomous nodes in the context of cooperative relaying. An important difference between our model and other related studies that feature payment for packet forwarding is that a packet maybe relayed by several nodes simultaneously, and, therefore, the payment is shared among several nodes that have participated in its delivery. We have shown that this variation leads to substantially different properties of the resulting game model. In particular, we have established that the game among the relay nodes (followers) possesses several kinds of Nash equilibria, including a unique symmetrical NE and at least one boundary NE. Furthermore, we have established that the game among the flows (leaders) always possesses a unique and strongly stable Stackelberg equilibrium if the followers respond in their symmetrical NE, but an equilibrium may not exist at all if the followers play in a boundary NE. Finally, we demonstrated that the resulting system equilibrium is reasonably efficient from a social perspective, particularly when the flows have very heterogeneous utilities.

Our work was motivated by cooperative relaying in wireless networks and presented in terms of modeling the competition among flows and relay nodes in that context. Nevertheless, the model we studied, and its novel aspect of shared payment, are generic and readily applicable to many other distributed systems where several "workers" may carry out the same "job" (e.g., for reliability purposes) and share the payment equally. Our study, in this paper, focused on the fundamental properties of the game, namely, existence and uniqueness of equilibria. Some important theoretical questions that remain to be investigated are the bounds on equilibrium efficiency (e.g., expressed as the price of anarchy), as well as extensions of the fundamental equilibrium properties to general asymmetrical systems. These important directions remain the subject for future work.

APPENDIX

Proof of Lemma 1. For convenience, we introduce a variable change of $y \triangleq 1 - K^f x$, and slightly abuse notation by referring to $g^f(y)$ and $h^f(y)$ as functions of y.

We compute the derivatives of $h^f(y)$:

$$\frac{dh^f(y)}{dy} = \frac{K^f}{R(1-y)^2}\left[1 + (R-1)y^R - Ry^{R-1}\right]; \qquad (24)$$

$$\frac{d^2h^f(y)}{dy^2} = \frac{K^f}{R(1-y)^3}\left[2 - (R-1)(R-2)y^R + 2R(R-2)y^{R-1} - R(R-1)y^{R-2}\right]. \qquad (25)$$

Denote the expressions in [brackets] in (24) and (25) by $A_1(y)$ and $A_2(y)$, respectively. Then,

1. Since $A_1(1) = 0$ and $\frac{dA_1(y)}{dy} = -R(R-1)y^{R-2}(1-y) < 0$ for $R > 1$ and $0 < y < 1$, it follows that $A_1(y)$ is strictly positive for all $0 < y < 1$. Hence, $h^f(y)$ is strictly increasing in y, i.e., $h^f(x)$ (and therefore $g^f(x)$ as well) is strictly decreasing in x.
2. Since $A_2(1) = 0$ and $\frac{dA_2(y)}{dy} = -R(R-1)(R-2)\cdot y^{R-3}(1-y)^2 < 0$ for $R > 1$ and $0 < y < 1$, it follows that $A_2(y)$ is strictly positive for all $0 < y < 1$. Hence, $\frac{dh^f(y)}{dy}$ is strictly increasing in y, which implies that $\frac{dh^f(x)}{dx} = -\frac{dh^f(y)}{dy}$ is strictly increasing in x.
3. To show that

$$\frac{dh^f(x)/dx}{\left[h^f(x)\right]^2}$$

is decreasing in x, or, equivalently, that

$$\frac{dh^f(y)/dy}{\left[h^f(y)\right]^2}$$

is decreasing in y, it suffices to show that

$$h^f(y)\frac{d^2h^f(y)}{dy^2} - 2\left(\frac{dh^f(y)}{dy}\right)^2 < 0.$$

By a straightforward calculation, this is shown to be equivalent to

$$(R+1)y + (R-1)y^{R+1} - (R+1)y^R - (R-1) < 0. \qquad (26)$$

Denote $A_3(y)$ to be the expression on the left-hand side of (26). Since $A_3(1) = 0$ and

$$\frac{dA_3(y)}{dy} = (R+1)\cdot\left[1 + (R-1)y^R - Ry^{R-1}\right] = (R+1)A_1(y) > 0$$

for all $0 < y < 1$, property 3 follows.

