Default Logics: A Unified View

Christine Froidevaux
Jérôme Mengin*
Laboratoire de Recherche en Informatique
Bat. 490 - Université Paris Sud
91405 ORSAY CEDEX
France
Tel.: 33 1 69 41 65 07
Fax: 33 1 69 41 65 86
e-mail: chris@lri.fr
September 30, 1992

Published in
Computational Intelligence **10(3)**, pp. 331–369
August 1994

*Work by the second author was partially performed during his stay at the Queen Mary and Westfield College, London, United Kingdom

Abstract: Default logic has been introduced for handling reasoning with incomplete knowledge. It has been widely studied, and various definitions have been proposed for it. Most of the variants have been defined by means of fixed points of some operator. We propose here another approach, which is based on a study of the way in which general rules with exceptions, used in a default reasoning process, can contradict one another. We then isolate sets of non-contradicting rules, as large as possible in order to exploit as much information as possible, and construct, for each of these sets of rules, the set of conclusions that can be deduced from it. We show that our framework encompasses most of the existing variants of default logic, allowing those variants to be compared from a knowledge representation point of view. Our approach also enables us to provide an operational definition of extensions in some interesting cases. Proof theoretical and semantical aspects are investigated.
Key-words: knowledge representation, nonmonotonic reasoning, default logic, operational approach.

1 Introduction

In many situations, a decision has to be made despite the lack of some relevant information. Nevertheless, thanks to some common sense knowledge, sensible conclusions can often be drawn from the information available. A part of this commonsense knowledge is contained in “rules of thumb”. Such a rule is for example “Generally birds fly, except penguins”: if one knows that an individual Tweety is a bird, and does not know that Tweety is a penguin, then the rule allows one to conclude that Tweety flies. Notice that the fact that Tweety is not a penguin does not have to be proved in order to use the rule. It is rather the absence of information, proving that Tweety is a penguin or that Tweety flies, that gives sense to the use of the rule. Therefore, this kind of reasoning is called *default reasoning*. Other forms of reasoning in the presence of incomplete information involve the closed world assumption, default values in frames, typicality, general rules with exceptions, the interpretation of negation in general logic programs, OUT-justifications in truth maintenance systems. The reader will find a comprehensive motivation for the use of default reasoning in Artificial Intelligence in Besnard (1989), Brewka (1991a), Etherington (1988), Lukaszewicz (1990), Reiter (1987) for instance.

Consider again the rule “Generally birds fly, except penguins”. As typical birds are not penguins, we normally use this rule to infer that a bird t flies, so long as we do not know that t is a penguin. The conclusion “t flies” may have to be withdrawn in the light of new information, if it proves that t is indeed a penguin. Therefore default reasoning is not monotonic: the set of conclusions does not grow with the set of premises. Thus classical first order logic, which is monotonic, cannot provide a formalization for default reasoning. Much work in the field of Artificial Intelligence is devoted to the search of logical formalisms which can be used to represent some forms of nonmonotonic reasoning. Let us cite, among others, McCarthy's circumscription (1980), McDermott and Doyle's modal nonmonotonic logics (1980), Reiter's default logic (1980), Moore's autoepistemic logic (1985), the approaches in terms of preferential models (Shoham 1988; Kraus, Lehman and Magidor 1990; Makinson and Gärdenfors 1990), and the study of properties of inference relations (Gabbay 1985; Makinson 1988; Kraus, Lehman and Magidor 1990; Makinson and Gärdenfors 1990).

Default logic is one of the most famous formalisms for the representation of general rules that have exceptions, and reasoning with them. Its original formulation by means of fixed points of some operator (Reiter 1980) is very natural, but hides some undesirable effects. In particular, the interaction between the general rules that have exceptions is difficult to handle. Several authors give other formulations of default logic, in order to remedy some drawbacks of Reiter's default logic. These flaws range from the absence of theorems for some theories or unexpected theorems to the fact that there is no constructive way to compute the sets of theorems. Other authors seek to obtain meta-logical properties that Reiter's default logic does not verify. These variants of default logic are still expressed in terms of fixed points of some operators. It makes them difficult to understand and to compare. More importantly, the kind of reasoning that these variants really express remains unclear.

In this paper, we present another approach to default logic, which is inspired by the work by Lévy (1991a, 1991b). It is based on the usual language of default logic. But, instead of using a fixed-point construction, we build *regular* and *saturated* paths of reasoning: a path of reasoning is regular if it does not contain any contradiction. It is saturated if it is as large as possible without breaking the regularity condition. Such regular and saturated paths of reasoning lead to what one might consider as valid and complete views of the world, given some default theory. By showing how most variants of default logic fit into our framework, we obtain a clear means of comparing these variants. In some cases, our approach also provides an operational definition of the sets of theorems of a theory.

The next section presents our interpretation of general rules that have exceptions. Sect. 3 surveys various definitions of default logic that appear in the literature. Most of these definitions are given in terms of fixed points of some operators. In Sect. 4, we introduce our own presentation of default logic, and show in Sect. 5 how this presentation relates to the usual definitions of default logic. Sect. 6 presents an operational definition of the extensions in some cases, as well as a semantics and a proof theory in the spirit of Lukaszewicz' semantics and proof theory.

All proofs can be found in appendices.

2 Defeasible rules

The aim of this section is to explain how general rules with exceptions are interpreted in our approach to default logic. As has been shown in the introduction, a conclusion deduced by means of such a rule might have to be retracted in the presence of new information. For this reason, these rules are called *defeasible*. More precisely, we call defeasible a rule which has the form: "Generally As are Cs, except Bs" (or, as a special case where there is no B, "Generally As are Cs"). Given an individual t, which is an A, the defeasible rule can be used to deduce that t is a C, so far as we have no reason to believe that t is exceptional with respect to the defeasible rule. We are going to specify three points in this interpretation of a defeasible rule: 1) What is an exceptional individual with respect to a defeasible rule ? 2) What can make us believe that an individual is exceptional with respect to a defeasible rule ? 3) How is a defeasible rule used to make a deduction about an individual which is not believed to be exceptional with respect to the rule ?

Concerning the first point, an individual can be an exception to a defeasible rule only if it verifies the premise of the rule. Moreover, an individual might be exceptional in two ways. Firstly, it can be one of the exceptions mentioned in the rule. That is, t might be a B. We shall call such a t an *explicit* exception to the rule. Secondly, an individual might not be one of the exceptions mentioned in the rule, but nevertheless not verify the conclusion of the rule. That is, although t is not a B, it is no C either. We shall speak in this case of an *implicit* exception to the rule. Therefore, we say that an individual is exceptional with respect to a defeasible rule if it is either an explicit or an implicit exception to the rule.

Before we discuss point 2), let us see how a defeasible rule is used when nothing makes us believe that an individual is an – implicit or explicit – exception to the rule (point 3). In this case, if the individual verifies the premise of the rule, we can infer that it verifies its conclusion. That is, if t is an A, we infer that t is a C. The defeasible rule works now like a special inference rule, $\frac{A(t)}{C(t)}$, or like an oriented implication $A(t) \rightarrow C(t)$. In particular, it should not be assimilated to the classical implication, which allows for contraposition. To illustrate this, consider the rule "Generally men do not have a beard". Given some bearded person, who is either a man or a woman, the contrapositive of this rule would make it possible to infer that she is not a man, therefore that she is a woman. It is clearly not what one expects from a typical bearded human being.

There remains to discuss point 2): what can make us believe that an individual is exceptional with respect to a defeasible rule ? Firstly, it can be non-defeasible knowledge that proves this. There is no choice in this case: the defeasible rule cannot be used for the said individual. Secondly, some other defeasible rule, say $R^\star$: "Generally $A^\star$s are $C^\star$s, except $B^\star$s" might make us believe that some individual t is exceptional with respect to the defeasible rule R: "Generally As are Cs, except Bs". That is, $C^\star$s are - explicit or implicit - exceptions to R. It is not possible to assume both that: a) t is an $A^\star$ not exceptional with respect the rule $R^\star$ (which would imply that t is a $C^\star$), and b) t is an A not exceptional with respect to the rule R. The two defeasible rules are incompatible. There are two plausible answers in this case: each answer consists in assuming that the individual is not exceptional with respect to one of the rules and making the corresponding inference. The other rule can then not be used any more for the said individual. Unless something not mentioned in the rules states that one rule should be preferred to the other one, we claim that we should separately infer two properties for the individual, in two different *extensions* of the theory. The remaining part of this section is devoted to this notion of extension.

More precisely, a knowledge base K that contains defeasible rules is first translated into a knowledge base K^i that contains only instantiated defeasible rules: for every rule "Generally As are Cs, except Bs" that belongs to K, and every individual t that appears in K, K^i contains the instantiated rule "If t is an A, if there is no evidence that t is a B or that it is not a C, infer that

t is a C". The non-defeasible part of K is left unchanged in K^i. Now, as we have just discussed, some of these instantiated defeasible rules may be incompatible with the non-defeasible part of the knowledge base or with one another. Therefore, sets of instantiated defeasible rules, maximal such that the rules they contain are compatible with the non-defeasible part of the knowledge base and with one another, are isolated. We will call *regular* a set of instantiated defeasible rules such that the rules it contains are compatible with the non-defeasible part of the knowledge base and with one another. A *maximally regular* set of instantiated defeasible rules is *saturated*: it is not possible to add any more instantiated defeasible rule to it without breaking the regularity condition. These regular and saturated sets are then used to generate the extensions: the instantiated rules that they contain are used, at that stage, like specific inference rules (point 3) and used so as to make deductions from the non-defeasible part of the knowledge base.

Notice that the instantiation of the defeasible rules is essential in this process: some individual t might be exceptional with respect to some defeasible rule R, while another individual t', which does not share the properties of t, is not. We do not mean that R cannot be used at all, because we want to use it to infer some property about t'. Neither do we mean that R should be used in all cases, as we certainly do not want to use it for t.

Now, there is an easy way to compute these extensions, by recursively constructing the set of instantiated defeasible rules that generate them. Starting from the empty set of instantiated defeasible rules, we choose an instantiated defeasible rule: a) such that its premise can be proved, from the non-defeasible part of the knowledge base, using the instantiated defeasible rules that have already been chosen, and b) which is compatible with these instantiated defeasible rules. When there are no such rules left, we have a set of instantiated defeasible rules, maximal such that: i) the rules it contains are compatible with one another and ii) these rules have their premises proved from the non-defeasible part of the knowledge base, using the rules that are in the set. The remaining instantiated defeasible rules either are incompatible with those already chosen, or have a premise which cannot be proved, and thus cannot be used anyway. By considering, at every step, all possible choices for the next instantiated defeasible rule to be used, we obtain all sets of instantiated defeasible rules that generate extensions.

Lastly, we have mentioned in the discussion of point 2) that, when there is a conflict between several instantiated defeasible rules, we should consider all possibilities by generating several extensions, "unless something not mentioned in the rules states that one rule should be preferred to the other ones". As will be seen in Sect. 4.4.3, some variants of default logic do indeed give priority to some instantiated defeasible rules over others. Roughly speaking, it works as follows: suppose that we want a rule $R^\star$ to be given priority over a rule R. This only makes sense in the case where the two rules are not compatible. So suppose that a set U of instantiated defeasible rules that would generate an extension is such that: a) U contains R, b) U does not contain $R^\star$, because $R^\star$ is incompatible with the rules already in U, and c) $R^\star$ would be compatible with the set of rules obtained from U by removing R. In this case, we reject the extension generated by U. Notice that circularities might appear in these priorities. Consequently, there are default theories without extension in the variants of default logic that use such a notion of priority.

3 Default Logic

Let us now give a brief review of default logic. We suppose that some logic L is given to formalize reasoning from non defeasible knowledge. We will denote by $\mathcal{L}$ its language. Defeasible rules are represented in default logic by *defaults*, of the form $\frac{a(x):b(x)}{c(x)}$, where $a(x)$, $b(x)$ and $c(x)$ are formulas of $\mathcal{L}$, and x is their vector of free variables. $a(x)$ is called the *prerequisite* of the default, $b(x)$ its *justification*[1] and $c(x)$ its *consequent*. It can be read for an individual t: if $a(t)$ is true and if $b(t)$ is consistent, then infer $c(t)$. A default is closed if its prerequisite, justification and consequent contain no free variable. A closed default theory is a pair $\Delta = (W, D)$, where W is a

[1]For the sake of clarity, we consider in this paper that the justification of a default consists of one formula only. However, we briefly describe in Sect. 4.6 how most results can be generalized to defaults that have multiple justifications.

set of closed formulas of $\mathcal{L}$, and D a set of closed defaults. We consider, in the rest of the paper, closed default theories only[2], such that W is consistent (most authors agree to say that when W is inconsistent, the theory has only one extension which is inconsistent, that is, the set $\mathcal{L}$ of all formulas). W contains the facts that are certain about a situation (Etherington (1988) calls them "hard facts"), whereas the defaults of D represent some general rules about a domain, rules that might have exceptions.

Example 1. Consider the knowledge base: "Generally birds fly", "All penguins are birds", "Tweety is a bird". It can be represented by the closed default theory $\Delta = (W, D)$, where $W = \{\forall x \ \mathsf{penguin}(x) \rightarrow \mathsf{bird}(x), \mathsf{bird}(\mathsf{Tweety})\}$ and $D = \{\frac{\mathsf{bird(Tweety):flies(Tweety)}}{\mathsf{flies(Tweety)}}\}$. D is the set of instantiations of the open default $\frac{\mathsf{bird}(x)\mathsf{:flies}(x)}{\mathsf{flies}(x)}$.

Before we proceed to the presentation of default logic, let us introduce some notation that we will use in the rest of this paper. Given a default d, we denote its prerequisite by $\mathrm{Pre}(d)$, its justification by $\mathrm{Jus}(d)$, and its consequent by $\mathrm{Con}(d)$, i.e. $d = \frac{\mathrm{Pre}(d):\mathrm{Jus}(d)}{\mathrm{Con}(d)}$. Similarly, if D is a set of defaults, we denote by $\mathrm{Pre}(D)$ the set $\{\mathrm{Pre}(d), d \in D\}$, by $\mathrm{Jus}(D)$ the set $\{\mathrm{Jus}(d), d \in D\}$, and by $\mathrm{Con}(D)$ the set $\{\mathrm{Con}(d), d \in D\}$. The symbol $\vdash$ denotes the consequence relation of the logic L. We suppose that this consequence relation is monotonic, reflexive, and compact. Given a set E of formulas, we denote by $\mathrm{Th}(E)$ the set of its logical consequences: $\mathrm{Th}(E) = \{f, E \vdash f\}$.

3.1 Reiter's original definition

The defaults, together with the usual inference rules and axioms of the logic L, are used to make deductions from W, whenever it is coherent to do so. Roughly, the difficulty is to formalize when it is coherent to use some defaults. In many works about default logic, this has been done by studying the admissibility of sets of formulas as extensions of a given default theory (W, D). That is, an operator is usually defined, that associates, to a set E of formulas, the set of formulas that is obtained by applying to W the defaults of D that are considered to be coherent with E. If the resulting set is E itself, then E is called an extension of the theory. The original definition of the extensions of a default theory, due to Reiter, is as follows:

Definition 1. (Reiter 1980) Let $\Delta = (W, D)$ be a default theory. For any set of formulas E, let $\Gamma_{\mathrm{R}}(E)$ be the smallest set of formulas $E' \subseteq \mathcal{L}$ satisfying the following three properties:

P1 $W \subseteq E'$,

P2 $\mathrm{Th}(E') = E'$,

R3 If $\frac{a:b}{c} \in D$ and $a \in E'$ and $\neg b \notin E$ then $c \in E'$.

A set of formulas E is a **R-extension**[3] for Δ if and only if E is a fixed point of the operator Γ_{R}.

A well-known result shown by Reiter gives another characterization of the extensions of a theory:

Proposition 1. *(Reiter 1980, Theorem 2.1) Let $\Delta = (W, D)$ be a default theory. For any set of formulas E, define:*

$$E_0 = W,$$

$$E_{i+1} = \mathrm{Th}(E_i) \cup \{c, \ s.t. \ \tfrac{a:b}{c} \in D, a \in E_i, \neg b \notin E\}.$$

Then E is an R-extension of Δ if and only if $E = \bigcup_{i \geq 0} E_i$.

[2]When there are open defaults, Reiter (1980) proposes to skolemize the formulas of the theory, and then to instantiate the defaults with the ground terms of the Herbrand universe. Baader and Hollunder (1992) discuss some problems of this method.

[3]Reiter simply says "extension"

Example 1 (continued) On the example about Tweety, which has been represented by the theory $W = \{\forall x\ \mathsf{penguin}(x) \to \mathsf{bird}(x), \mathsf{bird}(\mathsf{Tweety})\}$ and $D = \{\frac{\mathsf{bird(Tweety)}:\mathsf{flies(Tweety)}}{\mathsf{flies(Tweety)}}\}$, for any set E of formulas, $\Gamma_{\mathrm{R}}(E)$ must contain W. If E is to be a fixed point of Γ_{R}, then for any default $\frac{a:b}{c} \in D$, if $a \in E$ and $\neg b \notin E$ then $c \in E$. In particular, $\mathsf{bird(Tweety)} \in W \subseteq E$, thus E must contain either $\neg\mathsf{flies(Tweety)}$ or $\mathsf{flies(Tweety)}$. If E contains $\neg\mathsf{flies(Tweety)}$, then $\Gamma_{\mathrm{R}}(E) = \mathrm{Th}(W) \neq E$, thus no extension can contain $\neg\mathsf{flies(Tweety)}$. Therefore any extension of this theory must contain $\mathrm{Th}(W \cup \{\mathsf{flies(Tweety)}\})$. It is easy to check that this last set is indeed a fixed point of Γ_{R}. Moreover, as Reiter proved that extensions of a given theory cannot be strictly included in one another, $\mathrm{Th}(W \cup \{\mathsf{flies(Tweety)}\})$ is the only extension of the theory.

On the example above, $\frac{\mathsf{bird(Tweety)}:\mathsf{flies(Tweety)}}{\mathsf{flies(Tweety)}}$ is the only default that is taken into account when computing the extensions. It is also clear that the only extension of the theory results from the application of this default. However, the reason why this default has been applied in this case is hidden in the operator Γ_{R}. The next definition, still by Reiter, defines the set of defaults that generate some given extension.

