Joint Operator Pricing and Network Selection Game in Cognitive Radio Networks: Equilibrium, System Dynamics and Price of Anarchy

Jocelyne Elias, Fabio Martignon, Lin Chen, and Eitan Altman, *Fellow, IEEE*

***Abstract*—This paper addresses the joint pricing and network selection problem in cognitive radio networks (CRNs). The problem is formulated as a Stackelberg game, where the primary and secondary operators (POs and SOs) first set the network subscription price to maximize their revenue. Then, users perform the network selection process, deciding whether to pay more for a guaranteed service or to use a cheaper best-effort secondary network, where congestion and low throughput may be experienced. We derive optimal stable price and network selection settings. More specifically, we use the *Nash equilibrium* concept to characterize the equilibria for the price setting game. On the other hand, a *Wardrop equilibrium* is reached by users in the network selection game since, in our model, a large number of users must individually determine the network to which they should connect. Furthermore, we study network users' dynamics using a population game model, and we determine its convergence properties under replicator dynamics, which is a simple yet effective selection strategy. Numerical results demonstrate that our game model captures the main factors behind cognitive network pricing and network selection, thus representing a promising framework for the design and understanding of CR systems.**

***Index Terms*—Cognitive radio networks (CRNs), network selection, population game model, pricing, replicator dynamics, Stackelberg game.**

I. Introduction

COGNITIVE radio networks (CRNs), which are also referred to as *xG* networks, are envisioned to deliver high bandwidth to mobile users via heterogeneous wireless architectures and dynamic spectrum access techniques [1], [2]. In CRNs, a *primary* (or licensed) *user* (PU) has a license to operate in a certain spectrum band; his access is generally controlled by the primary operator (PO) and should not be affected by the operations of any other unlicensed user. On the other hand, the secondary operator (SO) has no spectrum license; therefore, *secondary users* (SUs) must implement additional functionalities to share the licensed spectrum band without interfering with primary users.

In this paper, we consider a CR scenario that consists of primary and secondary networks and a large set of cognitive users, and we focus on a fundamental issue concerning such systems, i.e., whether it is better for a CR user to act as a primary user, paying the PO for costlier dedicated network resources with quality-of-service (QoS) guarantees, or act as an SU (paying the SO), sharing the spectrum holes left available by licensed users and facing lower costs with degraded performance guarantees. At the same time, we consider the *pricing* problem of both POs and SOs, who compete with each other, setting access prices to maximize their revenues.

The joint pricing and CRN selection problem is modeled as a *Stackelberg* (leader–follower) game, where the POs and SOs first set their access prices to maximize their revenues. In this regard, we study both practical cases, where the POs and SOs fix access prices at the same time, and the PO exploits his dominant position by playing first, anticipating the choices of the SO.

Then, network users react to the prices set by the operators, choosing to which network they should connect, therefore acting either similar to the PU or SU.

The solution provides an insight on how rational users will distribute among existing access solutions (higher price primary networks versus lower price secondary networks), i.e., the proportion of players who choose different strategies.

We adopt a fluid queue approximation approach (as in [3]–[7]) to study the steady-state performance of these users, focusing on *delay* as QoS metric. In addition to considering *static* traffic equilibrium settings, we further formulate the network selection process of CR users as a *population game* [8], which provides a powerful framework for characterizing the strategic interactions among large numbers of agents, whose behavior is modeled as a *dynamic* adjustment process. More specifically, we study the cognitive users' behavior according to *replicator dynamics* [8], [9] since these users adapt their choices and strategies based on the observed network state.

We provide equilibrium and convergence properties of the proposed game and derive optimal stable price and network selection settings.

More specifically, we use the *Nash equilibrium* concept to characterize the equilibria of the pricing game between a finite number of decision makers (viz., the POs and SOs). In addition to that, we further determine the *Wardrop equilibrium* for the network selection game, in which a large number of users must individually choose the network to which they should connect.

Manuscript received October 20, 2012; revised March 24, 2013 and May 10, 2013; accepted May 12, 2013. Date of publication May 20, 2013; date of current version November 6, 2013. This work was supported in part by the French National Research Agency (ANR) under the ANR Green-Dyspan project. The review of this paper was coordinated by Dr. S. Zhong.

J. Elias is with Paris Descartes University, 75006 Paris, France (e-mail: jocelyne.elias@parisdescartes.fr).

F. Martignon and L. Chen are with the University of Paris-Sud, 91400 Orsay, France (e-mail: fabio.martignon@lri.fr; lin.chen@lri.fr).

E. Altman is with the National Research Institute in Computer Science and Control (INRIA), 06560 Sophia Antipolis, France (e-mail: Eitan.Altman@sophia.inria.fr).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TVT.2013.2264294

This equilibrium is characterized by two properties, namely, *traffic equilibrium* (the total costs perceived by users on all used networks are equal) and *system optimum principle* (the average delay/cost is minimum) [10].

Numerical results obtained in different network scenarios illustrate that our game captures the main factors behind cognitive network pricing and selection, thus representing a promising framework for the design and performance evaluation of CR systems.

In summary, in an effort to understand the pricing and networking selection issues that characterize CRNs, this paper makes the following contributions:

- the proposition of a novel game-theoretic model where POs and SOs set access prices and users select the network with which to connect, which are both based on the total delay and the experienced cost;
- the computation of equilibrium points for our game and relevant performance metrics, including the price of anarchy (PoA) and the price of stability (PoS);
- the analysis of a dynamic model, based on population games, which further illustrates how players converge to the equilibrium in a dynamic context under an easily implementable distributed strategy (viz., replicator dynamics), along with formal detailed proofs of its convergence.

The remainder of this paper is organized as follows. Related work is reviewed in Section II. The network model for the proposed joint pricing and network selection game is described in Section III. The equilibrium points of this game, as well as its PoA and PoS, are derived in Sections IV and V, respectively. The dynamic network selection model, which is based on population games and replicator dynamics, is presented in Section VI, and its convergence properties are demonstrated in Section VII. Numerical results are discussed in Section VIII, whereas Section IX concludes this paper.

II. Related Work

Here, we first review the most notable works on spectrum pricing and access in CRNs [3], [4], [11]–[18]. Then, we discuss relevant works that use evolutionary games to study the users behavior in CR and heterogeneous wireless networks [19]–[23].

In [11], the authors provide a systematic overview on CR networking and communications by looking at the key functions of the physical, medium-access-control, and network layers involved in a CR design and by studying how these layers are crossly related. In [3], the decision-making process of SUs who have the choice of either acquiring a dedicated spectrum (paying a price) or using the PU band for free are considered, and they characterize the resulting Nash equilibrium for the single-band case. This paper differs from ours in two main aspects: 1) The CR users already arrive at the system as SU or PU, and SUs have the choice between dedicated or PU band; and 2) the users' behavior is studied based on queueing theory. The work in [4] considers a CRN where multiple SUs contend for spectrum usage, using random access, over available PU channels, focusing on SUs' queueing delay performance. A fluid queue approximation approach is adopted to study the steady-state delay performance of SUs. In [12], the price competition between PUs who can lease out their unused bandwidth to SUs in exchange for a fee is analyzed, considering bandwidth uncertainty and spatial reuse. The problem of dynamic spectrum leasing in a secondary market of CRNs is considered in [14], where secondary service providers lease spectrum from spectrum brokers to provide service to SUs.

Recent works have considered *evolutionary games* to study the users' behavior in CR and heterogeneous wireless networks [19]–[23].

In [19], evolutionary game theory was used to investigate the dynamics of user behavior in heterogeneous wireless access networks (i.e., wireless metropolitan area networks, cellular networks, and wireless local area networks). The evolutionary game solution is compared with the Nash equilibrium, and a set of algorithms (i.e., population evolution and reinforcement learning algorithms) is proposed to implement the evolutionary network selection game model. In [20], the dynamics of a multiple-seller–multiple-buyer spectrum trading market is modeled as an evolutionary game, in which PUs want to sell and SUs want to buy spectrum opportunities. SUs evolve over time, buying the spectrum opportunities that optimize their performance in terms of transmission rate and price. In [21], the authors propose a distributed framework for spectrum access, with and without complete network information (i.e., channel statistics and user selections). In the first case, an evolutionary game approach is proposed, in which each SU compares its payoff with the system average payoff to evolve its spectrum-access decision over time. For the incomplete information case, a learning mechanism is proposed, in which each SU locally estimates its expected throughput and learns to adjust its channel selection strategy adaptively. The problem of opportunistic spectrum access in carrier-sense-multiple-access-with-collision-avoidance-based CRNs is also addressed in [22] from an evolutionary-game-theoretic angle.

In our preliminary works [24], [25], we addressed the pricing and network selection problems in CRNs. However, in [24], we assumed that the POs and SOs use *separate* frequency bands, which greatly simplifies the problem, and we did not study the impact of the order in which operators set prices on the quality of the reached equilibria. The work in [25] differs from the work presented here in that it considered uniquely POs and a finite set of SUs, which are characterized by elastic traffic demands that can be transmitted over one or multiple frequency spectra.

Unlike previous works, which study the interaction between two well-defined sets of users (primary and secondary users) who *already* performed the choice of using the primary or secondary network, this paper tackles a fundamental issue in CRNs. In fact, we model the users' decision process that occurs *before* such users enter the CRN, thus assessing the economic interest of deploying secondary (xG) networks. This choice depends on the tradeoff between *cost* and *performance guarantees* in these networks. At the same time, we derive the optimal price setting for both POs and SOs that play before network users to maximize their revenue. We use enhanced game-theoretic tools, which are derived from population game theory, to model the network selection dynamics, providing convergence conditions and equilibrium settings.

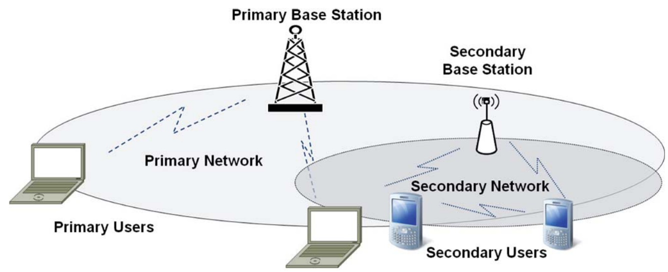


Fig. 1. CRN scenario with a primary network and a secondary (xG) network. Arriving users must decide whether to join the primary network, paying a subscription fee (p_1) for guaranteed QoS, or the xG network (which has a lower subscription cost $p_2 < p_1$ and less performance guarantees), based on the expected cost and congestion levels.

III. Network Model

We now detail the network model, which is shown in Fig. 1. A CR wireless system that consists of a secondary (xG) network that coexists with a primary network at the same location and on the same spectrum band.

We consider an *overlay model* (focusing on the "interference avoidance" approach [26], [27] to CR) as in [3], [20], [28], where SUs periodically sense the radio spectrum, intelligently detect occupancy in the different frequency bands, and then opportunistically communicate over the spectrum holes left available by PUs, thus avoiding interference with active PUs. In other words, our model is an overlay CR, where SUs opportunistically access PUs' spectrum only when it is not occupied. As in [3], we further consider perfect PU detection at the SUs and zero interference tolerance at each of the PUs and SUs.

