

Classification of P -oligomorphic groups, conjectures of Cameron and Macpherson

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Abstract. Let G be a group of permutations of a denumerable set E . The **profile** of G is the function φ_G which counts, for each n , the (possibly infinite) number $\varphi_G(n)$ of orbits of G acting on the n -subsets of E . Counting functions arising this way, and their associated generating series, form a rich yet apparently strongly constrained class. In particular, Cameron conjectured in the late seventies that, whenever the profile $\varphi_G(n)$ is bounded by a polynomial – we say that G is P -oligomorphic –, it is asymptotically equivalent to a polynomial. In 1985, Macpherson further asked whether the **orbit algebra** of G – a graded commutative algebra invented by Cameron and whose Hilbert function is φ_G – is finitely generated.

In this paper we establish a classification of (closed) P -oligomorphic permutation groups in terms of *finite permutation groups with decorated blocks*.

It follows from the classification that the orbit algebra of any P -oligomorphic group is isomorphic to (a straightforward quotient of) the invariant ring of some finite permutation group. This answers positively both Cameron's conjecture and Macpherson's question. The orbit algebra is in fact Cohen-Macaulay; therefore the generating series of the profile is a rational fraction whose numerator has positive coefficients, while the denominator admits a combinatorial description.

In addition, the classification provides a finite data structure for encoding closed P -oligomorphic groups, thereby enabling to compute with them and enumerate them. It also paves the way for proofs by structural induction. Finally, its relative simplicity gives hopes to extend the study to, e.g., the class of (closed) permutation groups with sub-exponential profile.

The proof exploits classical notions from group theory – notably block systems of permutation groups and their lattice properties –, commutative algebra, and invariant theory.

Keywords: Infinite permutation groups, orbit counting, profiles, generating series, orbit algebras, invariant theory, \aleph_0 -categorical complete theories, homogeneous relational structures

1 Introduction

1.1 A conjecture of Cameron and a question of Macpherson

Counting objects under a group action is a classical endeavor in algebraic combinatorics. If G is a permutation group acting on a finite set E , Burnside's lemma provides a formula for the number of orbits, while Pólya theory refines this formula to compute, for example, the **profile** of G , that is the function which counts, for each n , the number $\varphi_G(n)$ of orbits of G acting on subsets of size n of E .

In the seventies, Cameron initiated the study of the profile when G is instead a permutation group of an infinite set E . Of course the question makes sense mostly if $\varphi_G(n)$ is finite for all n ; in that case, the group is called **oligomorphic**, and the infinite sequence $\varphi_G = (\varphi_G(n))_n$ an **orbital profile**.

This setting includes, for example, counting integer partitions (with optional length and height restrictions) or graphs up to an isomorphism. It also relates naturally to logic: indeed Ryll-Nardzewski's Theorem [17] implies that orbital profiles (of oligomorphic groups) are exactly the profiles of \aleph_0 -categorical theories, that is *complete theories admitting a unique countable model* (see [3, Section 2.5] for further details).

The study of oligomorphic permutation groups has become a whole research subject; see [3, 4] for surveys. One central topic is the description of general properties of orbital profiles. It was soon observed that the potential growth rates exhibited jumps. For example, the profile either grows at least as fast as the partition function, or is bounded by a polynomial [11, Theorem 1.2]. In the latter case – we say that G is **P -oligomorphic** – it was conjectured to be asymptotically polynomial:

Conjecture 1.1 (Cameron [3]). *Let G be a P -oligomorphic permutation group. Then $\varphi_G(n) \sim an^k$ for some $a > 0$ and $k \in \mathbb{N}$.*

As a tool in this study, Cameron introduced early on the **orbit algebra** $\mathcal{QA}(G)$ of G , a graded connected commutative algebra whose Hilbert function coincides with φ_G . As proved in [14], it is always an integral domain (assuming empty kernel), implying that the profile is non decreasing.

Macpherson asked the following:

Question 1.2 (Macpherson [11] p. 286). *Let G be a P -oligomorphic permutation group. Is $\mathcal{QA}(G)$ finitely generated?*

The point is that, by standard commutative algebra, whenever $\mathcal{QA}(G)$ is finitely generated, its Hilbert function is asymptotically polynomial, as conjectured by Cameron. It is in fact *eventually a quasi polynomial* (with a constant leading coefficient). Equivalently, the generating series of the profile $\mathcal{H}_G = \sum_{n \in \mathbb{N}} \varphi_G(n)z^n$ is a rational fraction of the form

$$\mathcal{H}_G = \frac{P(z)}{\prod_{i \in I} (1 - z^{d_i})},$$

where $P(z)$ is a polynomial in $\mathbb{Z}[z]$ and the d_i 's are the degrees of the generators.

1.2 Main results

In the extended abstract [7], we had announced positive answers to both Cameron's conjecture and Macpherson's question with sketches of proof. In this paper we deliver a much stronger result with full proofs, namely the extension of the classification of the five oligomorphic groups with constant profile 1 [3] to a classification of all closed P -oligomorphic groups (Theorem 5.22): any such group G is uniquely classified by a finite piece of data Δ , encoded as a finite permutation group endowed with a block system whose blocks are suitably decorated. This data can be extracted from G by a detailed analysis of its lattice of *nested block systems* (block systems of block systems); conversely, the group can be reconstructed from the data using wreath products of finite permutation groups with one of the five (closed) highly homogeneous groups, direct products, and diagonal actions of finite permutation groups.

We then derive from the classification that the orbit algebra of any P -oligomorphic group G (closed or not¹) is isomorphic to (a straightforward quotient of) the invariant ring of some finite permutation group. By standard invariant theory, the latter is finitely generated and even Cohen-Macaulay.

It follows that the generating series of the profile is of the form

$$\mathcal{H}_G = \frac{P(z)}{\prod_{i \in I} (1 - z^{d_i})}, \quad (1.1)$$

where the fraction is irreducible and P is in $\mathbb{N}[z]$; therefore we have $\varphi_G(n) \sim an^{|I|-1}$ for some $a > 0$.

1.3 Connections with profiles and age algebras of relational structures

This research is part of a larger program initiated in the seventies: the study of the *profile of relational structures* [8, 13] and in general of the behavior of counting functions for hereditary classes of finite structures, like undirected graphs, posets, tournaments, ordered graphs, or permutations; see [10, 2] for surveys.

In this program, *jumps* in the set of potential growth rates of profiles are a ubiquitous phenomenon: not all growths are encountered. For instance, undirected graphs [1] and permutation classes [9] both exhibit such jumps: the counting functions, when bounded by a polynomial, are actually asymptotically equivalent to a polynomial. Besides, by [15, Theorem 1.7], the class of relational structures with *finite monomorphic dimension* – rough analogues of transitive groups with a finite number of infinite blocks – also exhibits

¹The profile and orbit algebra of an oligomorphic group and those of its closure coincide.

these jumps. Thanks to our proof of Cameron’s conjecture, the same is known to hold for homogenous relational structures, which correspond to oligomorphic permutation groups. This, together with evidence from many examples, suggests that a suitable generalization of the notion of block systems may enable to prove that large classes of relational structures exhibit such jumps in the possible profile growths.

In [5], Cameron extends the definition of orbit algebra to the general context of relational structures. The Cohen-Macaulay property holds when the profile is bounded (see [13, Theorem 26] and [15, Theorem 1.5]); it can fail as soon as the profile grows faster. Similarly, finite generation often fails; however, when the monomorphic dimension is finite, there exists a combinatorial characterization of when it holds [16].

1.4 Structure of the paper

This paper is structured as follows. In Section 2, we review the basic definitions of profiles and orbit algebras, and provide classical examples and operations. We then recall some group theory tools that we will be using a lot, such as the central notion of block systems and the subdirect product of groups, that will later help us handle interactions between the blocks of a given system.

In Section 3, we see that block systems provide lower bounds on the growth of the profile. This motivates the study of block systems and their lattice properties, which leads to the identification of a canonical system of blocks of blocks.

The classification of closed P -oligomorphic groups with a single block of blocks is established in Section 4. It was informed by an extensive computer exploration where P -oligomorphic groups were approximated by finite groups. The proof exploits towers of groups and subdirect products to suitably control synchronizations, and encapsulates the most technical aspects of this paper.

Section 5 builds on the previous one to extend the classification to all P -oligomorphic groups, after establishing the existence of a minimal finite index subgroup, which, together with a finite group acting diagonally, generates G .

Finally, in Section 6, we deduce from the classification that the orbit algebra is (a trivial quotient of) an invariant ring, providing positive answers to Cameron’s conjecture and Macpherson’s question, and further deducing that it is Cohen-Macaulay.

1.5 Perspectives

A notable outcome of the classification is that it provides a uniform finite description of any P -oligomorphic group; otherwise said a data structure. This prepares the ground for algorithms and computations with P -oligomorphic groups: constructing P -oligomorphic groups from their classification, combining them together (cartesian or wreath products), computing their properties such as the generating series of the profile, constructing

the orbit algebras and doing arithmetic with their elements, etc. The classification also paves the way for computations about the class of P -oligomorphic groups: there are indeed countably many of them and we may iterate through them, count them by kernel size and profile growth, etc. These algorithms are being implemented in a library on top of SageMath and are the topic of a follow-up publication; see the first author PhD dissertation [6] in the meantime.

The classification also enables proofs by structural induction: if a property holds for the five closed permutation groups with profile 1 and is stable under wreath products, cartesian products and taking finite index group extensions, then it holds for any closed P -oligomorphic group. It holds more generally for any permutation group obtained by such an inductive construction. This class of oligomorphic permutation groups would be worth studying for its own sake. At this stage, it seems plausible that it contains all oligomorphic permutation groups with subexponential profile growth, and that a classification thereof is in reach. On the other hand there is little hope beyond exponential growth, due to the appearance of new, more complicated primitive groups.

1.6 Acknowledgments

We would like to heartily thank Maurice Pouzet for suggesting to work on this conjecture, and Peter Cameron and Maurice Pouzet for enlightening discussions and benevolent encouragements throughout the project.

2 Preliminaries

2.1 The age, profile and orbit algebra of a permutation group

Let G be a permutation group, that is a group of permutations of some set E . Unless stated otherwise, E is denumerable and G is infinite. The action of G on the elements of E induces an action on the set \mathcal{E}_E of finite subsets of E .

The **age** of G is the set $\mathcal{A}(G)$ of the orbits of finite subsets in \mathcal{E}_E under this action. Within an orbit, all subsets share the same cardinality, which is called the **degree** of the orbit. This gives a grading of the age according to the degree of the orbits: $\mathcal{A}(G) = \sqcup_{n \in \mathbb{N}} \mathcal{A}(G)_n$; we will also use the notation $\mathcal{A}(G)^+ = \sqcup_{n \in \mathbb{N} \setminus \{0\}} \mathcal{A}(G)_n$. The **profile** of G is the function $\varphi_G : n \mapsto |\mathcal{A}(G)_n|$. In general, the profile may take infinite values; the group is called **oligomorphic** if it does not.

We call the **growth rate** of a profile bounded by a polynomial the smallest number r satisfying $\varphi_G(n) = O(n^r)$; for instance, the growth rate of $n^2 + n$ is 2. By extension, the **growth rate of a permutation group** G is that of its profile φ_G .

Definition 2.1. We say that a permutation group is **P -oligomorphic** if its profile is bounded by a polynomial.

Examples 2.2. Let G be the infinite symmetric group \mathfrak{S}_∞ . For each n there is a single orbit containing all subsets of size n , hence $\varphi_G(n) = 1$ for all n . We say that G is *highly homogeneous*, or, more informally, that it *has profile 1*.

Now take $E = E_1 \sqcup E_2$, where E_1 and E_2 are two copies of \mathbb{N} . Let G be the group acting on E by permuting the elements independently within E_1 and E_2 and by exchanging E_1 and E_2 : G is the **wreath product** $\mathfrak{S}_\infty \wr \mathfrak{S}_2$. In that case, the orbits of subsets of cardinality n are in bijection with the integer partitions of n with at most 2 parts. See Figure 1 and Figure 2 for other examples of wreath products.

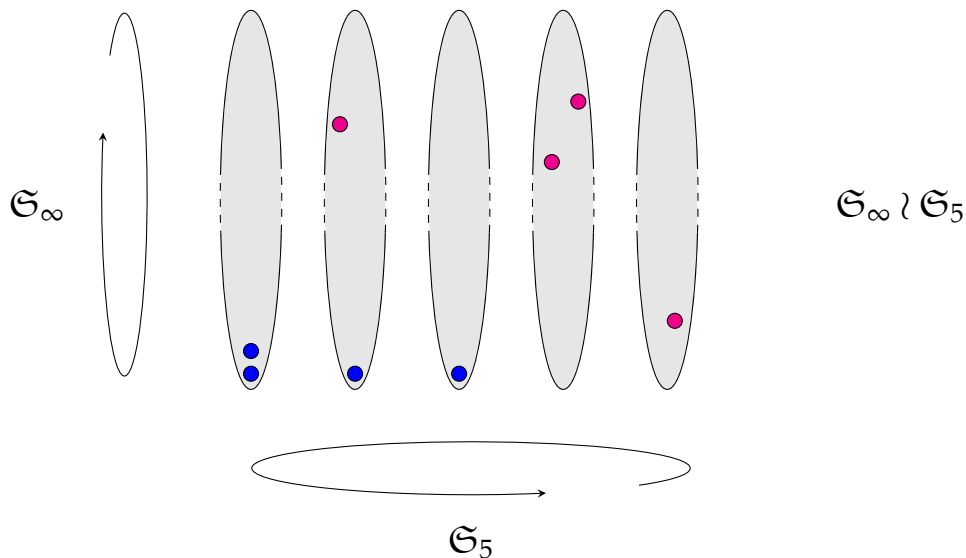


Figure 1: An example of a wreath product: $\mathfrak{S}_\infty \wr \mathfrak{S}_5$; the two highlighted subsets, in red and blue respectively, are in the same orbit.

Cameron's **orbit algebra** of G is the graded connected vector space $\mathcal{QA}(G)$ of formal finite linear combinations of elements of $\mathcal{A}(G)$; it is endowed with a commutative product as follows: let \mathcal{QE}_E be the *set algebra* of E , namely the vector space \mathcal{QE}_E of (possibly infinite) formal linear combinations of finite subsets of E , endowed with the *disjoint product* that maps two finite subsets to their union if they are disjoint and to 0 otherwise. Embed the orbit algebra $\mathcal{QA}(G)$ in the set algebra \mathcal{QE}_E through the linear morphism i_G that maps each orbit to the sum of its elements. The disjoint product stabilizes $i_G(\mathcal{QA}(G))$ and can thus be retracted to $\mathcal{QA}(G)$. Some care needs to be taken at each step to check that everything is well defined; see [3, 4] for details.

Example 2.3. • Assume, as an exception, that G is a finite permutation group. Then its orbit algebra is finitely generated (likely with redundancy) by its age, of which all elements are nilpotent of finite order (bounded by the degree of the finite group G). It is thus of Krull dimension 0.