Definition 2. (Reiter 1980) The set of generating defaults of an extension E of a default theory (W, D) is defined by:

$$\mathrm{GD}(E) = \{d \in D, \text{ s.t. } \mathrm{Pre}(d) \in E \text{ and } \neg\mathrm{Jus}(d) \notin E\}.$$

Reiter proves that an extension E is the set of theorems that can be deduced from W together with the consequents of the defaults of $\mathrm{GD}(E)$, that is: $E = \mathrm{Th}(W \cup \mathrm{Con}(\mathrm{GD}(E)))$. On the example above, the set of generating defaults of the only extension of the theory is of course $\{\frac{\mathsf{bird(Tweety)}:\mathsf{flies(Tweety)}}{\mathsf{flies(Tweety)}}\}$.

Let us now consider another example:

Example 2. Let $W = \emptyset$, and let $D = \{\frac{:p}{\neg p}\}$. This default can be read: "if the negation of p cannot be proved, then conclude the negation of p". Let E be a set of formulas. If E does not contain $\neg p$, then $\neg p \in \Gamma_{\mathrm{R}}(E)$, therefore E cannot be an extension of the theory. Thus any extension of the theory must contain $\neg p$. Suppose now that E contains $\neg p$. Then $\mathrm{Th}(W)$ verifies P1, P2, and the negation of the justification of the only default of D is in E, thus $\mathrm{Th}(W)$ verifies R3 with respect to E. Thus $\Gamma_{\mathrm{R}}(E) \subseteq \mathrm{Th}(W)$. Moreover, $\neg p \notin \mathrm{Th}(W)$, thus $\Gamma_{\mathrm{R}}(E) \subseteq \mathrm{Th}(W) \subset E$: E cannot be a fixed point of Γ_{R}. Therefore this theory has no extension.

Again, although it is clear that there is something bizarre with the default $\frac{:p}{\neg p}$, the reason why this default cannot be applied, and also prevents all axioms of classical logic from being logical consequences of this theory is hidden in the operator Γ_{R}.

3.2 Lukaszewicz' definition

Lukaszewicz (1988), motivated by the absence of R-extension for some theories, as well as by the willingness to give a proof theory, gives a new definition of extensions of a default theory. Let us recall Lukaszewicz' definition:

Definition 3. (Lukaszewicz 1988) E is an **L-extension** of a default theory Δ, **with respect to a set of formulas** F, if $E = \Gamma_{\mathrm{L},1}(E, F)$ and $F = \Gamma_{\mathrm{L},2}(E, F)$ where $\Gamma_{\mathrm{L},1}(E, F)$ and $\Gamma_{\mathrm{L},2}(E, F)$ are the smallest sets of formulas satisfying:

P1 $W \subseteq \Gamma_{\mathrm{L},1}(E, F)$,

P2 $\Gamma_{\mathrm{L},1}(E, F) = \mathrm{Th}(\Gamma_{\mathrm{L},1}(E, F))$,

L3 if $d \in D$, such that $\mathrm{Pre}(d) \in \Gamma_{\mathrm{L},1}(E, F)$ and for all $x \in F \cup \{\mathrm{Jus}(d)\}$, $E \cup \{x, \mathrm{Con}(d)\}$ is consistent, then $\mathrm{Con}(d) \in \Gamma_{\mathrm{L},1}(E, F)$ and $\mathrm{Jus}(d) \in \Gamma_{\mathrm{L},2}(E, F)$.

Consider again the theory of the previous example, which had no R-extension, this time in light of Lukaszewicz' definition:

Example 2 (continued) We have $W = \emptyset$ and $D = \{\frac{:p}{\neg p}\}$. Let $E = \mathrm{Th}(W)$, and $F = \emptyset$. It is clear that E verifies P1 and P2. Moreover, although the prerequisite of the only default of the theory is in E, the conjunction of its justification and its consequent is inconsistent with E. Therefore (E, F) verifies L3. Lastly, any set that verifies P1 and P2 must contain E, thus E is an L-extension of the theory.

Lukaszewicz proves that all R-extensions are L-extensions. Furthemore, he provides a proof theory for membership in L-extensions, as follows:

Definition 4. (Lukaszewicz 1988) Given a set W of formulas, a sequence of defaults (d_i) is **W-applicable** if (d_i) is the empty sequence or if:

- $\mathrm{Th}(W) \vdash \mathrm{Pre}(d_0)$ et $\mathrm{Th}(W \cup \{\mathrm{Con}(d_0)\}) \not\vdash \neg\mathrm{Jus}(d_0)$;
- for all $i > 0$
  - $\mathrm{Th}(W \cup \mathrm{Con}(\{d_0, \ldots, d_{i-1}\})) \vdash \mathrm{Pre}(d_i)$, and
  - $\mathrm{Th}(W \cup \mathrm{Con}(\{d_0, \ldots, d_i\})) \not\vdash \neg\mathrm{Jus}(d_j)$, for all $0 \leq j \leq i$.

Moreover, given some default theory (W, D) and a formula f, a finite sequence $d_0, \ldots, d_n$ of defaults of D is a **default proof** for f with respect to (W, D) if $d_0, \ldots, d_n$ is W-applicable and $W \cup \mathrm{Con}(\{d_0, \ldots, d_n\}) \vdash f$.

Theorem 2. *(Lukaszewicz 1988) A formula f is in an L-extension of a default theory (W, D) if and only if f has a default proof with respect to (W, D).*

Given some default theory (W, D), it is clear that the empty sequence of defaults is a default proof for any formula of W. It proves that any default theory has at least one extension. However, the next example describes a theory which seems to have too many L-extensions:

Example 3. Let $W = \emptyset$ and $D = \{\frac{:b}{c}, \frac{:c \wedge d}{c \wedge d}, \frac{:\neg b}{\neg b}\}$. This theory has only one R-extension, $\mathrm{Th}(\{c \wedge d, \neg b\})$, which is obtained by considering the two defaults $\frac{:c \wedge d}{c \wedge d}$ and $\frac{:\neg b}{\neg b}$. But it has another L-extension, $\mathrm{Th}(\{c \wedge d\})$, with respect to $F = \{b, c \wedge d\}$, corresponding to the application of the defaults $\frac{:c \wedge d}{c \wedge d}$ and $\frac{:b}{c}$. This is not an extension in the sense of Reiter, because the prerequisite of $\frac{:\neg b}{\neg b}$ is in $\mathrm{Th}(\{c \wedge d\})$, and the negation b of its justification is not, so the consequent $\neg b$ of this default should be in $\Gamma_{\mathrm{R}}(\mathrm{Th}(\{c \wedge d\}))$. But when computing $\Gamma_{\mathrm{L},1}(\mathrm{Th}(\{c \wedge d\}))$, we have to consider the consistency of the consequent of $\frac{:\neg b}{\neg b}$ with the justifications of other defaults, stored in F. Clearly, $\frac{:\neg b}{\neg b}$ is blocked here because of the justification b of $\frac{:b}{c}$. Therefore $\mathrm{Th}(\{c \wedge d\})$ is an L-extension of the theory.

The difference between the two extensions on this example comes from the conflicting defaults $\frac{:b}{c}$ and $\frac{:\neg b}{\neg b}$. However, as there is a third default, $\frac{:c \wedge d}{c \wedge d}$, whose consequent subsumes the consequent of $\frac{:b}{c}$, there is no need to use the redundant default $\frac{:b}{c}$. In view of this, the only valid extension of the theory should be its R-extension.

3.3 Rychlik's definition

Rychlik (1991), Krause et al. (1991), Moinard (1992) noticed that what interests us is the set of theorems that we can deduce from a generating set of defaults, rather than the set of defaults itself. Consequently, the defaults in a generating set of defaults, the consequents of which are subsumed by other defaults in the same set, can be omitted when generating the corresponding extension. Rychlik considers the defaults as arguments in favor of their conclusions; in this sense, one argument should suffice to establish a conclusion. Moreover, the justification of some redundant default may unnecessarily block other defaults from being in a generating set, as we have seen on example 3.

Rychlik's definition of extensions aims at avoiding the application of those defaults which are redundant because their consequent is subsumed by the consequents of other defaults, while keeping some good properties of L-extensions, like the existence property. Notice that his definition is motivated by representational issues, and that it does not provide a more efficient strategy to compute the extensions of a default theory. It is based on the following definition of the subsumption of a default by a set of defaults.

Definition 5. (Rychlik 1991) Let $\Delta = (W, D)$ be a default theory. A default d is said to be **weakly subsumed** by a subset D' of D if there exists a set of defaults $U \subseteq D'$ such that:

- $\forall d \in U$, there is a sequence $d_1, \ldots, d_n$ of elements of U such that $\text{Pre}(d_i) \in \text{Th}(W \cup \text{Con}(\{d_j \text{ s.t. } 1 \leq j < i\}))$ for $1 \leq i \leq n$, and $\text{Pre}(d) \in \text{Th}(W \cup \text{Con}(\{d_j \text{ s.t. } 1 \leq j \leq n\}))$, and
- $d \notin U$, and
- $\text{Con}(d) \in \text{Th}(W \cup \text{Con}(U))$.

Then d is said to be **subsumed** by $D' \subseteq D$ if there exists $D'' \subseteq D'$ such that d is weakly subsumed by D'' and no element of D'' is weakly subsumed by D'.

Rychlik's definition of extensions is then as follows:

Definition 6. (adapted from Rychlik (1991)) Let $\Delta = (W, D)$ be a default theory, and let E be a set of formulas and V a set of defaults. E is an **Ry-extension** of Δ w.r.t. V if (E, V) is a fixed point of the operator Γ_{Ry} which associates with (E, V), $\Gamma_{\text{Ry}}(E, V) = (\Gamma_{\text{Ry},1}(E, V), \Gamma_{\text{Ry},2}(E, V))$, where $\Gamma_{\text{Ry},1}(E, V)$ and $\Gamma_{\text{Ry},2}(E, V)$ are the smallest sets E' and V' respectively such that:

P1 $W \subseteq E'$, and

P2 $\text{Th}(E') = E'$, and

RY3 $\forall d \in D$, if $\text{Pre}(d) \in E'$ and $\forall x \in \text{Jus}(V \cup \{d\}), E \cup \{\text{Con}(d)\} \not\vdash \neg x$ and d is not subsumed by V, then $\text{Con}(d) \in E'$ and $d \in V'$.

3.4 De T. Guerreiro, Casanova and Hemerly's definition

De T. Guerreiro, Casanova and Hemerly (1990) define another type of extension. They are motivated by the lack of proof theory and semi monotonicity for Reiter's extensions, and by the fact that some theory have no R-extension. They propose the following definition:

Definition 7. (de T. Guerreiro et al. 1990) Let $\Delta = (W, D)$ be a default theory. A set of formulas E is a **E-set** of Δ if $E = \bigcup_{n \in \mathbf{N}} E_n$, where

$E_0 = W$,

$E_{i+1} = \text{Th}(E_i) \cup \text{Con}(\{d \in D, \text{Pre}(d) \in E_i, \neg\text{Jus}(d) \notin E, \text{Con}(d) \in E\})$.

Define $E(\Delta)$ to be the set of E-sets of Δ. A **C-extension** of Δ is a maximally deductively closed union of E-sets, i.e. a deductively closed union of E-sets which is not strictly included in any deductively closed union of E-sets.

3.5 Schaub's constrained extensions

Brewka (1991b) introduces a variant of default logic which is not based on the classical logic, but on an *assertion logic*. His aim is two-fold: he wants to introduce cumulativity in default logic, and to solve the following example:

Example 4. (Poole 1989) Let $W = \{\mathsf{Broken(left\text{-}arm)} \vee \mathsf{Broken(right\text{-}arm)}\}$, and $D = \{d_1, d_2\}$, where $d_1 = \frac{:\mathsf{Usable(left\text{-}arm)} \wedge \neg \mathsf{Broken(left\text{-}arm)}}{\mathsf{Usable(left\text{-}arm)}}$, $d_2 = \frac{:\mathsf{Usable(right\text{-}arm)} \wedge \neg \mathsf{Broken(right\text{-}arm)}}{\mathsf{Usable(right\text{-}arm)}}$. With any of the definitions given so far this theory would have one extension generated by $\{d_1, d_2\}$, and containing $\mathsf{Usable(left\text{-}arm)}$ and $\mathsf{Usable(right\text{-}arm)}$, although the justifications of the two defaults together are contradictory in this theory. It can seem counterintuitive to make both assumptions $\neg\mathsf{Broken(left\text{-}arm)}$ and $\neg\mathsf{Broken(right\text{-}arm)}$ when we know $\mathsf{Broken(left\text{-}arm)} \vee \mathsf{Broken(right\text{-}arm)}$.

Schaub (1991b) provides a semantical characterization of Brewka's assertional default logic. This leads him to introduce another variant of default logic:

Definition 8. (Schaub 1991b) Let $\Delta = (W, D)$ be a default theory. For any pair of sets of sentences (S, T) such that $S \subseteq T$, let $\Upsilon(S, T) = (S', T')$, where S' and T' are the smallest sets of sentences such that:

SC1 $W \subseteq S' \subseteq T'$,

SC2 $S' = \text{Th}(S')$ and $T' = \text{Th}(T')$,

SC3 for any $d \in D$, if $\text{Pre}(d) \in S'$ and $T \cup \{\text{Con}(d)\} \not\vdash \neg\text{Jus}(d)$, then $\text{Con}(d) \in S'$ and $\text{Jus}(d), \text{Con}(d) \in T'$.

A pair (E, C) is a **constrained extension** of Δ if and only if (E, C) is a fixed point of Υ.

The set T in this definition enables us to record both the consequent and the justification of all defaults that are used when constructing an extension. Let us illustrate this on the broken arm example.

Example 4 (continued) We will show that $(\text{Th}(W \cup \{\text{Con}(d_1)\}), \text{Th}(W \cup \{\text{Jus}(d_1), \text{Con}(d_1)\}))$ is a constrained extension of the theory. Let $S = \text{Th}(W \cup \{\text{Con}(d_1)\})$ and $T = \text{Th}(W \cup \{\text{Jus}(d_1), \text{Con}(d_1)\})$. Clearly, we have $W \subseteq S \subseteq T$. Moreover S and T are deductively closed. Consider d_2: its prerequisite is in S. But, as T contains W and the justification of d_1, it contains Broken(right-arm), that is, the negation of the justification of d_2. Therefore d_2 does not have to be considered when computing $\Upsilon(S, T)$. This is where lies the difference between Schaub's definition and the other definitions that we have seen. Consider now d_1: its prerequisite is in T and the negation of its justification cannot be proved from its consequent together with T, therefore its consequent has to be in S, and its justification and its consequent in T. It is indeed the case, therefore (S, T) verifies the three properties that define $\Upsilon(S, T)$. The same path of reasoning proves that any pair that verifies these same three properties with respect to (S, T) contains (S, T), therefore (S, T) is in fact the smallest pair which verifies them. Hence (S, T) is a fixed point of Υ. A similar study would show that $\{d_2\}$ also generates a constrained extension of the theory.

3.6 Delgrande and Jackson's extensions

Motivated by an example similar to Poole's "broken arm" example above, and by the fact that Reiter's definition of extensions is not constructive, Delgrande and Jackson (1991) propose another definition for the extensions of a semi-normal default theory, that is, a default theory whose defaults have the form $\frac{a:b \wedge c}{c}$: the justification of a semi-normal default entails its consequent.

Definition 9. (Delgrande and Jackson 1991) Let $\Delta = (W, D)$ be a semi-normal default theory, define

$E_0 = (E_{J,0}, E_{T,0}) = (\text{Th}(W), \text{Th}(W))$, and for $i \geq 0$,

$E_{i+1} = (E_{J,i+1}, E_{T,i+1}) = (\text{Th}(E_{J,i} \cup \{b \wedge c\}), \text{Th}(E_{T,i} \cup \{c\}))$, for some $\frac{a:b \wedge c}{c} \in D$ such that $a \in E_{T,i}$ and $\neg(b \wedge c) \notin E_{J,i}$.

Then E is a **J-extension** for Δ if and only $E = (\bigcup_{0 \leq i} E_{J,i}, \bigcup_{0 \leq i} E_{T,i})$ for some sequence $(E_{J,i}, E_{T,i})_{0 \leq i}$.

With this definition, starting from W, the defaults are applied one by one. The $E_{T,i}$'s record the consequents of the defaults, and it is the limit of this sequence which really gives the extension. The $E_{J,i}$'s record the justifications and consequents of the defaults, and it is against them that the consistency of the defaults to be used is checked.

Their definition is very close to the approach that we will advocate in Sect. 6.1. However, the following example shows a drawback in Delgrande and Jackson's definition:

Example 5. Let $W = \emptyset$, $D = \{\frac{:a}{a}, \frac{:a_n}{a_n}, n \in \mathbf{N}\}$. Define

$$E_0 = (E_{J,0}, E_{T,0}) = (\mathrm{Th}(W), \mathrm{Th}(W)),$$

$$E_{i+1} = (E_{J,i+1}, E_{T,i+1}) = (\mathrm{Th}(E_{J,i} \cup \{a_i\}), \mathrm{Th}(E_{T,i} \cup \{a_i\})).$$

By applying the defaults $\{\frac{:a_n}{a_n}, n \in \mathbf{N}\}$ according to the order of their indexes, we obtain a J-extension of Δ. But the last default $\frac{:a}{a}$ will in this case never be applied, although doing so would not show any inconsistency. There is just here a problem at the limit.

The solution that we will propose in Sect. 6.1 solves this problem at the limit.

4 An alternative definition

4.1 General principles

Our approach is based on the idea that defaults are applied one after another, according to certain conditions. More precisely, following our presentation of general rules with exceptions in Sect. 2, we propose to consider that W is extended by applying some defaults, according to the four general principles below:

Principle 1 A default can be applied only if it is *active*, i.e. if its prerequisite can be proved (from W using first order logic and maybe other defaults that can be applied too)[4].

Principle 2 A set of defaults that are applied together in order to build an extension must remain *regular*: this regularity condition varies from one variant of default logic to another, but in most of them it requires the resulting extension to be consistent. The regularity condition also specifies in which sense the justifications of the defaults used to build an extension have to be consistent. If the application of several defaults together leads to an irregular set of defaults, then several "smaller" extensions maybe built instead, by separately applying several regular sets of defaults.

Principle 3 An extension is *saturated*: any default which is applicable to an extension must be applied to it. The applicability condition is the other varying element (with the regularity) from one variant of default logic to another one (note that the first principle only gives a necessary condition to the applicability of a default). The saturation condition ensures that extensions are generated by maximally regular sets of defaults only. Moreover, a stronger saturation condition in some variants of default logic provides a way to give priority to some defaults over others.

Principle 4 An extension is *deductively closed* under provability in L.

Now, starting from the empty set of defaults, if we iteratively apply to it defaults which are applicable at each stage, we obtain a saturated set of defaults. If the applicability condition ensures that the set of defaults being constructed remains regular, the result is also regular, thus generates an extension. This will be the core of our operational approach in Sect. 6.1.