We assume that users arrive at this system following a Poisson process with rate λ, and the maximum achievable transmission rate of the wireless channel (licensed to the PO and opportunistically used by the SO) is denoted by C. The total traffic λ admitted in the network must not exceed its capacity C; this can be obtained, for example, using admission control techniques, which are out of the scope of this paper. All these assumptions are commonly adopted in several recent works such as [4]–[7].

Each arriving user must choose whether to join the primary network (paying a higher subscription cost) or the xG one (which has a lower subscription cost), based on criteria to be specified later, i.e., a combination of cost and QoS (service time/latency).

Finally, let us denote by λ_P the overall transmission rate of PUs (i.e., those who choose the primary network) and by λ_S the rate of SUs so that $\lambda = \lambda_P + \lambda_S$. Table I summarizes the basic notation used in our game model.

We now define users' *cost functions* and the *utility functions* of POs and SOs. We assume that the total cost incurred by a network user is a combination of the service time (delay or latency) experienced in the network and the cost for the player to access this network.

We underline that a similar model is used in [3], where the average cost incurred by an SU consists of two components: 1) the price $\tilde{C}$ of the dedicated spectrum band and 2) an average delay cost $1/\mu$, where μ is the service time. The average delay cost is weighted by parameter α, which represents the delay versus monetary cost tradeoff of the SUs. To further support our choice, another similar model is considered by Anshelevich *et al.* in [29] for a different networking context. The authors set the player's cost for using an edge e in the network as a combination of cost function $c_e(x)$ and latency function $d_e(x)$; the goal of each user in this game is to minimize the sum of his cost and latency. The same model is also used in [30]. Finally, note that, in [19], two components, namely, throughput (the allocated capacity to a player, which is obviously related to the delay experienced by such user) and the corresponding price (see [19, Eqs. (2) and (3)]) are considered.

TABLE I
Basic Notation

λ	Total traffic accepted in the network
C	Wireless channel capacity
α	Weighting parameter of delay wrt access cost
λ_P	Total traffic transmitted by Primary Users
λ_S	Total traffic transmitted by Secondary Users
X_P	Fraction of Primary Users
X_S	Fraction of Secondary Users
p_1, p_2	Price charged by the PO/SO
K	Constant, velocity of convergence

In this paper, we consider a fluid queue approximation approach, which permits to study the steady-state delay performance of both PUs and SUs. To this aim, and without loss of generality, we assume that the wireless channel is modeled as an $M|M|1$ queue, with service rate C and arrival rate λ. Recall that both the primary and secondary networks operate on the same channel; the POs and SOs fix the prices p_1 and p_2, respectively, for accessing their services. Therefore, the total cost perceived by PUs is given by

$$\mathrm{Cost}_{\mathrm{PU}} = \frac{\alpha}{C - \lambda_P} + p_1 \tag{1}$$

where parameter α weights the relative importance of the experienced delay with respect to the access cost. Note that PUs are *exclusively* affected by the traffic transmitted by PUs (λ_P) and not by the traffic of SUs (λ_S) since, usually, in a CRN, PUs have strict priority over SUs; these latter must therefore implement spectrum sensing and spectrum handover strategies to avoid any interference toward PUs and can transmit only in the spectrum holes left unoccupied by these ones.

As mentioned previously, we consider perfect PU detection at the SUs and zero interference tolerance at each of the PUs and SUs.

For this reason, SUs' performance is affected by the *whole traffic*, transmitted by both PUs and SUs; these users are characterized by the following cost function:

$$\mathrm{Cost}_{\mathrm{SU}} = \frac{\alpha}{C - (\lambda_P + \lambda_S)} + p_2 = \frac{\alpha}{C - \lambda} + p_2. \tag{2}$$

As for operators' utilities, they correspond to the total revenue obtained by pricing users. As a consequence, the PO's utility function is expressed as follows:

$$U_P = p_1 \lambda_P. \tag{3}$$

Correspondingly, the SO's utility function is

$$U_S = p_2 \lambda_S = p_2(\lambda - \lambda_P). \tag{4}$$

TABLE II
PU AND SU'S COST FUNCTIONS

Primary User (PU)	$Cost_{PU} = \frac{\alpha}{C-\lambda_P} + p_1$
Secondary User (SU)	$Cost_{SU} = \frac{\alpha}{C-\lambda} + p_2$

TABLE III
PO AND SO'S UTILITY FUNCTIONS

Primary Operator (PO)	$U_P = p_1\lambda_P$
Secondary Operator (SO)	$U_S = p_2\lambda_S$

To summarize, network users *minimize* the perceived cost, which is expressed as $\text{Cost}_{\text{PU}} = \alpha/C - \lambda_P + p_1$ [see (1)] if they choose the primary network, and $\text{Cost}_{\text{SU}} = \alpha/C - \lambda + p_2$ [see (2)] if they act as SUs. As for POs/SOs, they try to *maximize* the total revenue obtained by pricing PUs ($U_P = p_1\lambda_P$) or SUs ($U_S = p_2\lambda_S$), respectively. Users' cost functions and operators' utilities are also reported in Tables II and III, respectively.

IV. Equilibrium Computation

Here, we derive the equilibrium points of our game, namely, the equilibrium traffic sent by PUs and SUs, steady-state PO/SO's utilities, and equilibrium prices set by the PO/SO.

We consider two practical cases: 1) Both operators fix their access price *at the same time*, trying to maximize their own revenue (see Section IV-A); and 2) the PO plays *before* the SO, anticipating the strategy of this latter, thus exploiting his dominant position (see Section IV-B). We will refer to the first case as the *TOGETHER* scenario, whereas the latter will be referred to as the *BEFORE* scenario. Note that when the POs and SOs play at the same time, we have a Cournot duopoly competition between these operators. However, in the original Cournot duopoly, production quantities (outputs) and prices are linear, whereas in this paper, we consider a nonlinear system that requires nonstandard studies that cannot rely on existing results. On the other hand, when the PO plays before the SO, anticipating his choices, we have a Stackelberg game model between the operators.

The *Nash* equilibrium concept will be used for the price setting game since we have a finite number of decision makers, i.e., the two network operators. More precisely, a Nash equilibrium is a set of players' (here, operators') strategies, each of which maximizes the player's revenue, and such that none of the actors has an incentive to deviate unilaterally. For this reason, the corresponding network configurations are said to be stable.

On the other hand, a *Wardrop* equilibrium [31] is reached by CR users in the network selection game since, in our model, a large number of users must determine individually the network to which they should connect. This equilibrium satisfies the two Wardrop's principles, namely, traffic equilibrium (the total costs perceived by users on all used networks are equal) and system optimum principle (the average delay/cost is minimum).

Therefore, at Wardrop equilibrium, PUs and SUs will both experience the same cost, i.e., $\text{Cost}_{\text{PU}} = \text{Cost}_{\text{SU}}$, or

$$\frac{\alpha}{C-\lambda_P} + p_1 = \frac{\alpha}{C-(\lambda_P+\lambda_S)} + p_2 = \frac{\alpha}{C-\lambda} + p_2. \quad (5)$$

This permits computation of the equilibrium traffic[1] for the primary network as a function of the prices set by both the POs and SOs, i.e.,

$$\lambda_P = \frac{\alpha\lambda - C(C-\lambda)(p_1-p_2)}{\alpha - (C-\lambda)(p_1-p_2)} \quad (6)$$

with $0 \le \lambda_P \le \lambda$. The traffic sent by SUs, i.e., λ_S, will therefore be equal to $\lambda - \lambda_P$. Note that, in order for the equilibrium condition (5) to hold and for equilibrium traffic λ_P to be comprised in the $[0, \lambda]$ range, $p_1 - p_2$ must satisfy the condition $p_1 - p_2 < \alpha\lambda/C(C-\lambda)$. Furthermore, since there is a unique λ_P value, which satisfies condition (5), this value represents the unique Wardrop equilibrium point of the network selection game.

The corresponding equilibrium utility for the PO is given by the following expression:

$$U_P = p_1\lambda_P = p_1 \cdot \frac{\alpha\lambda - C(C-\lambda)(p_1-p_2)}{\alpha-(C-\lambda)(p_1-p_2)} \quad (7)$$

whereas the utility of the SO is

$$\begin{aligned} U_S &= p_2\lambda_S = p_2(\lambda - \lambda_P) \\ &= p_2\lambda + p_2\left[\frac{\alpha(C-\lambda)}{\alpha-(C-\lambda)(p_1-p_2)} - C\right]. \end{aligned} \quad (8)$$

Hereafter, we compute equilibrium prices for both our considered scenarios.

A. *POs and SOs Fix Their Prices Simultaneously (TOGETHER Scenario)*

In this scenario, both the POs and SOs fix their prices simultaneously, trying to maximize their own revenue. As a consequence, to maximize the utility function of the PO, it suffices to take the derivative of U_P with respect to p_1, imposing on it its equality to zero, i.e.,

$$\frac{\partial U_P}{\partial p_1} = C - \frac{\alpha(C-\lambda)[\alpha-(C-\lambda)(p_1-p_2)] + \alpha(C-\lambda)^2 p_1}{[\alpha-(C-\lambda)(p_1-p_2)]^2} = 0. \quad (9)$$

Hence, we can express the price p_1 as a function of p_2, i.e.,

$$p_1 = p_2 + \frac{\alpha}{C-\lambda}\left\{1 - \sqrt{\frac{(C-\lambda)}{\alpha C}[\alpha + (C-\lambda)p_2]}\right\}. \quad (10)$$

Similarly, the SO aims at maximizing his revenue U_S; by deriving U_S with respect to p_2 and imposing its equality to zero, we obtain

$$\frac{\partial U_S}{\partial p_2} = (\lambda - C) + \frac{\alpha^2(C-\lambda) - \alpha(C-\lambda)^2 p_1}{[\alpha-(C-\lambda)(p_1-p_2)]^2} = 0 \quad (11)$$

and the expression of p_2 as a function of p_1 is given by

$$p_2 = p_1 - \frac{1}{(C-\lambda)}\left\{\alpha - \sqrt{\alpha^2 - \alpha(C-\lambda)p_1}\right\}. \quad (12)$$

[1] With a slight abuse of notation, we will denote equilibrium flows still by λ_P and λ_S since, in the following, we will exclusively almost refer to equilibrium game conditions.

Finally, by combining (10) and (12), we obtain the equilibrium price values p_1 and p_2, which are function of α, C, and λ, i.e.,

$$p_1 = \alpha \frac{(3C^2 - \lambda^2) - (C-\lambda)^2 \sqrt{\frac{9C-5\lambda}{C-\lambda}}}{2(2C-\lambda)^2(C-\lambda)} \tag{13}$$

$$p_2 = \alpha \frac{C\sqrt{9C-5\lambda} - (3C-2\lambda)\sqrt{C-\lambda}}{2(2C-\lambda)^2\sqrt{C-\lambda}} \tag{14}$$

with $p_1 \geq 0$, and $p_2 \geq 0$.

B. PO Plays Before the Secondary (BEFORE Scenario)

In this case, we have a Stackelberg game between operators, in which the PO is the leader, whereas the SO is the follower.