- Take $G = \mathfrak{S}_m$ for some $m \in \mathbb{N}$. The profile is 1 until $n = m$ and 0 beyond, and the orbit algebra is finitely generated as a vector space: it is isomorphic to $\mathbb{Q}[y]$, where y is nilpotent of order $m + 1$. Take this to its infinite analogue, that is $G = \mathfrak{S}_\infty$; then, $\mathcal{QA}(G)$ is isomorphic to the algebra $\mathbb{Q}[x]$ of univariate polynomials. Indeed, if e_n denotes the unique orbit of all subsets of size n , then the morphism which maps x^n onto $n!e_n$ is a morphism of algebras (use that, the way the orbital product goes, we have $e_k e_\ell = \binom{k+\ell}{k} e_{k+\ell}$).
- Take on the other end the trivial permutation group on E . Each orbit consists of a single subset, and the profile counts for each n the subsets of size n of E . When E is infinite, the group is not at all P -oligomorphic nor even oligomorphic; its orbit algebra is the free algebra generated by the infinitely many singletons in E .

2.2 Basic properties and operations

We recall here a few technical basics about orbit algebras and profiles, in particular dealing with subgroups or restrictions, that one expects indeed to be able to manipulate in a natural way.

Lemma 2.4 (Relations between orbit algebras).

1. Let G be a permutation group acting on E , and F be a stable subset of E . Then, $\mathcal{QA}(G|_F)$ is both a subalgebra and a quotient of $\mathcal{QA}(G)$.
2. Let G be a permutation group acting on E , and H be a subgroup, both of which being oligomorphic. Then, $\mathcal{QA}(G)$ is a subalgebra of $\mathcal{QA}(H)$.

Proof. We exhibit the natural morphisms for each one of these cases.

1. We have the following commutative diagram

$$\begin{array}{ccc}
 \mathcal{QO}_F & \xleftarrow{\subseteq} & \mathcal{QO}_E \\
 i_{G|_F} \uparrow & & \uparrow i_G \\
 \mathcal{QA}(G|_F) & \xrightleftharpoons[\psi]{\phi} & \mathcal{QA}(G) \\
 i_{G|_F} \downarrow & & \downarrow i_G \\
 \mathcal{QO}_F & \xleftarrow{\pi} & \mathcal{QO}_E
 \end{array}$$

where π is the linear morphism mapping a subset of E to itself if it is a subset of F and to 0 otherwise. The injective morphisms i_G and $i_{G|_F}$ are reversible where needed, which allows to define the respectively injective and surjective morphisms ϕ and ψ by composition.

2. In the following diagram, i_G and i_H are the canonical embeddings of the orbit algebras into their set algebras.

$$\begin{array}{ccc} & & \mathcal{QA}(E) \\ & \nearrow i_G & \uparrow i_H \\ \mathcal{QA}(G) & \xrightarrow{\phi} & \mathcal{QA}(H) \end{array}$$

The orbits of G are unions of orbits (of same degree) of H , and since the groups are oligomorphic these unions are finite. The image of i_G is thus a subset of the image of i_H , and therefore the diagram is commutative. \square

Lemma 2.5 (Direct product). *Let G and H be permutation groups acting on E and F respectively. Take $G \times H$ endowed with its natural action on the disjoint union $E \sqcup F$. Then, $\mathcal{A}(G \times H) \simeq \mathcal{A}(G) \times \mathcal{A}(H)$, and $\mathcal{QA}(G \times H) \simeq \mathcal{QA}(G) \otimes \mathcal{QA}(H)$; it follows that $\mathcal{H}_{G \times H} = \mathcal{H}_G \mathcal{H}_H$.*

Lemma 2.6. *Let G be a permutation group and K be a normal subgroup of finite index. Then,*

$$\varphi_G(n) \leq \varphi_K(n) \leq |G : K| \varphi_G(n).$$

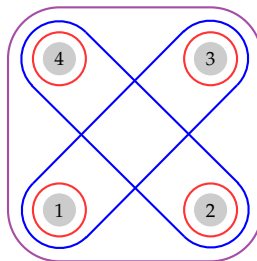
In particular, K and G share the same profile growth.

Proof. Let O be a G -orbit of elements. Since K is a normal subgroup, O splits into K -orbits on which G – and actually G/K – acts transitively by permutation; there are thus finitely many such K -orbits, all of the same size. In particular, infinite G -orbits split into infinite K -orbits, and similarly for finite ones. \square

2.3 Block systems and primitive groups

A key notion when studying permutation groups is that of **block systems**; they are the discrete analogues of quotient modules in representation theory. A **block system** is a partition of E into parts, called **blocks**, such that each $g \in G$ maps blocks onto blocks.

Example 2.7. Following is the list of all block systems of the cyclic permutation group \mathcal{C}_4 : $\{\{1, 2, 3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, $\{\{1\}, \{2\}, \{3\}, \{4\}\}$.



The partitions $\{E\}$ and $\{\{e\} \mid e \in E\}$ are always block systems and are therefore called the trivial block systems. A permutation group is **primitive** if it admits no non trivial block system. By extension, an orbit of elements is **primitive** if the restriction of the group to this orbit is primitive.

The following two theorems will be central in our study.

Theorem 2.8 (Macpherson [12] Theorem 1.1; see also [3] (3.21)). *The profile of an oligomorphic primitive permutation group is either the constant function 1 or bounded below by an exponential.*

In our study of Macpherson's question, all profiles are assumed to be bounded by a polynomial, and therefore primitive groups always have profile 1: in other words, they are highly homogeneous. These groups are classified (up to *closure*; see below).

Theorem 2.9 (Cameron [3] (Section 3.4)). *There are only five closed highly homogeneous groups:*

1. The automorphism group $\text{Aut}(\mathbb{Q})$ of the rational chain (order-preserving bijections on \mathbb{Q});
2. $\text{Rev}(\mathbb{Q})$ (generated by $\text{Aut}(\mathbb{Q})$ and a reflexion);
3. $\text{Aut}(\mathbb{Q}/\mathbb{Z})$, preserving the cyclic order (see \mathbb{Q}/\mathbb{Z} as a circle);
4. $\text{Rev}(\mathbb{Q}/\mathbb{Z})$, generated by $\text{Cyc}(\mathbb{Q}/\mathbb{Z})$ and a reflection;
5. \mathfrak{S}_∞ .

In the vocabulary of model theory, these are the groups preserving, respectively, the dense linear order, the betweenness order, the circular order, the separation relation, and a pure set.

The notion of closure refers here to the topology of simple convergence, described in Section 2.4 of [3]. Thanks to the following classical lemma, it plays only a minor role for our purposes.

Lemma 2.10. *A permutation group and its closure share the same profile and orbit algebra.*

Proof. Let G be a permutation group acting on a set E , and take two finite subsets A and A' that are in the same orbit for its closure: there exists a permutation σ of the closure that maps A to A' . By definition of the closure (for the simple convergence), there exists a sequence $(\sigma_i)_i$ of permutations in G that coincide with σ on more and more elements

of E . Eventually, σ_i will coincide with σ on A , and therefore A and $A' = \sigma_i(A)$ are in the same G -orbit. The age is thus the same for G and its closure, and with it the profile and the orbit algebra. \square

We make the following remark, that will prove crucial later on.

Remark 2.11. The set of five (closed) highly homogeneous permutation groups is stable under taking finite index normal subgroups. We will sometimes refer to the three of them having no proper finite index normal subgroup $\text{Aut}(\mathbb{Q})$, $\text{Aut}(\mathbb{Q}/\mathbb{Z})$ and \mathfrak{S}_∞ by “the three minimal highly homogeneous groups”.

Lemma 2.12. *Let G be a (closed) P -oligomorphic permutation group, endowed with a block system. If an infinite G -orbit of blocks is primitive and the blocks of the orbit are not singletons, then the action on these blocks is isomorphic to \mathfrak{S}_∞ .*

Sketch of proof. Using Theorem 2.9, the action on the set of blocks is given by one of the five closed highly homogeneous groups. Assume first it is $\text{Aut}(\mathbb{Q})$. Since there are at least 2 elements in each block, that makes at least two subsets which are not in the same orbit: simply consider one singleton and one pair of elements. Then every word using only the letters 1 and 2 can be related to at least one orbit of subsets: 1 means you take only one element in a given block and 2 that you take a pair. Since $\text{Aut}(\mathbb{Q})$ permutes the blocks, the exact block you take the subsets from does not matter, only the ordering of the blocks, which is the ordering of the letters. Several orbits of subsets might be a match for a given word, but picking just one will be good enough to us. Since there are 2^m words of length m , we have highlighted this way 2^m orbits of degrees between m and $2m$, each of them contributing by 1 to a value of the profile between m and $2m$. Therefore, we may bound the partial sum of the profile below: $\sum_{n=0}^m \varphi_G(n) \geq \sum_{n=0}^{\lfloor m/2 \rfloor} 2^n$. The second term is clearly exponential, which cannot be with a P -oligomorphic group.

Now, if the action on the blocks was described by another one of the four non symmetric highly homogeneous groups, the words we used would just have to be considered up to a reflexion or a cyclic permutation of the letters, which does not change the exponential growth. \square

2.4 Wreath products

Let G and H be permutation groups acting on E and F respectively. Intuitively, the **wreath product** $G \wr H$ acts on $|F|$ copies $(E_f)_{f \in F}$ of E , by permuting each copy of E independently according to G and permuting the copies according to H . By construction, the partition $(E_f)_{f \in F}$ forms a block system, and $G \wr H$ is not primitive (unless G or H is and F or E , respectively, is of size 1).

Examples 2.13 (Algebras of wreath products).

1. Let G be the wreath product $\mathfrak{S}_\infty \wr \mathfrak{S}_k$. The profile counts integer partitions with at most k parts. The orbit algebra is the algebra of symmetric polynomials over k variables, that is the free commutative algebra with generators of degrees $1, \dots, k$. The generating series of the profile is given by $\mathcal{H}_G = \frac{1}{\prod_{d=1, \dots, k} (1-z^d)}$.
See also Figure 1, on which the red and blue subsets are in the same orbit. The associated integer partition $(3, 1)$ can be read on the blue subset.
2. Let G' be a finite permutation group. Then, the orbit algebra of $G = \mathfrak{S}_\infty \wr G'$ is isomorphic to the **invariant ring** $\mathbb{Q}[X]^{G'}$, which consists of the polynomials in $\mathbb{Q}[X] = \mathbb{Q}[X_1, \dots, X_k]$ that are invariant under the action of G' .
3. Let G' be a finite permutation group. Then, the orbit algebra of $G = G' \wr \mathfrak{S}_\infty$ is the free commutative algebra generated by the the set $\mathcal{A}(G')^+$ of the G' -orbits of non trivial subsets. The generating series of the profile is given by $\mathcal{H}_G = \frac{1}{\prod_d (1-z^d)}$, where d runs through the degrees of $\mathcal{A}(G')^+$, taken with multiplicity.

Sketch of proof. The first item is a special case of the second one, that we examine now. Two subsets having the same number of elements in each infinite block are in the same orbit, so if one canonically embeds the orbit algebra of G into the set algebra of E , it is a subspace of that generated by the infinite sums

$$S_\alpha = \sum_{\text{card}(e \cap B_i) = \alpha_i} e, \quad \alpha = (\alpha_1, \dots, \alpha_k)$$

for each multi-index α of length the number of infinite blocks (that is the degree of G'), B_i being the i -th block and e the subsets of E . Now use the morphism $S_\alpha \mapsto X^\alpha = \prod_i X_i^{\alpha_i}$ to embed $\mathbb{Q}\mathcal{A}(G)$ into the algebra of polynomials. The action of G' on the blocks acts the variables the same way, so the image of $\mathbb{Q}\mathcal{A}(G)$ is the algebra of invariants of G' .

Set $G = G' \wr \mathfrak{S}_\infty$ in order to prove the third item. A canonical one-to-one correspondence can easily be established between G -orbits and multisets of $\mathcal{A}(G')^+$: a finite subset of E consists of a disjoint union of subsets that are included in the blocks, and thus each G -orbit is determined by the non trivial G' -orbits of these subsets, while the order does not matter. Since, the Hilbert series only depends on the structure of graded vector space, it is then $\frac{1}{\prod_d (1-z^d)}$, where the d 's are the orbital degrees of $\mathcal{A}(G')^+$, the set of generators.

Define now an alternative notion of degree δ , the number of blocks involved in (the representatives of) an orbit. In the orbit algebra $\mathbb{Q}\mathcal{A}(G)$, the product of two orbits O_1 and O_2 has one and only one dominant term for δ , followed by lower degree terms (we say that δ is a *filtration*, but we will not go into the details about this notion). It is easy to see that the dominant term is the orbit that corresponds to the multiset $\{O_1, O_2\}$.

Therefore, every G -orbit can be obtained as the dominant term of such a product of G' -orbits in $\mathcal{QA}(G)$, and since δ decreases on the other terms, it can actually be realized as a linear combination of products of G' -orbits (this can be argued by induction on δ). Hence $\mathcal{A}(G')^+$ generates all of $\mathcal{QA}(G)$.

On the other hand, and by homogeneity, the shape of the Hilbert series imply that it is also a free family of elements: one can consider the canonical morphism with the corresponding polynomial algebra (with indeterminates of the degrees of $\mathcal{A}(G')^+$), and deduce by dimension that it is an isomorphism. \square

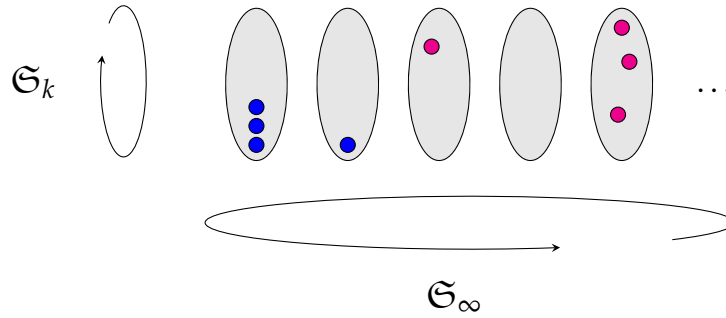


Figure 2: Example of a wreath product: $\mathfrak{S}_k \wr \mathfrak{S}_\infty$, with two subsets in the same orbit

We recall in the sequel some properties of wreath products that will prove helpful later on.

Proposition 2.14. *Let F_1, F_2, P_1 and P_2 be permutation groups such that F_1 (resp. P_1) is a normal subgroup of F_2 (resp. P_2). Then, $F_1 \wr P_1$ is a normal subgroup of $F_2 \wr P_2$, and we have $(F_2 \wr P_2)/(F_1 \wr P_1) \simeq (F_2/F_1) \wr (P_2/P_1)$.*

Sketch of proof. Denote each element of $(F_2 \wr P_2)$ as $g = ((g_i)_i, p)$, with i running through the domain of P_2 , g_i in F_2 for each i , and p in P_2 . Then check that the following defines a natural isomorphism:

$$\begin{cases} (F_2 \wr P_2)/(F_1 \wr P_1) & \longrightarrow (F_2/F_1) \wr (P_2/P_1) \\ g \cdot (F_1 \wr P_1) & \longmapsto ((g_i \cdot F_1)_i, p \cdot P_1) \end{cases} \quad \square$$

Corollary 2.15. *If P_1 and P_2 have finite degree m , we have*

$$[F_1 \wr P_1 : F_2 \wr P_2] = [F_1 : F_2]^m [P_1 : P_2].$$

If the degree of P_1 and P_2 is infinite, $F_1 \wr P_1$ is of infinite index in $F_2 \wr P_2$ as soon as F_1 is a proper subgroup of F_2 .