The rest of this section presents a formalization of the four principles above.

4.2 Grounded Sets of Defaults

The first principle, which gives the notion of a "directed path of reasoning", has been formalized by Levy (1991b) with the notion of "universes" of a default theory, and by Schwind (1990) with "grounded sets of defaults". We give below a definition of a grounded set of defaults which is equivalent to that of Schwind (1990).

[4] All the variants of default logic that we study in this paper agree on this principle. See e.g. Zhang and Marek (1989), Wilson (1990) for variants of default logic that do not agree on this principle.

Definition 10. A **default deduction** for a formula f from a set W of formulas, using a set U of defaults, is a finite sequence $(d_1, \dots, d_n)$ of defaults of U such that $\mathrm{Pre}(d_i) \in \mathrm{Th}(W \cup \mathrm{Con}(\{d_j \text{ s.t. } 1 \le j < i\}))$ for $1 \le i \le n$, and $f \in \mathrm{Th}(W \cup \mathrm{Con}(\{d_j \text{ s.t. } 1 \le j \le n\}))$. We denote, for a set of formulas W and a set of defaults U, by $\mathrm{Ground}(U)$ the subset of all the defaults of U such that their prerequisites have a default deduction from W using U. A set of defaults U is **grounded** in a set of formulas W if $U = \mathrm{Ground}(U)$. A default d is **active** with respect to a set of defaults U grounded in W if and only if $\mathrm{Pre}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(U))$.

A set of defaults is grounded in W if the prerequisites of all its elements can be proved from W and the consequents of other defaults in the same set, and there are no cycles in those proofs. Notice that if U is grounded in W, then a default d is active with respect to U if and only if $U \cup \{d\}$ is grounded in W.

The justifications of the defaults do not appear in this definition, which formalizes the purely deductive aspect of the defaults: "if the prerequisite of the default is true, ..., then infer its consequent". Justifications merely correspond to the hypothetical nature of the defaults ("... if the justification of the default is consistent...”). This aspect will be studied in the next section.

Example 6. Let $\Delta = (W, D)$, where $W = \{a\}$ and $D = \{\frac{a:b}{b}, \frac{b:c}{c}, \frac{d:e}{e}, \frac{e:d}{d}\}$. Then $U = \{\frac{a:b}{b}, \frac{b:c}{c}\}$ is grounded in W, but not $\{\frac{d:e}{e}, \frac{e:d}{d}\}$. The empty sequence of defaults is a default deduction from W using U for the prerequisite a of $\frac{a:b}{b}$. The sequence $(\frac{a:b}{b})$ is a default deduction from W using U for the prerequisite b of $\frac{b:c}{c}$

The following characterizations of grounded sets of defaults will be needed later on to make the link with the usual presentation of default logic:

Proposition 3. *Let W be a set of formulas, and let U be a set of defaults, then* $\mathrm{Ground}(U)$ *is the smallest subset U' of U such that:*

$$(F) \qquad \forall d \in U,\ \textit{if } \mathrm{Pre}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(U')) \textit{ then } d \in U'.$$

Define $\mathcal{L}_W^0(U) = \emptyset$ and

$$\mathcal{L}_W^{n+1}(U) = \{d \in U \text{ s.t. } \mathrm{Pre}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(\mathcal{L}_W^n(U)))\} \text{ for } n \ge 0,$$

then $\mathrm{Ground}(U) = \bigcup_{n \ge 0} \mathcal{L}_W^n(U)$. *Furthermore, any union of sets of defaults grounded in W is grounded in W. The function* Ground *is increasing and idempotent.*

Equation (F) in the proposition above can be compared to equation R3 in Reiter's definition (Def. 1) of the extensions of a defaults theory: in order to compute $\mathrm{Ground}(U)$ (respectively $\Gamma_{\mathrm{R}}(E)$), one has to consider all defaults that are in U (respectively whose justifications are not contradicted in E) and whose prerequisites can be proved from $\mathrm{Ground}(U)$ (respectively are in $\Gamma_{\mathrm{R}}(E)$). Similarly, the "layers" $\mathcal{L}_W^i(U)$ of U in the proposition above are obtained by applying at each stage the defaults of U whose prerequisites can be proved from the previous layer, and can be compared to the E_i's of Prop. 1.

Notice that if a set D of defaults contains some defaults d_1,. . .,d_n whose prerequisites all have a default proof, and whose consequents form an inconsistent set of formulas, then the prerequisite of any default in D has a default proof in D. Therefore, in this case $\mathrm{Ground}(D) = D$. However, we are going to study "regular" sets of defaults, that is, sets of defaults which are compatible with one another: the consequents of the defaults of such a set will not be inconsistent, therefore the notion of groundedness will not be trivial.

We discuss in subsequent sections the formalization of the notions of regularity and applicability (Principles 2 and 3). However, we can already formalize the notion of saturation, whatever the notion of applicability may be.

Definition 11. Let $\Delta = (W, D)$ be a default theory. A set $U \subseteq D$ of defaults, grounded in W, is **saturated** if all the defaults of D which are applicable to U are in U.

In this paper, we will focus on different grounded sets of defaults U characterized by various properties. But, whatever these properties may be, an extension is obtained as the set of theorems of $W \cup \mathrm{Con}(U)$ (in the sense of the logic L).

Definition 12. Let $\Delta = (W, D)$ be a default theory. A set E of formulas is **generated by** a set of defaults U, grounded in W, if $E = \mathrm{Th}(W \cup \mathrm{Con}(U))$. Moreover, E is an **extension** of Δ if E is generated by a regular and saturated set of defaults grounded in W.

4.3 Regularity

Regularity formalizes the notion of compatibility between defaults. In this section, we will propose three notions of regularity, that will be shown in Sect. 5 to underly some of the variants of default logic presented in Sect. 3.

4.3.1 Weak regularity

Weak regularity is the notion which underlies the first definitions of extensions, i.e. Reiter's (1980) and Lukaszewicz' (1988). Like all notions of regularity that we will study, the weak regularity of a grounded set of defaults requires the set of theorems that it generates to be consistent. The following example illustrates another required condition:

Example 7. Suppse that Noel is a nautilus. As nautilus are cephalopods, which themselves are molluscs, Noel is also a cephalopod and a mollusc. Moreover, normally molluscs have a shell, but cephalopods are abnormal in this respect, except for nautilus who have a shell. This can be encoded in the default theory $\Delta = (W, D)$ where $W = \{\mathsf{naut(Noel)}, \mathsf{ceph(Noel)}, \mathsf{mollusc(Noel)}\}$ and $D = \{\frac{\mathsf{mollusc(Noel)}:\neg\mathsf{ab(Noel)}}{\mathsf{shell(Noel)}}, \frac{\mathsf{ceph(Noel)}:\neg\mathsf{naut(Noel)}}{\mathsf{ab(Noel)}}\}$. The prerequisites of the two defaults are in W and $W \cup \mathrm{Con}(D)$ is consistent. However, the default $\frac{\mathsf{ceph(Noel)}:\neg\mathsf{naut(Noel)}}{\mathsf{ab(Noel)}}$ cannot be used to generate an extension because its justification is inconsistent with W.

The crucial point in the above example is that a grounded set of defaults can be considered valid only if for any default d in U, the justification of d is consistent with W and the consequents of the defaults in U. The intuitive meaning of a default $\frac{a:b}{c}$ is in this case: "if a is known, and if b is consistent with what is known, then infer c". As Delgrande and Jackson say (1991), "the justifications are individually consistent with the extension". According to Lukaszewicz' definition of a default proof, a sequence of defaults can be a default proof for some formula only if the justification of each of its elements is not contradicted by the consequents of the whole sequence. Levy has formalized this with the following definition:

Definition 13. (Levy 1991b) Let $\Delta = (W, D)$ be a default theory, a set of defaults $U \subseteq D$ grounded in W is said to be **weakly regular**[5] if $W \cup \mathrm{Con}(U) \cup \{\mathrm{Jus}(d)\}$ is consistent for all d in U.

4.3.2 Strong regularity

The second notion of regularity interprets the justification of a default as an assumption, which is implicitly concluded whenever the default is applied, even if not explicitly concluded. In this sense, it is not possible to assume that a proposition is true and to assume at the same time that it is false. More generally this notion of regularity requires that the justifications of the defaults in a grounded set of defaults are all together consistent with the generated set of theorems. We will illustrate it on the broken arm example:

[5] Levy simply says "regular"

Example 4 (continued) This theory has one weakly regular grounded set of defaults, $\{d_1, d_2\}$, which would generate an extension containing Usable(left-arm) and Usable(right-arm), although the justifications of the two defaults together are contradictory in this theory. As already discussed, it can seem counterintuitive to make both assumptions ¬Broken(left-arm) and ¬Broken(Right-Arm) when we know Broken(left-arm) ∨ Broken(right-arm).

The following notion of regularity solves this problem:

Definition 14. Let $\Delta = (W, D)$ be a default theory, a set of defaults $U \subseteq D$ grounded in W is said to be **strongly regular** if $W \cup \mathrm{Con}(U) \cup \mathrm{Jus}(U)$ is consistent.

Example 4 (continued) The grounded set of defaults $\{d_1, d_2\}$ is not strongly regular. Instead, $\{d_1\}$ and $\{d_2\}$ are two strongly regular grounded sets of defaults. They generate two extensions containing Usable(left-arm) and Usable(right-arm) respectively.

With this new definition of regularity, the intuitive meaning of a default $\frac{a:b}{c}$ becomes " if a is known, and if b is consistent with what is known, then *assume* b and infer c", where "assume b" means that $\neg b$ is not consistent in this extension. Although b is not inferred, there is a kind of commitment to the assumption b, which blocks the use of any other default, whose justification would be $\neg b$. Recall that Schaub's definition was partly motivated by the counter intuitive solutions given by Reiter's and Lukaszewicz' definitions on the broken-arm example. We will show in Sect. 5 that the latter definitions correspond to weak regularity, while the former one corresponds to strong regularity.

4.3.3 Subsumption of defaults

Rychlik's definition, aimed at avoiding the application of those defaults which are subsumed by other defaults (Def. 5), leads to the following notion:

Definition 15. Let $\Delta = (W, D)$ be a default theory. Suppose that some notion of regularity is given (either weak or strong). A set of defaults $U \subseteq D$ grounded in W is **concisely regular** if it is regular and for any $d \in U$, d is not subsumed by U.

4.3.4 Discussion

Let us first mention without proof the obvious relationships that exist between the previous notions of regularity:

Proposition 4. *Let $\Delta = (W, D)$ be a default theory. If a set of defaults $U \subseteq D$ grounded in W is strongly regular, then it is weakly regular.*

We have seen that the introduction of the strong regularity was motivated by a counterintuitive answer of the weak regularity on Poole's example about the broken arms. More generally, the difference between the two notions of regularity can be described as follows: if a grounded set of defaults U is such that the conjunction of the justifications of all the defaults in U is inconsistent with the set of theorems generated by U, but none of the justifications of the defaults in U is individually inconsistent with the set of theorems generated by U, then U is weakly regular but not strongly regular. As we already mentioned, the intuitive meaning of a default $\frac{a:b}{c}$ in the case of strong regularity is "if a is known, and if b is consistent, assume (i.e. implicitly conclude) b and infer c": although b is not inferred, there is a kind of commitment to the assumption b, which blocks the use of any other default the justification of which would be $\neg b$. The example below shows a case where we do not want commitment to the assumptions to be made.

Example 8. Suppose that the topic y appears in the curriculum of student x. If x knows that there will be an examination on y, then x will work hard on y, whereas if x knows that there will not be any examination on y, then x will skip over y. Now if x does not know whether there will be an examination on y or not, then x will do little work on y. We can represent this knowledge with the default theory $\Delta = (W, D)$, where

$$W = \left\{ \begin{array}{l} \forall x, y, \mathsf{student}(x) \wedge \mathsf{topic}(y) \wedge \mathsf{exam}(y) \rightarrow \mathsf{hard\text{-}work}(x, y), \\ \forall x, y, \mathsf{student}(x) \wedge \mathsf{topic}(y) \wedge \neg\mathsf{exam}(y) \rightarrow \mathsf{skip\text{-}over}(x, y) \end{array} \right\},$$

and

$$D = \left\{ \frac{\mathsf{student}(x) \wedge \mathsf{topic}(y) : \mathsf{exam}(y), \neg\mathsf{exam}(y)}{\mathsf{little\text{-}work}(x, y)} \right\}.$$

Commitment to the assumptions would mean here that the only default in D would never be applicable. This is clearly not what one might expect from this theory.

4.4 Applicability

We now study the notion of applicability of a default to a grounded set of defaults. Recall that a set of defaults is said to be saturated if it contains all defaults that are applicable to it. Saturation formalizes the fact that as many compatible defaults as possible have to be used when constructing an extension. We assume in this section that a notion of regularity of a grounded set of defaults is given. Both notions of weak and strong regularity give the same results for the examples of this section. We propose below two definitions of applicability, and then compare them.

4.4.1 Cautious applicability

The simplest notion of applicability of a default to a grounded set of defaults supposes of course that the default is active with respect to the set of defaults, and requires the resulting grounded set of defaults to be still regular. This can be formalized with the following definition:

Definition 16. Let $\Delta = (W, D)$ be a default theory, and suppose that we have defined a notion of regularity for the grounded sets of defaults of Δ. A default d is **cautiously applicable** to a grounded set of defaults U if d is active with respect to U and if the resulting grounded set of defaults $U \cup \{d\}$ is regular. A grounded set of defaults saturated for this notion of applicability is **cautiously saturated.**

With this definition, starting from an empty set of defaults and by iteratively applying to it defaults which are cautiously applicable to it, one always obtains a regular grounded set of defaults[6]. Moreover, a cautiously saturated grounded set of defaults is a grounded set of defaults which is maximally regular (for set inclusion). This property allows for a very constructive way of building extensions, as shown in Sect. 6.1. As a consequence, any default theory such that W is consistent always has an extension in this sense (maybe generated by the empty set of defaults).

Another property of this notion of applicability is that it leads to **semi-monotonic** default logic in the following sense:

Theorem 5. *Let D and D' be two sets of defaults, with $D \subseteq D'$, and W be a set of formulas. If U is a regular and cautiously saturated grounded set of defaults of (W, D) then there exists a regular and cautiously saturated grounded set of defaults U' of (W, D') such that $U \subseteq U'$.*

Notice that this notion of applicability makes possible the definition of a proof theory, in the spirit of the one given by Lukaszewicz: if a regular grounded set U of defaults is such that a formula f is provable from the consequents of the elements of U together with W, then there is at least one maximally regular grounded set of defaults containing U. This set generates an extension which contains f. We will see in Sect. 5 that Lukaszewicz' extensions correspond to this notion of applicablity.

It is also worth mentioning that, when considering cautious applicability, a default theory is equivalent to its semi-normal form:

[6] Therefore the name "cautious".

Proposition 6. *Let $\Delta = (W, D)$ be a default theory. Let $D' = \{\frac{a:b\wedge c}{c}, \frac{a:b}{c} \in D\}$. The sets of formulas generated by the weakly regular (respectively strongly regular) and cautiously saturated subsets of D grounded in W are the sets of formulas generated by the weakly regular (respectively strongly regular) and cautiously saturated subsets of D' grounded in W.*

4.4.2 Hazardous applicability

The definition of extensions given by Reiter (1980) involves another notion of applicability which we formalize as follows:

Definition 17. Let $\Delta = (W, D)$ be a default theory. A default d is **hazardously applicable** to a grounded set of defaults U if d is active with respect to U and if $W \cup \text{Con}(U) \cup \{\text{Jus}(d)\}$ is consistent. A grounded set of defaults U such that all defaults hazardously applicable to U are in U is **hazardously saturated**[7].

Example 9. Let $\Delta = (\{a\}, \{d_1, d_2\})$, with $d_1 = \frac{a:\neg c}{d}$ and $d_2 = \frac{a:b}{c}$. The strongly regular grounded set of defaults $\{d_1\}$ is not hazardously saturated: the default d_2 is such that $W \cup \{\text{Con}(d_1), \text{Jus}(d_2)\}$ is consistent, thus d_2 is hazardously applicable to $\{d_1\}$, but is not in $\{d_1\}$.

With this notion of applicability, a default $\frac{a:b}{c}$ is not always equivalent to its semi-normal form $\frac{a:b\wedge c}{c}$. Consider for example the theory $(\emptyset, \{\frac{:a}{\neg a}\})$. This theory has no hazardously saturated and regular grounded set of defaults, though the corresponding semi-normal default theory $(\emptyset, \{\frac{:a\wedge\neg a}{\neg a}\})$ has one hazardously saturated and regular grounded set of defaults, namely $\emptyset$. This example also shows that a default theory does not always have an extension generated by a hazardously saturated universe. Furthermore, default logic is in general not semi-monotonic when considering hazardous applicability.

4.4.3 Discussion

Let us start by giving the simple relationship that exists between these two notions of applicability.

Proposition 9. *Let $\Delta = (W, D)$ be a default theory. If a default $d \in D$ is cautiously applicable to a grounded set of defaults U of Δ, then d is also hazardously applicable to U. Consequently, a hazardously saturated grounded set of defaults is also cautiously saturated.*

It follows from the discussion of Sect. 2 (point 1) that a default $\frac{a:b}{c}$ might be used only if there is no evidence of $\neg b$, nor of $\neg c$. In view of this, a default $\frac{a:b}{c}$ should be equivalent to the semi-normal default $\frac{a:b\wedge c}{c}$: in both cases the default can be used only if there is no evidence of $\neg b$ nor of $\neg c$, and in both cases c is inferred. We have seen in the previous section that it is not always the case when the notion of hazardous applicability is considered.

In the remaining part of this section, however, we consider that all defaults are semi-normal (such that the justification entails the consequent in the sense of the logic L). As already shown, for instance by Reiter and Criscuolo (1981), Brewka (1991b), the notion of hazardous applicability, which underlies Reiter's definition of default logic, provides a way of giving priority to some defaults over others. Let us study when there is a difference between cautious applicability and hazardous applicability. For the sake of definiteness, we assume here that we work with the notion of weak regularity. Similar results hold when considering strong regularity, and have been described by Brewka (1991b). Suppose U is a maximally regular grounded set of defaults of a semi-normal default theory $\Delta = (W, D)$, and that the semi-normal default $d \in D$ is hazardously applicable to U, but not cautiously applicable to U: as we have semi-normal defaults, this is equivalent to saying that $W \cup \text{Con}(U \cup \{d\}) \cup \{\text{Jus}(d)\}$ is consistent, but there exists $d' \in U \cup \{d\}$ such that $W \cup \text{Con}(U \cup \{d\}) \cup \{\text{Jus}(d')\}$ is inconsistent: the default d is blocked by the presence of the justification of some other default in the grounded set of defaults U, because

[7] Hazardously saturated grounded sets of defaults are what Levy calls “complete universes”

it is inconsistent with the conclusion of d; in this case U is not hazardously saturated. The extension generated by the cautiously saturated grounded set of defaults U is rejected, according to the notion of hazardous saturation, because the consequent of d should have *priority* over the justifications of the defaults in U that cause the irregularity. Therefore hazardous saturation is indeed a stronger condition than cautious saturation (see principle 3, Sect. 4.1).