The PO will therefore anticipate the choice of the SO (who will set the price p_2 to maximize his utility), and will play his best strategy, setting the optimal value for p_1 taking into account the choice on p_2 operated by the SO.

To derive the equilibrium prices in this scenario, it suffices to take the derivative of U_S with respect to the price p_2, obtaining p_2 in function of p_1 [see (12)]. We next insert the expression of p_2 in (7), obtaining U_P as a function of p_1, i.e.,

$$U_P = p_1 \left\{ C + \frac{\alpha(\lambda - C)}{\sqrt{\alpha^2 - \alpha(C-\lambda)p_1}} \right\}.$$

Deriving U_P with respect to the price p_1, we obtain $C + \sqrt{\alpha}(\lambda - c)[2\alpha - (C-\lambda)p_1]/2[\alpha - (C-\lambda)p_1]^{3/2}$; then, imposing that this derivative is null, we obtain the equilibrium value for p_1, which has the following expression:

$$p_1 = \frac{\alpha}{C-\lambda} \left\{ 1 - \left(Z + \frac{h}{3} \right)^2 \right\} \tag{15}$$

where $Z = (h/4)^{1/3}[(\sqrt{1+4/27h^2}+1)^{2/3} + (\sqrt{1+4/27h^2} - 1)^{2/3}]$, and $h = C - \lambda/2C$.

If we combine this expression of p_1 with (12), we obtain the equilibrium price set by the SO, i.e.,

$$p_2 = \frac{\alpha}{C-\lambda} \left(Z + \frac{h}{3} \right) \left[1 - \left(Z + \frac{h}{3} \right) \right]. \tag{16}$$

C. Comments

Note that, in both the *TOGETHER* and *BEFORE* scenarios, equilibrium prices are unique. In fact, if we compute the second derivatives in both network scenarios ($\partial^2 U_P/\partial {p_1}^2$ and $\partial^2 U_S/\partial {p_2}^2$), they are both negative for all price values in the feasible region $p_1 - p_2 < \alpha\lambda/C(C-\lambda)$. Hence, the maximums, as well as the Nash equilibrium points, are unique.

Furthermore, equilibrium prices (p_1 and p_2) are directly proportional to α, whereas equilibrium flows (λ_P and λ_S) are independent of α; this can be seen by substituting, in (6), $p_1 - p_2$, which is proportional to α. As a consequence, operators' utilities grow proportionally to α. All these trends will be shown in more detail in Section VIII.

Finally, PUs' equilibrium traffic λ_P decreases with increasing C values, whereas SUs' traffic follows an opposite trend. As for operators' prices and utilities, they both decrease with C, as we will quantify in Section VIII.

V. Price of Anarchy and Price of Stability

We now investigate the efficiency of the equilibria reached by operators and users in our joint pricing and network selection game, through the determination of the PoA and the PoS. They both quantify the loss of efficiency as the ratio between the cost of a specific stable outcome/equilibrium and the cost of the optimal outcome, which could be designed by a central authority. In particular, the PoA, which is first introduced in [32], considers the worst stable outcome (that with the highest cost), whereas the PoS [29] considers the best stable equilibrium (that with the lowest cost). However, we observe that, in our game, these two performance metrics coincide due to the uniqueness of the equilibrium reached by network users. For this reason, in the following, we will exclusively refer to the first performance figure, i.e., the PoA, which has a particular importance in characterizing the efficiency of distributed game formulations.

To determine the optimal systemwide solution, we define the social welfare S as the weighted average of the delays experienced by PUs and SUs; S is therefore a function of the amount x of traffic sent by PUs, i.e.,

$$S(x) = \frac{\alpha x}{C-x} + \frac{\alpha(\lambda - x)}{C-\lambda}.$$

Note that p_1 and p_2 do not appear in the social welfare's expression since all the prices paid by PUs/SUs (which represent for them a *disutility* or *cost*) correspond to a symmetric *utility* or *gain* for the POs/SOs, who collect this income in exchange for the network services they offer.

To minimize this quantity, it suffices to derive with respect to x and impose its equality to zero, thus obtaining

$$\frac{dS(x)}{dx} = \frac{\alpha C}{(C-x)^2} - \frac{\alpha}{C-\lambda} = 0$$

which leads to $x_{\min} = C - \sqrt{C(C-\lambda)}$.

The optimal social welfare is therefore equal to

$$\begin{aligned} S(x_{\min}) &= \alpha \left[\frac{C - \sqrt{C(C-\lambda)}}{\sqrt{C(C-\lambda)}} + \frac{\lambda - C + \sqrt{C(C-\lambda)}}{C-\lambda} \right] \\ &= 2\alpha \left[\sqrt{\frac{C}{C-\lambda}} - 1 \right]. \end{aligned} \tag{17}$$

Recall that the total traffic transmitted by PUs at the Wardrop equilibrium is given by (6), and the equilibrium traffic for SUs is $\lambda_s = \lambda - \lambda_p$.

The (average) total delay experienced by PUs/SUs at equilibrium is therefore equal to

$$\mathrm{TD}_E = \alpha \frac{\lambda_p}{C - \lambda_p} + \alpha \frac{\lambda_s}{C-\lambda} \tag{18}$$

whereas the PoA is defined as the ratio between the cost of the worst (unique) equilibrium and the social optimum, i.e., PoA $= \mathrm{TD}_E / S(x_{\min})$.

Hereafter, we derive the closed-form expressions for the PoA in both the considered scenarios (i.e., the *TOGETHER* and *BEFORE* scenarios). To this aim, it is sufficient to use equilibrium expressions for λ_P and λ_S in both scenarios.

A. PoA for the TOGETHER Scenario (the POs and SOs Play Together)

The total delay of cognitive users at equilibrium TD_E^T can be expressed as follows:

$$\begin{aligned}\mathrm{TD}_E^T &= \alpha\frac{\lambda_p}{C-\lambda_p}+\alpha\frac{\lambda_s}{C-\lambda}=\frac{\alpha\lambda}{C-\lambda}-(p_1-p_2)\lambda_p\\ &=\frac{\alpha C(9C-5\lambda)-\alpha(3C-2\lambda)\sqrt{(C-\lambda)(9C-5\lambda)}}{(2C-\lambda)\left[(C-\lambda)+\sqrt{(C-\lambda)(9C-5\lambda)}\right]}. \quad (19)\end{aligned}$$

Therefore, the PoA can be calculated as (20), shown at the bottom of the page.

B. PoA for the BEFORE Scenario (the PO Plays Before the SO)

In this case, the total delay of cognitive users at equilibrium TD_E^B can be expressed as

$$\begin{aligned}\mathrm{TD}_E^B &= \alpha\frac{\lambda_p}{C-\lambda_p}+\alpha\frac{\lambda_s}{C-\lambda}=\frac{\alpha\lambda}{C-\lambda}-(p_1-p_2)\lambda_p\\ &=\alpha\left[-2+\frac{C}{C-\lambda}\left(Z+\frac{h}{3}\right)+\frac{1}{Z+\frac{h}{3}}\right] \quad (21)\end{aligned}$$

where

$$Z=\left(\frac{h}{4}\right)^{1/3}\left[\left(\sqrt{1+\frac{4}{27}h^2}+1\right)^{2/3}+\left(\sqrt{1+\frac{4}{27}h^2}-1\right)^{2/3}\right]$$

and $h = C-\lambda/2C$.

The PoA is therefore equal to

$$\begin{aligned}\mathrm{PoA}_B &= \frac{\mathrm{TD}_E^B}{S(x_{\min})}\\ &=\frac{\sqrt{C-\lambda}}{2(\sqrt{C-\lambda}-\sqrt{C})}\left[-2+\frac{C}{C-\lambda}\left(Z+\frac{h}{3}\right)+\frac{1}{Z+\frac{h}{3}}\right]. \quad (22)\end{aligned}$$

Note that (20) and (22) are independent of α.

VI. Cognitive Users' Behavior: Replicator Dynamics

After having characterized the *static* steady-state equilibria reached by network operators and users in the joint pricing and spectrum selection game, here, we further focus on modeling the *dynamic* behavior of network users.

To this aim, we use population dynamics (and, in particular, *replicator dynamics*) to model the behavior of users who decide to which network they should connect since these dynamics models network users who adapt their choices and strategies based on the observed state of the system (in terms of costs and congestion, in our case).

Before introducing replicator dynamics for our network selection game, we must first define some relevant game-theoretic concepts.

A. Introduction to Population Games and Replicator Dynamics

Hereafter, we briefly introduce population games and replicator dynamics; for more details, see [8].

1) Population Games: A population game G, with Q nonatomic classes of players (i.e., network users) is defined by a mass and a strategy set for each class, and a payoff function for each strategy. By a nonatomic population, we mean that the contribution of each member of the population is very small; this is the case in our game, where a large set of users compete for CRN's bandwidth resources. We denote the set of classes by $\mathcal{Q}=\{1,\ldots,Q\}$, where $Q\geq 1$. Class q has mass m^q. Let S^q be the set of strategies available for players of class q, where $S^q=\{1,\ldots,s^q\}$. These strategies can be thought of as the actions that members of q could possibly take (i.e., connecting to the primary or secondary network).

During game play, each player of class q selects a strategy from S^q. The mass of players of class q that choose the strategy $n\in S^q$ is denoted by x_n^q, where $\sum_{n\in S^q} x_n^q = m^q$. We denote the vector of strategy distributions being used by the entire population by $x=\{x^1,\ldots,x^Q\}$, where $x^i=\{x_1^i,\ldots,x_{s^i}^i\}$. Vector x can be thought of as the state of the system.

The marginal payoff function (per mass unit) of players of class q who play strategy n when the state of the system is x is denoted by $F_n^q(x)$, usually referred to as *fitness* in evolutionary game theory, which is assumed to be continuous and differentiable. The total payoff of the players of class q is therefore $\sum_{n\in S^q} F_n^q(x)x_n^q$.

2) Replicator Dynamics: The replicator dynamics describes the behavior of a large population of agents who are randomly matched to play normal form games. It was first introduced in biology in [33] to model the evolution of species, and it is used in the economics field. Recently, this dynamics has been applied to many networking problems, such as routing and resource allocation [34], [35].

Given x_n^q, which represents the proportion of players of class q that choose strategy n, as shown before, the replicator dynamics can be expressed as follows:

$$\dot{x}_n^q = x_n^q\left[F_n^q(x)-\frac{1}{m^q}\sum_{n\in S^q}F_n^q(x)x_n^q\right] \quad (23)$$

where $\dot{x}_n^q$ represents the derivative of x_n^q with respect to time.

In fact, the ratio $\dot{x}_n^q/x_n^q$ measures the evolutionary success (the rate of increase) of strategy n. This ratio can be also expressed as the difference in fitness $F_n^q(x)$ of strategy n and the average fitness $1/m^q\sum_{n\in S^q}F_n^q(x)x_n^q$ of class q.

$$\mathrm{PoA}_T=\frac{\mathrm{TD}_E^T}{S(x_{\min})}=\frac{C(9C-5\lambda)\sqrt{C-\lambda}-(3C-2\lambda)(C-\lambda)\sqrt{9C-5\lambda}}{2(2C-\lambda)[(C-\lambda)+\sqrt{(C-\lambda)(9C-5\lambda)}][\sqrt{C}-\sqrt{C-\lambda}]} \quad (20)$$

An important concept in population games and replicator dynamics is *Wardrop* equilibrium [31], which we introduced in Section IV. In this context, state $\hat{x}$ is a Wardrop equilibrium if, for any class $q \in \mathcal{Q}$, all strategies being used by the members of q yield the same marginal payoff to each member of q, whereas the marginal payoff that would be obtained by members of q is lower for all strategies not used by class q.