Proof. This is a direct consequence of Proposition 2.14 and the fact that a wreath product $F \wr P$ is of order $|F|^m |P|$ where m is the degree of the permutation group P . \square

2.5 Subdirect products

The previous two subsections dwelled on particular, well described and well understood kinds of P -oligomorphic groups. In Section 3 we will see that the support E of any P -oligomorphic group G splits up into pieces on each of which G essentially acts as one of those well understood groups. In the later sections, we will recover the desired properties of G (profile, orbit algebra) by controlling how the different pieces interact together using the classical notion of *subdirect product*, which we recall here.

Let us start with the intuition. Assume that E splits into G -stable subsets $(E_i)_i$. The actions of G on each E_i are not independent from each other in general: there may be partial or full synchronization between the actions, which has consequences on the profile and the orbit algebra. For two subsets this can be formalized by describing G as a subdirect product. The general case can then be treated by induction.

Definition 2.16. Let G_1 and G_2 be groups. A **subdirect product** of G_1 and G_2 is a subgroup of $G_1 \times G_2$ which projects onto each factor under the canonical projections.

For instance, assume G is a permutation group that has two orbits of elements E_1 and E_2 . Then G is a subdirect product of the groups G_1 and G_2 induced respectively on E_1 and E_2 . Denote $N_1 = \text{Fix}_G(E_2)$ and $N_2 = \text{Fix}_G(E_1)$, the pointwise stabilizers of E_2 and E_1 , respectively. Then, N_1 and N_2 are normal subgroups of G and their intersection is trivial, so that we have $\langle N_1, N_2 \rangle \simeq N_1 \times N_2$.

Definition 2.17. We call **synchronization** between G_1 and G_2 the following isomorphic quotients (expressed after restriction of N_i when needed):

$$\frac{G_1}{N_1} \simeq \frac{G}{N_1 \times N_2} \simeq \frac{G_2}{N_2}.$$

Intuitively, these quotients describe the parts of each group that are synchronized, whereas the N_i are the independent parts.

Proposition 2.18. Let G be a subdirect product of G_1 and G_2 . With the above notations, we have

$$G \simeq \{(g_1, g_2) \in G_1 \times G_2 \mid g_1 N_1 = g_2 N_2\}.$$

This proposition implies that a permutation group arising as a subdirect product is uniquely characterized by the associated groups G_1 , G_2 , N_1 and N_2 .

In addition, the possible synchronizations between two groups are directly linked to their normal subgroups. This observation combined with the classification of the groups of profile 1 (see Theorem 2.9) and Remark 2.11, lead to the following remark.

Remark 2.19. Consider the action of a group G on two infinite primitive orbits of points or of finite blocks. There is either:

1. no synchronization,
2. total synchronization,
3. a synchronized reflection in the cases of $\text{Rev}(\mathbb{Q})$ and $\text{Rev}(\mathbb{Q}/\mathbb{Z})$ (synchronization of order 2).

On non trivial finite blocks, only the first two situations can occur by Lemma 2.12.

3 The nested block system

Let G be a P -oligomorphic permutation group. In this section, we go back to the notion of block system and take a closer look at how we can exploit it for our purposes. First, we show that each block system provides a lower bound on the growth of the profile. Seeking to maximize this lower bound, we establish the nature of finite lattices of the posets of block systems, and use it to derive a construction of a special “block system” (for an extended version of the notion) satisfying appropriate properties. The later sections will show that this so-called nested block system minimizes synchronization and provides a tight lower bound.

3.1 How block systems provide a lower bound on the profile

We first consider the case where the block system is **transitive**, that is G acts transitively on its blocks. In this case, all the blocks are conjugated and thus share the same cardinality.

Lemma 3.1. *Let G be a P -oligomorphic permutation group, endowed with a transitive block system \mathcal{B} . Then,*

1. *Case 1: \mathcal{B} has finitely many infinite blocks, as in Example 2.13 (1) and (2). Then G is a subgroup of $\mathfrak{S}_\infty \wr \mathfrak{S}_k$ (where k is the number of blocks), and $\text{QA}(G)$ contains Sym_k which is a free algebra with generators of degrees $(1, \dots, k)$.*
2. *Case 2: \mathcal{B} has infinitely many finite blocks, as in Example 2.13 (3). Then, G is a subgroup of $G_{|\mathcal{B}} \wr \mathfrak{S}_\infty$, and $\text{QA}(G)$ contains the free algebra with generators of degrees given by that of the non trivial orbits of $G_{|\mathcal{B}}$.*

Note that the first case can be refined by stating that the orbit algebra contains the algebra of invariants of the finite group H acting on the blocks (which may be smaller than the full symmetric group \mathfrak{S}_k); this algebra is typically not free.

Sketch of proof. The blocks share the same size by transitivity, and if their size (resp. number) is infinite, then their number (resp. size) has to be finite in order to keep the group P -oligomorphic (indeed, G is otherwise a subgroup of $\mathfrak{S}_\infty \wr \mathfrak{S}_\infty$ and its profile is bounded below by the number of integer partitions). Use Lemma 2.4 and Examples 2.13 in each case. \square

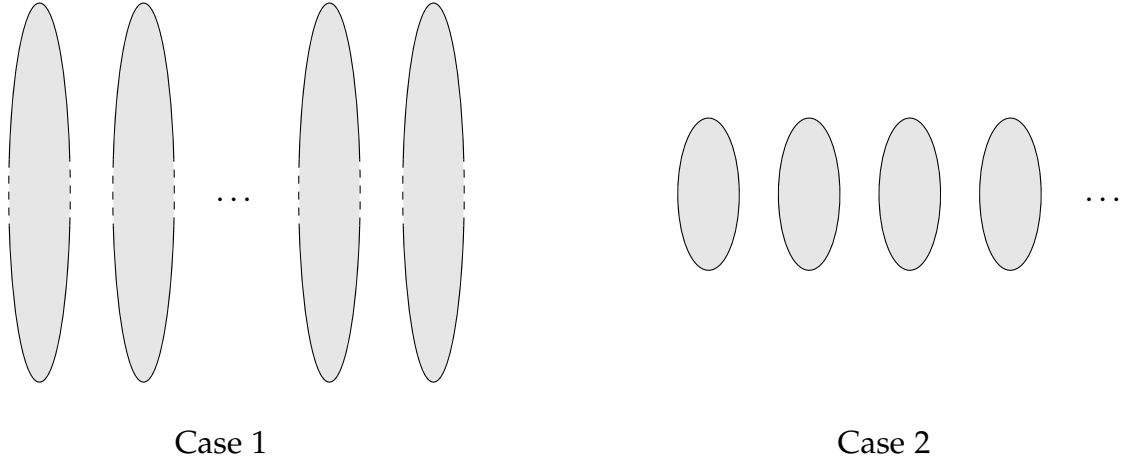


Figure 3: Essential cases for a *transitive* block system of a P -oligomorphic group

Assume that G is endowed with a block system. Then, the proof of the above lemma applies in the same fashion to the restrictions of G to (the support of) its orbits of blocks, leading the whole $\mathcal{QA}(G)$ (with just one more use of Lemma 2.4) to also contain the mentioned subalgebras (in the case with finitely many finite blocks, refer instead to the first item of Example 2.3). Recall also that in Case 2, we have the convenient property of Lemma 2.12.

Remark 3.2. Let G be an oligomorphic permutation group, and E_1, \dots, E_k be a partition of E such that each E_i is stable under G . In our use case, we have a block system \mathcal{B} , and each E_i is the support of one of the orbits of blocks in \mathcal{B} .

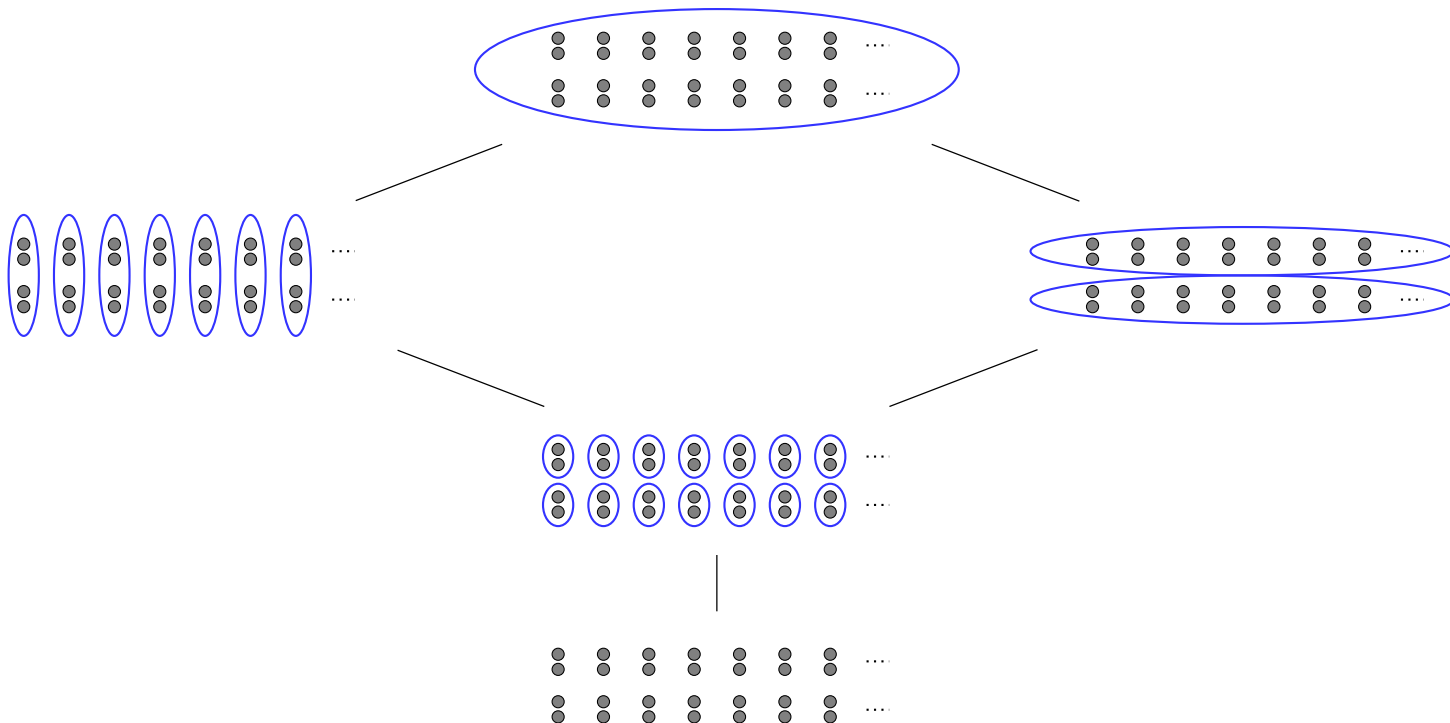
Then, G is a subgroup of $G|_{E_1} \times \dots \times G|_{E_k}$ (precisely, it is a *subdirect product* of this direct product). Therefore, by Lemma 2.4, $\mathcal{QA}(G)$ contains $\mathcal{QA}(G|_{E_1}) \otimes \dots \otimes \mathcal{QA}(G|_{E_k})$ as a subalgebra. In particular, the algebraic dimension of $\mathcal{QA}(G)$ is bounded below by the sum of the algebraic dimensions of the $\mathcal{QA}(G|_{E_i})$.

When, in addition, the actions of G on each E_i are completely independent, the containments above are equalities; then, $\mathcal{QA}(G)$ is finitely generated if and only if each $\mathcal{QA}(G|_{E_i})$ is.

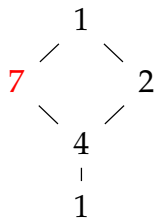
Remark 3.3. Combining Lemma 3.1 and Remark 3.2, each block system of G provides a lower bound on the algebraic dimension of $\mathcal{QA}(G)$, and therefore on the growth rate of the profile.

The following example illustrates that the lower bound on the profile highly depends on the chosen block system.

Example 3.4. Let $G = (\mathfrak{S}_2 \times \mathfrak{S}_2) \wr \mathfrak{S}_\infty$. Following is the poset of all its block systems, ordered by refinement:



The picture below displays the lower bounds on the algebraic dimension that can be deduced respectively from each of these block systems, using Lemma 3.1 and Remark 3.2:



For instance, for the block system with two orbits of blocks of size 2, the lower bound on the algebraic dimension is $4 = 2 + 2$ since we have $G|_B = \mathfrak{S}_2$ in each orbit; for the block system with finite blocks of size 4, the lower bound is 7, for $G|_B = \mathfrak{S}_2 \times \mathfrak{S}_2$ has this many orbits of non empty subsets. This latter lower bound is obviously tight since the inclusion $G|_B \wr \mathfrak{S}_\infty \subset G$ is an equality: the algebraic dimension of $\text{QA}(G)$ is 7 and the growth rate of the profile of G is 6.

This example suggests that better lower bounds are obtained when maximizing the size of the finite blocks (and then maximizing the number of infinite blocks; consider also the example $\mathfrak{S}_\infty \wr \text{Id}_n$ for that).

Nevertheless, the bound provided by this heuristic alone can be improved at rather low cost, as advertised by the following example.

Example 3.5 (Towards blocks of blocks). Consider the permutation group $G = \mathcal{C}_4 \wr (\mathfrak{S}_\infty \wr \mathcal{C}_3)$; we use here the parentheses regardless of the associativity to emphasize the action of G on its natural system of infinitely many (maximal) blocks of size 4.

By Remark 3.3, this block system provides a lower bound of 4 on the algebraic dimension. As we will see, it is very crude; a lot of information was lost when embedding the action on the blocks $\mathfrak{S}_\infty \wr \mathcal{C}_3$ into \mathfrak{S}_∞ . This action was not even primitive to begin with: one can form 3 infinite blocks (of 4-blocks). Let us exploit that information. Consider the stabilizer S of the three infinite blocks of finite blocks. This is a normal subgroup of finite index of G , and therefore it has the same algebraic dimension using Lemma 2.6. But now that these infinite blocks of blocks are stable parts of the domain, their contributions to the algebraic dimension can be treated separately; which hands a bound of $3 * 4 = 12$ for S , and thus for G .

Let us step toward a generalization and a formalization of the phenomenon observed in the above example.

Let G be a P -oligomorphic permutation group. Take a block system $\mathcal{B}^{<\infty}$ of finite blocks only. Assume further that these finite blocks are maximal: G does not have any strictly coarser system of finite blocks. This choice is motivated by the earlier observations (see Example 3.4); we will see in Subsection 3.2 that $\mathcal{B}^{<\infty}$ always exists and is unique.

If G has some finite orbits of elements, their union forms a stable finite block which contains all other stable finite blocks (a stable block is a union of orbits, that are obviously finite if the block is).