4.5 Normal defaults

A default is *normal* if its consequent is logically equivalent to its justification. An example of normal default is $\frac{\mathsf{bird(Tweety)}:\mathsf{flies(Tweety)}}{\mathsf{flies(Tweety)}}$ (cf. Example 1). Recall that the weak regularity of a grounded set of defaults means that the justification of each default in this set is consistent with the generated set of formulas, whereas the strong regularity means that the conjunction of the justifications of all the defaults in the set has to be consistent with the generated set of formulas. When the defaults are normal, the justifications of the defaults are equivalent to their consequents and are all in the generated set of formulas. Therefore, both notions of regularity reduce in this case to the consistency of the generated set of formulas. Moreover, a default d is hazardously applicable to a grounded set of defaults U if and only if it is active with respect to U and if its justification is not contradicted by the consequents of the elements of U. If d is normal, its hazardous applicability to U means that its consequent (equivalent to its justification) is consistent with the generated set of theorems. Therefore, in this case the normal default is also cautiously applicable to U. We have seen that in all cases, cautious applicability entails hazardous applicability. Hence, when a default theory contains normal defaults only, cautious applicability and hazardous applicability are equivalent.

Proposition 10. *Let $\Delta = (W, D)$ be a normal default theory, and U a grounded set of defaults of Δ, then U is weakly regular if and only if U is strongly regular, if and only if $W \cup \mathrm{Con}(U)$ is consistent.*

Proposition 11. *Let $\Delta = (W, D)$ be a normal default theory, and let some notion of regularity be given (either weak or strong). Then a grounded subset U of D is cautiously saturated if and only if U is hazardously saturated, if and only if (for all $d \in D$, if d is active with respect to U and $U \cup \{d\}$ is regular then $d \in U$).*

4.6 Multiple justifications

As we mentioned at the beginning of this section, the original form of a default is $d = \frac{a:b_1,\ldots,b_n}{c}$, where a, b_1,. . ., b_n and c are logical formulas. The defaults that we have considered until now are defaults whose set of justifications has a unique element. Let us now study how we can extend the various notions of regularity and applicability to the case of defaults having multiple justifications.

In Reiter's definition of extensions, the meaning of such a default $\frac{a:b_1,\ldots,b_n}{c}$ is: "if a is known, and if for all i, b_i is consistent with what is known, then infer c". This is closely related to the notion of weak regularity. For example, the default theory $(\emptyset, \{\frac{:b,\neg b}{c}\})$ has two weakly regular grounded sets of defaults, $\{\frac{:b,\neg b}{c}\}$ and $\emptyset$. So we should rewrite the notion of weak regularity as follows:

> A set of defaults U will be said to be *weakly regular* if for any default $\frac{a:b_1,\ldots,b_n}{c} \in U$, $W \cup \mathrm{Con}(U) \cup \{b_i\}$ is consistent for any $1 \leq i \leq n$.

But if we consider the notion of strong regularity, then no grounded set of defaults containing the default $\frac{:b,\neg b}{c}$ in any default theory should be strongly regular, because the conjunction of the justifications $b \wedge \neg b$ of this default is inconsistent. We can rewrite the definition of strong regularity for this more general form of default:

> A grounded set of defaults U is *strongly regular* if $W \cup \mathrm{Con}(U) \cup (\bigcup_{d \in U} \{b_i \in \mathrm{Jus}(d)\})$ is consistent.

For this second notion it is equivalent to replace a default $\frac{a:b_1,\ldots,b_n}{c}$ by the one which has only one justification $\frac{a:b_1\wedge\ldots\wedge b_n}{c}$ (this is not the case with Reiter's definition of extensions, as pointed out by Besnard (1989, Prop. 6.1.8.).

As the cautious applicability does not involve the justifications of the defaults excepted through the notion of regularity, it does not have to be modified for defaults having multiple justifications. The hazardous applicability can be expressed as follows:

> Let $\Delta = (W, D)$ be a default theory, a default $d = \frac{a:b_1,\ldots,b_n}{c}$ is *hazardously applicable* to a grounded set of defaults U if d is active with respect to U and if it is true that $W \cup \mathrm{Con}(U) \cup \{b_i\}$ is consistent for $1 \le i \le n$.

5 A classification of the various definitions of default logic

5.1 Equivalence between fixed points and regular and saturated grounded sets of defaults

The aim of this section is to show that Reiter's default logic and most of its variants fit in our framework, by exhibiting the regularity and saturation conditions that correspond to extensions in each of them. Most of these variants have been introduced as fixed points of some operator Γ which depends on a set D of defaults.

In Reiter's original version, this operator maps sets of formulas to sets of formulas: given a set E of formulas, $\Gamma_{\mathrm{R}}(E)$ is the smallest set of formulas E' which is deductively closed, which contains W, and such that for any default d whose prerequisite is in E' and whose justification is not contradicted in E, the conclusion of d is in E'. We will see that $\Gamma_{\mathrm{R}}(E)$ is in fact the set of theorems generated by the grounded part of the set of all defaults of D whose justification is not contradicted in E.

Lukaszewicz introduced an operator on a space of pairs of the form (E, F), where E and F are sets of formulas. On the other hand, Rychlik operates on a space of pairs of the form (E, V), where E is a set of formulas and V a set of defaults. In both cases, the operator on (E, F) (respectively (E, V)) generates the smallest deductively closed set E' of formulas containing W such that for any default d whose prerequisite is in E' and which verifies some condition with respect to (E, F) (respectively (E, V)), the consequent of d is in E'. An extension E is then a fixed point of this operator, where F is the set of justifications of defaults used in generating E (respectively, V is the set of those defaults).

More generally, in the earlier approaches to default logic, an operator Γ is defined on some space of the form $\mathcal{E} = 2^{\mathcal{L}} \times 2^{\mathcal{H}}$, where $\mathcal{L}$ is the logical language, and $\mathcal{H}$ is the logical language or the set of defaults, depending on the variant of default logic considered. The extensions of a default theory are defined as follows:

Definition 18. Let $\Delta = (W, D)$ be a default theory. A set E of formulas is an extension of Δ if there exists $H \subseteq \mathcal{H}$ such that (E, H) is a fixed point of the operator Γ, which associates (E', H') with (E, H), where E' and H' are the smallest sets which verify:

P1 $W \subseteq E'$,

P2 $\mathrm{Th}(E') = E'$,

$\mathbf{P3}_{(E,H)}$ $\forall d \in D$, if $\mathrm{Pre}(d) \in E'$ and $\mathbf{Q}((E, H), d)$ holds then $\mathrm{Con}(d) \in E'$ and $f(d) \in H'$,

where

$f(d)$ returns the information about d that must be stored in H',

$\mathbf{Q}$ is some predicate on $\mathcal{E} \times D$ which defines the defaults that are used to construct $\Gamma(E, H)$.

Both the mapping f and the predicate $\mathbf{Q}$ depend on the variant of default logic considered.

In the case of Reiter's definition, where the operator is defined on the set of subsets of the logical language itself, we can take $\mathcal{H}$ to be the empty set, or equivalently consider that $\mathcal{E} = 2^{\mathcal{L}}$. The corresponding predicate $\mathbf{Q}_R$ is then defined on $2^{\mathcal{L}} \times D$: $\mathbf{Q}_R(E, d) \equiv \neg\text{Jus}(d) \notin E$. There is no mapping f in this case. As another example, in the case of Lukaszewicz' definition, H is the set of the justifications of the defaults that are used to construct the extension, and the corresponding predicate is $\mathbf{Q}_L((E, H), d) \equiv \forall x \in H \cup \{\text{Jus}(d)\}, E \cup \{x, \text{Con}(d)\}$ is consistent.

The first theorem that we prove shows that $\Gamma(E, H)$ is the pair generated by the grounded part of the set of the defaults of D that are compatible with (E, H), i.e. for which $\text{Q}((E, H), d)$ holds. We denote by $\text{Comp}(E, H)$ this set, i.e.

$$\text{Comp}(E, H) = \{d \in D \text{ s.t. } \text{Q}((E, H), d) \text{ holds}\}.$$

Given some grounded set U of defaults, we denote by $\sigma(U)$ the pair of $\mathcal{E}$ generated by U, that is:

$$\sigma(U) = (\text{Th}(W \cup \text{Con}(U)), \{f(d), d \in U\}).$$

Theorem 12. *For all* (E, H), $\Gamma(E, H) = \sigma(\text{Ground}(\text{Comp}(E, H)))$.

The next theorem shows the link between regular and saturated grounded sets of defaults of a theory, and the fixed points of the operator Γ above.

Theorem 13. *Let* $\Delta = (W, D)$ *be a default theory. A pair* (E, H) *is a fixed point of* Γ *if and only if there exists a subset* U *of* D *such that* $E = \text{Th}(W \cup \text{Con}(U))$ *and* $H = f(U)$ *and for all* d *in* D*:*

$$d \in U$$
$$\Leftrightarrow$$
$$U \cup \{d\} \text{ is grounded in } W \text{ and } \mathbf{Q}((\text{Th}(W \cup \text{Con}(U)), f(U)), d) \text{ holds.}$$

The "if and only if" condition in the above theorem leads to a characterization of extensions in terms of regular and saturated grounded sets of defaults. In one direction, the implication expresses the regularity condition and the groundedness requirement (for all $d \in U$, $U \cup \{d\} = U$ is grounded and $\mathbf{Q}(\sigma(U), d)$ holds), whilst in the other direction it expresses the saturation condition.

The next corollary will also be used in the following:

Corollary 16. *Let* $\Delta = (W, D)$ *be a default theory. An element* (E, H) *of* $\mathcal{E}$ *is a fixed point of* Γ *if and only if* $(E, H) = \bigcup_{n \in \mathbf{N}} (E_n, H_n)$*, where:*

$$(E_0, H_0) = (\text{Th}(W), \emptyset),$$

$$U_{n+1} = \{d \in D, \text{Pre}(d) \in E_n, \mathbf{Q}((E, H), d) \text{ holds }\},$$
$$(E_{n+1}, H_{n+1}) = \sigma(U_{n+1}), \text{ for } n \geq 0.$$

The following sections show the notions of regularity and saturation that correspond to each condition $\mathbf{Q}$ which occurs in the literature on default logic.

5.2 The variants

Reiter's extensions The following theorem shows that Reiter's extensions correspond to the notions of weak regularity (Def. 13) and hazardous applicability (Def. 17). Analogous theorems have been proved by Levy (1991a) and independantly by Schwind and Risch (1991).

Theorem 17. *Let* $\Delta = (W, D)$ *be a default theory, then a set of formulas* E *is an R-extension of* Δ *if and only if there exists a weakly regular and hazardously saturated subset* U *of* D *grounded in* W *which generates* E*.*

Lukaszewicz' extensions In this case, we use the notion of cautious applicability (Def. 16).

Theorem 18. *Let $\Delta = (W, D)$ be a default theory, and let E be a set of formulas. Then E is an L-extension of Δ if and only if E is generated by a weakly regular and cautiously saturated grounded set of defaults.*

Rychlik's extensions The concept of concise regularity has been introduced in Def. 15 for dealing with Rychlik's extensions. Recall that a grounded set of defaults is concisely regular if it is regular and if it does not subsume any of its elements.

Theorem 19. *Let $\Delta = (W, D)$ be a default theory, and let E be a set of formulas. Then E is an Ry-extension of Δ with respect to U if and only if U is such that $E = \mathrm{Th}(W \cup \mathrm{Con}(U))$ and U is grounded in W and concisely weakly regular and for any $d \in D$, if d is U-active and $U \cup \{d\}$ is weakly regular and if d is not subsumed by U, then $d \in U$.*

De T. Guerreiro, Casanova and Hemerly's extensions

Proposition 20. *Let $\Delta = (W, D)$ be a default theory. A set E of formulas is a E-set of Δ if and only if E is generated by a weakly regular grounded set of default.*

The following theorem characterizes the C-extensions in the case where the theory has a finite set of E-sets:

Theorem 22. *Let $\Delta = (W, D)$ be a default theory, such that Δ has a finite number of E-sets. Then the C-extensions of Δ are the maximal sets of theorems generated by weakly regular and cautiously saturated grounded sets of defaults.*

This theorem does not hold in the case of an infinite set of E-sets. The following example illustrates this:

Example 10. (de T. Guerreiro et al. 1990) Let $D = \{\frac{:\neg p(f^n(a))}{q(a)}, \frac{:p(f^n(a))}{p(f^n(a))}, n \geq 0\}$, and let $W = \emptyset$. The theory (W, D) has a unique C-extension, $E = \mathrm{Th}(\{q(a), p(f^n(a)), n \geq 0\})$, which is the union of the E-sets generated (among other E-sets) by the weakly regular and cautiously saturated grounded sets of defaults $U_n = \{\frac{:\neg p(f^n(a))}{q(a)}, \frac{:p(f^m(a))}{p(f^m(a))}, m \neq n\}$ for all $n \geq 0$. However, no weakly regular and cautiously saturated grounded set of defaults generates E.

Schaub's constrained extensions The following theorem gives the regularity and saturation notions that correspond to Schaub's constrained extensions.

Theorem 23. *Let $\Delta = (W, D)$ be a default theory. A pair (E, C) is a constrained extension of Δ if and only if there exists a strongly regular and cautiously saturated grounded set of defaults U which generates E and such that $C = \mathrm{Th}(W \cup \mathrm{Jus}(U) \cup \mathrm{Con}(U))$.*

5.3 A classification of various definitions of default logic

The application of the theorem 13 to the various definitions of default logic that can be found in the literature yields the classification shown in Table 1.

As the definitions of Delgrande and Jackson (1991) and de T. Guerreiro et al. (1990) are not given in terms of fixed points of some operator, they do not exactly correspond to saturated grounded sets of defaults. However, when the set of defaults is *finite*, their extensions are the sets of theorems generated by the regular and saturated grounded sets of defaults of the theory, with the regularity and applicability notions shown in the array above (in the case of de T. Guerreiro et al., only the maximal sets of theorems generated by some weakly regular and cautiously saturated grounded sets of defaults are considered). In this case, the definitions by Lukaszewicz and by de T. Guerreiro et al. are equivalent, as are the definitions by Delgrande and Jackson, and Schaub.

U weakly regular	$\equiv$	$\forall d \in U, W \cup \mathrm{Con}(U) \cup \{\mathrm{Jus}(d)\}$ is consistent
U strongly regular	$\equiv$	$W \cup \mathrm{Con}(U) \cup \mathrm{Jus}(U)$ is consistent
U concisely regular	$\equiv$	U is regular and no default of U is subsumed by U
d cautiously applicable to U	$\equiv$	$U \cup \{d\}$ regular and d is active w.r.t. U
d hazardously applicable to U	$\equiv$	$W \cup \mathrm{Con}(U) \cup \{\mathrm{Jus}(d)\}$ consistent and d is active w.r.t. U

Variant	**Regularity condition**	**Applicability condition**
Reiter's R-extensions (1980)	weak regularity	hazardous applicability
Lukaszewicz' L-extensions (1988)	weak regularity	cautious applicability
de T. Guerreiro et al.'s C-extensions (1990)	weak regularity	cautious applicability
Delgrande and Jackson's J-extensions (1991) (semi normal default theories)	strong regularity	cautious applicability
Schaub's constrained extensions(1991b)	strong regularity	cautious applicability
Brewka's CDL-extensions (1991b)	strong regularity	cautious applicability
Brewka's priority preserving CDL-extensions (1991b)	strong regularity	d is cautiously applicable to U, or $W \cup \mathrm{Con}(U \cup \{d\}) \cup \{\mathrm{Jus}(d)\}$ is consistent and $W \cup \mathrm{Con}(U \cup \{d\}) \cup \mathrm{Jus}(U)$ is inconsistent.
Rychlik's Ry-extension (1991)	concise weak regularity	$U \cup \{d\}$ is weakly regular and d is not subsumed by U.

Table 1: Classification according to regularity and saturation criteria

The notions of regularity and saturation corresponding to Brewka's CDL-extensions and priority preserving CDL-extensions have been extablished by Froidevaux and Mengin (1992b), where Brewka's method for defining cumulative variants of default logic is generalized.

It is interesting to note that Brewka's priority preserving CDL-extensions correspond to the notion of strong regularity combined with a modified version of hazardous applicability. Brewka (1991b, Sect. 3) discusses in detail his mechanism for preserving the "priority of the consequent over the justification".

5.4 Related works

Gelfond et al. (1991), motivated by Poole's example too, and willing to give a semantics to disjunctive logic programs, propose to introduce a new form of defaults. They use *disjunctive defaults*, of the form $\frac{a:b}{c_1|\ldots|c_n}$. The intended meaning of this default is: "Given some extension E, if a is true, and if there is no evidence of $\neg b$, then one of the c_is at least must be in E". Because they use a different form of defaults, a translation of their definition of extensions into some regularity and saturation conditions on sets of disjunctive defaults cannot be obtained by means of the tools provided in Sect. 4.

Several authors propose to consider the fixed point of Γ^2 for some operator Γ. Przymusinska and Przymusinski (1992) are concerned by Reiter's Γ, while Pereira et al. (1992) and Dix (1992) propose to use other operators. The tools given in Sect. 4 cannot be used in an immediate way to express these definitions of extensions in terms of regularity and saturation conditions on sets of defaults.

6 Applications

6.1 Operational definition of the extensions

Our operational approach is in the same spirit as Lukaszewicz' (1988) approach for a semantical account of default logic and as Krause, Byers, Hajnal and Cozens (1991) algorithm for computing extensions.

Let $\mathcal{L}$ be a countable propositional language. Default theories (W, D) contain a countable set of defaults D defined on $\mathcal{L}$, that is, $\mathrm{Con}(D) \cup W \subseteq \mathcal{L}$. It is well known that the lattice of all subsets of a given set X is complete for inclusion: for any family A of subsets $S_a \subseteq X$, $\mathrm{Inf}_{\subseteq}(A) = \cap_{a\in A} S_a$ and $\mathrm{Sup}_{\subseteq}(A) = \cup_{a\in A} S_a$. Hence $(2^D, \subseteq)$ is a complete lattice.