B. Cognitive Users' Behavior in the Network Selection Game: Replicator Dynamics

Having reviewed the mathematical tools that we will rely on, we now focus on the CR scenario shown in Section III, introducing replicator dynamics for the network selection game. In particular, we consider a population game G with a nonatomic set of players $(q = 1)$, which is defined by a *strategy set* denoted by $\mathcal{S} = \{s_p, s_s\}$, identical for all players, and a *payoff function* for each strategy; s_p means that the player chooses the *primary* network, and s_s means that the player chooses the *secondary* network, using the spectrum holes left free by PUs.

Our goal is to determine the dynamic network selection settings (X_P and $X_S = 1 - X_P$), i.e., the fraction of players that choose the primary and secondary networks, respectively, based on the equilibrium prices set by POs and SOs. Hence, the total traffic accepted in the primary network is equal to $\lambda_P = \lambda X_P$, and the one accepted in the secondary network is $\lambda_S = \lambda X_S$.

The proposed replicator dynamics provides a means to analyze how players can "learn" about their environment, and converge toward an equilibrium choice. Replicator dynamics is also useful to investigate the speed of convergence of strategy adaptation to reach a stable solution in the game. A mathematical analysis to bound such speed is provided in Section VII. In this case, CR users need to know some information, viz., the total cost (the service delay plus the price charged by the PO/SO, respectively) and the size of the populations (X_P, X_S) that already performed such selection, before undertaking the best choice based on the system state.

As shown in Section III, the goal of each CR user is to *minimize* a weighted sum of his delay (latency) and the price paid to the network operator (either primary or secondary), with α being the parameter that permits to give more weight to delay with respect to the paid price. Hence, we can formalize the network selection game as follows:

$$\begin{aligned}\dot{X}_P =& KX_P\left[\frac{-\alpha}{C-\lambda X_P} - p_1 - \left(\frac{-\alpha X_P}{C-\lambda X_P} - X_P \cdot p_1 \right.\right.\\ &\left.\left. - (1-X_P)\left(\frac{\alpha}{C-\lambda} + p_2\right)\right)\right]\\ =& KX_P(1-X_P)\left[-p_1 + p_2 + \frac{\alpha}{C-\lambda} - \frac{\alpha}{C-\lambda X_P}\right] \qquad (24)\end{aligned}$$

where $\dot{X}_P$ represents the derivative of X_P with respect to time.

This equation has the same structure as the replicator dynamics [see (23)]: The first term $(F_n^q(x) \equiv -\alpha/C - \lambda X_P - p_1)$ corresponds to the total cost (the service delay plus the price charged by the PO) perceived by users that choose to connect to the primary network, using a $M|M|$ 1 approximation; the second term $(1/m^q \sum_{n \in S^q} F_n^q(x) x_n^q \equiv -\alpha X_P/C - \lambda X_P - X_P \cdot p_1 - (1 - X_P)(\alpha/C - \lambda + p_2))$ represents the average cost/delay incurred by the fraction X_P of PUs and by the fraction X_S of SUs (recall that p_1 and p_2 are the prices charged by the POs and SOs, respectively).

In particular, the speed of variation of X_P is proportional to the population size X_P (via the proportionality coefficient K), which models the willingness of the population to change strategy.

A similar equation can be written for SUs; thus, we can express the replicator dynamics for such SUs as follows:

$$\begin{aligned}\dot{X}_S = KX_S(1 - X_S)&\left[p_1 - p_2 - \frac{\alpha}{C-\lambda}\right.\\ &\left. + \frac{\alpha}{(C-\lambda) + \lambda X_S}\right]. \qquad (25)\end{aligned}$$

Obviously, by comparing these two expressions, it can be verified that condition $X_p + X_s = 1$ holds.

It can be demonstrated [8] that Wardrop equilibria are the stationary points of (24) and (25). As we will show in the following, it can be easily proven that the unique nontrivial fixed point of such dynamics coincides with the Wardrop equilibrium point of the CR users' network selection game already determined in Section IV.

VII. Convergence Analysis of Replicator Dynamics

Here, we provide an in-depth analysis on the replicator dynamics given by (24).[2] To this end, we rewrite it in a discretized version as follows:

$$X_P(t+1) = X_P(t) + kX_P(t)\left[1 - X_P(t)\right]\left[A - \frac{1}{B - X_P(t)}\right] \qquad (26)$$

where $k = K\alpha/\lambda$, $A = \lambda(-p_1/\alpha + p_2/\alpha + 1/C - \lambda)$, and $B = C/\lambda$.

The given dynamics has three fixed points, among which 0 and 1 are *trivial* fixed points corresponding to the case where all users either act as SUs or PUs, respectively. $X_P^* = B - 1/A$ is the only nontrivial fixed point, which is also the Wardrop equilibrium of the game; its expression is equal to $X_P^* = \lambda_P/\lambda$, where λ_P is the equilibrium flow already derived for the static game in Section IV [see (6)].

In the subsequent analysis, we investigate the convergence of the replicator dynamics to X_P^*. We start by establishing the following auxiliary lemma.

Lemma 1: Under the condition that $K(A - 1/B - 1) \leq 1$, the following holds.

- $X_P(t+1)$ is nondecreasing with respect to $X_P(t)$ for $X_P(t) \in [0, X_P^*)$ and nonincreasing with respect to $X_P(t)$ for $X_P(t) \in (X_P^*, 1]$.
- $X_P(t+1) > X_P(t)$, $\forall X_P(t) < X_P^*$, and $X_P(t+1) < X_P(t), \forall X_P(t) > X_P^*$.

[2] Note that the same analysis can be conducted for (25).

Proof: The proof of the first part is straightforward by checking the derivative $\partial X_P(t+1)/\partial X_P(t)$. Specifically, it can be checked that, under the condition that $K(A - 1/B - 1) \leq 1$, $\partial X_P(t+1)/\partial X_P(t) > 0$, when $X_P(t) \in [0, X_P^*)$ and $\partial X_P(t+1)/\partial X_P(t) < 0$ when $X_P(t) \in (X_P^*, 1]$. The second part follows readily from (26). □

The following theorem establishes the convergence of the replicator dynamics to the nontrivial fixed point X_P^*.

Theorem 1: Under the condition that $K(A - 1/B - 1) \leq 1$, the replicator dynamics depicted in (26) converges to the nontrivial fixed point X_P^* for any initial state $0 < X_P(0) < 1$.

Proof: Consider an arbitrary sequence of update steps commencing from an initial vector $X_P(0)$. We distinguish the following two cases.

- *Case 1*: $0 < X_P(0) \leq X_P^*$. In this case [recall that X_P^* is a fixed point of (26)], it follows from Lemma 1 that 1) $X_P(t) \leq X_P^*$ $\forall t$ and 2) that $X_P(0) \leq X_P(1) \leq \cdots \leq X_P(t-1) \leq X_P(t) \leq \cdots$, i.e., $X_P(t)$ is a nondecreasing sequence. Since $X_P(t)$ is also bounded by X_P^*, it follows that it must converge to a limit. Since there is no fixed point other than X_P^* in the range $(0, X_P^*]$, this limit must be X_P^*.
- *Case 2*: $X_P^* < X_P(0) < 1$. This case can be proven in a similar manner. In fact [recall that X_P^* is a fixed point of (26)], it follows from Lemma 1 that 1) $X_P(t) > X_P^*$ $\forall t$ and 2) that $X_P(0) \geq X_P(1) \geq \cdots \geq X_P(t-1) \geq X_P(t) \geq \cdots$, i.e., $X_P(t)$ is a nonincreasing sequence. Since $X_P(t)$ is also bounded by X_P^*, it follows that it must converge to a limit. Since there is no fixed point other than X_P^* in the range $[X_P^*, 1)$, this limit must be X_P^*.

By combining the given analysis, the replicator dynamics is ensured to converge to the nontrivial fixed point X_P^* for any initial state $0 < X_P(0) < 1$. □

The given theorem essentially shows that, with a conservative strategy (i.e., small K), the replicator dynamics is ensured to converge to the Wardrop equilibrium.

Remark: The given theorem establishes the sufficient condition for the convergence of the replicator dynamics to the unique nontrivial fixed point, which is also the Wardrop equilibrium. It straightforwardly follows that, under the same condition, the equilibrium is also stable in that any deviated point from it will be dragged back under the replicator dynamics. In fact, X_P^* is an evolutionary stable equilibrium. Meanwhile, it follows from the theorem that the two trivial fixed points 0 and 1 are not stable, in the sense that any deviation from them will drag the system to X_P^*.

It is also worth pointing out that Theorem 1 provides only a *sufficient* condition for the convergence and may be too stringent in some cases.

We further investigate the stability and the convergence speed of the replicator dynamics in the following theorem, following the guidelines of [36].

Theorem 2: Under the condition that $K(A - 1/B - 1) < 1$, the nontrivial fixed point X_P^* is exponentially stable under the replicator dynamics depicted in (26), i.e., there exists $0 \leq k' < 1$, such that $|X(t) - X_P^*| \leq (k')^t |X(0) - X_P^*|$.

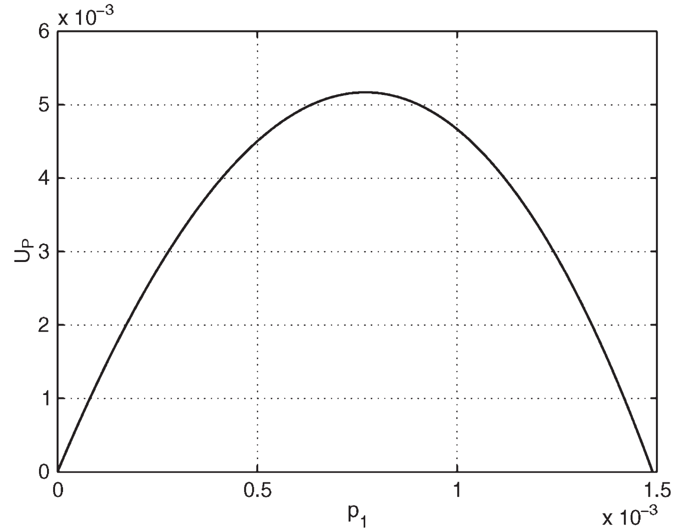


Fig. 2. PO's utility U_P as a function of the imposed price p_1 in the *TOGETHER* scenario. Price p_2 has been fixed to the Nash equilibrium value.

Proof: We show that the replicator dynamics $X_P(t) \to X_P(t+1)$ in (26) is a contraction.

The contraction is defined as follows. Let $(X,\, d)$ be a metric space, $f\colon X \to X$ is a contraction if there exists a constant $k' \in [0, 1)$, such that $\forall x, y \in X$, $d(f(x), f(y)) \leq k' d(x,\, y)$, where $d(x,\, y) = \|x - y\| = \max_i \|x_i - y_i\|$.