By definition of a block system, G acts on the set of blocks of $\mathcal{B}^{<\infty}$. Furthermore, this induced action does not admit any non trivial finite block, for else the blocks of $\mathcal{B}^{<\infty}$ would not be maximal. It has no special reason to be primitive though: its block systems will just have infinite blocks only (plus possibly one singleton, if $\mathcal{B}^{<\infty}$ has a stable block) – and finitely many of them, for the same reasons as in Lemma 3.1. By choosing one such system, we end up with two nested block systems: an inner one with finite blocks, and an outer one with (finitely many) infinite blocks; in other words, *a finite system of infinite blocks of finite blocks*.

Remark 3.6. A lower bound can be obtained from such a double, nested block system by first stabilizing the infinite blocks of finite blocks (which does not change the growth of the group, as stated by Lemma 2.6), and then applying the same method as in Remark 3.3. For the same choice of (maximal) finite blocks, the lower bound L provided by this method is better than the one deduced from the matching simple block system *via* Remark 3.3.

This is pretty much obvious if you consider the typical case highlighted in Example 3.5.

As for the choice of the infinite blocks of blocks, the general intuition remains the same as with classical, simple block systems: we feel that the more the better, as far as the lower bound is concerned.

The next subsection formalizes these intuitions to construct a canonical block system (in fact a system of blocks of blocks) that will hopefully maximize the lower bound.

3.2 Optimizing the lower bound through lattice structures

As suggested by the previous subsection, maximizing the lower bound will involve maximizing or minimizing block systems with certain properties. To this end, we will exploit the lattice structure of the poset of block systems, which we recall now.

Proposition 3.7. *Let G be a permutation group G acting on a set E , finite or infinite. The poset $\mathcal{L}(G)$ of all its block systems, endowed with the refinement order, is a sublattice of the lattice of set partitions of E . Its maximum and minimum are respectively the trivial block systems $\top = \{E\}$ and $\perp = \{\{e\} \mid e \in E\}$.*

Proof. Take two block systems \mathcal{B} and \mathcal{B}' , and consider their meet in the lattice of set partitions, namely the set partition:

$$\mathcal{B} \wedge \mathcal{B}' = \{B \cap B' \mid B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}.$$

It is straightforward to check that this is still a block system for the group. Hence this is the meet of \mathcal{B} and \mathcal{B}' in $\mathcal{L}(G)$.

Similarly, consider the join $\mathcal{B} \vee \mathcal{B}'$ in the lattice of set partitions. It is obtained by taking the equivalence classes of the closure of the relation “being in the same block in \mathcal{B} or in \mathcal{B}' ”. There remains to check that $\mathcal{B} \vee \mathcal{B}'$ is a block system: if x and y are in the same part and σ is an element of G , then $\sigma(x)$ and $\sigma(y)$ are in the same part as well. To this end, consider a sequence x_0, \dots, x_k such that we have $x_0 = x$, $x_k = y$ and any two consecutive elements in the same block for either \mathcal{B} or \mathcal{B}' ; then the same holds for the sequence $\sigma(x_0), \dots, \sigma(x_k)$.

In conclusion, $\mathcal{L}(G)$ is stable under both join and meet operations, and therefore a sublattice of the lattice of set partitions of E . \square

In the sequel, we will consider block systems with only finite blocks (resp. only infinite blocks, up to kernel); the following propositions state that those block systems form finite sublattices. This will provide us with a canonical maximal (resp. minimal) block system from which we will derive bounds.

Proposition 3.8. *Let G be an oligomorphic permutation group, and $\mathcal{L}^{<\infty}(G)$ be the subposet of block systems consisting of finite blocks only. Then, $\mathcal{L}^{<\infty}(G)$ is a sublattice of $\mathcal{L}(G)$, with the trivial block system as minimum. If, in addition, G is P -oligomorphic, then $\mathcal{L}^{<\infty}(G)$ is finite, with a maximum $\mathcal{B}^{<\infty}$.*

Proposition 3.9. *Let G be a P -oligomorphic permutation group, and $\mathcal{L}^\infty(G)$ be the subposet of block systems consisting of infinite blocks only; if the kernel of G is non trivial, then finite blocks contained in the kernel are allowed as well. Then, $\mathcal{L}^\infty(G)$ is a finite sublattice of $\mathcal{L}(G)$, with a minimum and the trivial block system as maximum.*

Proving Propositions 3.8 and 3.9 will require a couple of lemmas.

Lemma 3.10. *Let G be a P -oligomorphic group, and L be the function that maps a block system onto the associated lower bound described in the previous subsection (Remark 3.3). Let $\mathcal{B} < \mathcal{B}'$ be a cover (not involving the kernel) in the lattice $\mathcal{L}^{<\infty}(G)$, then we have $L(\mathcal{B}) < L(\mathcal{B}')$.*

If instead $\mathcal{B} < \mathcal{B}'$ is a cover (not involving the kernel) in the lattice $\mathcal{L}^\infty(G)$, we have $L(\mathcal{B}) > L(\mathcal{B}')$.

Proof. Assume that $\mathcal{B} < \mathcal{B}'$ is a cover in $\mathcal{L}^{<\infty}(G)$. Pick one of the finite blocks B in \mathcal{B}' that splits into two new blocks B_1 and B_2 in \mathcal{B} ; by conjugation, the same can be said about all the other blocks in the orbit of B . There are two cases: either B_1 and B_2 may swap or they may not.

If not, then the support O_B of the orbit of B is the union of the supports O_1 and O_2 of the orbits of B_1 and B_2 (resp.), and the age of $G|_{O_B}$ contains the (disjoint) ages of the restrictions G_1 and G_2 to O_1 and O_2 (resp.). It also contains the additional orbits of subsets that have non empty intersections with both B_1 and B_2 , so the inclusion is strict. Using Lemma 3.1, the provided bound on the profile is strictly better with the coarser system (since we took a cover, the situation in \mathcal{B} and \mathcal{B}' is the same everywhere else).

If B_1 and B_2 do swap, then we get a single orbit of (small) blocks in \mathcal{B} , just as in \mathcal{B}' ; except that if one denotes by H the restriction of G to one of the small blocks in \mathcal{B} , the restriction to one block of \mathcal{B}' is $H \wr \mathfrak{S}_2$, which has a strictly larger age. Hence, \mathcal{B}' provides a better bound.

As for $\mathcal{L}^\infty(G)$, the result is rather obvious from Lemma 3.1. □

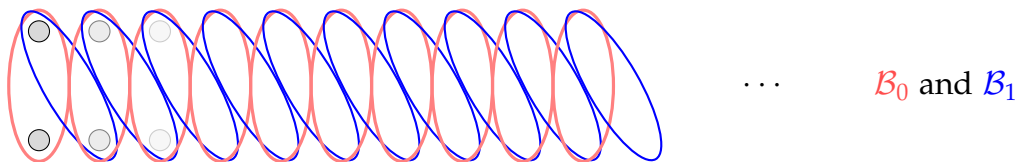
Lemma 3.11. *Let G be an oligomorphic permutation group. The poset $\mathcal{L}^{<\infty}(G)$ is closed under taking joins (as defined in the lattice of set partitions).*

The following simple example illustrates that this statement may fail without the oligomorphic condition.

Example 3.12. Recall that the (non oligomorphic) permutation group $\text{Aut}(\mathbb{Z})$ is generated by the translation $x \mapsto x + 1$, and take $G = \text{Aut}(\mathbb{Z}) \times \text{Aut}(\mathbb{Z})$, acting on two copies of \mathbb{Z} : $E = \{1, 2\} \times \mathbb{Z}$. This group admits an infinite family of block systems $(\mathcal{B}_j)_{j \in \mathbb{Z}}$ with non trivial finite blocks of size 2:

$$\mathcal{B}_j := \{ \{ (1, i), (2, i + j) \} \mid i \in \mathbb{Z} \}.$$

The following picture illustrates the block systems \mathcal{B}_0 and \mathcal{B}_1 ; their join is the trivial block system with a single infinite block.



In general, the join of two of block systems \mathcal{B}_i and \mathcal{B}_j with $i \neq j$ is composed of infinite blocks.

Proof of Lemma 3.11. Assume that the join of two systems of finite blocks \mathcal{B} and \mathcal{B}' from $\mathcal{L}^{<\infty}(G)$ contains at least one infinite block. This block is thus a union of infinitely many blocks from both \mathcal{B} and \mathcal{B}' , in which every block from one system intersects at least one block from the other one. If all of the blocks of \mathcal{B} involved were singletons, each of them would be included in one block from \mathcal{B}' and so the join would not have an infinite block; hence at least one of them, call it B_0 , is not.

Consider the stabilizer S_0 of this B_0 (in red in the center of Figure 4). In this subgroup, the union of the blocks from \mathcal{B}' having a non empty intersection with B_0 (in blue) is also stable, so as well is their set difference with B_0 . One can iterate the argument with the union of blocks from \mathcal{B} intersecting this stable domain (the outer crown of red blocks), and so on.

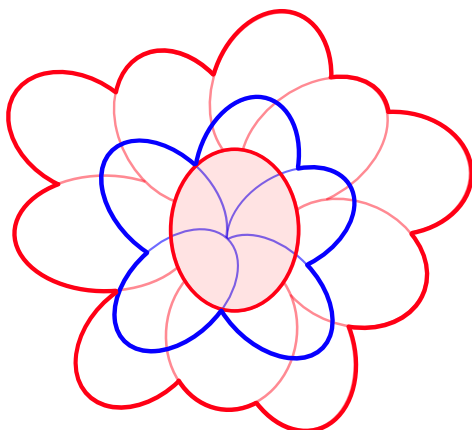


Figure 4: Nested stable areas arising when stabilizing one block of \mathcal{B}

This reveals an infinite sequence of finite disjoint (by taking the set difference every time) domains that are stable under the action of S_0 , and of which the first item is B_0 . Take now two distinct subsets A_1 and A_2 of E , each of them consisting of two elements in B_0 and just one in any of the other S_0 -stable domains. An element of G mapping A_1 to A_2 , if there is any, necessarily belongs to S_0 , since the pair included in B_0 has no other choice but to be mapped onto the corresponding pair of A_2 . Therefore, changing the S_0 -stable domain in which we take the singleton for A_2 (or A_1) exhibit infinitely many non isomorphic subsets of size 3 for G , which is to say infinitely many orbits of degree 3, and makes G a non oligomorphic group. \square

Proof of Proposition 3.8. Thanks to Lemma 3.11, we already know that $\mathcal{L}^{<\infty}(G)$ is table under taking the join. We will successively prove that $\mathcal{L}^{<\infty}(G)$ is stable under meets, locally finite, and that it admits no infinitely increasing chain. We will then conclude that it is bounded and finite.

Take two block systems \mathcal{B} and \mathcal{B}' in $\mathcal{L}^{<\infty}(G)$. Consider their meet in the lattice of block systems:

$$\mathcal{B} \wedge \mathcal{B}' = \{B \cap B' \neq \emptyset \mid B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}.$$

By construction, it has again finite blocks, which proves that $\mathcal{L}^{<\infty}(G)$ is stable under taking either joins or meets. In addition to this, the trivial block system $\perp = \{\{e\} \mid e \in E\}$ is obviously its minimal element.

Let \mathcal{B} be an element of $\mathcal{L}^{<\infty}(G)$. Consider the interval $[\perp, \mathcal{B}]$, and take a block system \mathcal{B}' in that interval. The way a block B in \mathcal{B} splits into blocks in \mathcal{B}' forces the way the blocks in the same orbit split in \mathcal{B}' themselves. Since the blocks of B are finite, and there are finitely many orbits thereof for G is oligomorphic, there are finitely many ways of splitting them. Therefore the interval $[\perp, \mathcal{B}]$ is finite, and the same holds for any interval: $\mathcal{L}^{<\infty}(G)$ is locally finite.

Take a strict chain C in $\mathcal{L}^{<\infty}(G)$. Using the local finiteness, embed this chain in a strict chain C' where each step is a cover. Thanks to Lemma 3.10, L is strictly increasing along that chain. Since G is P -oligomorphic, L is also bounded, and it follows successively that C' and C are finite.

This ensures the existence of a maximum $\mathcal{B}^{<\infty}$, for else we could construct an infinite chain by starting with an element and then recursively take the join with an incomparable element. We conclude by remarking that $\mathcal{L}^{<\infty}(G) = [\perp, \mathcal{B}^{<\infty}]$ is finite. \square

Proof of Proposition 3.9. The poset $\mathcal{L}^\infty(G)$ obviously has $\top = \{E\}$ as maximal element, and it is stable under joins: take indeed two block systems \mathcal{B} and \mathcal{B}' in $\mathcal{L}^{<\infty}(G)$: their blocks are infinite or included in the kernel. It is straightforward to check that the blocks of their join satisfy the same property.

Let us prove that $\mathcal{L}^\infty(G)$ is stable under meet. Consider the meet of two block systems in $\mathcal{L}^\infty(G)$:

$$\mathcal{B} \wedge \mathcal{B}' = \{B \cap B' \mid B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}.$$

It has finitely many blocks. The union of all the finite ones is finite and stable under G ; it is therefore included in the kernel of G . It follows that the blocks of $\mathcal{B} \wedge \mathcal{B}'$ are either infinite or included in the kernel of G , as desired.

Consider an interval $[\mathcal{B}, \top]$ in $\mathcal{L}^\infty(G)$. Every block system \mathcal{B}' from the interval is obtained by merging together some of the finitely many blocks of \mathcal{B} . Hence this interval is finite, and $\mathcal{L}^\infty(G)$ is locally finite.

We conclude as in the proof of Proposition 3.8: there are no infinite chains in $\mathcal{L}^\infty(G)$ (a bit of care needs to be taken since $L(\mathcal{B})$ may not be strictly increasing at steps where

two finite blocks are merged; but there can be only finitely many such steps). This in turn ensures the existence of a minimal element \mathcal{B}^∞ and the finiteness of $\mathcal{L}^\infty(G)$. \square

3.3 The nested block system

We may now use the structure of finite lattice on the block systems of a P -oligomorphic group to select a block system of a special kind (actually a system of blocks of blocks), which we expect to maximize the associated lower bound, and thus to provide a best fitted set-up for the study of the group.

Definition 3.13. Let G be a P -oligomorphic permutation group. Take:

1. the maximal (coarsest) element $\mathcal{B}^{<\infty}$ of $\mathcal{L}^{<\infty}(G)$
2. the minimal (finest) element \mathcal{B}^∞ of the lattice of block systems for the induced action of G on $\mathcal{B}^{<\infty}$.

We call the pair formed by the nested two partitions of E defined this way the **nested block system** $\mathcal{B}_B(G)$ of G .

Definition 3.14. We call an infinite primitive block of maximal finite blocks a **superblock**.

Note that, under the preliminary assumption of maximality, the primitivity requirement is equivalent to asking that the infinite block be minimal, so the above results on lattice structures imply that there is no “choice” for such superblocks— as opposed to the many choices of blocks, for the classical notion. From this point of view, the following (straightforward from the construction process of the nested block system) proposition offers an alternative definition of the superblocks, as the blocks of blocks in $\mathcal{B}_B(G)$.

Proposition 3.15 (Structure of the nested system). *The nested block system consists of finitely many superblocks, and maybe one stable finite block.*

Besides providing a competitive lower bound on the profile growth, the nested block system can pride itself on some pleasant properties of manageability.