Let $\Delta = (W, D)$ be a default theory and let U be a grounded set of defaults. We define an **operator** APP_Δ which associates with each grounded set of defaults U the set of defaults which are applicable to U and which are not already in U, as follows:

Definition 19. $\mathrm{APP}_\Delta U = \{d \in D, d \notin U$ and d is applicable to $U\}$.

We also define a **nondeterministic choice operator** TC_Δ which, given a grounded set of defaults U, selects a default among the defaults applicable to U which are not already in U:

Definition 20.

$$\mathrm{TC}_\Delta(U) = \begin{cases} U \text{ if } \mathrm{APP}_\Delta U = \emptyset, \\ U \cup \{d\} \text{ for some } d \in \mathrm{APP}_\Delta U \text{ otherwise.} \end{cases}$$

If the default theory Δ is understood, we will omit the subscript Δ in TC_Δ.

Example 5 (continued) Let $W = \emptyset$, $D = \{\frac{:a}{a}, \frac{:a_n}{a_n}, n \in \mathbf{N}\}$. Consider the sequence $(\mathrm{TC}^n(\emptyset))_{n\in\mathbf{N}}$, where $\mathrm{TC}^{n+1}(\emptyset)$ is obtained by applying $\frac{:a_n}{a_n}$ for all $n \in \mathbf{N}$. Then $\bigcup_{n\in\mathbf{N}} \mathrm{TC}^n(\emptyset) = \{\frac{:a_n}{a_n}, n \in \mathbf{N}\}$ does not contain $\frac{:a}{a}$. However, D is the only maximally regular subset of D.

This example shows that, in order to ensure that any infinite iteration of the application of TC will lead to a maximally regular set of defaults, we have to iterate TC beyond the first limit ordinal ω and consider the ordinal powers[8] of TC.

Definition 21. The ordinal powers of TC are inductively defined as follows:

$\mathrm{TC}^0 = \emptyset$,

$\mathrm{TC}^\alpha = \mathrm{TC}(\mathrm{TC}^{\alpha-1})$ if α is a successor ordinal,

$\mathrm{TC}^\alpha = \mathrm{Sup}_{\subseteq}\{\mathrm{TC}^\beta, \beta < \alpha\} = \cup_{\beta<\alpha} \mathrm{TC}^\beta$ if α is a limit ordinal.

TC^α is a **limit** if for all $\beta \geq \alpha$, $\mathrm{TC}^\beta = \mathrm{TC}^\alpha$.

We denote by $(\mathrm{TC}^\alpha_\rho)_{\alpha\leq\beta}$ the transfinite sequence of ordinal powers until β , under some *default selection strategy* ρ, which gives the default to be selected in order to get the successor element in the sequence.

Proposition 25. *Let $\Delta = (W, D)$ be a default theory.*

1) $\forall \alpha, \beta$, if $\alpha < \beta$ then $\mathrm{TC}^\alpha \subseteq \mathrm{TC}^\beta$; moreover if $\alpha < \beta$ and $\mathrm{APP}(\mathrm{TC}^\alpha) \neq \emptyset$ then $\mathrm{TC}^\alpha \subset \mathrm{TC}^\beta$.

2) There exists γ such that $\mathrm{APP}(\mathrm{TC}^\gamma) = \emptyset$.

3) TC has at least one limit. Moreover any limit is a saturated grounded set of defaults.

It is worth noticing that the existence of a limit does not depend on the strategy ρ chosen. However, we are not sure that the limit is an extension, as the regularity condition might not be verified. Thus in the general case there remains a need to check that limits are regular grounded sets of defaults. This is not always the case, but if the applicability condition is the cautious one, clearly it is.

[8]A brief account of transfinite arithmetic is given in appendix. See e.g. Lloyd (1987) for a more detailed presentation.

Theorem 26. *Let $\Delta = (W, D)$ be a default theory. Assume that* TC *is a choice operator that selects defaults under a cautious applicability condition and that extensions of Δ are defined using this same notion of applicability (whatever the notion of regularity may be). Then:*

i) Any limit of TC *is an extension of Δ.*

ii) Δ has at least one extension.

iii) Any extension of Δ can be generated by a limit of TC.

Theorem 26 holds for Lukaszewicz' extensions and for Brewka's extensions, but does not hold for Reiter's extensions. However, we can use the operational approach for Reiter's extensions, insofar as we can construct Lukaszewicz' extensions as limits of TC with respect to weak regularity, and then verify whether the limits obtained are hazardously saturated or not. This last condition is analogous to the stability test mentioned by Moinard (1992), and introduced (as the 'viability' test) by Krause et al. (1991).

Note that we could obtain the same results by considering all sequences that exhaust the countable set of defaults. However, ordinal powers of TC allow us to apply defaults one after another in any order (provided that it respects the groundedness condition), without having to construct all those sequences exhausting the set of defaults. The maximally regular subset of D in Example 5 could be obtained by any sequnce that exhausts D, since such a sequence contains $\frac{:a}{a}$ and is grounded.

From the point of view of logic programming, Bidoit and Froidevaux (1987; 1991) have shown how logic programs with negation in the premises can be seen as particular default theories associated with them. The semantics obtained (default models semantics or equivalently stable semantics, Gelfond and Lifschitz 1988) is defined by Herbrand models of the Reiter's extensions. Our operational approach provides a way of defining these default models by iteration with a "stability test". We can take into account any restriction on the order in which logic rules have to be selected by considering a well-suited strategy. For example, stratified logic programs lead us to select defaults in one stratum after the other.

Rule-based systems have also been formally specified by means of default logic, by considering some particular default theories (Krause et al. 1991). The operational definition is closely related to the way of applying production rules: rules are fired only one at a time. The set of applicable defaults coincides with the conflict set as defined for production rules. This approach also bears some similarities with Niemela's approach to autoepistemic reasoning (1992).

6.2 Proof theory

We can extend Lukaszewicz' default proof theory to the extensions defined by regular and cautiously saturated grounded sets of defaults.

Definition 22. Let $\Delta = (W, D)$ be a default theory. Recall that we suppose W to be consistent. Let f be a formula. A finite set of defaults U of D is *a default proof* for f with respect to Δ if and only if there exists a sequence $d_1, \ldots, d_n$ of all the defaults in U such that:

- d_i is cautiously applicable to $\{d_1, \ldots, d_{i-1}\}$ for all $i \geq 0$, and
- $W \cup \mathrm{Con}(\{d_1, \ldots, d_n\}) \vdash f$.

We immediately get the following result:

Theorem 28. *Let Δ be a default theory, where extensions are generated by regular and cautiously saturated grounded sets of defaults. A formula f has a default proof with respect to Δ if and only if there is some extension of Δ that contains f.*

6.3 Semantics

We now provide a semantical account of a default theory $\Delta = (W, D)$. This follows Lukaszewicz' approach, by considering the grounded set of defaults U that generates an extension E for Δ as restricting the family of models of W, so that we obtain exactly all models of the extension E.

Let (W, D) be a default theory, and let U be a grounded and regular subset of D. U generates a set of theorems, $\mathrm{Th}(W \cup \mathrm{Con}(U))$. Let Φ be the set of models of $\mathrm{Th}(W \cup \mathrm{Con}(U))$. A default d is active with respect to U if and only if its prerequisite is true in all models of Φ. Moreover, $U \cup \{d\}$ is weakly regular (respectively strongly regular) if and only if $W \cup \mathrm{Con}(U \cup \{d\}) \cup \mathrm{Jus}(\{d'\})$ is consistent for any $d' \in U \cup \{d\}$ (respectively if and only if $W \cup \mathrm{Con}(U \cup \{d\}) \cup \mathrm{Jus}(U \cup \{d\})$ is consistent). That is, if and only if for all $d' \in U \cup \{d\}$, there exists a model of Φ that satisfies $\mathrm{Con}(d) \wedge \mathrm{Jus}(d')$ (respectively if and only if there exists a model of Φ that satisfies $\mathrm{Con}(\{d\}) \cup \mathrm{Jus}(U \cup \{d\})$).

Given a set of interpretations Φ and a set of defaults U, we will define the set of defaults that are applicable to them, with respect to some notion of regularity, denoted by $\mathrm{w-app}(\Phi, U)$ for the weak regularity, and $\mathrm{s-app}(\Phi, U)$ for the strong regularity.

Definition 23. Let $\Delta = (W, D)$ be a default theory. Let Φ be a family of interpretations and let U be a set of defaults of D.

$\mathrm{w-app}(\Phi, U) = \{d \in D, \forall \phi \in \Phi, \phi \models \mathrm{Pre}(d), \text{ and } \forall d' \in U \cup \{d\}, \exists \phi \in \Phi, \phi \models \mathrm{Con}(d) \cup Jus(d')\}$,

$\mathrm{s-app}(\Phi, U) = \{d \in D, \forall \phi \in \Phi, \phi \models \mathrm{Pre}(d), \text{ and } \exists \phi \in \Phi, \phi \models \mathrm{Con}(d) \cup Jus(U \cup \{d\})\}$.

Example 11. Let $\Delta = (\{a\}, \{\frac{a:b}{c}\})$. Let $U = \emptyset$. The models of $\mathrm{Th}(\{a\} \cup \mathrm{Con}(U))$ are the models of a. It is clear that all these models satisfy the prerequisite a of $\frac{a:b}{c}$. Moreover, at least one of these models satisfies $b \wedge c$. Therefore, $\frac{a:b}{c}$ is w-applicable to (Φ, U) and s-applicable to (Φ, U), where Φ is the set of models of $\mathrm{Th}(\{a\} \cup \mathrm{Con}(U))$.

It is easy to establish the links with the notions of regularity concerned.

Proposition 29. *Let $\Delta = (W, D)$ be a default theory and let U be a grounded set of defaults of Δ. Let Φ be the family of all models of $W \cup \mathrm{Con}(U)$, then:*

i) $\forall d \in D$, $d \in \mathrm{w-app}(\Phi, U)$ *iff d is U-active and $U \cup \{d\}$ is weakly regular;*

ii) $\forall d \in D$, $d \in \mathrm{s-app}(\Phi, U)$ *iff d is U-active and $U \cup \{d\}$ is strongly regular.*

As a consequence of the proposition, d is applicable with respect to $\mathrm{w-app}$ (respectively $\mathrm{s-app}$) to some grounded set of defaults U such that $d \notin U$ iff d is cautiously applicable to U with respect to APP for the notion of weak regularity (respectively strong regularity).

The following nondeterministic operator restricts the set of interpretations and constructs a set of defaults, by applying some default. We consider that some notion of regularity has been defined.

Definition 24. Let $\Delta = (W, D)$ be a default theory. Let Φ be a family of interpretations and let U be a set of defaults of D, and let $\mathrm{APP}(U)$ denote the set of defaults that are cautiously applicable to U. We define $\mathrm{CC}(\Phi, U) = (\mathrm{CC}_1(\Phi, U), \mathrm{CC}_2(\Phi, U))$, where

$$\begin{cases} \mathrm{CC}_1(\Phi, U) = \{\phi \in \Phi, \phi \models \mathrm{Con}(d)\} \\ \mathrm{CC}_2(\Phi, U) = U \cup \{d\}, \end{cases}$$

for some $d \in \mathrm{APP}(U)$, and $\mathrm{CC}_1(\Phi, U) = \Phi$, $\mathrm{CC}_2(\Phi, U) = U$ if $\mathrm{APP}(U) = \emptyset$.

The **ordinal powers of** CC are defined as usual:

$\mathrm{CC}^0(\Phi, U) = (\Phi, U)$,

$$\mathrm{CC}^\alpha(\Phi, U) = \mathrm{CC}(\mathrm{CC}^{\alpha-1}(\Phi, U)), \text{ if } \alpha \text{ is a successor ordinal},$$

$$\mathrm{CC}^\alpha(\Phi, U) = (\cap_{\beta<\alpha}\mathrm{CC}_1^\beta(\Phi, U), \bigcup_{\beta<\alpha}\mathrm{CC}_2^\beta(\Phi, U)) \text{ if } \alpha \text{ is a limit ordinal}.$$

$\mathrm{CC}^\alpha(\Phi, U)$ is a **limit** if $\forall \beta \geq \alpha$, $\mathrm{CC}^\beta(\Phi, U) = \mathrm{CC}^\alpha(\Phi, U)$.

As previously, we can use a strategy for applying defaults. We will denote by $(\mathrm{CC}_\rho^n(\Phi, U))_{n\geq 0}$ the sequence obtained by iteratively mapping CC on (Φ, U), for a sequence of defaults $(d_1, \ldots, d_n, d_{n+1}, \ldots)$ chosen under the strategy ρ. The following completeness result holds:

Theorem 30. *Let $\Delta = (W, D)$ be a default theory. Let Ψ be the family of all models of W. The family of interpretations Φ is the family of all models of some extension for Δ generated by a regular and cautiously saturated grounded set of defaults U if and only if there is some strategy ρ such that $(\Phi, U) = \mathrm{CC}_\rho^\omega(\Psi, \emptyset)$ and is a limit.*

From this theorem, we have a way of providing a semantics for extensions generated by strongly regular and cautiously saturated grounded sets of defaults, like the extensions in Brewka's assertional default logic or Schaub's (1991b) constrained extensions.

Notice that if we define an ordering $\preceq$ on the sets of pairs of the form (Φ, U) by $(\Phi, U) \preceq (\Phi', U')$ if there is some default d such that $(\Phi', U') = \mathrm{CC}(\Phi, U)$, then the extensions are characterized by the maximal pairs which are greater than $(\Psi, \emptyset)$.

Another semantics for constrained extensions can be found in (Schaub 1991a). It involves pairs of sets of models of the form $(\Pi, \breve{\Pi})$: Π has a meaning similar to our Φ in (Φ, U), since it is the set of models that one obtains after the application of some defaults to W. Furthermore, $\breve{\Pi}$ enables us to record of the justifications of the defaults that have been used, and can thus be compared to our U.

Etherington's (1988) semantics for R-extensions involves a relation among sets of models (instead of pairs of sets of models). However, a set of models maximal for this relation characterizes an extension generated by some hazardously saturated grounded sets of defaults. An additional condition on this set of defaults ensures that it is weakly regular, thus that the extension is indeed a R-extension.

Besnard and Schaub (1992) introduce an ordering among sets of Kripke structures: let D be some set of defaults, if $\mathcal{M}$ and $\mathcal{M}'$ are sets of Kripke structures, $\mathcal{M} \geq_D \mathcal{M}'$ if there exists $d \in D$ such that $\mathcal{M} = \{m \in \mathcal{M}', m \models \mathrm{Con}(d) \land \Box\mathrm{Con}(d) \land \Diamond\mathrm{Jus}(d)\}$, $\mathcal{M}' \models \mathrm{Pre}(d)$, and $\mathcal{M}' \not\models \Box\neg\mathrm{Jus}(d)$, where $\Box$ (respectively $\Diamond$) denotes the modal necessity operator (respectively its dual, the possibility operator). They show that the $\geq_D$-maximal sets of Kripke structures, above the set $\mathcal{M}_W$ of Kripke structures that satisfy W, that is $\mathcal{M}_W = \{m \models f \land \Box f, \forall f \in W\}$, characterize the R-extensions of (W, D). They give similar relations among sets of Kripke structures to characterize Schaub's constrained extensions and Lukaszewicz' L-extensions.

In general, there are several 'levels' of truth that appear in default logic: the justifications of the defaults do not have the same status as the the consequents, with either weak or strong regularity. Therefore, a simple relation among sets of models of classical logic cannot give a semantics for default logic in general, and a relation among more complex structures is required. However, in the case of normal defaults, where the justification and the consequent of a default are equivalent, a relation on sets of models of classical logic is enough to characterize the extensions: let (W, D) be some normal default theory, and define

$$\mathrm{CC}_{\mathrm{nor}}(\Phi) = \{\phi \in \Phi, \phi \models \mathrm{Con}(d)\}$$

for some $d \in D$ such that $\forall \phi \in \Phi, \phi \models \mathrm{Pre}(d)$ and $\exists \phi \in \Phi, \phi \models \mathrm{Con}(d)$. Then the limits $\mathrm{CC}_{\mathrm{nor}}^\omega(\Psi)$, where Ψ is the set of models of W, characterize the extensions of (W, D).

7 Conclusion

In our approach, the study of extensions is carried out by means of the characterization of the sets of generating defaults, instead of an analysis of the fixed points of an operator Γ.

Three basic notions have emerged: groundedness, regularity and saturation. Our framework is general enough to allow for the introduction of other concepts, such as the minimality of the extensions. It encompasses several other definitions of extensions, and provides interesting grounds on which to compare them. The results of the last section (operational definition of the extensions, proof theoretical and semantical aspects), show some of the interesting possibilities offered by our approach to default logic. Notice however that these results apply mainly when using the notion of cautious applicability. We have described (Froidevaux and Mengin 1992b) how it is possible to modify these definitions in order to obtain a cumulative inference relation, generalizing Brewka's work on cumulative default logic. Our approach is especially interesting from the theorem-proving point of view. It has already given rise to three theorem provers: one, based on CAT-resolution, is presented by Levy (1991b); another one is based on an A.T.M.S. (Levy 1991a). The third one, based on the tableaux method, is proposed by Schwind and Risch (1990; 1991).

As future work, a study of necessary and sufficient conditions on the regularity and saturation conditions, to have properties such as semi-monotonicity, cumulativity, and some kind of proof theory, could give some further information on the nature of default entailment. It might also be interesting to investigate how fundamental the concepts of groundedness, regularity and saturation are for the purpose of logic programing semantics.

8 Acknowledgments

The authors wish to thank DRUMS partners, especially Mike Clarke, Mike Hopkins and Nick Wilson for their support and fruitful discussions. The authors are also greatly indebted to Paul Krause, Simon Parsons and anonymous referees for their valuable comments on earlier drafts. This work was partially supported by the Esprit - Basic Research Action project DRUMS 3085.

A References

J. Allen, R. Fikes, and E. Sandewall, editors, 1991. *Proceedings of the 2nd International Conference on Principles of Knowledge Representation (KR'91).* Morgan Kaufmann.

F. Baader and B. Hollunder, 1992. Embedding defaults into terminological knowledge representation formalisms. In *Proceedings of the third Conference on Principles of Knowledge Representation*, pages 306–317.

P. Besnard, 1989. *An Introduction to default logic.* Springer-Verlag.

P. Besnard and T. Schaub, 1992. Possible world semantics for default logic. In *Proceedings of the Canadian conference on artificial intelligence.* to appear.