To that end, note that

$$d\left(f(x),\, f(y)\right) = \|f(x) - f(y)\| \leq \left\| \frac{\partial f}{\partial x} \right\| \cdot \|x - y\| = \left\| \frac{\partial f}{\partial x} \right\| d(x,\, y).$$

If the Jacobian $\|\partial f / \partial x\| \leq k'$, then f is a contraction.

By some algebraic operations, we can bound the Jacobian as

$$\|J\|_\infty = \max_{X_P(t) \in (0,1)} \left| \frac{\partial X_P(t+1)}{\partial X_P(t)} \right| \leq 1 - K\left(A - \frac{1}{B-1}\right).$$

Hence, since the condition $K(A - 1/B - 1) < 1$ holds, i.e., $\|J\|_\infty \leq k' \triangleq 1 - K(A - 1/B - 1) < 1$, X_P^* is exponentially stable where k' is the exponential convergence speed. □

VIII. Numerical Results

Here, we analyze and discuss the numerical results obtained from solving our joint pricing and spectrum-access game model in different CR scenarios. More in detail, we measure the sensitivity of the operators' utilities and prices, and users' equilibrium flows and costs, to different parameters such as the total traffic λ accepted in the network and the channel capacity C.

Before doing so, let us first consider an example of a PO utility function U_P. Fig. 2 shows this latter as a function of price p_1 set by the PO (price p_2 has been fixed to the Nash equilibrium value), with $\alpha = 1$, $C = 100$, and $\lambda = 10$. By simply deriving and using the second-order derivative test, it can be proven that the PO's revenue has a global maximum, as shown in the figure, since for small p_1 values, the incoming primary traffic is priced too low, resulting in a low PO revenue, whereas for high p_1 values, few users choose the primary network, thus diminishing its profitability.

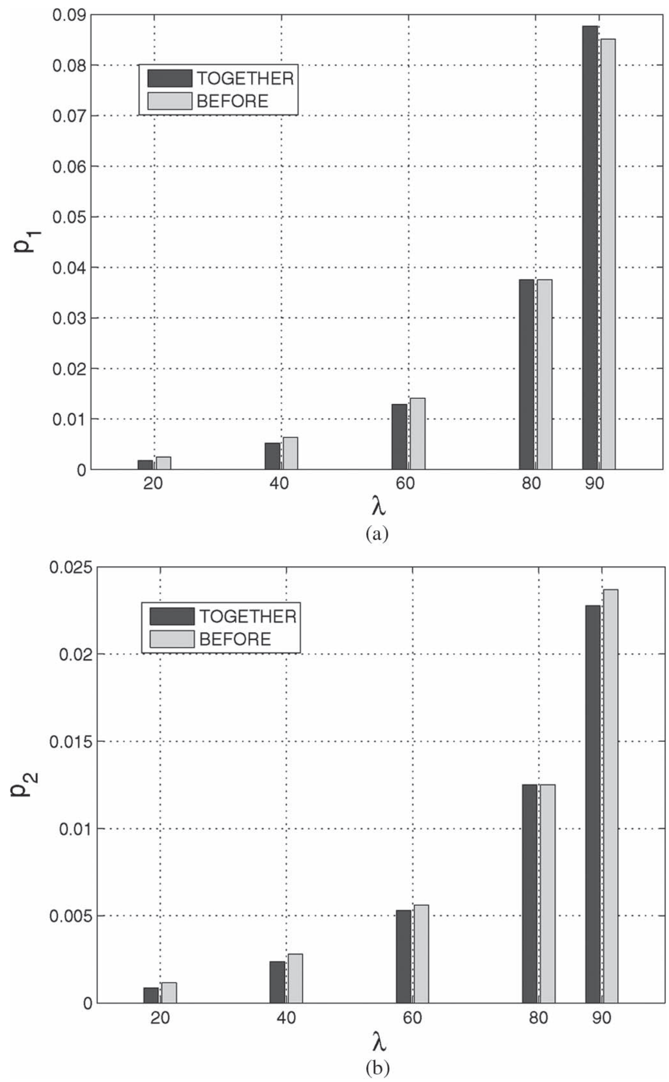


Fig. 3. (a) Equilibrium price p_1 set by the PO. (b) Equilibrium price p_2 set by the SO as a function of the total traffic λ offered to the network for both the *BEFORE* and *TOGETHER* scenarios.

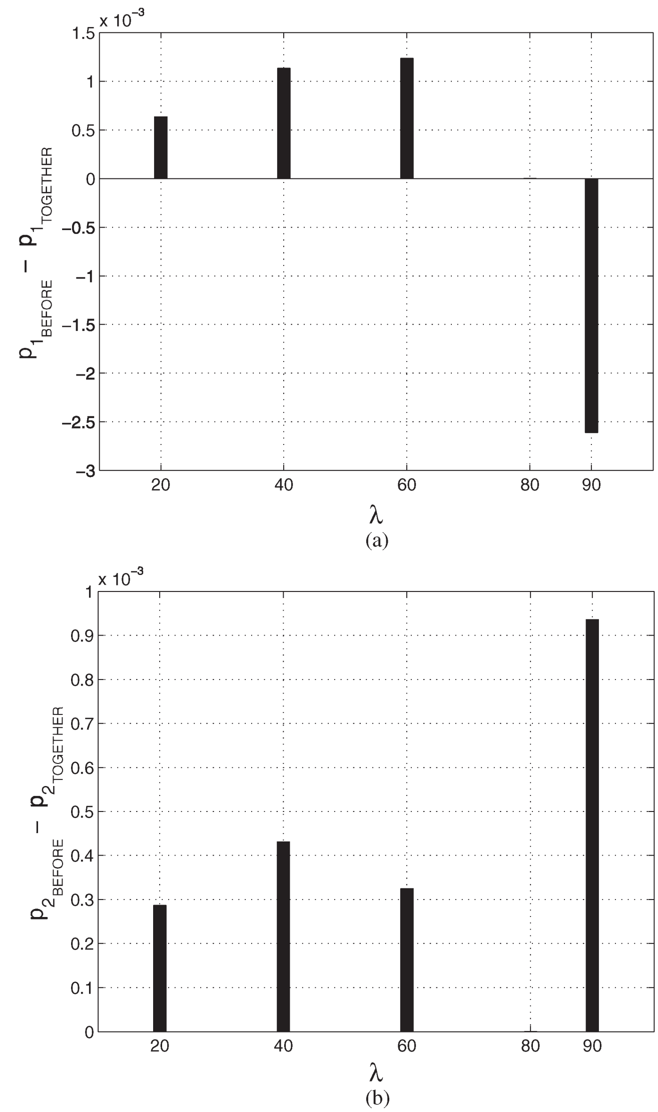


Fig. 4. (a) Difference in the equilibrium prices p_1 set by the PO in the *TOGETHER* and *BEFORE* scenarios. (b) Difference in the equilibrium prices p_2 set by the SO in the same scenarios.

A. *Effect of the Traffic Accepted in the Network* λ

We first consider a CRN scenario with maximum channel capacity $C = 100$ and total accepted traffic λ varying in the [0, 100] range. Parameter α, which expresses the relative importance of the experienced delay with respect to the access cost, is set to 1, unless otherwise stated.

Fig. 3(a) and (b) shows the prices set at the Nash equilibrium by the PO (p_1) and the SO (p_2), respectively, in the two considered scenarios (the PO and SO play *TOGETHER*, and the PO plays *BEFORE* the SO, anticipating the choices of this latter). The difference between the prices set by the operators in these two scenarios can be better appreciated in Fig. 4(a) and (b) for the POs and SOs, respectively. All numerical results shown in Figs. 3 and 4 are summarized in Table IV.

It can be observed [see Fig. 4(a)] that in the *BEFORE* scenario, the PO sets a higher price than in the *TOGETHER* scenario, until the network is overloaded $(\lambda \leq 80)$; above this threshold, the price that is set by the PO in the former scenario is lower than that in the latter. As for the price that is set by the SO [see Fig. 4(b)], it is always higher in the *BEFORE* than in the *TOGETHER* scenario, and such a difference increases consistently for increasing λ values. This is the reason why the PO in the *BEFORE* scenario can lower his price while still attracting the large majority of network users, as we will show in the following.

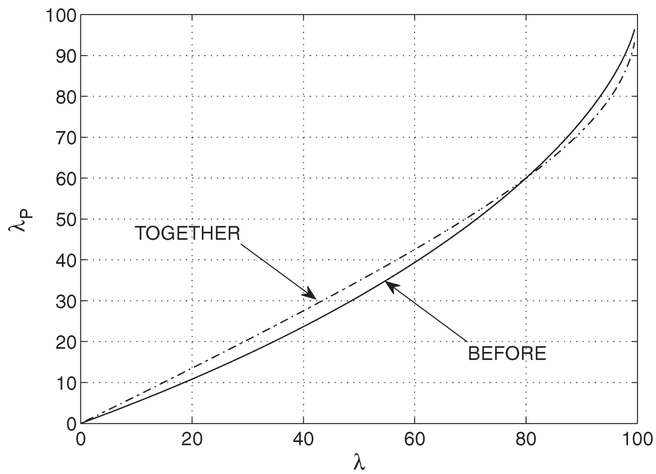
The corresponding equilibrium traffic sent by PUs (λ_P) and SUs (λ_S) is shown in Fig. 5(a) and (b) as a function of λ for both the considered scenarios.

We can observe the following.