Lemma 3.16. *Take two stable superblocks; the actions induced by G on their sets of maximal finite blocks are independent (up to taking a normal subgroup of finite index).*

Proof. By definition, the actions on the finite blocks of each superblock are isomorphic to one of the five highly homogeneous groups. Recall then Remark 2.19, and if need be take the finite index subgroup in which the actions of type $\text{Rev}(\mathbb{Q})$ and $\text{Rev}(\mathbb{Q}/\mathbb{Z})$ are replaced by $\text{Aut}(\mathbb{Q})$ and $\text{Aut}(\mathbb{Q}/\mathbb{Z})$ to avoid synchronizations of order 2. Now the maximality of the finite blocks allows to eliminate the case of total synchronizations, which leaves none possible. \square

Put otherwise, superblocks in $\mathcal{B}_{\mathcal{B}}(G)$ are not that far from independence, which would allow to use Remark 3.2. This paper will eventually clarify what “not that far” actually means.

Lemma 3.17. *Let G be a P -oligomorphic permutation group, and K be a finite index subgroup of G . Then we have $\mathcal{B}_{\mathcal{B}}(K) = \mathcal{B}_{\mathcal{B}}(G)$.*

Proof. We aim to prove that K has the same superblocks as G . Observe first that blocks of imprimitivity of any permutation group are still blocks for any subgroup, as a direct consequence of the definition. Let BB be a superblock, and \mathfrak{M} be the action of G (implicitly after stabilization and restriction to the support of BB) on the set of finite blocks of BB . Then, \mathfrak{M} is one of the five highly homogeneous groups; as the action of K on the same set of finite blocks is necessarily a finite index subgroup of \mathfrak{M} , it is highly homogeneous as well. We now just need to justify that the maximal finite blocks of G are still maximal for K . Assume some of them are not, then there exists $m \geq 2$ superblocks $(BB^{(j)})_{1 \leq j \leq m}$ in $\mathcal{B}_{\mathcal{B}}(G)$ and an ordering of their respective finite blocks $(B_i^{(j)})_i$ such that the unions $\cup_j B_i^{(j)}$ form new blocks for K (up to taking the join in $\mathcal{L}^{<\infty}(K)$). This can only happen if some of the actions of K on distinct $(B_i^{(j)})_i$ fully synchronize for $1 \leq j \leq m$. Since they are infinite (highly homogeneous) actions, this is in contradiction with K 's being of finite index. (Indeed, the action of K on the blocks would be of infinite index in that of G , which is not possible.) \square

The reader has probably already wondered at this point why to stop here. We already have blocks of blocks, why not blocks of blocks of blocks... etc.? The blocks of blocks of the nested block system allow a good description of wreath products of type $F_1 \wr P \wr F_2$, where F_1 and F_2 are two finite groups that may be trivial and P is an infinite permutation group (recall Example 3.5). But what if we add a layer of wreath product: $F_1 \wr P \wr F_2 \wr G$? Well, it simply turns out that if G is not finite then the group is not P -oligomorphic anymore; and if it is, we are actually back to the same configuration as earlier (by associativity). Of course, if we had not made any hypothesis on the growth of the profile, it would be relevant to consider any number of layers of blocks.

4 Classification in the case of a single superblock

In this section, we consider the class of closed P -oligomorphic permutation groups G with a single superblock, of which we denote by B_1, B_2, \dots the maximal finite blocks. This class includes wreath products $G = H \wr \mathfrak{S}_{\infty}$ where H is finite. In Subsection 4.1 we construct other examples by direct products; then, by combining wreath products and direct products, we introduce a family of permutation groups that subsumes all these

examples. We show that their orbit algebras are invariant rings of permutation groups, hence finitely generated and Cohen-Macaulay.

In Subsection 4.2 we announce a classification theorem: any instance of this class is isomorphic to exactly one permutation group in the family. This answers positively Macpherson's question for this class of permutation groups.

The next subsections undertake the proof of the classification theorem: Subsection 4.3 handles the action on the set of blocks; Subsection 4.4 introduces the tower of G in order to deal with the action within the blocks, an object that will be the key tool in the rest of the proof, and that turns out to be classified; finally, Subsection 4.5 shows that this classification can be lifted to the groups themselves.

4.1 A family of examples beyond wreath products

Definition 4.1. We call **direct product on blocks** of two permutation groups H and S and denote by $H \square S$ the permutation group defined by the action of $H \times S$ on $\deg(S)$ blocks of size $\deg(H)$ by

$$b_{r,i} \cdot (\tau, \sigma) = b_{\tau(r), \sigma(i)},$$

where $b_{1,i}, \dots, b_{m,i}$ is an arbitrary ordering of the elements of each block B_i . It is isomorphic to the natural action of $H \times S$ on the cartesian product of the supports.

This can be thought of as the action of permuting (by H and S respectively) the rows and columns of a potentially infinite matrix.

As opposed to the wreath product, for which every action of H within a block is independent from what happens within the others, here there is actually only one diagonal action of H , on all finite blocks at once. These two cases are in this regard the two opposite ends of the spectrum of all possible synchronizations between blocks.

It is then natural to think of a class of groups that would complete the spectrum. We introduce such groups, as hybrids of wreath products and direct products.

Definition 4.2. Let $H \triangleleft H_0$ and \mathfrak{M} be three permutation groups, with H and H_0 finite. Denote by $[H_0, H^\infty, \mathfrak{M}]$ the permutation group generated by the elements of $H \wr \mathfrak{M}$ and $H_0 \square \mathfrak{M}$. For short, denote by $[H_0, H^\infty] = [H_0, H^\infty, \mathfrak{S}_\infty]$.

Remark 4.3. The group $[H_0, H^\infty, \mathfrak{M}]$ is P -oligomorphic if and only if we have $\mathfrak{M} = \mathfrak{S}_\infty$ or $H_0 = H = \text{Id}_1$: indeed, $[H_0, H^\infty, \mathfrak{S}_\infty]$ contains $H \wr \mathfrak{S}_\infty$ as a subgroup; it is therefore P -oligomorphic; the other implications are trivial using Lemma 2.12.

Lemma 4.4. *The permutation group $G = [H_0, H^\infty, \mathfrak{M}]$ contains $H \wr \mathfrak{M}$ as a normal subgroup of finite index $[H : H_0]$.*

Proof. First note that G can be defined equivalently as the group generated by $H \wr \text{Id}_\infty = \langle H^\infty, \text{Id} \wr \mathfrak{M} \rangle$, and the finite group $H_0 \square \text{Id}_\infty$. For the sake of notations, and when

there is no ambiguity, we identify an element h_0 of H_0 with the element (h_0, h_0, \dots) of $H_0 \square \text{Id}_\infty$, and identify H_0 with $H_0 \square \text{Id}_\infty$.

Note that h_0 commutes with the elements of $\text{Id} \wr \mathfrak{M}$ and, by normality of H in H_0 , skew-commutes with those of $H \wr \text{Id}_\infty$, meaning $H^\infty h_0 = h_0 H^\infty$. It follows that we have

$$G = \bigcup_{h_0 \in H_0} h_0 \cdot (H \wr \mathfrak{M}).$$

This union becomes a decomposition into cosets if the range is restricted to some collection of representatives of the cosets of H in H_0 . Therefore $H \wr \mathfrak{M}$ is normal and of finite index $[H : H_0]$ in G , as desired. \square

We now describe the orbit algebra of $[H_0, H^\infty, \mathfrak{M}]$ as an invariant ring of a finite permutation group. Recall that the orbit algebra $\text{QA}(H \wr \mathfrak{M})$ of $H \wr \mathfrak{M}$ is the free commutative algebra $\mathbb{Q}[X]$, with $X = (X_{\bar{A}})_{\bar{A}}$ where \bar{A} ranges through the non-trivial H -orbits, and $X_{\bar{A}}$ denotes the $H \wr \mathfrak{M}$ -orbit of A , seen as an element of the orbit algebra. Finally, lift the action of H_0 on the H -orbits \bar{A} to an action on the variables $X_{\bar{A}}$.

Proposition 4.5. *With the above notations, the orbit algebra $\text{QA}(G)$ of $G = [H_0, H^\infty, \mathfrak{M}]$ is isomorphic to the invariant ring $\mathbb{Q}[X]^{H_0}$.*

Proof. That an element $h_0 \in H_0$ and A a subset of the support of H . Check that the $H \wr \mathfrak{M}$ -orbit of A is mapped onto another such $H \wr \mathfrak{M}$ -orbit, as prescribed by the announced action of H_0 on the variables $X_{\bar{A}}$. \square

Remark 4.6. The variables of invariant rings are commonly taken of degree 1; this is not the case here: the degree of the variable $X_{\bar{A}}$ is given by $|A|$. This must be taken into account when computing the Hilbert series using Molien's formula or Pólya enumeration.

4.2 Classification and application to Macpherson's conjecture

We may now state the main theorem of this section, which includes the classification of the trivial case of highly homogeneous groups \mathfrak{M} (case $H_0 = H = \text{Id}_1$ below); nevertheless, the core of this section is about the case of non trivial finite blocks, in which \mathfrak{M} is necessarily \mathfrak{S}_∞ .

Theorem 4.7 (Classification on one superblock). *Let G be a closed P -oligomorphic permutation group such that $\mathcal{B}_B(G)$ consists of a single superblock. Then G is isomorphic as a permutation group to $[H_0, H^\infty, \mathfrak{M}]$, where $H \triangleleft H_0$ are two finite permutation groups and \mathfrak{M} is one of the five (closed) highly homogeneous groups. In addition, H , H_0 , and \mathfrak{M} are unique, and satisfy the condition of Remark 4.3.*

Proof. The statement is obvious if the finite blocks are singletons. Otherwise, by Lemma 2.12, G acts on the set of finite blocks as $\mathfrak{M} = \mathfrak{S}_\infty$. Use the upcoming Proposition 4.16 to classify the action of G on its blocks (the *tower of G*) and the upcoming Proposition 4.19 to lift this classification to G itself. \square

A positive answer to Macpherson's question follows immediately thanks to the description of the orbit algebras of the groups $[H_0, H^\infty, \mathfrak{M}]$ from Proposition 4.5.

Corollary 4.8 (Macpherson on one superblock). *Let G be a closed P -oligomorphic permutation group such that $\mathcal{B}_B(G)$ consists of a single superblock. Then, $\mathcal{QA}(G)$ is an invariant ring of permutation group, hence finitely generated, Cohen-Macaulay, and of algebraic dimension the number of H -orbits (of non trivial subsets), where H is defined by the classification.*

Remark 4.9. Until now, the lower bound provided by the nested block system evoked in Remark 3.6 was calculated using Example 2.13 when it came to stable superblocks in $\mathcal{B}_B(G)$. With the notations of this section, it was based on the (possibly infinite index) supergroup $H_0 \wr \mathfrak{M}$: namely, the provided lower bound for the algebraic dimension was the cardinality of the age of H_0 . Corollary 4.8 hands a refinement of this bound, that is based on the subgroup $H \wr \mathfrak{M}$ and tight on the relevant restriction of the group.

In Section 5 the strategy to tackle a group G with several superblocks will be to consider the restrictions of G on each of its superblocks, and patch together their properties. This will use the following technical corollary.

Corollary 4.10. *Let G be a closed P -oligomorphic permutation group such that $\mathcal{B}_B(G)$ consists of a single superblock; write it as $G = [H_0, H^\infty, \mathfrak{M}]$ using the classification of Theorem 4.7, and let M be the minimal finite index normal subgroup of \mathfrak{M} . Then, any finite index normal subgroup \tilde{G} of G is of the form $[\tilde{H}_0, H^\infty, \tilde{M}]$, with $H \leq \tilde{H}_0 \leq H_0$ and $M \leq \tilde{M} \leq \mathfrak{M}$. In particular, $H \wr M$ is the minimal finite index normal subgroup of G .*

Proof. Since \tilde{G} is of finite index, its nested block system is still equal to $\mathcal{B}_B(G)$ by Lemma 3.17, and its action on the maximal finite blocks is a normal subgroup of finite index of \mathfrak{M} . Using the classification of Theorem 4.7, \tilde{G} is of the form $[\tilde{H}_0, \tilde{H}^\infty, \tilde{M}]$, with the expected group inclusions: $\tilde{H}_0 \triangleleft H_0$, $\tilde{H} \triangleleft H$, and $\tilde{M} \triangleleft \mathfrak{M}$. Lemma 4.4 also states that it contains $\tilde{H} \wr \tilde{M}$ as a finite index normal subgroup, while G contains $H \wr \mathfrak{M}$ and thus $H \wr \tilde{M}$ as finite index normal subgroups. Considering Lemma 2.15, we need to have $\tilde{H} = H$ for \tilde{G} to be of finite index in G . \square

4.3 Action on the set of blocks

The sequel of Section 4 is devoted to the statement and proof of the two propositions used in the proof of Theorem 4.7.

From now on, we assume that G acts on the set of finite blocks as $\mathfrak{M} = \mathfrak{S}_\infty$. The following two technical lemmas strengthen this assumption by showing that, for an appropriate enumeration of the elements within in each block, G can permute the blocks while preserving that enumeration.

Lemma 4.11. *Take any finite collection $(B_{i_1}, \dots, B_{i_k})$ of blocks; then $\text{Fix}_G(B_{i_1}, \dots, B_{i_k})$ acts on the remaining blocks as \mathfrak{S}_∞ .*

Proof. Take k in \mathbb{N} . As $\text{Fix}_G(B_{i_1}, \dots, B_{i_k})$ is a normal subgroup of finite index of $\text{Stab}_G(B_{i_1}, \dots, B_{i_k})$, it acts on the remaining blocks as a subgroup of finite index of \mathfrak{S}_∞ , which may only be \mathfrak{S}_∞ itself. By conjugation of the blocks, the same holds for any collection $(B_{i_1}, \dots, B_{i_k})$ of blocks. \square

Lemma 4.12. *There exists an ordering $b_{1,i}, \dots, b_{m,i}$ of the elements within each block B_i such that (the closure of) G contains $\text{Id}_m \square \mathfrak{S}_\infty = \text{Id}_m \wr \mathfrak{S}_\infty$ as a permutation subgroup.*

Proof. Since G acts by \mathfrak{S}_∞ on the blocks, there exists for each $i > 1$ a permutation $\tau_{1,i}^{(0)} \in G$ that swaps B_1 and B_i and stabilizes all the other blocks. Take now $k \geq 0$; using Lemma 4.11 there exists a permutation $\tau_{1,i}^{(k)}$ that not only swaps B_1 and B_i , but also fixes all the (other) blocks in B_2, \dots, B_k .

Take an infinite sequence $\tau_{1,i}^{(0)}, \dots, \tau_{1,i}^{(k)}, \dots$. Noting that there are only finitely many possibilities for the restriction of $\tau_{1,i}^{(k)}$ to $B_1 \cup B_i$, we can extract a subsequence with always the same restriction. Thus, using the property of closure, there exists in G a permutation $\tau_{1,i}$ which swaps B_1 and B_i and fixes all the other blocks. This permutation need not be of order 2 though.

Say that $\tau_{1,i}$ and $\tau_{1,j}$ are equivalent if their restrictions to $B_1 \cup B_i$ and $B_1 \cup B_j$ coincide up to renaming the elements of B_i (or B_j).

