On the enumeration of P-oligomorphic groups

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Abstract

We describe an algorithm to enumerate (closed) P-oligomorphic permutation groups per profile growth, up to kernel, and we announce its implementation as an ongoing work. This work is based on the recent classification of P-oligomorphic groups from [1].

Keywords Infinite permutation groups · profile · P-oligomorphic groups · blocks · computer algebra · enumeration

1 Introduction

Given a permutation group G of a set E, the *profile* of G is the sequence that counts, for every nonnegative integer n, the G-orbits of degree n: that is, the orbits of the induced action of G on the subsets of size n of E. In the seventies, Cameron initiated the study of infinite permutation groups of countably infinite sets whose profile took only finite values, calling them *oligomorphic groups*[2].

When, in addition, the profile is bounded by a polynomial, the group may be called P*-oligomorphic*. In that case, as once conjectured by Cameron, the profile has been recently shown by the author and Thiéry to be asymptotically equivalent to a polynomial [3], and the *profile growth* refers to the degree of this polynomial.

Along with the resolution of the conjecture, these groups have been classified [1]: basically, a P-oligomorphic group is uniquely and entirely described by a finite permutation group endowed with a block system, each block of which is decorated by a pair of groups — one finite, the other infinite — satisfying some explicit conditions.

This extended abstract presents an algorithm, based on this classification, that allows to enumerate all closed P-oligomorphic groups per profile growth, up to kernel (for else there would be infinitely many groups for each growth), the closure notion refering to the simple convergence topology. It is being implemented using the software SageMath [4] and features from GAP-system [5]. The obtained counting sequence will hopefully be presented at the ALGOS2020 conference.

This paper is dedicated to Maurice Pouzet on the occasion of his 75th birthday. Oligomorphic groups are a particular case of relational structures, which are one of his domains of predilection, and he is at the origin of my work on oligomorphic groups.

2 Enumeration of (closed) P-oligomorphic groups

2.1 Generation of finite permutation groups

As it is the first brick in the classification, the first step is to generate all finite permutation groups F, up to permutation group isomorphism. They are counted, per degree, by the following sequence (A000638):

$$1, 1, 2, 4, 11, 19, 56, 96, 296, 554, 1593, 3094, 10723, \ldots$$

We implemented this using the GAP Data Library "Transitive Groups", thanks to the property that any intransitive group is a *subdirect product* of transitive groups.

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The method used is essentially that described in [6]. Roughly speaking, for each domain size N, one needs to enumerate all partitions of N, which represent the possible sets of orbits; then, for each one of them, choose a transitive group F_i on each orbit, and finally compute the subdirect products of these groups in order to generate all permutation groups with these orbits. Some care can be taken at different stages of the generation in order to limit the production of isomorphic groups, yet the author could not spare a final conjugacy test. As the references consulted about this did not include any code, we created a repository to share our implementation with SageMath. The enumeration of P-oligomorphic groups should be available there as well by the time of the conference.

2.2 Enumeration of P-oligomorphic groups using their classification

2.2.1 Block systems of intransitive groups

Once the finite permutation groups have been generated, there remains to enumerate their *block systems*: generalizing the usual definition to intransitive groups, a block system is a set partition of the domain that is globally stable under the action of the group.

As for the previous one, this step needs some implementation work. Indeed, the implementation of block systems in GAP (and thereby SageMath) requires the group to be transitive, and therefore requires to be extended. This involves considering the automorphism group of the finite permutation group F that is being processed, more precisely its action on the orbit restrictions F_i.

There are some technical issues to handle along the way: for instance, the fact that trivial block systems are not included in the output of `AllBlocks`, or that fixed points are removed when GAP computes an action. In addition, one must manually consider the systems involving blocks that are union of orbits — all that up to isomorphism.

2.2.2 The decorations and profile growth

The final layer, in the enumeration as in the classification, is the choice of *decorations* for each one of the orbits of blocks of the finite permutation group F: on the one hand, a normal subgroup H_i of the induced action of F on one of the blocks of this orbit; on the other hand, a (closed) *highly homogeneous group*. There are five of this latter kind of groups, but only one (the infinite symmetric group) is possible if the blocks are not singletons.

This steps corresponds to choosing the behaviour of the final P-oligomorphic group inside each one of its *superblocks*. It must again be carried out up to isomorphism.

When a P-oligomorphic group is finally obtained this way, one determines its profile growth using the profiles of the H_i's (the profile has been previously implanted by the author as a method for finite permutation groups). Indeed, it corresponds to the total number of orbits of subsets of the H_i's, minus one. This ensures that it is (tightly) bounded above by the degree of F minus one, so when asking for the number of P-oligomorphic groups up to growth r, one needs to consider finite groups F up to degree $r + 1$.

Note that the whole enumeration is much easier when counting only transitive P-oligomorphic groups. An implementation can be found in the same repository, and hands the sequence:

$$1, 5, 6, 14, 33, 32, 114, 47, 323, 260, 338, 50, 2108, 58, 430, 940, 12470, 60, 7361, 64, 12136, \ldots$$

which is unknown by OEIS.

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