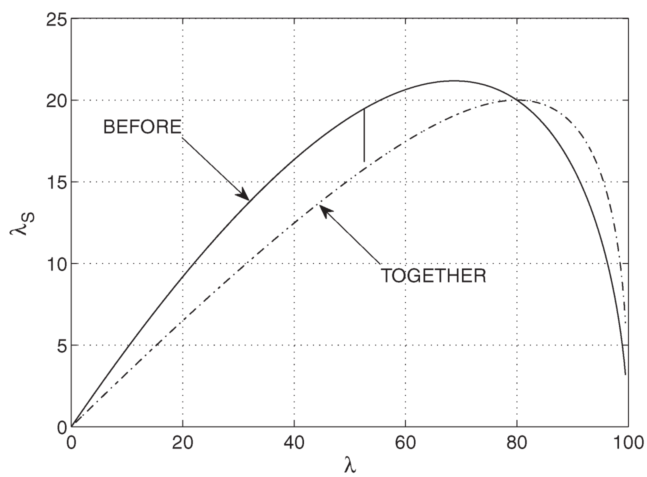
- The traffic accepted (and, consequently, the overall fraction of users) in the primary network, i.e., λ_P, always increases with the offered traffic until, finally, when $\lambda \to C$, all users choose the primary network. This is due to the superior attractiveness of such network (in terms of the delay experienced by users) with respect to the secondary network since resources are licensed to PUs and SUs always observe a higher delay than PUs.
- Furthermore, concerning λ_P, in the *BEFORE* scenario, the PO admits (slightly) less traffic than the SO when

TABLE IV
EQUILIBRIUM PRICES p_1 AND p_2 SET BY THE PO/SO (AND THEIR DIFFERENCE) FOR DIFFERENT VALUES OF THE TOTAL TRAFFIC λ OFFERED TO THE NETWORK FOR BOTH THE *BEFORE* AND *TOGETHER* SCENARIOS

λ	$p1_{TOGETHER} \times 10^{-3}$	$p1_{BEFORE} \times 10^{-3}$	$p2_{TOGETHER} \times 10^{-3}$	$p2_{BEFORE} \times 10^{-3}$	$\Delta p_1 \times 10^{-3}$	$\Delta p_2 \times 10^{-3}$
20	1.806	2.441	0.868	1.154	0.635	0.286
40	5.242	6.375	2.374	2.805	1.133	0.431
60	12.885	14.122	5.288	5.613	1.237	0.325
80	37.5	37.5	12.5	12.5	0	0
90	87.724	85.112	22.761	23.697	- 2.612	0.936

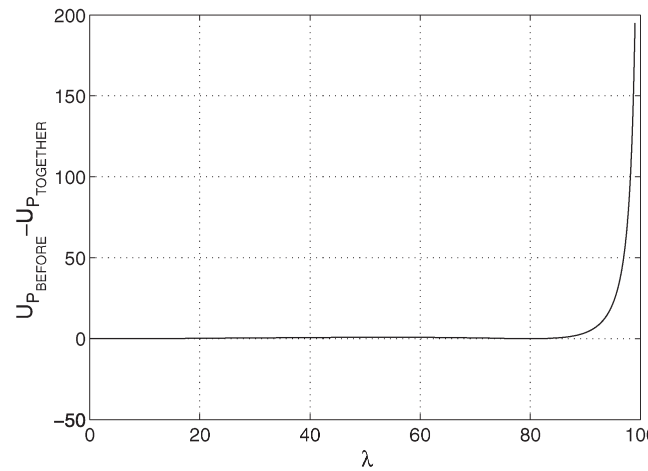


(a)

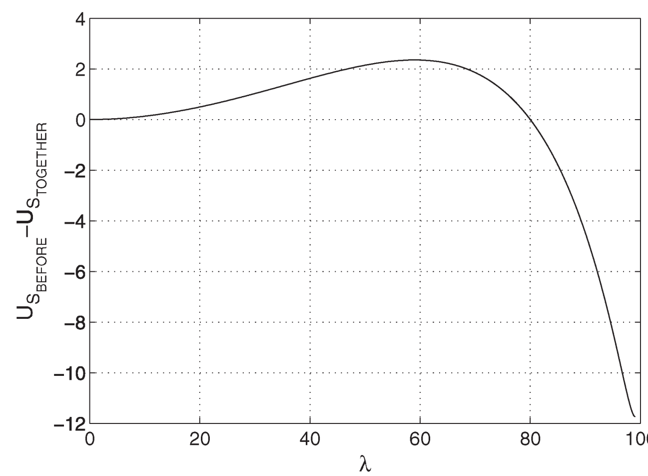


(b)

Fig. 5. Equilibrium traffic sent by PUs (λ_P) and SUs (λ_S) as a function of the total traffic λ accepted in the network for both the *TOGETHER* and *BEFORE* scenarios.



(a)



(b)

Fig. 6. (a) Difference in utilities U_P of the PO when he plays *BEFORE* and *TOGETHER* with the SO. (b) Difference in utilities U_S of the SO in the same scenarios.

$\lambda < 80\%$ of the total capacity C [see Fig. 5(a)]; this is due to the fact that the equilibrium price p_1 set by the PO in this scenario is higher than in the *TOGETHER* case [see Fig. 4(a)], which in turn makes λ_P decrease. In the high-traffic regime, the PO increasingly attracts more traffic due to the significantly lower delay experienced in the primary network, whereas the SO increases p_2 in an effort to increase his utility, in spite of the customer rush toward the primary network (more specifically, fewer clients choose the SO, who reacts by raising his access price p_2 to increase his revenue, which is a reaction that, in turn, accentuates this phenomenon).

- Concerning λ_S, its derivative with respect to λ is always decreasing: It is increasingly less attractive to be a SU than a PU since, for increasing λ values, the delay tends to dominate in the total cost perceived by the user.

We now focus our analysis on operators' utility, which we recall is defined as the product of the price that is set by the operator and the total flow transmitted by users that choose such operator. Fig. 6(a) and (b) shows, respectively, the difference in utilities for the PO (ΔU_P) and SO (ΔU_S) in the *TOGETHER* and *BEFORE* scenarios.

It can be observed that it is increasingly more convenient for the PO to be a leader, anticipating the SO, and this is reflected in the utility, which consistently grows for increasing λ values. At the same time, for low and medium λ values $(\lambda < 0.8\ C)$, even the SO obtains a higher utility in the *BEFORE* scenario. This means that, in such scenario, both operators achieve an

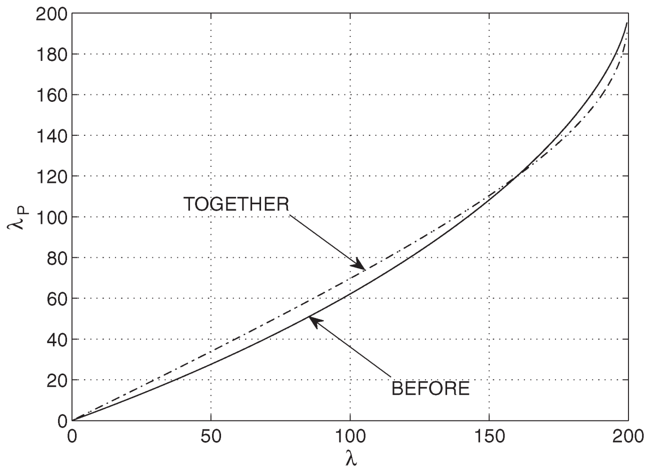


Fig. 7. Equilibrium traffic sent by PUs λ_P as a function of the total traffic λ accepted in the network, for both the *TOGETHER* and *BEFORE* scenarios. The total channel capacity is $C = 200$.

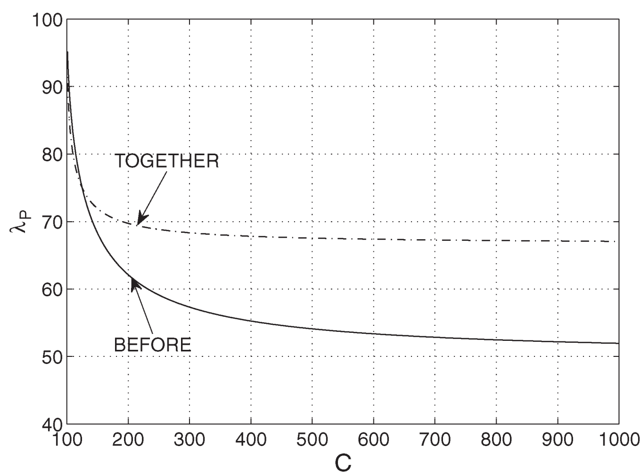


Fig. 8. Equilibrium traffic sent by PUs λ_P as a function of the channel capacity C for both the *TOGETHER* and *BEFORE* scenarios. The total traffic offered to the network λ is fixed and equal to 100.

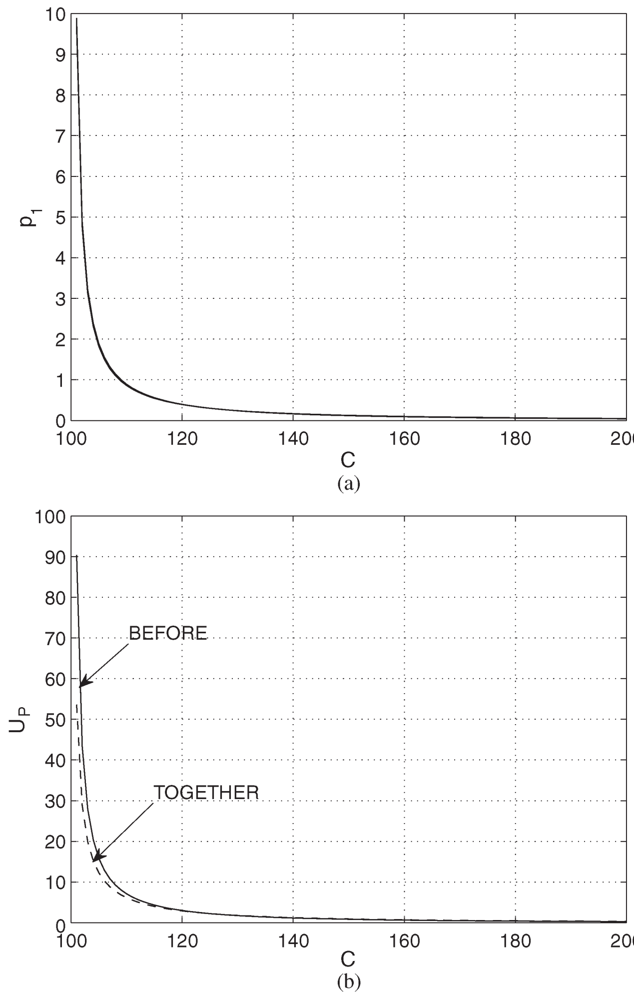


Fig. 9. (a) POs price p_1 and (b) utility U_P as a function of the channel capacity C for both the *TOGETHER* and *BEFORE* scenarios. The total traffic offered to the network λ is fixed and equal to 100. Note that prices p_1 practically overlap in the two considered scenarios.

economic advantage at the expense of the total price paid by CR users.

B. Effect of the Channel Capacity C

We now consider a variation of this network scenario, doubling the channel capacity C to 200; the total traffic admitted in the primary network is shown in Fig. 7. The trend is the same as already shown in Fig. 5(a), and a similar behavior can be observed for the secondary traffic, which is not reported for the sake of brevity.

On the other hand, Fig. 8 shows the equilibrium traffic sent by PUs as a function of the wireless channel capacity C, with λ fixed to 100. It can be observed that λ_P tends to $\lambda/2$ $(= 50$ in this case) in the *BEFORE* scenario and to $2\lambda/3(\approx 66.6)$ in the *TOGETHER* scenario.[3] This behavior is in line with what is already observed in Fig. 7 since, when λ is consistently lower than C, the PO who plays before the SO (*BEFORE* scenario) tends to admit less traffic than the latter.

We further show in Fig. 9 the chosen price and the utility perceived by the PO, in both the considered scenarios, for increasing values of the channel capacity C and a total accepted traffic λ fixed to 100 (note that the prices p_1 set by the PO, which is shown in Fig. 9(a), almost overlap in the two considered scenarios). A similar trend can be observed for both the price and utility of the SO (see Fig. 10).

In summary, as the available capacity increases, operators fix increasingly lower prices, achieving a lower total revenue.

The impact of C on the PoA is further investigated in Section VIII-B.

C. Efficiency of the Reached Equilibria: PoA

We now measure the efficiency of the equilibria reached by the system. The PoA, which in our game coincides with the PoA due to the uniqueness of the equilibria reached by operators and users, is shown in Fig. 11 for both the *TOGETHER* (PoA_T) and *BEFORE* scenarios (PoA_B).