Now consider the map $i \mapsto \tau_{1,i}$. It takes finitely many values, and therefore there exists i and j such that $\tau_{1,i}$ and $\tau_{1,j}$ are equivalent. Define $\tau'_{i,j} = \tau_{1,i} \tau_{1,j}^{-1} \tau_{1,i}$.

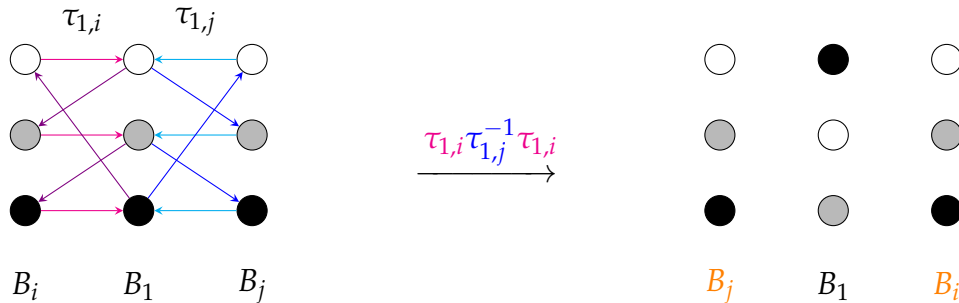


Figure 5: Example: Straight swap of B_i and B_j (also permuting B_1)

Now check that

- $\tau'_{i,j}$ swaps B_i and B_j “straightforwardly”: that is its restriction to $B_i \cup B_j$ is of order 2 (see Figure 5);
- $\tau'_{i,j}$ stabilizes B_1 ;
- $\tau'_{i,j}$ fixes all the other blocks (pointwise).

We may then conjugate $\tau'_{i,j}$ to stabilize some block B_k instead of B_1 , with k as large as desired, and still swap straightforwardly B_i and B_j while fixing the remaining blocks.

Therefore there exists in G , which we recall is assumed to be closed, a permutation $\tau_{i,j}$ of order 2 that swaps B_i and B_j and fixes all the other blocks. By conjugation, we can find for each n a similar permutation τ_n swapping B_n and B_{n+1} .

Choose an arbitrary ordering $b_{1,1}, \dots, b_{m,1}$ of B_1 . Define the ordering $b_{1,2}, \dots, b_{m,2}$ of B_2 so that τ_1 is the trivial swap, meaning that it swaps $b_{1,r}$ and $b_{2,r}$ for each r . Proceed similarly to order the elements of B_3 so that τ_2 is the trivial swap, and so on (Figure 6 shows the stage $k - 1$).

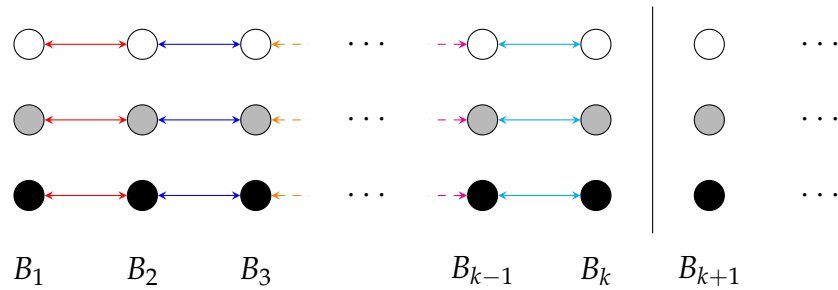


Figure 6: Straight swaps between the first k blocks

Conclusion: the τ_n 's generate $\text{Id}_m \square \mathfrak{S}_\infty$ as a permutation subgroup of G , as desired. \square

4.4 Towers and their classification

While the previous subsection dealt with the way the finite blocks could permute, this subsection is going to focus on what can happen within the blocks when they do not permute (the results obtained above state that the actions on and within the blocks can be decorrelated anyway).

Definition 4.13. Let $S_{\mathcal{B}} = S_{\mathcal{B}}^G = \text{Stab}_G(\mathcal{B})$ be the kernel of the morphism that maps G onto its induced action on the set of blocks, and, for $i \geq 0$, set $H_i = H_i^G = \text{Fix}_{S_{\mathcal{B}}}(B_1, \dots, B_i)_{|B_{i+1}}$. We call the sequence $H_0, H_1, H_2 \dots$ the **tower** of G with respect to the block system \mathcal{B} . The groups H_i are considered up to a permutation group isomorphism.

Remark 4.14. • By conjugation, using Lemma 2.12, the tower does not depend on the ordering B_1, B_2, \dots of the blocks. In others words, H_i can be obtained by fixing (pointwise) any i blocks and taking the restriction to any other block.

- Each H_{i+1} can be naturally considered a subgroup of H_i . Indeed, choose a block “far enough”, or take an arbitrary block out of the numbering by relabelling the blocks, call it say B_0 , and consider all H_i 's on this block: $H_i = \text{Fix}_{S_B}(B_1, \dots, B_i)|_{B_0}$. This realizes the H_i 's as a chain of decreasing subgroups (that can be renormalized, if not already, to the domain $\{1, 2, \dots, m\}$, where m is the cardinality of the blocks). Furthermore, each H_i is normal in H_{i-1} .

The above definition and remark also apply to a permutation group of a finite set, as long as it acts on the (finitely many) blocks as the full symmetric group.

Example 4.15 (Fundamental examples). Let H be a finite permutation group. The tower of $H \wr \mathfrak{S}_\infty$ (resp. $H \square \mathfrak{S}_\infty$) for its natural block system is $H, H, H, H \dots$ (resp. $H, \text{Id}, \text{Id} \dots$). The tower of $[H_0, H^\infty]$ is $H_0, H, H, H \dots$.

$$\begin{array}{ccc}
 H_0 \wr \mathfrak{S}_\infty & \longleftrightarrow & H_0, H_0, H_0, H_0 \dots \\
 \vdots & & \\
 [H_0, H^\infty] & \longleftrightarrow & H_0, H, H, H \dots \\
 \mid < \infty & & \\
 H \wr \mathfrak{S}_\infty & \longleftrightarrow & H, H, H, H \dots \\
 \vdots & & \\
 H \square \mathfrak{S}_\infty & \longleftrightarrow & H, \text{Id}, \text{Id}, \text{Id} \dots
 \end{array}$$

We aim to prove that these are the only possibilities for a tower (so there is actually only one prototype of a tower, since the first two examples are a specialization of the third one).

Proposition 4.16. *Let G be a closed P -oligomorphic permutation group with $\mathcal{B}_B(G)$ consisting of a single superblock. Then, the tower of G has the form $H_0, H, H, H \dots$, where H_0 is a finite permutation group and H is a normal subgroup of H_0 .*

Proof. Consider, for any $i \in \mathbb{N}$, the restriction G_i of $\text{Fix}_G(\cup_{j < i} B_j)$ to the four next blocks. The tower of this permutation group (for a natural extension of the notion to finite groups of the adequate shape) is $H_i, H_{i+1}, H_{i+2}, H_{i+3}$. We aim to show that $H_{i+1} = H_{i+2}$, which will conclude the proof.

An element s of the blockwise stabilizer S_i of G_i is determined by its action on each block, which we write as a quadruple. Let g be an element of H_{i+1} . Then S_i has an element x that may be written $(1, g, h, l)$, with h and l also in H_{i+1} . Let σ be an element of G_i that permutes “straightforwardly” the first two blocks and fixes the other two (Lemma 4.12 states that such an element actually exists). By conjugating x with σ in G_i , we get an element y in S_i that we may write $(g, 1, h, l)$, so that $x^{-1}y = (g, g^{-1}, 1, 1)$. Hence, using Remark 4.14, g is actually in H_{i+2} . \square

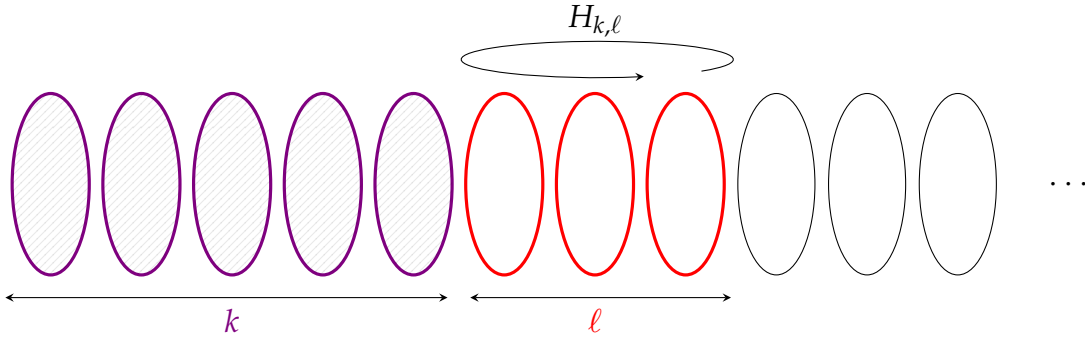
4.5 Lifting of the classification from towers to groups

The two results to follow will show that G is uniquely defined by its tower, by first recovering the blockwise stabilizer of the group from the tower, and then using the result of “straightforward” permutation we proved in Lemma 4.12.

Lemma 4.17. *The tower of G w.r.t. \mathcal{B} uniquely determines its blockwise stabilizer $S_{\mathcal{B}}$.*

Proof. Let $(H_i)_i$ be the tower of G w.r.t. \mathcal{B} and $S_{\mathcal{B}}$ be the blockwise stabilizer of \mathcal{B} .

Using that S is closed, it is sufficient to prove that, for any $l \geq 0$, the restriction S_{ℓ} of S to the first ℓ blocks is determined by the tower; or equivalently to any ℓ blocks (recall that the order of the blocks is irrelevant). To this end, we will show that S_{ℓ} admits an expression that involves only explicit subdirect products and the H_i 's (which will do the job thanks to Proposition 2.18).



In order to proceed by induction on ℓ , we consider the larger family $(H_{k,\ell})_{k \geq 0, \ell > 0}$, where $H_{k,\ell}$ is the restriction on l blocks of the fixator of k other blocks in S . Of course, $H_{0,\ell} = S_{\ell}$.

First, note that we have $H_{k,1} = H_k$ for all k . This gives the base case for the induction. We now take $\ell > 1$, and express $H_{k,\ell}$ as a subdirect product involving only $H_{k',\ell'}$ with $\ell' < \ell$ (and incidentally also $k' + \ell' \leq k + \ell$).

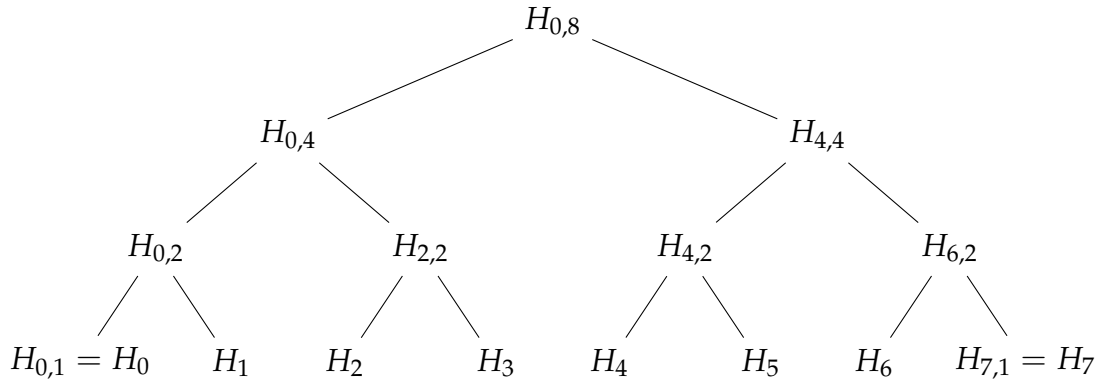
Write $l = \ell_1 + \ell_2$ with $\ell_1 > 0$ and $\ell_2 > 0$, partition the ℓ blocks into ℓ_1 and ℓ_2 blocks, and let E_1 and E_2 be their respective union. Considering the action of $H_{k,\ell}$ on E_1 and E_2 provides the desired expression:

$$H_{k,\ell} = \text{Subdirect}((G_1, G_2), (N_1, N_2)),$$

where:

$$\begin{aligned}
G_1 &= H_{k,\ell}|_{E_1} &= H_{k,\ell_1} \\
G_2 &= H_{k,\ell}|_{E_2} &= H_{k,\ell_2} \\
N_1 &= \text{Fix}_{H_{k,\ell}}(E_2)|_{E_1} &= H_{k+\ell_1,\ell_2} \\
N_2 &= \text{Fix}_{H_{k,\ell}}(E_1)|_{E_2} &= H_{k+\ell_2,\ell_1} \quad \square
\end{aligned}$$

Remark 4.18. If desired, more explicit formulae can be obtained, by imposing the partition. For instance, even splittings have the pleasant property that $G_1 = G_2$ and $N_1 = N_2$, which allows to illustrate the process by a binary tree: following is for example a recursion tree to express $H_{0,8}$ as a subdirect product of H_i 's, assuming that the left (resp. right) hand child of a group is the G_i (resp. N_i) of the subdirect product making this group (so a group is determined by its two children). This recursion tree generalizes immediately to any $H_{0,2^n}$, which is sufficient to retrieve S by closure.



Proposition 4.19. *The permutation group G is the natural semidirect product of its blockwise stabilizer $S_{\mathcal{B}}$ and $L = \text{Id}_m \square \mathfrak{S}_{\infty}$. In particular, it is uniquely defined by its tower w.r.t. \mathcal{B} .*

Proof. Use first Lemma 4.12 to state that G contains $L = \text{Id}_m \square \mathfrak{S}_{\infty}$ (for “ladder”) as a permutation subgroup.

Take $k > 0$, the stabilizer $\text{Stab}_G(B_1, \dots, B_k)$ of the first k blocks is isomorphic to $\text{Stab}_G(B_1, \dots, B_k)$ by Lemma 4.17.

Now, the group generated by $L|_{B_1 \cup \dots \cup B_k}$ and $\text{Stab}_G(B_1, \dots, B_k)|_{B_1 \cup \dots \cup B_k}$ is a subgroup of the restriction of G (actually of $\text{Stab}_G(B_1 \cup \dots \cup B_k)$) to the same domain. Moreover, the latter is of size $|\mathfrak{S}_k| \cdot |\text{Stab}_G(B_1, \dots, B_k)|$ (consider the morphism that projects onto the action on the blocks); therefore the two groups are equal.

Finally, since G acts on the blocks as the symmetric group, which is the closure of the group of all finitely supported permutations, each of its elements is also the simple limit of a sequence of finitely supported permutations. Since we just showed that the restrictions to any finite number of blocks are uniquely defined by the tower, so is the whole (closed) group G . \square

This hands the final element to the proof of the theorem of classification in the particular case of this section, and we are now free to move on to the general case.