4. Noticing that

$$-\frac{h'^f(x)}{h^f(x)}\frac{K^f x}{(1-K^f x)^{R-1}} = \frac{\frac{dh'^f(y)}{dy}}{h^f(y)}\frac{1-y}{y^{R-1}} = \frac{1 + (R-1)y^R - Ry^{R-1}}{(1-y^R)y^{R-1}}, \qquad (27)$$

it suffices to show that the derivative of the above with respect to y is nonpositive, i.e.,

$$-(R-1) + (3R-2)y^R - R^2y^{2R-1} + (R-1)^2y^{2R} \le 0. \qquad (28)$$

Denote the expression on the left-hand side of (28) by $A_4(y)$. Since $A_4(1) = 0$, it suffices to show that $\frac{dA_4}{dy} \ge 0$, which is equivalent to

$$(3R-2) - R(2R-1)y^{R-1} + 2(R-1)^2y^R \ge 0.$$

However,

$$\begin{aligned}(3R-2) &- R(2R-1)y^{R-1} + 2(R-1)^2y^R \\ &\ge (3R-2)Ry^{R-1} - (3R-2)(R-1)y^R \\ &\quad - R(2R-1)y^{R-1} + 2(R-1)^2y^R \\ &= R(R-1)(y^{R-1} - y^R) = A_1(y),\end{aligned}$$

and $A_1(y) \ge 0$ for $0 < y < 1$ has already been shown in the proof of property 1 above. The proof of the lemma is thus complete. □

Proof of Lemma 2. The continuity of λ with respect to C^f is immediate from (12)-(14) and the continuity of g^f. To establish the monotonicity, suppose to the contrary that $\lambda_1 = \Lambda(C_1^f) > \Lambda(C_2^f) = \lambda_2$ for some $C_1^f < C_2^f$. Then, $\lambda_1 > \lambda_2 \ge 0$ implies that, in the equilibrium corresponding to

C_1^f, $\sum_{f'\in\mathcal{F}} \rho^{f'} = 1$. Therefore, the set $\mathcal{F}' = \{f' | \rho^{f'} > 0\}$ is nonempty, and all flows $f' \in \mathcal{F}'$ satisfy $g^{f'}(\rho^{f'}) = \lambda_1$ (since $\rho^{f'} > 0$ implies $\mu^{f'} = 0$). In the second equilibrium (corresponding to C_2^f), since $\lambda_2 < \lambda_1$, it follows that each $g^{f'}(\rho^{f'})$ must be smaller than in the first equilibrium. However, we note that for any $f' \neq f$ the function $g^{f'}$ has not changed, and for f itself the function g^f even increased (since $C_1^f < C_2^f$). Since $g^{f'}(\rho^{f'})$ are strictly decreasing functions (Lemma 1), it follows that $\rho^{f'}$ must have strictly increased in the second equilibrium for all $f' \in \mathcal{F}'$, which is obviously impossible. □

Proof of Lemma 3. Again, the continuity of ρ^f with respect to C^f is immediate from the continuity of g^f and (12)-(14). We now prove the monotonicity. Assume $C_{\min}^f \leq C_1^f < C_2^f \leq C_{\max}^f$. Consider the following alternatives:

- $\Lambda(C_2^f) = 0$. By Lemma 2, this implies $\Lambda(C_1^f) = 0$ as well. In both equilibria (corresponding to C_1^f and to C_2^f), since $g^f(0) \geq 0$ by the definition of $C_{\min}^f$, it follows that (12) is satisfied with $\mu^f = 0$ and $g^f(\rho^f) = C^f h^f(\rho^f) - e^f = 0$. It follows that $C_1^f h^f(F(C_1^f)) = C_2^f h^f(F(C_2^f))$, which implies $F(C_1^f) < F(C_2^f)$ by the monotonicity of h^f (Lemma 1).
- $\Lambda(C_2^f) > 0$. Thus, in the equilibrium corresponding to C_2^f, $\rho^f + \sum_{f'\neq f} \rho^{f'} = 1$; moreover, by the definition of $C_{\max}^f$, the set $\mathcal{F}' = \{f' | \rho^{f'} > 0\}$ contains at least one flow $f' \neq f$, and all flows $f' \in \mathcal{F}' \setminus \{f\}$ satisfy $g^{f'}(\rho^{f'}) = \Lambda(C_2^f)$. Since the functions $g^{f'}$ for all flows $f' \in \mathcal{F}' \setminus \{f\}$ are the same in both equilibria and are strictly decreasing in the respective $\rho^{f'}$, and $\Lambda(C_1^f) \leq \Lambda(C_2^f)$ by Lemma 2, it follows that, for all $f' \in \mathcal{F}' \setminus \{f\}$, $\rho^{f'}$ in the equilibrium of C_2^f are not higher than in that of C_1^f. Therefore, $F(C_1^f) \geq F(C_2^f)$. Since $h^f(\rho^f)$ is decreasing in ρ^f and $C_1^f < C_2^f$, this leads to

$$\begin{aligned}\Lambda(C_1^f) &= C_1^f h^f(F(C_1^f)) - e^f \\ &< C_2^f h^f(F(C_2^f)) - e^f = \Lambda(C_2^f).\end{aligned}$$

Consequently, the $\rho^{f'}$ of all flows $f' \in \mathcal{F}' \setminus \{f\}$ in equilibrium 2 are, after all, strictly lower than in equilibrium 1; therefore, $F(C_1^f) < F(C_2^f)$. □

Proof of Lemma 4. From Lemma 3, it follows that, for every $0 < \rho^f < 1$, there exist unique $C^f = F^{-1}(\rho^f)$, λ, and $\{\rho^{f'}, f' \neq f\}$ that define a symmetrical equilibrium together with ρ^f. Therefore, we can view these quantities as functions of ρ^f, and consider their derivatives with respect to ρ^f.

We rewrite (12) as follows:

$$C^f h^f(\rho^f) - e^f = \lambda, \tag{29}$$

and, for any flow $f' \neq f$ such that $\rho^{f'} > 0$,

$$C^{f'} h^{f'}(\rho^{f'}) - e^{f'} = \lambda. \tag{30}$$

Taking the derivative of both sides in (29) and (30) with respect to ρ^f, we obtain, respectively,

$$\frac{dC^f}{d\rho^f} h^f(\rho^f) + C^f \frac{dh^f(\rho^f)}{d\rho^f} = \frac{d\lambda}{d\rho^f}, \tag{31}$$

$$C^{f'} \frac{dh^{f'}(\rho^{f'})}{d\rho^{f'}} \frac{d\rho^{f'}}{d\rho^f} = \frac{d\lambda}{d\rho^f}, \tag{32}$$

or, rearranging (32),

$$\frac{d\rho^{f'}}{d\rho^f} = \frac{\frac{d\lambda}{d\rho^f}}{C^{f'} \frac{dh^{f'}}{d\rho^{f'}}}. \tag{33}$$

We now distinguish between two subregions of λ. If $\rho^f + \sum_{f'\neq f} \rho^{f'} < 1$, then $\lambda = 0$ in a vicinity of ρ^f. Thus, $\frac{d\lambda}{d\rho^f} = 0$; also, $g^f(\rho^f) = C^f h^f(\rho^f) - e^f = 0$. From (31), we thus obtain

$$\frac{dC^f}{d\rho^f} = -\frac{C^f dh^f(\rho^f)/d\rho^f}{h^f(\rho^f)} = -\frac{e^f dh^f(\rho^f)/d\rho^f}{[h^f(\rho^f)]^2}, \tag{34}$$

which, by Lemma 1, is nondecreasing in ρ^f with $e^f \geq 0$.