[3] It suffices to compute the limit for $C \to \infty$ of λ_P in (6), substituting the equilibrium values p_1 and p_2 for both the considered scenarios. Note that such a limit is independent of α.

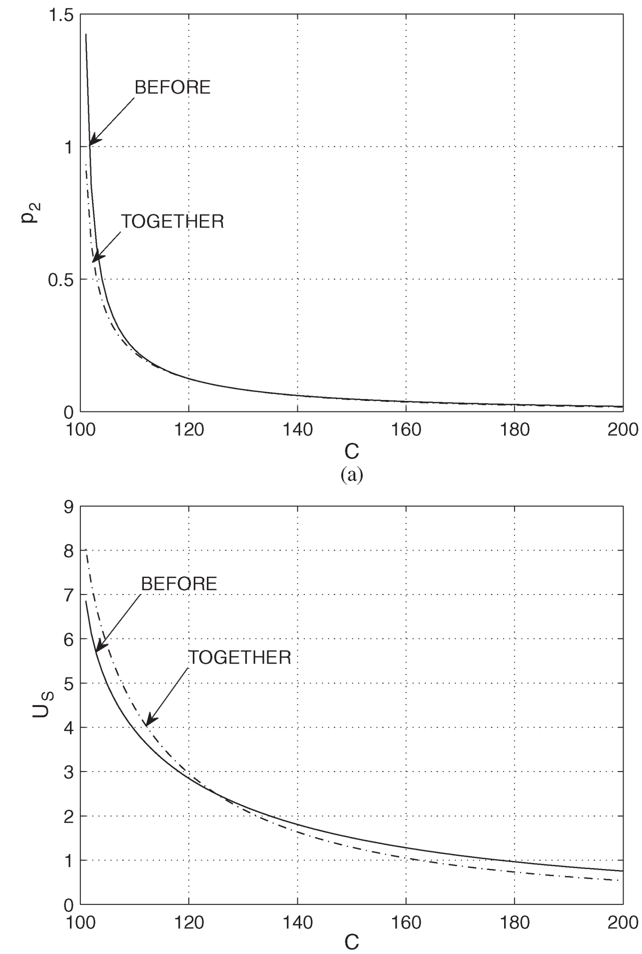


Fig. 10. (a) SO's price p_2 and (b) utility U_S as a function of the channel capacity C for both the *TOGETHER* and *BEFORE* scenarios. The total traffic offered to the network λ is fixed and equal to 100.

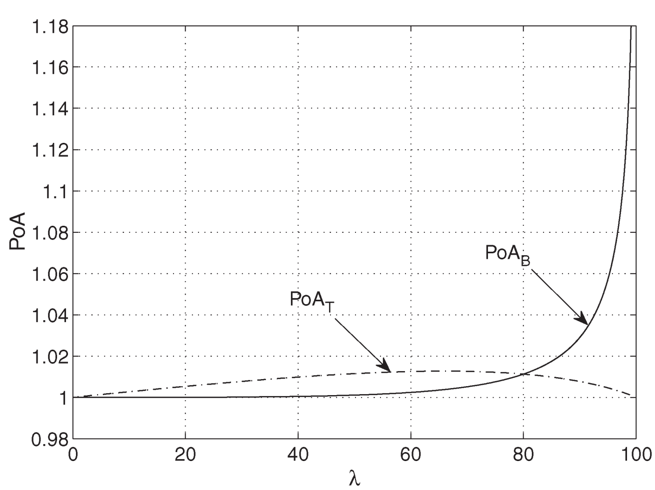


Fig. 11. PoA as a function of the total traffic offered to the network λ in both the *TOGETHER* (PoA_T) and *BEFORE* (PoA_B) scenarios.

When both operators play together, the PoA is equal to 1 for both extreme cases ($\lambda = 0$ and $\lambda = C$). Furthermore, it has a maximum equal to 1.0127 for $\lambda/C = 2/3$, which means that, in such a scenario, the equilibrium reached by the system is only $\approx$1.3% worse (in terms of the overall experienced delay) with respect to the socially optimal solution. In the *BEFORE* scenario, the PoA is also low, but the trend exhibited by such performance figure differs from the previous scenario since the PoA tends to infinity for λ approaching the channel capacity C. This is due to the fact that the total cost for users at equilibrium significantly increases faster than the social welfare, particularly for high λ values.

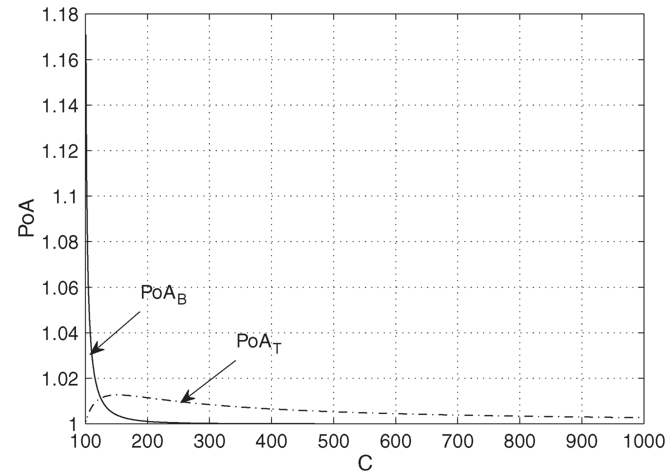


Fig. 12. PoA as a function of the channel capacity C for both the *TOGETHER* (PoA_T) and *BEFORE* (PoA_B) scenarios. The total traffic offered to the network λ is fixed and equal to 100.

As a consequence, this situation should be avoided by market controllers either by 1) controlling the admitted traffic λ, imposing on it that it does not exceed a predefined fraction of the available channel capacity; or 2) preventing the *BEFORE* scenario to occur, imposing on it antitrust policies to limit dominant position abuse.

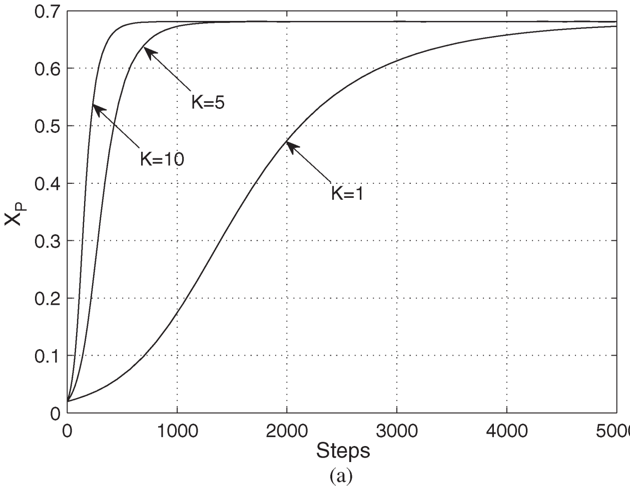
Fig. 12 further reports the PoA as a function of the channel capacity C for both the considered scenarios; λ is fixed and equal to 100. It is not surprising that both curves rapidly decrease with C, since, as already observed in Fig. 11, when λ is consistently lower than C, the PoA $\rightarrow 1$ in both scenarios.

In summary, we can conclude that, apart from the limiting case shown before for very high traffic loads, the quality of the reached equilibria is indeed excellent: When the system is loaded at less than 95%, which is a reasonable operating region, the PoA is always less than 1.1, which means a loss of efficiency of 10% with respect to the social optimum. The system hence converges to a stable state, which is globally very efficient.

D. Replicator Dynamics for the Network Selection Game

We now analyze the convergence of the proposed replicator dynamics, fixing $\lambda = 30$ and $C = 100$. Fig. 13 shows this convergence (expressed in steps needed in the replicator dynamics) of network users to a stationary solution, for different values of the parameter K in (24), namely, 1, 5, and 10. More specifically, the figure reports the fraction X_P of users that choose the primary network. We consider both cases where the initial fraction of these users is close to zero [see Fig. 13(a)] and one [see Fig. 13(b)].

Note that the speed of convergence to the unique stable equilibrium point of the dynamics ($X_P^* \approx 0.68$, in this scenario) increases for increasing K values. Furthermore, when p_1 and p_2 are equilibrium price values, we observe that the convergence



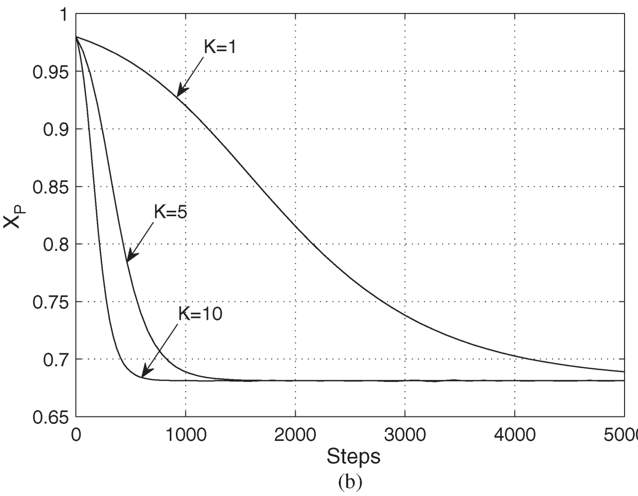


Fig. 13. Convergence of PUs to the stationary point ($X_P^* \approx 0.68$). The initial point is (a) *lower* or (b) *higher* than the equilibrium.

conditions demonstrated in Theorems 1 and 2 for our proposed replicator dynamics (see the previous section) are always satisfied.

IX. Conclusion

In this paper, we tackled a fundamental problem related to CRNs, i.e., the joint pricing and primary/secondary network selection process. More specifically, we considered a CRN scenario that is composed of primary/secondary networks and a set of CR users who must decide whether to subscribe to the primary network for guaranteed bandwidth or to access the secondary network, paying a lower price at the expense of possible service degradation (in terms of experienced delay and congestion). At the same time, we studied the pricing game between the POs and SOs, considering two practical cases where such operators fix their access price simultaneously and where the PO anticipates the SO strategy, exploiting his dominant position.

We computed optimal stable pricing values and network selection settings; furthermore, we studied network users' dynamics using a population game model, and we determined its convergence properties under replicator dynamics. Numerical results demonstrate that our game model captures the main factors behind cognitive network pricing and access network selection, thus representing a promising framework for the design and understanding of CR systems.

A key finding of this paper is that the advantage for the PO to play before the SO can be significant, particularly in a high-traffic regime; this has an adverse impact on customers' choices since, in this situation, the equilibria reached by CR users drift away from the social optimum, and the PoA tends to infinity. It is therefore important (e.g., for government and regulatory authorities) to implement actions that prevent or limit this dominant position abuse, if possible.

Apart from this limiting case, which exclusively occurs for very high traffic regimes, we observe that the quality of the reached equilibria is excellent: When the system is loaded at less than 95%, which seems to be a reasonable operating region, the PoA is always less than 1.1 (regardless of the order in which operators fix their price), which means a loss of efficiency of 10% with respect to the social optimum. Hence, the system is guaranteed to converge to a stable state that is very efficient from a social point of view.

