Mechanizing the Odd Order Theorem: Local Analysis

In honour of Gérard Berry and Jean-Jacques Lévy

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February 8, 2011

Finite group theory

Groups are algebraic structures closed under an associative, invertible law.

Finite group theory

- Elements of a group combine thanks to the group law:

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- And most of the theory is developed forgetting about the points.

Decomposition

Theorem (Existence of prime decomposition)

A number always admits a decomposition into a product of prime numbers.

Theorem (Existence of composition series)

For any finite group G, there exists a sequence of subgroups:

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G = G_n$$

such that for all k, G_{k+1}/G_k is simple.

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Such a sequence is called a composition series for G.
Such a quotient G_{k+1}/G_k is called a factor.

Uniqueness

Theorem (Prime decomposition uniqueness)

The decomposition of any number into a product of primes is unique up to permutations.

Theorem (Jordan-Hölder uniqueness)

For any group G, two composition series for G have the same length, and the same factors up to isomorphism and permutation.

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- Base case: $\sharp G = 0$.
  - trivial (a group has at least one element: the neutral)
- Inductive case: $\sharp G > 0$:
  - If G has an empty series: then it is trivial and all its series are empty.
  - If G is simple: then all its series are trivial.
  - Else let (N_i) and (M_j) be two (non empty) composition series of G.

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$$G \triangleright N_1 \triangleright N_2 \triangleright \cdots \triangleright N_r = 1$$

$$I = N_1 \cap M_1$$

$$G \triangleright M_1 \triangleright M_2 \triangleright \cdots \triangleright M_s = 1$$

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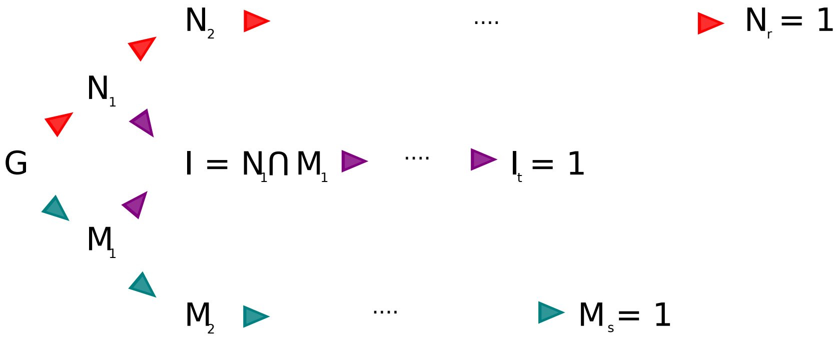

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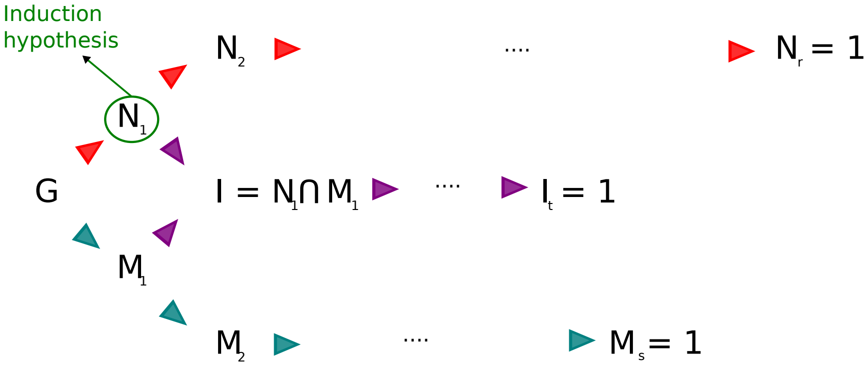