N. Bidoit and C. Froidevaux, 1987. Minimalism subsumes default logic and circumscription in stratified logic programming. In *Proceedings of the IEEE International Conference on Logic in Computer Science, Ithaca*, pages 89–97.

N. Bidoit and C. Froidevaux, 1991. General logic databases and programs: Default logic semantics and stratification. *Information and Computation*, 91:15–54.

G. Brewka, 1991a. *Nonmonotonic reasoning: Logical foundations of commonsense.* Cambridge University Press.

G. Brewka, 1991b. Cumulative default logic: in defense of nonmnotonic inference rules. *Artificial Intelligence*, 50:183–205.

J. P. Delgrande and W. Ken Jackson, 1991. Default logic revisited. In Allen et al. (1991), pages 118–127.

J. Dix, 1992. Default theories of Poole-type and a method for constructing cumulative versions of default logic. In Neumann (1992), pages 289–293.

D. W. Etherington, 1988. *Reasoning with Incomplete Information.* Pitman, London.

C. Froidevaux and J. Mengin, 1992b. A framework for default logics. In Pearce and Wagner (1992), pages 154–173.

D. M. Gabbay, 1985. Theoretical foundations for non-monotonic reasoning in expert systems. In K. R. Apt, editor, *Logics and Models of Concurrent systems*, pages 439–457. Springer-Verlag.

M. Gelfond and V. Lifschitz, 1988. The stable model semantics for logic programming. In *Proceedings of the International Conference on Logic Programming, Seattle*, pages 1070–1080. MIT Press.

M. Gelfond, V. Lifschitz, H. Przymusinska, and M. Truszczynski, 1991. Disjunctive defaults. In Allen et al. (1991), pages 230–237.

S. Kraus, D. Lehmann, and M. Magidor, 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44:167–207.

P. J. Krause, P. Byers, S. Hajnal, and J. Cozens, 1991. The formal specification of a database extension management system. Technical Report 116, Biomedical Computing Unit, Imperial Cancer Research Fund, P.O. Box No 123, Lincoln's Inn Fiels, London WC2A 3PX, UK.

R. Kruse and P. Siegel, editors, 1991. *Symbolic and Quantitative Approaches for Uncertainty, European Conference ECSQAU*, volume 548 of *Lecture Notes in Computer Science*. Springer Verlag.

F. Levy, 1991a. An A.T.M.S. for default theories. Research Report 91-8, C.S.P., Université Paris Nord, Avenue J.B. Clément, 93430 Villetaneuse, France.

F. Levy, 1991b. Computing extensions of default theories. In Kruse and Siegel (1991), pages 219–226.

J.W. Lloyd, 1987. *Foundations of logic programming.* Springer-Verlag, New York, 2nd edition.

W. Lukaszewicz, 1988. Considerations on default logic: an alternative approach. *Computational Intelligence*, 4:1–16.

W. Lukaszewicz, 1990. *Non-Monotonic Reasoning.* Ellis Horwood.

D. Makinson, 1988. General theory of cumulative inference. In M. Reinfrank, J. de Kleer, M. L. Ginsberg, and E. Sandewall, editors, *Proceedings of the 2nd International Workshop on Nonmonotonic Reasoning, Grassau, Germany*, volume 346 of *Lecture Notes in Artificial Intelligence*, pages 1–18. Springer-Verlag.

D. Makinson and P. Gärdenfors, 1990. Relations between the logic of theory change and nonmonotonic reasoning. In A. Fuhrmann and M. Morreau, editors, *The logic of theory change*, volume 465 of *Lecture Notes in Computer Science*, pages 185–205. Springer Verlag.

J. McCarthy, 1980. Circumscription–a form of non-monotonic reasoning. *Artificial Intelligence*, 13:27–39.

D. McDermott and J. Doyle, 1980. Non-monotonic logic 1. *Artificial Intelligence*, 13:41–72.

Y. Moinard, 1992. Unifying various approaches to default logic. In *Proceedings of the international conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, pages 61–64.

R. C. Moore, 1985. Semantical considerations on nonmonotonic logics. *Artificial Intelligence*, 25:75–94.

B. Neumann, editor, 1992. *Proceedings of the 10th European Conference on Artificial Intelligence (ECAI 92).* ECCAI, John Wiley & Sons Ldt.

I.N.F. Niemela, 1992. On the decidability and complexity of autoepistemic reasoning, to appear in Fundamenta Informaticæ.

D. Pearce and G. Wagner, editors, 1992. *Logics in AI, Proceedings of the European workshop JELIA'92, Berlin, Germany*, volume 633 of *Lecture Notes in Artificial Intelligence*. Springer-Verlag.

L. M. Pereira, J. J. Alferes, and J. N. Aparicio, 1992. Default theory for well founded semantics with explicit negation. In Pearce and Wagner (1992), pages 339–373.

D. Poole. What the lottey paradox tells us about default reasoning. In *Proceedings of the 1st International Conference on Principles of Knowledge Representation and Reasoning*, Toronto, pages 333–340.

H. Przymusinska and T. Przymusinski, 1992. Stationary default extensions. In *Working Notes of the 4th International Workshop on Nonmonotonic Reasoning, Plymouth, Vermont*, pages 179–193.

R. Reiter, 1980. A logic for default reasoning. *Artificial Intelligence*, 13:81–132.

R. Reiter and G. Criscuolo, 1981. On interacting defaults. In *Proceedings of the 7th International Joint Conference on Artificial Intelligence*, pages 270–276.

R. Reiter, 1987. Nonmonotonic reasoning. *Annual Review of Computer Science*, 2:147–186. (Annual Reviews, Inc., Palo Alto, CA).

P. Rychlik, 1991. Some varations on default logic. In *Proceedings of the National Conference on Artificial Intelligence (AAAI 91)*, pages 373–378. AAAI.

T. Schaub, 1991a. Assertional default theories: a semantical view. In Allen et al. (1991), pages 496–506.

T. Schaub, 1991b. On commitment and cumulativity in default logic. In Kruse and Siegel (1991), pages 305–309.

C. Schwind, 1990. A tableau-based theorem prover for a decidable subset of default logic. In *Proceedings of the 10th International Conference on Automated Deduction*, volume 449 of *Lecture notes in computer science*, pages 541–546. Springer Verlag.

C. Schwind and V. Risch, 1991. A tableau-based characterization for default logic. In Kruse and Siegel (1991), pages 310–317.

Y. Shoham, 1988. *Reasoning about Change.* MIT Press.

R. A. de T. Guerreiro, M. A. Casanova, and Andrea. S. Hemerly, 1990. Contributions to a proof theory for generic defaults. In L. C. Aiello, editor, *Proceedings of the 9th European Conference on Artificial Intelligence (ECAI 90)*, pages 213–218. ECCAI.

N. Wilson, 1990. Rules, belief functions and default logic. In *Proceedings of the International Conference on Uncertainty in Artificial Intelligence.*

A. Zhang and W. Marek, 1989. On the classification and existence of structures in default logic. In J.P. Martin and E.M. Morgado, editors, *Proceedings of the 4th Portuguese Conference on Artificial Intelligence*, volume 390 of *Lecture Notes in Computer Science*, Springer-Verlag, pages 129–140.

B A few results on ordinals

The definition of the ordinal powers of an operator uses the notion of ordinal. We give here some elements about ordinals.

The first ordinal 0 is defined as being equal to $\emptyset$. The other ordinals are defined to be $\{\emptyset\}$ (denoted by 1), $\{\emptyset, \{\emptyset\}\}$ (denoted by 2), $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ (denoted by 3), and so on. The first infinite ordinal is ω and is defined to be $\{0, 1, 2, 3, ...\}$, that is, the set of all non-negative integers, often denoted by $\mathbf{N}$. Two kinds of ordinals have to be distinguished. An ordinal is a *successor ordinal* if it is of the form $n \cup \{n\}$, where n is an ordinal. In this case, $n \cup \{n\}$ is the least ordinal greater than n, and is the successor of n. An ordinal is a limit ordinal if it is not the successor of any ordinal. The first limit ordinal is of course 0. The next limit ordinal is ω. The successor of ω is $\omega + 1$. The third limit ordinal is denoted by $\omega 2$ and defined to be equal to $\omega \cup \{\omega + n, n \in \omega\}$.

We also use the following *principle of transfinite induction*:

> Let $P(\alpha)$ be a property of ordinals. Assume that for all ordinals β, if $P(\gamma)$ holds for all $\gamma < \beta$, then $P(\beta)$ holds. Then $P(\alpha)$ holds for all ordinals α.

C Proofs of the theorems

Proposition 3. *Let W be a set of formulas, and let U be a set of defaults, then* Ground(U) *is the smallest subset U' of U such that:*

$$(F) \qquad \forall d \in U, \ \textit{if } \mathrm{Pre}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(U')) \textit{ then } d \in U'.$$

Define $\mathcal{L}_W^0(U) = \emptyset$ and

$$\mathcal{L}_W^{n+1}(U) = \{d \in U \ s.t. \ \mathrm{Pre}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(\mathcal{L}_W^n(U)))\} \textit{ for } n \geq 0,$$

then Ground$(U) = \bigcup_{n\geq 0} \mathcal{L}^n_W(U)$*. Furthermore, any union of sets of defaults grounded in* W *is grounded in* W*. The function* Ground *is increasing and idempotent.*

Proof . First, we prove that Ground$(U) = \bigcup_{n\geq 0} \mathcal{L}^n_W(U)$. As Th() is monotonic, an easy induction proves that the $\mathcal{L}^n_W(U)$s form an increasing sequence. Suppose that $d \in$ Ground(U), and let $(d_1, \ldots, d_n)$ be a default deduction for d from W using U. Then an easy induction proves that $d_i \in \mathcal{L}^i_W(U)$ for $1 \leq i \leq n$, and thus that $d \in \mathcal{L}^{n+1}_W(U)$. Thus Ground$(U) \subseteq \bigcup_{n\geq 0} \mathcal{L}^n_W(U)$. Conversely, we prove by induction that $\mathcal{L}^n_W(U) \subseteq$ Ground(U) for all $n \geq 0$: it is obvious for $\mathcal{L}^0_W(U) = \emptyset$. Assume now that $\mathcal{L}^n_W(U) \subseteq$ Ground(U), and let $d \in \mathcal{L}^{n+1}_W(U)$: since Pre$(d) \in$ Th$(W \cup$ Con$(\mathcal{L}^n_W(U)))$, and because Th() is compact, there exist $\delta_1, \ldots, \delta_p \in \mathcal{L}^n_W(U)$ such that Pre$(d) \in$ Th$(W \cup$ Con$(\{\delta_1, \ldots, \delta_p\}))$. By induction hypothesis, each δ_j $(1 \leq j \leq p)$ has a default deduction $(d_{j,1}, \ldots, d_{j,n_j})$ from W using U. Then the sequence $(d_{1,1}, \ldots, d_{1,n_1}, \delta_1, \ldots, d_{p,1}, \ldots, d_{p,n_p}, \delta_p)$ is clearly a default deduction for d from W using U. Thus $\mathcal{L}^{n+1}_W(U) \subseteq$ Ground(U). Therefore, for all $n \geq 0$, $\mathcal{L}^n_W(U) \subseteq$ Ground(U), so $\bigcup_{n\geq 0} \mathcal{L}^n_W(U) \subseteq$ Ground(U).

Let us now prove that $\bigcup_{n\geq 0} \mathcal{L}^n_W(U)$ is the smallest subset U' of U which verifies (F). We first prove that $\bigcup_{n\geq 0} \mathcal{L}^n_W(U)$ verifies property (F): let $d \in U$, such that Pre$(d) \in$ Th$(W \cup$ Con$(\bigcup_{n\geq 0} \mathcal{L}^n_W(U)))$. As Th() is compact, there exists $N \geq 0$ such that Pre$(d) \in$ Th$(W \cup$ Con$(\bigcup_{0\leq n\leq N} \mathcal{L}^n_W(U)))$. Moreover, as the $\mathcal{L}^n_W(U)$s form an increasing sequence, $\bigcup_{0\leq n\leq N} \mathcal{L}^n_W(U) = \mathcal{L}^N_W(U)$, thus Pre$(d) \in$ Th$(W \cup$ Con$(\mathcal{L}^N_W(U)))$, therefore $d \in \mathcal{L}^{N+1}_W(U) \subseteq \bigcup_{n\geq 0} \mathcal{L}^n_W(U)$. Hence $\bigcup_{n\geq 0} \mathcal{L}^n_W(U)$ verifies (F), thus $\bigcup_{n\geq 0} \mathcal{L}^n_W(U)$ contains the smallest U' which verifies (F). Conversely, let us prove that any U' which verifies (F) contains $\bigcup_{n\geq 0} \mathcal{L}^n_W(U)$. We prove by induction that $\mathcal{L}^n_W(U) \subseteq U'$ for all n: it is clear that $\mathcal{L}^0_W(U) \subseteq U'$, because $\mathcal{L}^0_W(U) = \emptyset$. Assume that $\mathcal{L}^n_W(U) \subseteq U'$. Let $d \in \mathcal{L}^{n+1}_W(U)$, i.e. $d \in U$ and Pre$(d) \in$ Th$(W \cup$ Con$(\mathcal{L}^n_W(U)))$. Then, as $\mathcal{L}^n_W(U) \subseteq U'$, Pre$(d) \in$ Th$(W \cup$ Con$(U'))$, hence as U' verifies property (F), $d \in U'$. Thus $\mathcal{L}^{n+1}_W(U) \subseteq U'$, for all n such that $\mathcal{L}^n_W(U) \subseteq U'$; as $\mathcal{L}^0_W(U) \subseteq U'$, for all $n \geq 0$, $\mathcal{L}^n_W(U) \subseteq U'$, thus $\bigcup_{n\geq 0} \mathcal{L}^n_W(U) \subseteq U'$. This is true for any U' which verifies (F), and in particular for the smallest one. Thus $\bigcup_{n\geq 0} \mathcal{L}^n_W(U)$ is included in the smallest subset U' of U which verifies (F).

Suppose now that $(U_i)_{i\in I}$ is a family of sets of defaults that are grounded in W. Then any default of $\bigcup_{i\in I} U_i$ has a default deduction from W using U_i, for some $i \in I$. U_i is included in $\bigcup_{i\in I} U_i$, thus $\bigcup_{i\in I} U_i$ is grounded in W.

In order to prove that Ground is increasing, consider $U \subseteq V \subseteq D$: if $d \in$ Ground(U), then d has a default deduction from W using U. Clearly, it is also a default deduction from W using V, thus $d \in$ Ground(V).

Lastly, we prove that Ground is idempotent: Ground(U) is the set of defaults of U that have a default deduction from W using U. But clearly each default in such a default deduction has itself a default deduction from W using U, namely the sequence of the defaults that appear before it in the default deduction, and thus is also in Ground(U). Hence a default deduction from W using U is also a default deduction from W using Ground(U). Therefore, Ground(Ground$(U)) =$ Ground(U).

Theorem 5. *Let* D *and* D' *be two sets of defaults, with* $D \subseteq D'$*, and* W *be a set of formulas. If* U *is a regular and cautiously saturated grounded set of defaults of* (W, D) *then there exists a regular and cautiously saturated grounded set of defaults* U' *of* (W, D') *such that* $U \subseteq U'$*.*

Proof . U is a regular grounded set of defaults of (W, D), thus it is a regular grounded set of defaults of (W, D'). Hence there exists a maximally regular grounded set of defaults U' of (W, D') which contains U.

Proposition 6. *Let* $\Delta = (W, D)$ *be a default theory. Let* $D' = \{\frac{a:b\wedge c}{c}, \frac{a:b}{c} \in D\}$*. The sets of formulas generated by the weakly regular (respectively strongly regular) and cautiously saturated subsets of* D *grounded in* W *are the sets of formulas generated by the weakly regular (respectively strongly regular) and cautiously saturated subsets of* D' *grounded in* W*.*

Let us first define, for any default $\frac{a:b}{c}$, sn$(\frac{a:b}{c})$ to be the semi normal form of $\frac{a:b}{c}$: sn$(\frac{a:b}{c}) = \frac{a:b\wedge c}{c}$. Furthermore, given a set U of defaults, sn(U) will denote the set of the semi normal forms of the defaults which are in U, so that sn$(D) = D'$. The following lemmas hold:

Lemma 7. *Let $\Delta = (W, D)$ be a default theory. A subset U of D is grounded in W if and only if* $\mathrm{sn}(U)$ *is grounded in W.*

Proof . The fact that U is grounded depends on the prerequisites and the consequents of the defaults which are in U only, not on their justifications. Thus it is the same to consider the set of semi-normal forms of the defaults of U.

Lemma 8. *Let $\Delta = (W, D)$ be a default theory. A subset U of D is weakly regular (respectively strongly regular) if and only if* $\mathrm{sn}(U)$ *is weakly regular (respectively strongly regular).*

Proof . Let U be a set of defaults, then U is weakly regular if and only if for all $\frac{a:b}{c} \in U$, $W \cup \mathrm{Con}(U) \cup \{b\}$ is consistent. As for any default $\frac{a:b}{c} \in U$, $W \cup \mathrm{Con}(U) \cup \{b\}$ is logically equivalent to $W \cup \mathrm{Con}(U) \cup \{b \wedge c\}$, U is weakly regular if and only for any $\frac{a:b\wedge c}{c} \in \mathrm{sn}(U)$, $W \cup \mathrm{Con}(U) \cup \{b \wedge c\}$ is consistent, i.e. if and only if $\mathrm{sn}(U)$ is weakly regular.

Similarly, U is strongly regular if and only if $W \cup \mathrm{Con}(U) \cup \mathrm{Jus}(U)$ is consistent. As $W \cup \mathrm{Con}(U) \cup \mathrm{Jus}(U)$ is logically equivalent to $W \cup \mathrm{Con}(\mathrm{sn}(U)) \cup \mathrm{Jus}(\mathrm{sn}(U))$, U is strongly regular if and only if $\mathrm{sn}(U)$ is strongly regular.

Proof of the proposition. Just remark that sn is a one-to-one mapping from D onto D', such that, from the lemmas above, $U \subseteq D$ is grounded in W (respectively regular) if and only if $\mathrm{sn}(U)$ is grounded in W (respectively regular). The result comes from the fact that the regular and cautiously saturated subsets of D (respectively $\mathrm{sn}(D)$) grounded in W are the maximally regular and cautiously saturated subsets of D (respectively $\mathrm{sn}(D)$) grounded in W.

Proposition 9. *Let $\Delta = (W, D)$ be a default theory. If a default $d \in D$ is cautiously applicable to a grounded set of defaults U of Δ, then d is also hazardously applicable to U. Consequently, a hazardously saturated grounded set of defaults is also cautiously saturated.*

Proof . Straightforward.