Otherwise, if $\rho^f + \sum_{f'\neq f} \rho^{f'} = 1$, then $\sum_{f'} \frac{d\rho^{f'}}{d\rho^f} = -1$ in the vicinity of ρ^f. Together with (33), this implies

$$\frac{d\lambda}{d\rho^f} = -\sum_{f'\neq f} C^{f'} \frac{dh^{f'}(\rho^{f'})}{d\rho^{f'}}, \tag{35}$$

which can be fed back into (31) to yield

$$\begin{aligned}\frac{dC^f}{d\rho^f} &= -\frac{1}{h^f(\rho^f)}\left[C^f \frac{dh^f(\rho^f)}{d\rho^f} + \sum_{f'\neq f} C^{f'} \frac{dh^{f'}(\rho^{f'})}{d\rho^{f'}}\right] \\ &= -\frac{(\lambda + e^f) dh^f(\rho^f)/d\rho^f}{[h^f(\rho^f)]^2} - \frac{1}{h^f(\rho^f)} \sum_{f'\neq f} C^{f'} \frac{dh^{f'}(\rho^{f'})}{d\rho^{f'}}.\end{aligned} \tag{36}$$

We observe that, since λ is increasing in ρ^f and $g^{f'}(\rho^{f'})$ is decreasing in $\rho^{f'}$, it follows that each $\rho^{f'}$ is decreasing in ρ^f. Therefore, $\frac{dh^{f'}(\rho^{f'})}{d\rho^{f'}}$, which is increasing in $\rho^{f'}$ by Lemma 1, is decreasing in ρ^f. It follows that (36) is increasing in ρ^f.

Combining our findings that both (34) and (36) are nondecreasing in ρ^f, and noticing that the jump in $\frac{dC^f}{d\rho^f}$ at the boundary between the two subregions (namely, the difference between (36) at $\lambda = 0$ and (34)) is positive, we conclude that the function $C^f = F^{-1}(\rho^f)$ is convex, and, therefore, $F(C^F)$ is concave in the entire range $C_{\min}^f \leq C^f \leq C_{\max}^f$. □

Proof of Theorem 4 (Uniqueness). We repeat the observation that, in an equilibrium, for every flow $f_0 \in \mathcal{F}$, unless $\rho^{f_0} = 0$, the equation $\frac{\partial U_{f_0}}{\partial C^{f_0}} = 0$ must be satisfied. Using (22) and further applying either (34) or (36) (depending on whether the regime of $\sum_{f\in\mathcal{F}} \rho^f < 1$ or $\sum_{f\in\mathcal{F}} \rho^f = 1$ holds in the equilibrium), it can be verified by a straightforward simplification that this equation is equivalent to either

$$\frac{RK^{f_0}(u'_{f_0} - C^{f_0})(1 - K^{f_0}\rho^{f_0})^{R-1}}{\rho^{f_0}} = -C^{f_0}h'^{f_0}(\rho^{f_0}) = -\frac{(e^{f_0} - \mu^{f_0})h'^{f_0}(\rho^{f_0})}{h^{f_0}(\rho^{f_0})}, \tag{37}$$

if $\sum_{f\in\mathcal{F}} \rho^f < 1$, or

$$\frac{RK^f(u'_{f_0} - C^{f_0})(1 - K^{f_0}\rho^{f_0})^{R-1}}{\rho^{f_0}} = -\sum_{j\in\mathcal{F}} C^j h'^j, \tag{38}$$

in the case of $\sum_{f\in\mathcal{F}} \rho^f = 1$.

Assume, by contradiction, that there exist two different equilibria a and b, and denote the respective quantities by $\{C_a^f, \rho_a^f, \lambda_a, \mu_a^f\}$ and $\{C_b^f, \rho_b^f, \lambda_b, \mu_b^f\}$, where $C_a^f \neq C_b^f$ for some $f \in \mathcal{F}$. Without loss of generality, assume $\sum_f \rho_a^f \geq \sum_f \rho_b^f$.