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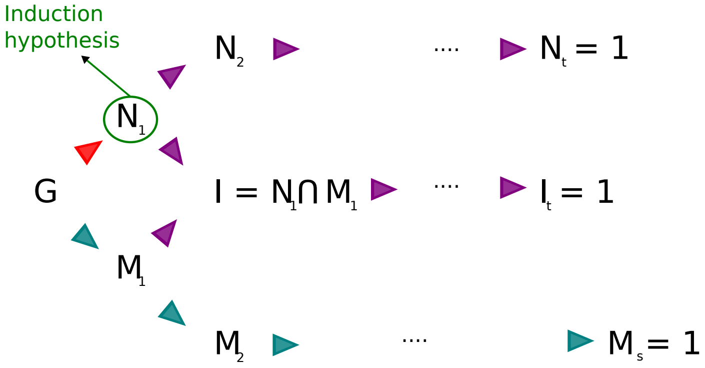

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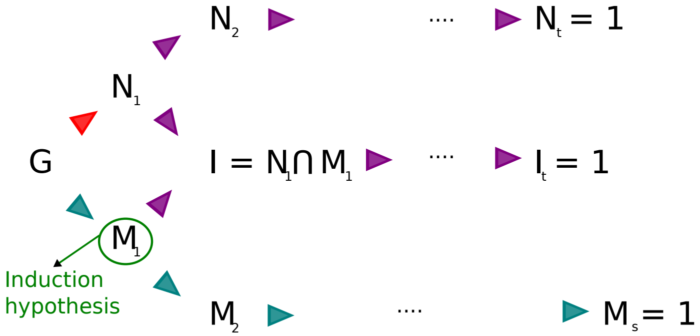

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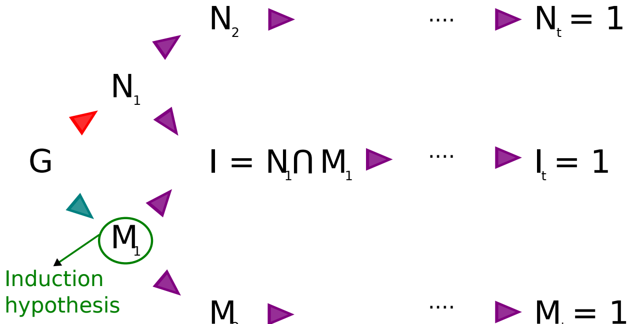

Jordan Hölder Theorem

We have obtained:

$$
\begin{tikzcd}
 & & N_2 \arrow[r, phantom, "\triangleright"] & \cdots \arrow[r, phantom, "\triangleright"] & N_t = 1 \\
 & N_1 \arrow[ur, phantom, "\triangleright" sloped] \arrow[dr, phantom, "\triangleright" sloped] & & & \\
G \arrow[ur, phantom, "\triangleright" sloped] \arrow[dr, phantom, "\triangleright" sloped] & & I = N_1 \cap M_1 \arrow[r, phantom, "\triangleright"] & \cdots \arrow[r, phantom, "\triangleright"] & I_t = 1 \\
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$$M_1 \triangleright I$$

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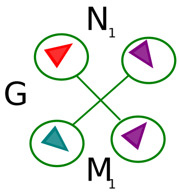
Jordan Hölder Theorem

We conclude by a "butterfly lemma":



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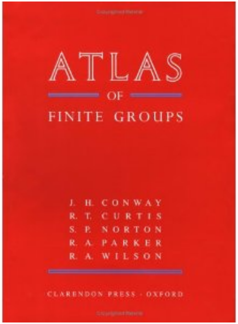
- Two numbers with the same multiset of prime factors are equal.
- Two groups with the same (up to isomorphism) multiset of prime factors are not necessarily equal (not even isomorphic).

$\Rightarrow$ Classifying finite groups is much more difficult than "classifying" numbers.

The Atlas of Finite Groups

The classification of all simple finite groups
aka. The Enormous Theorem



has been considered achieved in 1983 and revised in 2005.

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- (Wikipedia 01/02/2011) "It takes a professional group theorist about a year of hard work to understand the proof completely."
- The proof mixes many theories, non only combinatorics but also linear algebra, Galois theory, characters,...