Proposition 10. *Let $\Delta = (W, D)$ be a normal default theory, and U a grounded set of defaults of Δ, then U is weakly regular if and only if U is strongly regular, if and only if $W \cup \mathrm{Con}(U)$ is consistent.*

Proof . U is weakly regular if and only if for each $d \in U$, $W \cup \mathrm{Con}(U) \cup \{\mathrm{Jus}(d)\}$ is consistent. As for any normal default $d \in U$, $\mathrm{Jus}(d) = \mathrm{Con}(d) \in \mathrm{Con}(U)$, $W \cup \mathrm{Con}(U) \cup \{\mathrm{Jus}(d)\} = W \cup \mathrm{Con}(U)$, thus U is weakly regular if and only if $W \cup \mathrm{Con}(U)$ is consistent.

Similarly, U is strongly regular if and only if $W \cup \mathrm{Con}(U) \cup \mathrm{Jus}(U)$ is consistent, and as the defaults are normal, $W \cup \mathrm{Con}(U) \cup \mathrm{Jus}(U) = W \cup \mathrm{Con}(U)$.

Proposition 11. *Let $\Delta = (W, D)$ be a normal default theory, and let some notion of regularity be given (either weak or strong). Then a grounded subset U of D is cautiously saturated if and only if U is hazardously saturated, if and only if (for all $d \in D$, if d is active with respect to U and $U \cup \{d\}$ is regular then $d \in U$).*

Proof . U is hazardously saturated if and only if for any default d which is U-active, if $W \cup \mathrm{Con}(U) \cup \{\mathrm{Jus}(d)\}$ is consistent, then $d \in U$. As d is normal, $W \cup \mathrm{Con}(U) \cup \{\mathrm{Jus}(d)\} = W \cup \mathrm{Con}(U \cup \{d\})$. Thus from the result of the previous proposition, we conclude that d is hazardously applicable to U if and only if d is U-active and $U \cup \{d\}$ is regular.

Theorem 12. *For all (E, H), $\Gamma(E, H) = \sigma(\mathrm{Ground}(\mathrm{Comp}(E, H)))$.*

Proof . Clearly, σ is an increasing mapping from D on $\mathcal{E}$, i.e. if $U \subseteq U'$, and if $(E, H) = \sigma(U)$, $(E', H') = \sigma(U')$, then $E \subseteq E'$ (because Th() is increasing), and $H \subseteq H'$.

By definition, $\mathrm{Ground}(\mathrm{Comp}(E, H))$ is the smallest U' such that

$$\forall d \in \mathrm{Comp}(E, H), \text{ if } \mathrm{Pre}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(U')) \text{ then } d \in U'.$$

Moreover, $\Gamma(E,H)$ is the pair of the smallest E' and H' such that $W \subseteq E'$, $\mathrm{Th}(E') = E'$ and

$$\forall d \in \mathrm{Comp}(E,H), \text{ if } \mathrm{Pre}(d) \in E' \text{ then } \mathrm{Con}(d) \in E' \text{ and } f(d) \in H'.$$

Let (E',H') be an element of $\mathcal{E}$ such that $W \subseteq E'$, $\mathrm{Th}(E') = E'$ and $\forall d \in \mathrm{Comp}(E,H)$, if $\mathrm{Pre}(d) \in E'$ then $\mathrm{Con}(d) \in E'$ et $f(d) \in H'$. Define

$$\rho(E',H') = \{d \in D \text{ s.t. } d \in \mathrm{Comp}(E,H) \text{ and } \mathrm{Pre}(d) \in E'\}.$$

As (E',H') verifies (P3), $\mathrm{Con}(\rho(E',H')) \subseteq E'$. Moreover, $W \subseteq E'$ and $\mathrm{Th}(E') = E'$, therefore $\mathrm{Th}(W \cup \mathrm{Con}(\rho(E',H'))) \subseteq E'$. Let $U' = \rho(E',H')$, then if $d \in \mathrm{Comp}(E,H)$ such that $\mathrm{Pre}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(U')) \subseteq E'$, $d \in U'$ by definition of $\rho(E',H')$. Hence U' verifies (F), therefore $\mathrm{Ground}(\mathrm{Comp}(E,H)) \subseteq U'$. Moreover, if $d \in U'$, then as (E',H') verifies (P3), $f(d) \in H'$. Hence $f(U') \subseteq H'$, and we have already seen that $\mathrm{Th}(W \cup \mathrm{Con}(U')) \subseteq E'$, thus $\sigma(U') \subseteq (E',H')$. Hence, as σ is increasing, $\sigma(\mathrm{Ground}(\mathrm{Comp}(E,H))) \subseteq (E',H')$, for any pair (E',H') such that $W \subseteq E'$, $\mathrm{Th}(E') = E'$ and which verifies (P3). In particular, $\sigma(\mathrm{Ground}(\mathrm{Comp}(E,H))) \subseteq \Gamma(E,H)$.

Conversely, consider a set of defaults U' which verifies (F). Let $(E',H') = \sigma(U')$: if $d \in \mathrm{Comp}(E,H)$, such that $\mathrm{Pre}(d) \in E'$, then $d \in U'$ thus $\mathrm{Con}(d) \in E'$ and $f(d) \in H'$. By definition of σ, it is obvious that $W \subseteq E'$ and $\mathrm{Th}(E') = E'$, and as (E',H') verifies (P3), $\Gamma(E,H) \subseteq (E',H')$. Hence for any U' that verifies (F), $\Gamma(E,H) \subseteq \sigma(U')$. In particular, this is true for $U' = \mathrm{Ground}(\mathrm{Comp}(E,H))$, therefore $\Gamma(E,H) \subseteq \sigma(\mathrm{Ground}(\mathrm{Comp}(E,H)))$.

Hence $\Gamma(E,H) = \sigma\mathrm{Ground}\mathrm{Comp}(E,H)$.

Theorem 13. *Let $\Delta = (W,D)$ be a default theory. A pair (E,H) is a fixed point of Γ if and only if there exists a subset U of D such that $E = \mathrm{Th}(W \cup \mathrm{Con}(U))$ and $H = f(U)$ and for all d in D:*

$$d \in U$$
$$\Leftrightarrow$$
$$U \cup \{d\} \textit{ is grounded in } W \textit{ and } \mathbf{Q}((\mathrm{Th}(W \cup \mathrm{Con}(U)), f(U)), d) \textit{ holds.}$$

Lemma 14. *Define for any set U of defaults,*

$$\nabla(U) = \mathrm{Ground}(\mathrm{Comp}(\sigma(U))).$$

(E,H) is a fixed point of Γ if and only if there exists U such that $(E,H) = \sigma(U)$ and U is a fixed point of ∇.

Proof. If $(E,H) = \Gamma(E,H)$, let $U = \mathrm{Ground}(\mathrm{Comp}(E,H))$, $\Gamma(E,H) = \sigma(U)$, thus $\sigma(U) = (E,H)$, hence $U = \mathrm{Ground}(\mathrm{Comp}(\sigma(U))) = \nabla(U)$.

Conversely, if $U = \nabla(U)$, let $(E,H) = \sigma(U)$, then $U = \mathrm{Ground}(\mathrm{Comp}(\sigma(U))) = \mathrm{Ground}(\mathrm{Comp}(E,H))$, thus $(E,H) = \sigma(U) = \Gamma(E,H)$.

Hence (E,H) is a fixed point of Γ if and only if there exists U such that $(E,H) = \sigma(U)$ and U is a fixed point of ∇.

Lemma 15. *Let $U \subseteq D$, then $U = \nabla(U)$ if and only if for all $d \in D$,*

$$d \in U \Leftrightarrow (U \cup \{d\} \textit{ is grounded and } \mathrm{Q}(\sigma(U), d) \textit{ holds}).$$

Proof. Let $U \subseteq D$ such that $U = \nabla(U)$. Then as Ground is idempotent (cf. proposition 3), $\mathrm{Ground}(U) = U$, thus U is grounded. Moreover $\mathrm{Ground}(\mathrm{Comp}(\sigma(U))) \subseteq \mathrm{Comp}(\sigma(U))$, thus $U \subseteq \mathrm{Comp}(\sigma(U))$, and U is grounded, thus $\mathrm{Ground}(U) \subseteq \mathrm{Ground}(\mathrm{Comp}(\sigma(U)))$. Let $d \in D$ such that $U \cup \{d\}$ is grounded and $\mathrm{Q}(\sigma(U), d)$ holds. As $U \cup \{d\}$ is grounded, and as $\mathrm{Ground}(U) \subseteq \mathrm{Ground}(\mathrm{Comp}(\sigma(U)))$, $\mathrm{Ground}(\mathrm{Comp}(\sigma(U))) \cup \{d\}$ is grounded; moreover $d \in \mathrm{Comp}(\sigma(U))$, thus $d \in \mathrm{Ground}(\mathrm{Comp}(\sigma(U))) = U$. Conversely, if $d \in U$, then as $\mathrm{Ground}(\mathrm{Comp}(\sigma(U))) \subseteq \mathrm{Comp}(\sigma(U))$, $\mathrm{Q}(\sigma(U), d)$ holds. Furthermore $U \cup \{d\} = U$ is grounded because $\mathrm{Ground}(U) = U$. Hence $d \in U$ if and only if $U \cup \{d\}$ is grounded and $\mathrm{Q}(\sigma(U), d)$ holds.

Assume now that U is such that for all $d \in D$, $d \in U$ if and only if $U \cup \{d\}$ is grounded and $\mathrm{Q}(\sigma(U), d)$ holds. If $U = \emptyset$, then obviously U is grounded. If $U \neq \emptyset$, let $d \in U$. Clearly, $U \cup \{d\} = U$ is grounded. Hence $U = \mathrm{Ground}(U)$. Furthermore for all $d \in U$, $\mathrm{Q}(\sigma(U), d)$ holds, thus $U \subseteq \mathrm{Comp}(\sigma(U))$, therefore $U = \mathrm{Ground}(U) \subseteq \mathrm{Ground}(\mathrm{Comp}(\sigma(U)))$, because Ground is increasing (cf. proposition 3). Therefore $U \subseteq \nabla(U)$. Lastly, if $d \in D$, such that $\mathrm{Q}(\sigma(U), d)$ holds and $\mathrm{Pre}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(U))$, then $d \in U$. Thus as $\mathrm{Ground}(\mathrm{Comp}(\sigma(U)))$ is the smallest set of defaults which verifies this property, $\mathrm{Ground}(\mathrm{Comp}(\sigma(U))) \subseteq U$, i.e. $\nabla(U) \subseteq U$. Hence $\nabla(U) = U$.

Proof of the theorem. Follows from the lemmas above.

Corollary 16. *Let $\Delta = (W, D)$ be a default theory. An element (E, H) of $\mathcal{E}$ is a fixed point of Γ if and only if $(E, H) = \bigcup_{n \in \mathbf{N}} (E_n, H_n)$, where:*

$$(E_0, H_0) = (\mathrm{Th}(W), \emptyset),$$

$$U_{n+1} = \{d \in D, \mathrm{Pre}(d) \in E_n, \mathbf{Q}((E, H), d) \textit{ holds } \},$$
$$(E_{n+1}, H_{n+1}) = \sigma(U_{n+1}), \textit{ for } n \geq 0.$$

Proof . We have proved that $\Gamma(E, H) = \sigma(\mathrm{Ground}(\mathrm{Comp}(E, H))) = \sigma(\bigcup_{n \geq 0} \mathcal{L}^n_W(\mathrm{Comp}(E, H)))$. Clearly, $(E_0, H_0) = \sigma(\emptyset) = \sigma(\mathcal{L}^0_W(\mathrm{Comp}(E, H)))$. Moreover, if $U_n = \mathcal{L}^n_W(\mathrm{Comp}(E, H))$, then by definition $U_{n+1} = \mathcal{L}^{n+1}_W(\mathrm{Comp}(E, H))$ and $(E_{n+1}, H_{n+1}) = \sigma(\mathcal{L}^{n+1}_W(\mathrm{Comp}(E, H)))$. Thus $(E_n, H_n) = \sigma(\mathcal{L}^n_W(\mathrm{Comp}(E, H)))$ for all $n \geq 0$, thus $\bigcup_{n \in \mathbf{N}} (E_n, H_n) = \sigma(\bigcup_{n \geq 0} \mathcal{L}^n_W(\mathrm{Comp}(E, H))) = \Gamma(E, H)$. The result follows.

Theorem 17. *Let $\Delta = (W, D)$ be a default theory, then a set of formulas E is an R-extension of Δ if and only if there exists a weakly regular and hazardously saturated subset U of D grounded in W which generates E.*

Proof . Reiter's definition is an instantiation of definition 18, with $\mathcal{E} = 2^{\mathcal{L}}$ (thus there is no mapping f, $\sigma(U) = \mathrm{Th}(W \cup \mathrm{Con}(U))$). The predicate corresponding to Reiter's definition is $\mathbf{Q}_R(E, d) \equiv \neg\mathrm{Jus}(d) \notin E$. Hence, by theorem 13, E is an R-extension of Δ if and only if there exists a grounded set of defaults U of Δ such that $E = \mathrm{Th}(W \cup \mathrm{Con}(U))$ and for all $d \in D$, $d \in U$ if and only if (d is U-active and $\neg\mathrm{Jus}(d) \notin \mathrm{Th}(W \cup \mathrm{Con}(U))$). If E is an R-extension, we conclude that U is weakly regular (if $d \in U$, then $\neg\mathrm{Jus}(d) \notin \mathrm{Th}(W \cup \mathrm{Con}(U))$), and hazardously saturated (if d is U-active and $\neg\mathrm{Jus}(d) \notin \mathrm{Th}(W \cup \mathrm{Con}(U))$, then $d \in U$). Conversely, if U is a weakly regular and hazardously saturated grounded set of defaults, then if d is U-active and $\neg\mathrm{Jus}(d) \notin \mathrm{Th}(W \cup \mathrm{Con}(U))$ we have $d \in U$ (because U is hazardously saturated), and if $d \in U$ then d is U-active (U is a grounded set of defaults) and $\neg\mathrm{Jus}(d) \notin \mathrm{Th}(W \cup \mathrm{Con}(U))$ (U is weakly regular). Therefore E is an R-extension.

Theorem 18. *Let $\Delta = (W, D)$ be a default theory, and let E be a set of formulas. Then E is an L-extension of Δ if and only if E is generated by a weakly regular and cautiously saturated grounded set of defaults.*

Although a slightly different theorem has already been proved by Schwind and Risch (1991), we give here a proof which uses the result of the previous section.

Proof . The definition of Lukaszewicz is an instantiation of definition 18, with $\mathcal{E} = 2^{\mathcal{L}} \times 2^{\mathcal{L}}$, $f(d) = \mathrm{Jus}(d)$, and $\mathbf{Q}_L((E, F), d) \equiv \forall x \in F \cup \{\mathrm{Jus}(d)\}, E \cup \{x, \mathrm{Con}(d)\}$ is consistent. From theorem 13, the L-extensions of Δ are generated by the grounded sets of defaults U such that: for all $d \in D$, $d \in U$ if and only if (d is U-active and $\forall d' \in U \cup \{d\}, W \cup \mathrm{Con}(U \cup \{d\}) \cup \{\mathrm{Jus}(d')\}$ is consistent). This is equivalent to: for all $d \in D$, $d \in U$ if and only if (d is U-active and $U \cup \{d\}$ is weakly regular). Hence the result.

Theorem 19. *Let $\Delta = (W, D)$ be a default theory, and let E be a set of formulas. Then E is an Ry-extension of Δ with respect to U if and only if U is such that $E = \mathrm{Th}(W \cup \mathrm{Con}(U))$ and U is grounded in W and concisely weakly regular and for any $d \in D$, if d is U-active and $U \cup \{d\}$ is weakly regular and if d is not subsumed by U, then $d \in U$.*

Proof . Rychlik's definition of extensions is an instantiation of definition 18 with $\mathcal{E} = 2^{\mathcal{L}} \times 2^{D}$, $f(d) = d$, and the predicate $\mathbf{Q}_{Ry}((E, V), d) \equiv (\forall d' \in V \cup \{d\}, E \cup \{\mathrm{Con}(d), \mathrm{Jus}(d')\}$ is consistent and d is not subsumed by $V)$. From theorem 13, the Ry-extensions of a default theory are generated by the grounded sets of defaults such that for all $d \in D$, $d \in U$ if and only if (d is U-active and $\forall d' \in U \cup \{d\}, W \cup \mathrm{Con}(U \cup \{d\}) \cup \{\mathrm{Jus}(d')\}$ is consistent and d is not subsumed by U). This is equivalent to: for all $d \in D$, $d \in U$ if and only if (d is U-active and $U \cup \{d\}$ is weakly regular and d is not subsumed by U).

Proposition 20. *Let $\Delta = (W, D)$ be a default theory. A set E of formulas is a E-set of Δ if and only if E is generated by a weakly regular grounded set of default.*

In the following, we will say that a default d is *C-applicable* to a grounded set of defaults U if d is U-active, $\neg\mathrm{Jus}(d) \notin \mathrm{Th}(W \cup \mathrm{Con}(U))$ and $\mathrm{Con}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(U))$.

We first prove the following lemma:

Lemma 21. *Let $\Delta = (W, D)$ be a default theory. A set of formulas E is a E-set of Δ if and only if E is generated by a grounded set of defaults U of Δ which is weakly regular and such that for all $d \in U$, if d is C-applicable, then $d \in U$.*

Proof . From corollary 16 and theorem 13, we deduce that E is a E-set if and only if E is generated by a grounded set of defaults U of Δ such that for any $d \in D$, $d \in U$ if and only if (d is U-active and $\neg\mathrm{Jus}(d) \notin \mathrm{Th}(W \cup \mathrm{Con}(U))$ and $\mathrm{Con}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(U))$).

If U generates a E-set, then for all $d \in U$, $\neg\mathrm{Jus}(d) \notin \mathrm{Th}(W \cup \mathrm{Con}(U))$, thus U is weakly regular. Moreover if d is U-active and $\neg\mathrm{Jus}(d) \notin \mathrm{Th}(W \cup \mathrm{Con}(U))$ and $\mathrm{Con}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(U))$, then $d \in U$.

Conversely, suppose that U is a weakly regular and C-saturated grounded set of defaults. Then for all $d \in U$, $\neg\mathrm{Jus}(d) \notin \mathrm{Th}(W \cup \mathrm{Con}(U))$ (U is weakly regular), d is U-actif (U is a grounded set of defaults), and $\mathrm{Con}(d) \in \mathrm{Th}(W \cup \mathrm{Con}(U))$ (by definition of $\mathrm{Th}(W \cup \mathrm{Con}(U))$.