At this stage, we establish two auxiliary lemmas that are used later in the proof.

Lemma 8. *There exists an $f \in \mathcal{F}$ such that $\rho_a^f > \rho_b^f$.*

Proof. Assume to the contrary that $\rho_a^f = \rho_b^f$ for all $f \in \mathcal{F}$, and, without loss of generality, that $C_a^f > C_b^f$ for some $f \in \mathcal{F}$. According to the definition of the best-response function $B^f(\mathbf{C^{-f}})$, if $\rho_a^f = \rho_b^f = 0$, then $C_a^f = C_b^f = u'_f(0)$. Therefore, there must exist a flow f_0 with $C_a^{f_0} > C_b^{f_0}$ and $\rho_a^{f_0} = \rho_b^{f_0} > 0$. Consequently, $\mu_a^{f_0} = \mu_b^{f_0} = 0$. This leads to $\lambda_a = C_a^{f_0}h^{f_0}(\rho_a^{f_0}) - e^{f_0} > C_b^{f_0}h^{f_0}(\rho_b^{f_0}) - e^{f_0} = \lambda_b$. It follows that $C_a^{f'} > C_b^{f'}$ for *all* $f' \in \mathcal{F}$ such that $\rho_a^{f'} = \rho_b^{f'} > 0$. Furthermore, either λ_a or λ_b must be nonzero, implying that $\sum_{f\in\mathcal{F}} \rho_{a,b}^f = 1$.

It follows that (38) must be satisfied in both equilibria, which can be seen to be impossible. Indeed, $C_a^{f_0} > C_b^{f_0}$ implies that the left-hand side of (38) is smaller in equilibrium a than in b; on the other hand, since $C_a^f \geq C_b^f$ for all $f \in \mathcal{F}$, the right-hand side of (38) is larger in equilibrium a than in b. This contradiction completes the proof of the lemma. □

Lemma 9. *If $\lambda_a \geq \lambda_b$ and $\rho_a^f > \rho_b^f$ for some $f \in \mathcal{F}$, then $C_a^f > C_b^f$.*

Proof. Since $\rho_a^f > \rho_b^f \geq 0$ implies $\mu_a^f = 0$, we have (from (12))

$$C_a^f h^f(\rho_a^f) - e^f = \lambda_a \geq \lambda_b - \mu_b^f = C_b^f h^f(\rho_a^f) - e^f.$$

Since $h^f(\rho^f)$ is a monotonically decreasing function (Lemma 1), the above inequality readily implies $C_a^f > C_b^f$. □

We now turn to prove the SNE uniqueness by considering the following possible three cases.

Case 1. $\sum_f \rho_a^f < 1$ (and therefore $\sum_f \rho_b^f < 1$, implying that $\lambda_a = \lambda_b = 0$). Consider the flow f_0 such that $\rho_a^{f_0} > \rho_b^{f_0}$; therefore $\mu_a^{f_0} = 0$ and $C_a^{f_0} > C_b^{f_0}$ by Lemma 9.

Since (37) must hold at both equilibria, applying Lemma 1 (property 4), we get

$$RK^{f_0}\left(u'_{f_0}\Big|_a - C_a^{f_0}\right) = \frac{e^{f_0}h'^{f_0}(\rho_a^{f_0}) \cdot \rho_a^{f_0}}{h^{f_0}(\rho_a^{f_0}) \cdot (1 - K^{f_0}\rho_a^{f_0})^{R-1}} > \frac{(e^{f_0} - \mu_b^{f_0})h'^{f_0}(\rho_b^{f_0}) \cdot \rho_b^{f_0}}{h^{f_0}(\rho_b^{f_0}) \cdot (1 - K^{f_0}\rho_b^{f_0})^{R-1}} = RK^{f_0}\left(u'_{f_0}\Big|_b - C_b^{f_0}\right),$$

which implies that $u'_{f_0}\Big|_a > u'_{f_0}\Big|_b$, contradicting the concavity of u_{f_0}.