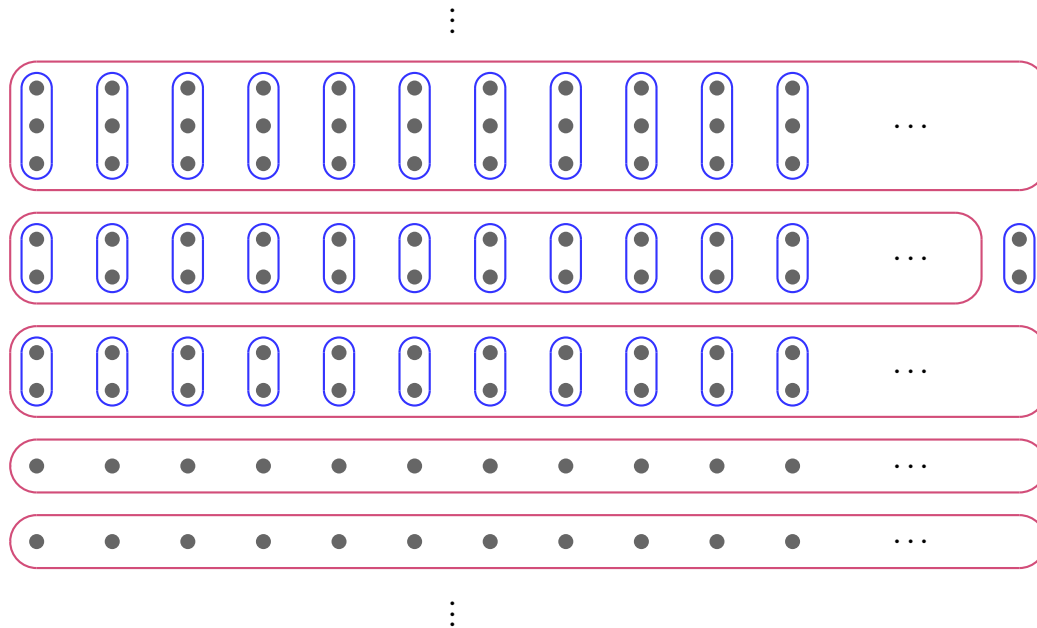
5 Classification of (closed) P -oligomorphic groups

Let G be a closed P -oligomorphic group. In Subsection 5.1 we exploit the results from Subsection 3 on the blocks systems of G and the classification of closed P -oligomorphic groups with one single superblock of Section 4 to give a constructive description of the minimal finite index subgroup K of G . This subgroup is the first piece of the classification of G .

In Subsection 5.2, we explicit G as a semidirect product of K and a finite permutation group acting diagonally on its nested block system.

Finally, in Subsection 5.3, we classify (closed) P -oligomorphic groups, using a finite data structure involving some finite group and a choice of decorated block system for it. We first handle the case when G does not act as $\text{Rev}(\mathbb{Q})$ or $\text{Rev}(\mathbb{Q}/\mathbb{Z})$ on any of its blocks, and then show how to be inclusive.

5.1 The minimal finite index subgroup



Following Definition 3.13, let $\mathcal{B}_B(G)$ be the nested block system of G ; recall that it consists in a partition of the set of maximal finite blocks (but the kernel of G) into finitely many superblocks $(BB^{(j)})_j$. Let $\text{Stab}_G(\mathcal{B}_B(G))$ be the stabilizer of the superblocks; it

is a finite index normal subgroup of G and therefore, by Lemma 3.17, has the same superblocks.

We now consider the restriction $G^{(j)} = \text{Stab}_G(\mathcal{B}_B)|_{E^{(j)}}$ of $\text{Stab}_G(\mathcal{B}_B(G))$ on the support $E^{(j)}$ of each superblock $BB^{(j)}$. It admits a single superblock, so that we can use the classification result of Section 4. Consider the minimal finite index normal subgroup $K^{(j)} = H^{(j)} \wr M^{(j)}$ of $G^{(j)}$ of Corollary 4.10.

To be concrete, use Lemma 4.12 to choose a *coherent enumeration* of the elements of each block $B_i^{(j)}$ of $BB^{(j)}$: for each i, i' there exists $g \in G$ that maps $B_i^{(j)}$ to $B_{i'}^{(j)}$ while preserving the enumeration. From now on, we use this chosen enumeration to implicitly identify elements of $B_i^{(j)}$ and of $B_{i'}^{(j)}$ when meaningful. Recall that $M^{(j)}$ is obtained by considering the homomorphic image \mathfrak{M} of $G^{(j)}$ acting on the blocks in $BB^{(j)}$ and, if needed, taking its minimal finite index normal subgroup; $H^{(j)}$ can be obtained by picking arbitrarily two blocks $B_0^{(j)}$ and $B_1^{(j)}$ in $BB^{(j)}$, and taking the restriction to $B_1^{(j)}$ of the subgroup of G that fixes $B_0^{(j)}$ and stabilizes $B_1^{(j)}$. Recall also that $M^{(j)}$ is \mathfrak{S}_∞ whenever the blocks are non trivial; otherwise, $H^{(j)}$ is the trivial permutation group on one element, and $K^{(j)} = M^{(j)}$ is one of \mathfrak{S}_∞ , $\text{Aut}(\mathbb{Q})$ or $\text{Aut}(\mathbb{Q}/\mathbb{Z})$.

In addition, set by convention $B_0^{(0)} = \ker G$, $BB^{(0)} = \{\ker G\}$, and $K^{(0)} = \text{Id}_{\ker G}$.

Remark 5.1. The groups $K^{(j)}$ and $K^{(j')}$ are conjugate whenever the superblocks $BB^{(j)}$ and $BB^{(j')}$ are in the same G -orbit.

Proposition 5.2. *Let G be a closed P -oligomorphic group. Then $K = \prod_j K^{(j)}$ is the minimal finite index normal subgroup of G .*

Let us start by proving the following result.

Lemma 5.3. *A permutation group K of the above form admits no (proper) finite index normal subgroup.*

Proof. If a subgroup \tilde{K} of K is normal and of finite index, Lemma 3.17 states that K 's superblocks are still superblocks for \tilde{K} ; and since they are stable under K , they are also stable under any subgroup of K . Using the classification of Corollary 4.10, the restrictions of K to any superblocks have no finite index normal subgroup, so \tilde{K} has the same restrictions and is thus a subdirect product of these. There remains to show that there is no synchronization between these parts. Again, the $K^{(j)}$ have no finite index normal subgroups, so no finite synchronization (that would be linked to a proper normal subgroup of finite index) is to consider; and the case of infinite synchronizations is excluded by Lemma 3.16. \square

Proof of Proposition 5.2. We are going to reduce G down to K by applying two successive reductions to a normal subgroup of finite index, which will conclude the proof using Lemma 5.3.

Recall that the intersection of two finite index normal subgroups is again a finite index normal subgroup. Hence the finite index normal subgroups of G form a lattice. It is not guaranteed *a priori* to have a minimal element though.

We consider the nested block system $\mathcal{B}_{\mathcal{B}}(G)$ introduced in Section 3. By Lemma 3.17, at each step, the nested block system of the finite index normal subgroup will still be $\mathcal{B}_{\mathcal{B}} = \mathcal{B}_{\mathcal{B}}(G)$. In particular, the kernel will not grow bigger. Denote as earlier by $(\mathcal{B}_{\mathcal{B}}^{(j)})_j$ the superblocks of $\mathcal{B}_{\mathcal{B}}(G)$ and by $(E^{(j)})_j$ their respective supports.

Let $\text{Stab}_G(\mathcal{B}_{\mathcal{B}})$ be the finite index normal subgroup of G that stabilizes the superblocks in $\mathcal{B}_{\mathcal{B}}(G)$, which is the first reduction.

Assume first $j \neq 0$. We may apply the classification result of Lemma 4.4: $\text{Stab}_G(\mathcal{B}_{\mathcal{B}})|_{E^{(j)}}$ contains as a finite index normal subgroup some wreath product $H^{(j)} \wr M^{(j)}$, where $M^{(j)}$ is given by the action of $\text{Stab}_G(\mathcal{B}_{\mathcal{B}})$ on the set of maximal finite blocks of $E^{(j)}$ (possibly up to taking an index 2 normal subgroup), while $H^{(j)}$, which acts within the blocks, is given by H in the tower H_0, H, H, H, \dots of $\text{Stab}_G(\mathcal{B}_{\mathcal{B}})|_{E^{(j)}}$. This subgroup is isomorphic to $K^{(j)}$, which also implies that it contains no proper normal subgroup of finite index; therefore, thanks to the aforementioned lattice structure, it is the minimal finite index normal subgroup of $\text{Stab}_G(\mathcal{B}_{\mathcal{B}})|_{E^{(j)}}$.

The same conclusions can be reached trivially for $j = 0$ (recall that $E^{(0)}$ is the kernel of G and $K^{(0)}$ is the trivial group thereupon).

Now is the time for the second reduction. Consider the finitely many cosets of $K^{(j)}$ in $\text{Stab}_G(\mathcal{B}_{\mathcal{B}})|_{E^{(j)}}$; the latter (and therefore $\text{Stab}_G(\mathcal{B}_{\mathcal{B}})$) acts by permutation on these cosets. Now denote by \tilde{K} the kernel of its simultaneous action on the whole set of cosets for all j . At this point, for each given j , the restriction of \tilde{K} to $E^{(j)}$ is a subgroup of $K^{(j)}$; it could be a proper subgroup at first glance, due to the constraints inherited from the action on the other sets of cosets (the cosets from other $E^{(j)}$, for different j). However, thanks to the minimality of $K^{(j)}$, we do have $\tilde{K}|_{E^{(j)}} = K^{(j)}$, for every j : \tilde{K} is a subdirect product of the direct product K . Conclude the proof by replaying that of Lemma 5.3 to show that \tilde{K} is actually the whole direct product, namely K itself, and use Lemma 5.3 and the lattice structure to state its status of minimum. \square

Remark 5.4. From Remark 3.2, $\text{QA}(K)$ is a free algebra, possibly tensored with some finite dimensional diagonal algebra, which is finitely generated. Explicitely, we may write:

$$\text{QA}(K) = \bigotimes_i \text{QA}(K_i) \otimes \text{QA}(\ker K) = \mathcal{A}_f \otimes \text{QA}(\ker K) = \bigoplus_k e_k \mathcal{A}_f$$

the e_k being the subsets of the kernel of K ; in other words, $\text{QA}(K)$ is a Cohen-Macaulay algebra over the free subalgebra \mathcal{A}_f .

Corollary 5.5. *The lower bound provided by the nested block system according to Remark 4.9 is tight.*

Proof. The algebraic dimension of the algebra of K is the sum of dimensions of the \mathcal{A}_f 's, which coincide with the lower bound handed by the nested system; and the algebraic dimension of $\mathcal{QA}(G)$ is the same by finiteness of the index of K . \square

5.2 Semidirect product structure and diagonal action

In this section, we generalize the notion of diagonal action of Section 4, and prove that G is a product FK , where F is a finite permutation group acting diagonally on $\mathcal{B}_B(G)$, or, equivalently, that G is a semidirect product of K with some natural quotient $F/K_{<\infty}$.

Informally, a permutation acts diagonally if it acts consistently within each block of a given superblock, and possibly permutes the superblocks but not the blocks within each superblock. Considering Figure 5.1, one may intuitively picture it as a purely vertical action, that permutes the rows formed by the elements of E on the figure — and preserves the superblock structure of course. There remains to define formally the vertical alignment.

An indexing of the $B^{(j)}$ within each superblock $BB^{(j)}$ is *coherent* if, for any two superblock $BB^{(j)}$ and $BB^{(j')}$ in the same orbit, there exists some permutation $g_{j,j'}$ in G which maps each block $B_i^{(j)}$ to $B_i^{(j')}$.

Lemma 5.6. *A P -oligomorphic group always admits a coherent indexing of the blocks within each superblock.*

Proof. We may, for example, proceed as follow: for each orbit of superblock, pick one superblock $BB^{(j_0)}$ and choose any indexing for this superblock; then, for each superblock $BB^{(j)}$ in its orbit, pick some permutation $g_{j_0,j} \in G$ mapping $BB^{(j_0)}$ to $BB^{(j)}$; use $g_{j_0,j} \in G$ to transport the indexing of the blocks in $BB^{(j_0)}$ to $BB^{(j)}$; finally, for each $BB^{(j)}$ and $BB^{(j')}$ in the orbit of $BB^{(j_0)}$, define $g_{j,j'} = g_{j_0,j'} g_{j_0,j}^{-1}$. \square

From now on, we fix a coherent indexing of the blocks within the superblock. We further assume without loss of generality that the chosen coherent enumeration of the elements within the blocks is preserved by the $g_{j,j'}$'s: the enumeration of the elements of a block $B_i^{(j)}$ is mapped by $g_{j,j'}$ to the enumeration of the elements of $B_i^{(j')}$. To achieve this, we may proceed as above: choose the coherent enumerations in the superblock $BB^{(j_0)}$, and then use the $g_{j_0,j}$ to transport them to the other superblocks in the same orbit.

It will be convenient to index globally the elements of E . To this hand, choose one block $U^{(j)}$ within each superblock in such a way that, whenever $U^{(j)}$ and $U^{(j')}$ are in the same orbit, they have the same index in $BB^{(j)}$ and $BB^{(j')}$ respectively. Let $U = \sqcup_j U^{(j)}$. Now any element e of E can be uniquely described by two coordinates: “horizontally” the index i of the block $B^{(j)}$ containing it; “vertically”, the element $u \in U$ which corresponds to e in the coherent enumeration of $U^{(j)}$. For convenience, we identify e with the pair (u, i) .

Definition 5.7. A permutation $g \in G$ acts *diagonally* on E if any pair of elements $((u, i_1), (u, i_2))$ is mapped to some pair $((u', i_1), (u', i_2))$.

Let respectively $G_{<\infty}$ (resp. $K_{<\infty}$) be the collection of the permutations of G (resp. K) that acts diagonally on E .

Remark 5.8. The collections $G_{<\infty}$ and $K_{<\infty}$ are groups and can be canonically identified with finite permutation groups of U . In addition, $K_{<\infty}$ is the direct product of each $H^{(j)}$ acting on $U^{(j)}$. Finally, $G_{<\infty}$ as a finite permutation group does not depend on the choice of the coherent indexing of the blocks within the superblocks and, up to isomorphism, does not depend on the choice of U or of the coherent enumeration of the elements within the blocks.

We may now state the main result of this section.

Proposition 5.9. (i) $G = KG_{<\infty}$;

(ii) G/K is isomorphic to $G_{<\infty}/K_{<\infty}$;

(iii) G is a semi-direct product of $G_{<\infty}/K_{<\infty}$ and K .

Proof. (i) We take $g \in G$ and aim to write it as a product in $KG_{<\infty}$.

Consider some superblock $BB^{(j)}$. Let j' be such that g maps $BB^{(j')}$ to $BB^{(j)}$, and let $h = gg_{j,j'}$; by construction, h stabilizes $BB^{(j)}$. Using Theorem 4.7, we may choose $k^{(j)}$ in $K^{(j)}$ such that $k^{(j)}h_{|E^{(j)}}$ acts diagonally on $E^{(j)}$. Hence $k^{(j)}g_{|E^{(j)'}}$ maps $BB^{(j')}$ to $BB^{(j)}$ while preserving the coherent indexing of blocks and enumeration within the blocks.

Define $k = \prod k^{(j)} \in K$ by extending each $k^{(j)}$ by the identity on the other superblocks. Then, kg acts diagonally, as desired.