Formalization issues: sets vs. types

Not all distinct sets should be modelled as distinct types

- A type fixes common requirements for its inhabitants.
- Type inhabitants are objects we want to observe and combine.

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- Operations on groups remain homogeneous.
- Only change type when a new group law is really needed

quotient groups...

Formalisation issues: notations

Let $M \in M_n(F)$,

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In a proof assistant, one would like to write:

```
Definition determinant n (A : 'M[R]_n) : R :=
  \sum_(s : 'S_n) (-1) ^+ s * \prod_i A i (s i).
```

Formalisation issues: notations

$$\sum \quad \prod \quad \bigcup \quad \bigcap \quad \oplus \ldots$$

This requires:

- Concise and uniform notations: $\displaystyle\sum_{i=0}^{n} \quad \bigcup_{A \notin E} \quad \bigcap_{A|P(A)} \ldots$
- Generic toolkit: $\displaystyle\bigcap_{A \in E} = \bigcap_{A \in E \cap B} \cap \bigcap_{A \in E \cap B^c} \ldots$
- Implicit properties of associated operators:

$$\prod_{i=1}^{n} \sum_{j=1}^{m} = \sum \prod$$

This is possible using:

- A generic operator to program these expressions;
- Higher order type inference (à la type classes);
- Coq notation mechanism.

Formalization issues : a page in finite group theory

Finite groups, vol 2. VIII.5.9 - Huppert Blackburn (excerpt):

Since $\mathfrak{A}/\mathfrak{U} = (\mathfrak{K}/\mathfrak{U})(\mathfrak{B}/\mathfrak{U})$, it follows that:

$$\mathfrak{A} = \mathfrak{K}\mathfrak{B} = \mathfrak{D}\mathfrak{U}\mathfrak{B} = \mathfrak{D}\mathfrak{B}$$

Also $\mathfrak{K} \cap \mathfrak{B} = \mathfrak{U}$ and $\mathfrak{U}\mho^p(\mathfrak{K}) \cap \mathfrak{D} = \mho^p(\mathfrak{K})$, so

$$\mathfrak{D} \cap \mathfrak{B} = \mathfrak{D} \cap \mathfrak{K} \cap \mathfrak{B} = \mathfrak{D} \cap \mathfrak{U} = \mathfrak{D} \cap \mathfrak{U}\mho^p(\mathfrak{K}) \cap \mathfrak{U} = \mho^p(\mathfrak{K}) \cap \mathfrak{U} = 1$$

Thus $\mathfrak{A} = \mathfrak{D} \times \mathfrak{B}$. Then $\mathfrak{K} = \mathfrak{D} \times (\mathfrak{B} \times \mathfrak{K}) = \mathfrak{D} \times \mathfrak{U}$, so $\mathfrak{D} \simeq \mathfrak{K}/\mathfrak{U}$ is homocylic and $\mathfrak{D}/\Phi(\mathfrak{D})$ is $\mathfrak{X}$-irreducible. Also $\mathfrak{D} \neq 1$ so the inductive hypothesis may be applied to $\mathfrak{B}$ and the theorem follows at once.

Formalization issues : a page in finite group theory

Fortunately a computer scientist is used to α-conversion:

Since $A/U = (K/U)(B/U)$, it follows that:

$$A = KB = DUB = DB$$

Also $K \cap B = U$ and $U\mho^p(K) \cap D = \mho^p(K)$, so

$$D \cap B = D \cap K \cap B = D \cap U = D \cap U\mho^p(K) \cap U = \mho^p(K) \cap U = 1$$

Thus $A = D \times B$. Then $K = D \times (B \times K) = D \times U$, so $D \simeq K/U$ is homocylic and $D/\Phi(D)$ is X-irreducible. Also $D \neq 1$ so the inductive hypothesis may be applied to B and the theorem follows at once.

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Yet as usual, the devil is in the details:

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