Case 2. $\sum_f \rho_a^f = \sum_f \rho_b^f = 1$. Without loss of generality, assume $\lambda_a \geq \lambda_b$. We divide $\mathcal{F}$ into two subsets, $\mathcal{F}_1 = \{f | \rho_a^f > \rho_b^f\}$ and $\mathcal{F}_2 = \{f | \rho_a^f \leq \rho_b^f\}$. Note that both $\mathcal{F}_1$ and $\mathcal{F}_2$ must be nonempty, and that it follows from Lemma 9 that $C_a^f > C_b^f$ for all $f \in \mathcal{F}_1$.

In this case, (38) must be satisfied for every $f_0 \in \mathcal{F}_1$ in both equilibria. Since $\rho_a^{f_0} > \rho_b^{f_0}$, $C_a^{f_0} > C_b^{f_0}$, and the function u_{f_0} is concave, it is clear that the left-hand side of (38) is greater in equilibrium b than in a. As a result, the same must be true for the right-hand side of (38) as well.

Now, consider (38) for a flow $f_0 \in \mathcal{F}_2$. Since the right-hand side of (38) is the same for all flows (it does not depend on f_0), the left-hand side must again be greater in equilibrium b than in a. Since, for $f_0 \in \mathcal{F}_2$, $\rho_a^{f_0} \leq \rho_b^{f_0}$ (and again taking into account the concavity of u_{f_0}), it follows that $C_a^{f_0} \geq C_b^{f_0}$. In other words, $C_a^f \geq C_b^f$ for all $f \in \mathcal{F}$, which contradicts the fact that the right-hand side of (38) is greater at equilibrium b.

Case 3. $\sum_f \rho_a^f = 1 > \sum_f \rho_b^f$. This case is similar to the previous two, except that now (38) must hold at equilibrium a and (37) applies in equilibrium b.

Again, consider a flow f_0 such that $\rho_a^{f_0} > \rho_b^{f_0}$. Clearly, $\lambda_a \geq \lambda_b = 0$. Lemma 9 therefore implies that $C_a^{f_0} > C_b^{f_0}$. Combining (37) and (38), and applying property 3 from Lemma 1, we obtain

$$\begin{aligned}\frac{RK^{f_0}\left(u'_{f_0}\big|_b - C_b^{f_0}\right)\left(1 - K^{f_0}\rho_b^{f_0}\right)^{R-1}}{\rho_b^{f_0}} &= -\frac{e^{f_0}h'^{f_0}(\rho_b^{f_0})}{\left[h^{f_0}(\rho_b^{f_0})\right]^2} \\ < -\frac{(\lambda_a + e^{f_0})h'^{f_0}(\rho_a^{f_0})}{\left[h^{f_0}(\rho_a^{f_0})\right]^2} &< \sum_{f\in\mathcal{F}} \frac{(\lambda_a + e^f)h'^f(\rho_a^{f_0})}{\left[^f(\rho_a^{f_0})\right]^2} \\ = \frac{RK^{f_0}\left(u'_{f_0}\big|_a - C^{f_0}\right)\left(1 - K^{f_0}\rho_a^{f_0}\right)^{R-1}}{\rho_a^{f_0}}&,\end{aligned}$$

which contradicts with $\rho_a^{f_0} > \rho_b^{f_0}$ and $C_a^{f_0} > C_b^{f_0}$.

The uniqueness of the fixed point of $\mathbf{B}(\mathbf{C})$, and therefore of the system equilibrium, is thus established. □

ACKNOWLEDGMENTS

Part of this work was done while the first author (Lin Chen) was a visiting research student at NICTA, 13 Garden St., Eveleigh, NSW 1430, Australia. The second author (Lavy Libman) is supported by the Australian Research Council Discovery Project grant DP0987782. Part of this work was done while the second author (Lavy Libman) was with the Networked Systems Research Group, NICTA, 13 Garden St., Eveleigh, NSW 1430, Australia.

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