Proof of the proposition. On the one hand, we have seen that any E-set is generated by a weakly regular grounded set of defaults. On the other hand, suppose that U is a weakly regular grounded set of defaults , and define $U_C = \{d \in D, d$ is C-applicable to $U\}$. Obviously, $\mathrm{Th}(W \cup \mathrm{Con}(U)) = \mathrm{Th}(W \cup \mathrm{Con}(U_C))$, thus U_C is grounded, weakly regular and C-saturated. Hence U generates a E-set.

Theorem 22. *Let $\Delta = (W, D)$ be a default theory, such that Δ has a finite number of E-sets. Then the C-extensions of Δ are the maximal sets of theorems generated by weakly regular and cautiously saturated grounded sets of defaults.*

Proof . Notice that if a finite union $\bigcup_{1 \le i \le n} E_i$ of deductively closed sets of formulas is deductively closed, then $\bigcup_{1 \le i \le n} E_i = E_{i_0}$ for some $1 \le i_0 \le n$. Thus if we have a finite number of E-sets, the C-extensions are the maximal E-sets. We show that these maximal E-sets are generated by weakly regular and cautiously saturated grounded sets of defaults: suppose E is some maximal E-set, let U be a weakly regular grounded set of defaults that generates E. Let U' be a weakly regular and cautiously saturated grounded sets of defaults containing U: then $\mathrm{Th}(W \cup \mathrm{Con}(U'))$ is a E-set containing E, thus $E = \mathrm{Th}(W \cup \mathrm{Con}(U'))$. Therefore E is generated by a weakly regular and cautiously saturated grounded sets of defaults.

Theorem 23. *Let $\Delta = (W, D)$ be a default theory. A pair (E, C) is a constrained extension of Δ if and only if there exists a strongly regular and cautiously saturated grounded set of defaults U which generates E and such that $C = \mathrm{Th}(W \cup \mathrm{Jus}(U) \cup \mathrm{Con}(U))$.*

Altough the definition above does not exactly correspond to the pattern of theorem 13, it is clear that the set T' in the definition of Υ does not need to be logically closed. The following lemma gives the link with theorem 13:

Lemma 24. *Let $\Delta = (W, D)$ be a default theory. Define, for any pair of formulas (S, T), $\Gamma_S(S, T)$ to be the pair composed of the smallest sets S' and T' respectively such that:*

P1 $W \subseteq S'$,

P2 $\mathrm{Th}(S') = S'$,

S3 *if $d \in D$, such that $\mathrm{Pre}(d) \in S'$ and $T \cup W \cup \{\mathrm{Con}(d)\} \not\vdash \neg\mathrm{Jus}(d)$, then $\mathrm{Con}(d) \in S'$ and $\mathrm{Jus}(d) \wedge \mathrm{Con}(d) \in T'$.*

Then a pair (E, T) is a fixed point of Γ_S if and only if $(E, \mathrm{Th}(W \cup T)$ is a fixed point of Υ.

Proof . Straightforward

Proof of the theorem. Using the lemma above, we instantiate theorem 13: $f(d) = (\mathrm{Con}(d), \mathrm{Jus}(d) \wedge \mathrm{Con}(d))$, $\mathbf{Q}_S((E, T), d) \equiv T \cup W \cup \{\mathrm{Con}(d)\} \not\vdash \neg\mathrm{Jus}(d)$, and $\sigma(U) = (\mathrm{Th}(W \cup \mathrm{Con}(U)), \{\mathrm{Jus} d \wedge \mathrm{Con}(d), d \in U\})$. Thus the constrained extensions are generated by the grounded sets of defaults U such that: for all $d \in D$, $d \in U$ if and only if (d is U-active and $\mathrm{Jus}(U \cup \{d\}) \cup \mathrm{Con}(U \cup \{d\}) \cup W$ is consistent), i.e. if and only if (d is U-active and $U \cup \{d\}$ is strongly regular).

Proposition 25. *Let $\Delta = (W, D)$ be a default theory.*

1) $\forall \alpha, \beta$, if $\alpha < \beta$ then $\mathrm{TC}^\alpha \subseteq \mathrm{TC}^\beta$; moreover if $\alpha < \beta$ and $\mathrm{APP}(\mathrm{TC}^\alpha) \neq \emptyset$ then $\mathrm{TC}^\alpha \subset \mathrm{TC}^\beta$.

2) There exists γ such that $\mathrm{APP}(\mathrm{TC}^\gamma) = \emptyset$.

3) TC has at least one limit. Moreover any limit is a saturated grounded set of defaults.

Proof . **1)** It is clear that $U \subseteq \mathrm{TC}(U)$, and that $U \subset \mathrm{TC}(U)$ if $\mathrm{APP}(\mathrm{TC}) \neq \emptyset$. The proof succeeds by transfinite induction on β. Two cases are to be considered:

If β is a successor ordinal, then $\alpha \leq \beta - 1$. If $\alpha = \beta - 1$, then $\mathrm{TC}^\alpha \subseteq \mathrm{TC}^\beta$. Moreover if $\mathrm{APP}(\mathrm{TC}^\alpha)$ is not empty, $\mathrm{TC}^\beta = \mathrm{TC}^\alpha \cup \{d\}$, for some $d \in \mathrm{APP}(\mathrm{TC}^\alpha)$. Thus $\mathrm{TC}^\alpha \subset \mathrm{TC}^\beta$. Otherwise, $\alpha < \beta - 1$, and by induction hypothesis, $\mathrm{TC}^\alpha \subseteq \mathrm{TC}^{\beta-1}$. Since $\mathrm{TC}^{\beta-1} \subseteq \mathrm{TC}(\mathrm{TC}^{\beta-1}) = \mathrm{TC}(\beta)$, we get $\mathrm{TC}^\alpha \subseteq \mathrm{TC}^\beta$. Now, if $\mathrm{APP}(\mathrm{TC}^\alpha)$ is not empty, then $\mathrm{TC}^\alpha \subset \mathrm{TC}^{\beta-1}$, by induction hypothesis, and $\mathrm{TC}^\alpha \subset \mathrm{TC}^\beta$.

If β is a limit ordinal , then $\mathrm{TC}^\beta = \bigcup_{\gamma<\beta} \mathrm{TC}^\gamma$. Therefore if $\alpha < \beta$ then clearly $\mathrm{TC}^\alpha \subseteq \mathrm{TC}^\beta$. On the one hand, if $\alpha < \beta$ then $\alpha + 1 < \beta$, and $\mathrm{TC}^{\alpha+1} \subseteq \mathrm{TC}^\beta$. On the other hand, if $\mathrm{APP}(\mathrm{TC}^\alpha)$ is not empty, then $\mathrm{TC}^\alpha \subset \mathrm{TC}^{\alpha+1}$, and thus $\mathrm{TC}^\alpha \subset \mathrm{TC}^\beta$.

2) Let α be the least ordinal, the cardinal of which is strictly greater than the cardinal of D, and assume that $\forall \gamma < \alpha$, $\mathrm{APP}(\mathrm{TC}^\gamma)$ is not empty. Let us define a mapping f from the ordinal α into D as follows: $f(\gamma) = \mathrm{TC}^\gamma$. Let γ_1, γ_2 be two ordinals such that $\gamma_1 < \gamma_2 < \alpha$. Since $\mathrm{APP}(\mathrm{TC}^{\gamma_1}) \neq \emptyset$, $\mathrm{TC}^{\gamma_1} \subset \mathrm{TC}^{\gamma_2}$. Therefore the mapping f is injective from α into D, a contradiction with the definition of cardinals. In conclusion there exists γ such that $\mathrm{APP}(\mathrm{TC}^\gamma) = \emptyset$.

3) Let γ be such that $\mathrm{APP}(\mathrm{TC}^\gamma) = \emptyset$ (such a γ exists by the preceding result). Let us show that for each $\beta > \gamma$, $\mathrm{TC}^\beta = \mathrm{TC}^\gamma$ and $\mathrm{APP}(\mathrm{TC}^\beta) = \emptyset$. We proceed by a transfinite induction on β. If β is a successor ordinal, then $\beta - 1 \geq \gamma$. If $\gamma < \beta - 1$, then by induction hypothesis, $\mathrm{TC}^{\beta-1} = \mathrm{TC}^\gamma$ and $\mathrm{APP}(\mathrm{TC}^{\beta-1}) = \emptyset$. Hence, $\mathrm{TC}^\beta = \mathrm{TC}^{\beta-1} = \mathrm{TC}^\gamma$ and $\mathrm{APP}(\mathrm{TC}^\beta) = \emptyset$. If β is a limit ordinal, then $\mathrm{TC}^\beta = \bigcup_{\alpha<\beta} \mathrm{TC}^\alpha$. From the first part of the proposition, it results that $\bigcup_{\alpha\leq\gamma} \mathrm{TC}^\alpha \subseteq \mathrm{TC}^\gamma$, and thus $\mathrm{TC}^\beta = \bigcup_{\gamma\leq\alpha<\beta} \mathrm{TC}^\alpha$. By induction hypothesis, for $\gamma < \alpha < \beta$, $\mathrm{TC}^\alpha = \mathrm{TC}^\gamma$. Consequently, $\mathrm{TC}^\beta = \mathrm{TC}^\gamma$. Moreover, $\mathrm{APP}(\mathrm{TC}^\beta) = \emptyset$. In conclusion, there exists some γ such that TC^γ is a limit. Let γ be an ordinal such that TC^γ is a limit. Assume that TC^γ is not saturated. Then there is some default d that is applicable to it and that is not in TC^γ. Therefore, $\mathrm{APP}(\mathrm{TC}^\gamma) \neq \emptyset$, and $\mathrm{TC}^\gamma \subset \mathrm{TC}^{\gamma+1}$, so that TC^γ cannot be a limit. In conclusion, any limit is saturated.

Theorem 26. *Let $\Delta = (W, D)$ be a default theory. Assume that* TC *is a choice operator that selects defaults under a cautious applicability condition and that extensions of Δ are defined using this same notion of applicability (whatever the notion of regularity may be). Then:*

i) Any limit of TC *is an extension of Δ.*

ii) Δ has at least one extension.

iii) Any extension of Δ can be generated by a limit of TC.

We first establish the following lemma:

Lemma 27. *Let $(U_i)_{0\leq i}$ be an increasing sequence of weakly (respectively strongly) regular grounded sets of defaults. Then $\bigcup_{0\leq i} U_i$ is a weakly (respectively strongly) regular grounded set of defaults.*

Proof . We have already seen that any union of grounded sets of defaults is a grounded set of defaults (cf. proposition 3).

a) We assume first that $(U_i)_{0\leq i}$ is an increasing sequence of weakly regular grounded sets of defaults, and show that $\bigcup_{0\leq i} U_i$ is a weakly regular grounded set of defaults. Otherwise, there exists some default d in $\bigcup_{0\leq i} U_i$, such that $W \cup \mathrm{Con}(\bigcup_{0\leq i} U_i) \vdash \neg\mathrm{Jus}(d)$. Since $\neg\mathrm{Jus}(d)$ admits a finite proof and that $(U_i)_{0\leq i}$ is an increasing sequence, there is some i_0 such that $W \cup \mathrm{Con}(U_{i_0}) \vdash \neg\mathrm{Jus}(d)$. Let j_0 be such that $d \in U_{j_0}$, and let $j = \max(i_0, j_0)$. Then, $W \cup \mathrm{Con}(U_j) \vdash \neg\mathrm{Jus}(d)$, with $d \in U_j$, a contradiction with the fact that U_j is weakly regular.

b) The proof is analoguous for strongly regular grounded sets of defaults.

Proof of the theorem. **i)** Let TC^γ be a limit of TC. We have proved that TC^γ is saturated. We show now by induction on γ that TC^γ is regular. If γ is a successor ordinal, then $\mathrm{TC}^\gamma = \mathrm{TC}(\mathrm{TC}^{\gamma-1})$, where $\mathrm{TC}^{\gamma-1}$ is a regular grounded set of defaults (induction hypothesis). As TC selects some default d under cautious applicability to $\mathrm{TC}^{\gamma-1}$, TC^γ is still regular. If γ is a limit ordinal, then $\mathrm{TC}^\gamma = \bigcup_{\alpha<\gamma} \mathrm{TC}^\alpha$. By induction hypothesis, every TC^α is a regular grounded set of defaults, and the sequence $(\mathrm{TC}^\alpha)_{\alpha<\gamma}$ is increasing. From the previous lemma we conclude that the union is a regular grounded set of defaults.

ii) As TC has at least one limit, it is clear that Δ has at least one extension.

iii) Let E be an extension of Δ generated by some cautiously saturated and regular grounded set of defaults U. Let us show that U can be obtained as a limit of TC. Since U is a grounded set of defaults, $U = \bigcup_{0\leq i} U_i$. Let $(d_1, \ldots, d_n, d_{n+1}, \ldots)$ be an enumeration of U such that for any d_p and d_q, $p < q$ if and only if there are i and j with $d_p \in U_i \setminus U_{i-1}$, $d_q \in U_j \setminus U_{j-1}$, and $i \leq j$. Let us consider the strategy ρ under which the default d_i is selected at step i: on the one hand, since U is regular, every subset of U is also regular; on the other hand, since the enumeration of U is compatible with the sequence $(U_i)_{0\leq i}$, any default d_n is active with respect to $\{d_1, \dots, d_{n-1}\}$. Therefore any default d_n is cautiously applicable to $\{d_1, \dots, d_{n-1}\}$. It is then straightforward to see that $U = \mathrm{TC}_\rho^\omega$, where ω denotes the first limit ordinal. As U is cautiously saturated, TC_ρ^ω is clearly a limit of TC.

Theorem 28. *Let Δ be a default theory, where extensions are generated by regular and cautiously saturated grounded sets of defaults. A formula f has a default proof with respect to Δ if and only if there is some extension of Δ that contains f.*

Proof . It is enough to notice that any regular grounded set of defaults is included into a regular and cautiously saturated grounded set of defaults. The set $\{d_1, \dots, d_n\}$ in the definition is grounded and regular, hence the result.

Proposition 29. *Let $\Delta = (W, D)$ be a default theory and let U be a grounded set of defaults of Δ. Let Φ be the family of all models of $W \cup \mathrm{Con}(U)$, then:*

i) $\forall d \in D$, $d \in \mathrm{w} - \mathrm{app}(\Phi, U)$ *iff d is U-active and $U \cup \{d\}$ is weakly regular;*

ii) $\forall d \in D$, $d \in \mathrm{s} - \mathrm{app}(\Phi, U)$ *iff d is U-active and $U \cup \{d\}$ is strongly regular.*

Proof . If Φ is the family of all models of $W \cup \mathrm{Con}(U)$, then saying that $\forall \phi \in \Phi, \phi \models \mathrm{Pre}(d)$ is equivalent to saying that d is active with respect to U, by completeness of propositional calculus. Moreover, under the same hypothesis, saying that $\exists \phi \in \Phi, \phi \models \mathrm{Con}(d) \cup \mathrm{Jus}(d')$ is equivalent to saying that $\mathrm{Con}(U \cup \{d\}) \cup \mathrm{Jus}(d') \cup W$ is consistent, while saying that $\exists \phi \in \Phi, \phi \models \mathrm{Con}(d) \cup \mathrm{Jus}(U \cup \{d\})$ is equivalent to saying that $\mathrm{Con}(U \cup \{d\}) \cup \mathrm{Jus}(U \cup \{d\}) \cup W$ is consistent. The results follow from the definitions of weak and strong regularity.

Theorem 30. *Let $\Delta = (W, D)$ be a default theory. Let Ψ be the family of all models of W. The family of interpretations Φ is the family of all models of some extension for Δ generated by a regular and cautiously saturated grounded set of defaults U if and only if there is some strategy ρ such that $(\Phi, U) = \mathrm{CC}_\rho^\omega(\Psi, \emptyset)$ and is a limit.*

Proof . Let $(\Phi_n, U_n) = \mathrm{CC}_\rho^n(\Psi, \emptyset)$. By a straightforward induction on n, we can show that Φ_n is the class of all models of $W \cup \mathrm{Con}(U_n)$. It is easy to see that $(U_n)_{n\geq 0}$ is an increasing sequence while $(\Phi_n)_{n\geq 0}$ is a decreasing one. Therefore, $\mathrm{CC}_\rho^\omega(\Psi, \emptyset) = (\cap_{n\geq 0}\Phi_n, \bigcup_{n\geq 0} U_n)$ and $\cap_{n\geq 0}\Phi_n$ is the class of all models of $W \cup \mathrm{Con}(\bigcup_{n\geq 0} U_n)$. Moreover, from lemma 27, since for each n, U_n is a regular grounded set of defaults, $U = \bigcup_{n\geq 0} U_n$ is a regular grounded set of defaults.

$\Rightarrow$ Assume first that there is some grounded set of defaults U generating an extension for Δ: U is a regular and cautiously saturated grounded set of defaults. Let $(d_1, \dots, d_n, \dots)$ be some enumeration of U and let ρ be the strategy that selects d_n at step n, such that $W \cup \mathrm{Con}(\{d_1, \dots, d_i\}) \vdash \mathrm{Pre}(d_{i+1})$ for $i \geq 0$. Every default d_n of U is applicable for CC. Clearly, $U = \bigcup_{n\geq 0} U_n$ and $\cap_{n\geq 0}\Phi_n$ is the class of all models of $W \cup \mathrm{Con}(U)$, so that $\Phi = \cap_{n\geq 0}\Phi_n$. Moreover, $\mathrm{CC}_\rho^\omega(\Psi, \emptyset)$ is a limit for CC, since U is cautiously saturated, and $\mathrm{app}(\Phi, U) = \mathrm{APP}(U) = \emptyset$, by proposition 29.

$\Leftarrow$ Now we assume that there is some strategy ρ such that $(\Phi, U) = \mathrm{CC}_\rho^\omega(\Psi, \emptyset)$ is a limit. Since $\cap_{n\geq 0}\Phi_n$ is the class of all models of $W \cup \mathrm{Con}(\bigcup_{n\geq 0} U_n)$, Φ is the class of all models of $W \cup \mathrm{Con}(U)$. It remains to prove that U generates an extension. On the one hand, every U_n is regular, so that U is also regular. On the other hand, since $\mathrm{CC}_\rho^\omega(\Psi, \emptyset)$ is a limit, $\mathrm{app}(\Phi, U) = \mathrm{APP}(U) = \emptyset$, so that U is cautiously saturated. U is clearly grounded, therefore, U generates an extension.