References

[1] I. F. Akyildiz, W. Y. Lee, M. C. Vuran, and S. Mohanty, "A survey on spectrum management in cognitive radio networks," *IEEE Commun. Mag.*, vol. 46, no. 4, pp. 40–48, Apr. 2008.

[2] J. Elias, F. Martignon, A. Capone, and E. Altman, "Non-cooperative spectrum access in cognitive radio networks: A game theoretical model," *Comput. Netw.*, vol. 55, no. 17, pp. 3832–3846, Dec. 2011.

[3] K. Jagannathan, I. Menache, E. Modiano, and G. Zussman, "Non-cooperative spectrum access—The dedicated vs. free spectrum choice," *IEEE J. Sel. Areas Commun.*, vol. 30, no. 11, pp. 2251–2261, Dec. 2012.

[4] S. Wang, J. Zhang, and L. Tong, "Delay analysis for cognitive radio networks with random access: A fluid queue view," in *Proc. IEEE Infocom*, San Diego, CA, USA, Mar. 2010, pp. 1–9.

[5] Q. Zhao, L. Tong, A. Swami, and Y. Chen, "Decentralized cognitive MAC for opportunistic spectrum access in ad hoc networks: A POMDP framework," *IEEE J. Sel. Areas Commun.*, vol. 25, no. 3, pp. 589–600, Apr. 2007.

[6] X. Li, Q. Zhao, X. Guan, and L. Tong, "On the performance of cognitive access with periodic spectrum sensing," in *Proc. ACM Workshop Cognit. Radio Netw.*, Beijing, China, Sep. 20–25, 2009, pp. 31–36.

[7] O. Habachi and Y. Hayel, "Optimal opportunistic sensing in cognitive radio networks," *IET Commun.*, vol. 6, no. 8, pp. 797–804, May 2012.

[8] W. H. Sandholm, *Population Games and Evolutionary Dynamics*. Cambridge, MA, USA: MIT Press, Jan. 2011.

[9] T. L. Vincent and J. S. Brown, *Evolutionary Game Theory, Natural Selection, and Darwinian Dynamics*. Cambridge, U.K.: Cambridge Univ. Press, 2005.

[10] E. Altman, T. Boulogne, R. El-Azouzi, T. Jimenez, and L. Wynter, "A survey on networking games in telecommunications," *Comput. Oper. Res.*, vol. 33, no. 2, pp. 286–311, Feb. 2006.

[11] Y. C. Liang, K. C. Chen, G. Y. Li, and P. Mahonen, "Cognitive radio networking and communications: An overview," *IEEE Trans. Veh. Technol.*, vol. 60, no. 7, pp. 3386–3407, Sep. 2011.

[12] G. S. Kasbekar and S. Sarkar, "Spectrum pricing games with spatial reuse in cognitive radio networks," *IEEE J. Sel. Areas Commun.*, vol. 30, no. 1, pp. 153–164, Jan. 2012.

[13] D. Xu, X. Liu, and Z. Han, "Decentralized bargain: A two-tier market for efficient and flexible dynamic spectrum access," *IEEE Trans. Mobile Comput.*, May 31, 2012, IEEE Computer Society Digital Library, IEEE Computer Society. [Online]. Available: http://doi.ieeecomputersociety.org/10.1109/TMC.2012.130

[14] K. Zhu, D. Niyato, P. Wang, and Z. Han, "Dynamic spectrum leasing and service selection in spectrum secondary market of cognitive radio networks," *IEEE Trans. Wireless Commun.*, vol. 11, no. 3, pp. 1136–1145, Mar. 2012.

[15] R. Xie, F. R. Yu, and H. Ji, "Dynamic resource allocation for heterogeneous services in cognitive radio networks with imperfect channel

sensing," *IEEE Trans. Veh. Technol.*, vol. 61, no. 2, pp. 770–780, Feb. 2012.
[16] T. Elkourdi and O. Simeone, "Spectrum leasing via cooperation with multiple primary users," *IEEE Trans. Veh. Technol.*, vol. 61, no. 2, pp. 820–825, Feb. 2012.
[17] Y. Li, S. K. Jayaweera, M. Bkassiny, and K. A. Avery, "Optimal myopic sensing and dynamic spectrum access in cognitive radio networks with low-complexity implementations," *IEEE Trans. Wireless Commun.*, vol. 11, no. 7, pp. 2412–2423, Jul. 2012.
[18] W. Saad, Z. Han, H. V. Poor, T. Basar, and J. B. Song, "A cooperative bayesian nonparametric framework for primary user activity monitoring in cognitive radio networks," *IEEE J. Sel. Areas Commun.*, vol. 30, no. 9, pp. 1815–1822, Oct. 2012.
[19] D. Niyato and E. Hossain, "Dynamics of network selection in heterogeneous wireless networks: An evolutionary game approach," *IEEE Trans. Veh. Technol.*, vol. 58, no. 4, pp. 2008–2017, May 2009.
[20] D. Niyato, E. Hossain, and Z. Han, "Dynamics of multiple-seller and multiple-buyer spectrum trading in cognitive radio networks: A game-theoretic modeling approach," *IEEE Trans. Mobile Comput.*, vol. 8, no. 8, pp. 1009–1022, Aug. 2009.
[21] X. Chen and J. Huang, "Evolutionarily stable spectrum access," *IEEE Trans. Mobile Comput.*, vol. 12, no. 7, pp. 1281–1293, Jul. 2013.
[22] S. Iellamo, L. Chen, and M. Coupechoux, "Imitation-based spectrum access policy for CSMA/CA-based cognitive radio networks," in *Proc. IEEE WCNC*, Paris, France, Apr. 2012, pp. 2780–2785.
[23] X. Chen and J. Huang, "Spatial spectrum access game: Nash equilibria and distributed learning," in *Proc. MobiHoc*, Hilton Head Island, SC, USA, Jun. 2012, pp. 205–214.
[24] J. Elias, F. Martignon, and E. Altman, "Joint pricing and cognitive radio network selection: A game theoretical approach," in *Proc. WiOpt*, Paderborn, Germany, May 2012, pp. 49–53.
[25] J. Elias and F. Martignon, "Joint spectrum access and pricing in cognitive radio networks with elastic traffic," in *Proc. IEEE ICC*, Cape Town, South Africa, May 2010, pp. 1–5.
[26] S. Jafar and S. Srinivasa, "Capacity limits of cognitive radio with distributed and dynamic spectral activity," *IEEE J. Sel. Areas Commun.*, vol. 25, no. 3, pp. 529–537, Apr. 2007.
[27] S. Srinivasa and S. Jafar, "Cognitive radios for dynamic spectrum access-the throughput potential of cognitive radio: A theoretical perspective," *IEEE Commun. Mag.*, vol. 45, no. 5, pp. 73–79, May 2007.
[28] S. Srinivasa and S. Jafar, "How much spectrum sharing is optimal in cognitive radio networks?" *IEEE Trans. Wireless Commun.*, vol. 7, no. 10, pp. 4010–4018, Oct. 2008.
[29] E. Anshelevich, A. Dasgupta, J. Kleinberg, E. Tardos, T. Wexler, and T. Roughgarden, "The price of stability for network design with fair cost allocation," in *Proc. 45th Annu. Symp. Foundations Comput. Sci.*, Rome, Italy, Oct. 17–19, 2004, pp. 295–304.
[30] S. Albers, S. Eilts, E. Even-Dar, Y. Mansour, and L. Roditty, "On nash equilibria for a network creation game," in *Proc. 17th Annu. ACM-SIAM Symp. Discrete Algorithm*, Miami, FL, USA, Jan. 2006, pp. 89–98.
[31] J. G. Wardrop, "Some theoretical aspects of road traffic research," *Proc. Inst. Civil Eng., Part II*, vol. 1, no. 36, pp. 325–378, 1952.
[32] E. Koutsoupias and C. Papadimitriou, "Worst-case equilibria," in *Proc. 16th Annu. Symp Theor. Aspects Comput. Sci.*, Trier, Germany, Mar. 4–6, 1999, pp. 404–413.
[33] P. D. Taylor and L. B. Jonker, "Evolutionary stable strategies and game dynamics," *Math. Biosci.*, vol. 40, no. 1/2, pp. 145–156, Jul. 1978.
[34] S. Shakkottai, E. Altman, and A. Kumar, "Multihoming of users to access points in WLANs: A population game perspective," *IEEE J. Sel. Areas Commun*, vol. 25, no. 6, pp. 1207–1215, Aug. 2007.
[35] E. Altman, Y. Hayel, and H. Kameda, "Evolutionary dynamics and potential games in non-cooperative routing," in *Proc. WiOpt*, Limassol, Cyprus, Apr. 16–20, 2007, pp. 1–5.
[36] D. Niyato and E. Hossain, "Competitive spectrum sharing in cognitive radio networks: A dynamic game approach," *IEEE Trans. Wireless Commun.*, vol. 7, no. 7, pp. 2651–2660, Jul. 2008.

Jocelyne Elias received the Ph.D. degree in information and communication technology from the Polytechnic University of Milan, Milan, Italy, in 2009.

From 2009 to 2010, she held a postdoctoral position with the Department of Information and Mathematical Methods, University of Bergamo, Bergamo, Italy. Since September 2010, she has been an Associate Professor with Paris Descartes University, Paris, France. Her main research interests include network optimization and, in particular, modeling and performance evaluation of networks (cognitive radio, wireless, overlay, and wired networks) and the application of game theory to resource allocation, spectrum access, and pricing problems.

Fabio Martignon received the M.S. and Ph.D. degrees in telecommunication engineering from the Polytechnic University of Milan, Milan, Italy, in October 2001 and May 2005, respectively.

He was an Associate Professor with the University of Bergamo, Bergamo, Italy. He is currently a Full Professor with the Laboratory for Computer Science, University of Paris-Sud, Orsay, France. His current research interests include cognitive radio networks, content-centric networks, network planning, and game theory applications to networking problems.

Lin Chen received the B.E. degree in radio engineering from Southeast University, Nanjing, China, in 2002; the Dipl.-Ing. degree from Telecom ParisTech, Paris, France, in 2005; and the M.S. degree in networking from the Pierre and Marie Curie University, Paris.

He is currently an Associate Professor with the Department of Computer Science, University of Paris-Sud, Orsay, France. His main research interests include modeling and control for wireless networks, security and cooperation enforcement in wireless networks, and game theory.

Eitan Altman (F'10) received the Ph.D. degree in electrical engineering from the Technion–Israel Institute of Technology, Haifa, Israel, in 1990.

Since 1990, he has been a Researcher with the National Research Institute in Computer Science and Control (INRIA), Sophia-Antipolis, France. His research interests include network engineering games and social networks and their control.

Dr. Altman has served in the editorial boards of several scientific journals, including *Wireless Networks*, *Computer Networks*, *Computer Communications*, the *Journal of Discrete Event Dynamic Systems–Theory and Applications*, the *SIAM Journal on Control and Optimization*, *Stochastic Models*, and the *Journal of Economy Dynamic and Control*. He received the Best Paper Award at the 2006 Networking Conference, the 2007 IEEE Global Communications Conference, the 2009 International Federation for Information Processing Wireless Days Conference, and the 2011 International Conference on Network and Service Management. He received the Grand Prix de France Telecom from the French Academy of Sciences in 2012.