(ii) Consider now the morphism $\phi : G_{<\infty} \mapsto G/K$ obtained through the embedding of $G_{<\infty}$ in G and the canonical projection of the latter on G/K . Thanks to (i), ϕ is surjective. In addition $\ker \phi = G_{<\infty} \cap K = K_{<\infty}$. Therefore, ϕ is the desired isomorphism between $G_{<\infty}/K_{<\infty}$ and G/K .

(iii) is a reformulation of (ii). □

5.3 Classification of closed P -oligomorphic groups

We now have all the ingredients to classify (closed) P -oligomorphic groups. We first use the previous sections to extract from a P -oligomorphic group G a finite piece of information $\text{Data}(G)$. It consists of a finite permutation group, endowed with a block system where each block is decorated with a permutation group of its elements and one of the five highly homogeneous permutation groups (or the trivial group). Conversely, we show that, starting from such a permutation group with decorated blocks Δ , one can construct an oligomorphic permutation group $\text{Group}(\Delta)$.

We check that, up to a natural isomorphism, G can be reconstructed from $\text{Data}(G)$ by Group . More generally, we check that, up to an isomorphism, Data and Group give a one-to-one correspondence between finite permutation groups with decorated blocks and P -oligomorphic permutation groups. This concludes the classification.

5.3.1 Classification of Rev-free closed P -oligomorphic groups

For the sake of simplicity of exposition, we first tackle the subclass of *Rev-free groups*; that is groups that do not act by $\text{Rev}(\mathbb{Q})$ or $\text{Rev}(\mathbb{Q}/\mathbb{Z})$ on any of their superblocks. This is actually sufficient to classify ages of P -oligomorphic groups. In the following section, we detail how $\text{Data}(G)$ can be extended to also preserve this piece of information.

Let G be a closed P -oligomorphic group. Take again the notations introduced at the beginning of the previous subsection: $\mathcal{B}_B(G) = \{BB^{(j)}\}_j$, $K = \prod_j K^{(j)}$, with $K^{(j)} = H^{(j)} \wr M^{(j)}$, and finally the finite blocks $(B_0^{(j)})_j$ arbitrarily picked in each superblock.

As in Subsection 5.2, consider also the subgroup of G that stabilizes $\sqcup_j B_0^{(j)}$ and take its restriction to this union of finite blocks; call it $G_{<\infty}$. Similarly, consider the same restriction but calculated from K , and call it $K_{<\infty}$. Observe that $K_{<\infty}$ is a subgroup of $G_{<\infty}$ that is isomorphic to $\prod_j H^{(j)}$.

Definition 5.10. Define $\text{Data}(G) = (G_{<\infty}, (B_0^{(j)})_j, (H^{(j)})_j, (M^{(j)})_j)$.

Definition 5.11. A **permutation group with decorated blocks** $(F, B, (H^{(j)})_j, (M^{(j)})_j)$ consists of a finite permutation group F endowed with a block system $B = \{B^{(j)}\}_j$ together with the choice, for each block $B^{(j)}$, of

- a normal subgroup $H^{(j)}$ of the restriction $\text{Fix}_F(\cup_{i \neq j} B^{(i)})|_{B^{(j)}}$ of the pointwise stabilizer of the other blocks,
- $M^{(j)}$, one of the three minimal highly homogeneous groups, or the trivial group,

satisfying the following constraints:

- the choices must be the same for $i \neq j$ whenever $B^{(i)}$ and $B^{(j)}$ are in the same F -orbit,
- $M^{(j)}$ is \mathfrak{S}_∞ whenever $B^{(j)}$ is not of size 1,
- at most one $M^{(j)}$ is trivial, and when it is $K^{(j)} = H^{(j)}$ is trivial too.

Remark 5.12. Let G be a closed P -oligomorphic permutation group. Then $\text{Data}(G)$ is a permutation group with decorated blocks.

Definition 5.13. Let $\Delta = (F, B, (H^{(j)})_j, (M^{(j)})_j)$ be a permutation group with decorated blocks, and E the disjoint union $\sqcup_j E^{(j)}$, where $E^{(j)}$ is the cartesian product of $B^{(j)}$ and the domain of $M^{(j)}$. For each j , take the wreath product $K^{(j)} = H^{(j)} \wr M^{(j)}$ acting naturally on $E^{(j)}$. Finally, let K be the direct product $\prod_j K^{(j)}$, acting on E .

We define $\text{Group}(\Delta)$ as the smallest permutation group on E containing both K and F acting diagonally on $\sqcup_j E^{(j)}$. Denote additionally $H = \prod_j H^{(j)} = K_{<\infty}$.

Proposition 5.14. *Let Δ be a permutation group with decorated blocks. Define $G = \text{Group}(\Delta)$, and use the notations above. Then, G is a P -oligomorphic permutation group.*

Proof. The subgroup K of G is the direct product of the wreath products $H^{(j)} \wr M^{(j)}$, and therefore P -oligomorphic. This implies the result by Lemma 2.6. □

We proceed by defining the notion of isomorphism for groups with decorated blocks, and checking that it matches the classical notion of isomorphism for P -oligomorphic groups.

Definition 5.15. Let Δ and Δ' be two permutation groups with decorated blocks. Then, Δ and Δ' are isomorphic if there exists an isomorphism between the underlying groups F and F' that transports the block system B and the groups $H^{(j)}$ and $M^{(j)}$ to their equivalents in Δ' .

Lemma 5.16. *Let Δ be a permutation group with decorated blocks. Then $\text{Group}(\Delta)$ is the natural semidirect product $K \rtimes F/H$ with the notations of Definition 5.13, and $\Delta' = \text{Data}(\text{Group}(\Delta))$ is isomorphic to Δ .*

Proof. Denote $G = \text{Group}(\Delta)$. The groups K , F and $K_{<\infty}$ obtained in Subsection 5.2 from G correspond to the groups of same name from Definition 5.13, so the results from that subsection apply and the first part is immediate. In particular, K is a normal subgroup of G of finite index, which in addition, by Proposition 5.2, admits no finite index normal subgroup and is therefore the unique minimal finite index normal subgroup of G . This in turn implies the uniqueness of all the other pieces of $\text{Data}(G)$: the nested block system of G is given by $\mathcal{B}_{\mathcal{B}}(G) = (BB^{(j)})_j$, with $BB^{(j)} = (B_i^{(j)})_i$ and, for i in the support of $M^{(j)}$, $B_i^{(j)} = B^{(j)} \times \{i\}$; the permutation subgroups $G_{<\infty}$ and $K_{<\infty}$ induced respectively by G and K on $\sqcup_j B_0^{(j)}$ are respectively trivially isomorphic to F and H . □

Lemma 5.17. *Let G be a Rev-free P -oligomorphic group. Then, $G' = \text{Group}(\text{Data}(G))$ is isomorphic to G .*

Proof. We use the coherent enumeration to identify the elements of each $B_i^{(j)}$ with that of $B^{(j)} \times \{i\}$. Through this identification and by construction, we have $K' = K$; since the finite groups acting diagonally are the same as well, and using Proposition 5.9, we have indeed $G' = G$. □

Theorem 5.18. *Rev-free P -oligomorphic permutation groups are classified by finite permutation groups with decorated blocks through the Data and Group reciprocal correspondences.*

Proof. Lemma 5.16 together with Lemma 5.16 asserts that Data and Group are reciprocal correspondences, as desired. \square

5.3.2 Extending the classification to all P -oligomorphic groups

We start with an example illustrating that the straightforward extension of Data to all P -oligomorphic groups does not give a proper correspondence.

Example 5.19. Consider the P -oligomorphic group $G = \text{Rev } \mathbb{Q} \times \text{Rev } \mathbb{Q}$, and the index 2 subgroup generated by $G' = \text{Aut } \mathbb{Q} \times \text{Aut } \mathbb{Q}$ on the one hand and the reversal acting simultaneously on the two copies of \mathbb{Q} on the other hand.

Let us try to define Data on G and G' as before; in both cases, we get the same data:

$$(\text{id}(\{1, 2\}), (\{i\})_{i=1,2}, (\text{id}(\{i\}))_{i=1,2}, (\text{Rev}(\mathbb{Q}))_{i=1,2}) .$$

The information about the synchronization of the reversal on the two superblock is lost.

We now tweak the definition of Data to keep track of reversals in the finite group $G_{<\infty}$. To achieve this, each copy of $R = \mathbb{Q}$ (or of $R = \mathbb{Q}/\mathbb{Z}$) where a reversal can occur will be compressed into a block of two points instead of a single one.

Let $BB^{(j)}$ be a superblock; if its blocks are of size 1 and G acts on them by $M^{(j)} = \text{Aut}(\mathbb{Q})$ or $M^{(j)} = \text{Aut}(\mathbb{Q}/\mathbb{Z})$, then define $\bar{B}_0^{(j)}$ by choosing any two points of $E^{(j)}$; note that $\bar{B}_0^{(j)}$ is not a block anymore, but this is fine. Otherwise, define $\bar{B}_0^{(j)}$ as $B_0^{(j)}$.

Define $\bar{G}_{<\infty}$ as before, but using $\sqcup_j \bar{B}_0^{(j)}$ instead of $\sqcup_j B_0^{(j)}$.

Example 5.20. With G and G' as in the previous example, $\bar{G}_{<\infty}$ and $\bar{G}'_{<\infty}$ both act on $\{1, 2\} \sqcup \{1, 2\}$. However $\bar{G}_{<\infty}$ is of size 4, permuting independently the two blocks, whereas $\bar{G}'_{<\infty}$ is of size 2, permuting simultaneously the two blocks.

Definition 5.21. Define $\overline{\text{Data}}(G) = (\bar{G}_{<\infty}, (\bar{B}_0^{(j)})_j, (H^{(j)})_j, (M^{(j)})_j)$.

The definition of permutation group with decorated blocks must be extended accordingly: each $M^{(j)}$ can now be any one of the five closed highly homogeneous groups; however $\bar{B}^{(j)}$ must be of size 2 whenever $M^{(j)}$ is of the form $\text{Rev}(R)$ and of size 1 whenever $M^{(j)}$ is of the form $\text{Aut}(R)$.

The definition of Group must be adjusted as well: if $M^{(j)}$ is of the form $\text{Rev}(R)$ and therefore $B^{(j)}$ is of size 2, then $E^{(j)}$ consists of a single copy of the support of $M^{(j)}$. Also, the diagonal action of an element f of F on E must be adjusted: assumes that $M^{(j)}$ is of the form $\text{Rev}(R)$ and therefore $B^{(j)}$ is of size 2; let j' be such that f maps $B^{(j)}$ to $B^{(j')}$. Then, f maps the elements of $E^{(j)}$ onto those of $E^{(j')}$, with a reversal whenever the elements of $B^{(j)}$ are swapped by f in $B^{(j')}$.

Theorem 5.22. *P-oligomorphic permutation groups are classified by finite permutation groups with (extended) decorated blocks through the $\overline{\text{Data}}$ and $\overline{\text{Group}}$ reciprocal correspondences.*

Proof. Follow the steps of the proof of Theorem 5.18, mutatis-mutandis. \square

6 Resolution of the conjectures and the Cohen-Macaulay property

Here, we use the classification previously obtained to prove that the orbit algebra of G is isomorphic to (a simple quotient of) the invariant ring of a finite permutation group F . We deduce that Macpherson's conjecture holds: $\text{QA}(G)$ is finitely generated, and even Cohen-Macaulay. In addition, the minimal finite index subgroup K prescribes the algebraic dimension of the orbit algebra of G and provides a natural system of parameters, and thus (a choice of) the degrees appearing in the denominator of 1.1, the Hilbert series.

Let G be a (closed) P -oligomorphic group, $K = \prod_j K^{(j)}$ its minimal subgroup of finite index, and use again the notations of the previous section. Let D_G be the set of degrees of the non zero degree elements of the ages $\mathcal{A}(H^{(j)})$ of the $H^{(j)}$'s.

Theorem 6.1. *Let G be a permutation group whose profile is bounded by a polynomial. Then, $\text{QA}(G)$ is isomorphic to the algebra of invariants of some finite permutation group acting on variables of degrees D_G , quotiented by the relations $x^2 = 0$ for some of the variables.*

Proof. For each superblock $BB^{(j)}$, let S_j be the collection of all the non-trivial subsets of all blocks of $BB^{(j)}$. Let $S = \sqcup_j S_j$. By the definition of block systems, K acts on each S_j and on S . Denote by $(\theta_{i,j})_i$ the K -orbits in S_j and observe that they are in bijection with the positive degree part $\mathcal{A}(G)^+(H^{(j)})$ of the age of $H^{(j)}$.

As in Example 2.13, the orbit algebra $\text{QA}(K^{(j)})$ of $K^{(j)}$, for $j \neq 0$, is the free algebra $\mathbb{Q}[(\theta_{i,j})_i]$; for $j = 0$, the orbit algebra is the finite dimensional algebra $\mathbb{Q}[(\theta_{i,0})_i] / (\theta_{i,0}^2 = 0 \forall i)$ instead. The orbit algebra of K itself is the tensor product $\otimes_j \text{QA}(K^{(j)})$, generated by $(\theta_{i,j})_{i,j}$.

The group G itself also acts on S ; since K is normal in G , this lifts to an action on the finitely many K -orbits $(\theta_{i,j})_{i,j}$ in S . Let G_0 be the finite permutation group induced by this action, and let $\mathbb{Q}[(\theta_{i,j})_{i,j}]^{G_0}$ be its invariant ring. Then, $\text{QA}(G)$ is the following quotient thereof:

$$\text{QA}(G) = \mathbb{Q}[(\theta_{i,j})_{i,j}]^{G_0} / (\theta_{i,0}^2 = 0 \forall i).$$

\square

An immediate corollary is a positive resolution of Macpherson's question and therefore Cameron's conjecture.

Corollary 6.2. *The orbit algebra $\mathcal{QA}(G)$ of a P -oligomorphic permutation group G is finitely generated.*

Proof. By Hilbert theorem, the invariant ring of a finite group is finitely generated (see e.g. [18, Theorem 1.2]) and $\mathcal{QA}(G)$ is a quotient thereof. \square

Invariant rings of permutation groups are not only finitely generated and but also Cohen-Macaulay (see e.g. [18, Theorem 3.2]). The quotient of a Cohen-Macaulay algebra is not Cohen-Macaulay in general, but the statement and proof of [18, Theorem 3.2] can be generalized to suitable quotients of invariant rings to deduce that orbit algebras are Cohen-Macaulay.

Let us first recall the following classical fact about graded commutative algebras.

Fact 6.3. Let $R = \bigoplus_{d \in \mathbb{N}} R_d$ be a graded connected commutative algebra (over a field \mathbb{K}) and take a free graded R -module A . Then, from any homogeneous \mathbb{K} -basis of A , we may extract a subfamily which is an R -basis of A .

Proof. Write $R^+ = \bigoplus_{d > 0} R_d$, and consider a family $(v_k)_{k \in K}$ of secondary invariants for A : $A = \bigoplus_{k \in K} v_k R$; let then $(\beta_j)_j$ be a vector basis of A , each β_j being chosen homogeneous. Proceed by degree d : note that the elements v_k of degree d form a vector basis of a supplementary in A_d of the homogeneous component $(R^+ A)_d$ of the ideal $R^+ A$; extract from $(\beta_j)_j$ a basis of some other supplementary. We obtain a subfamily $(\mu_k)_{k \in K}$ which spans M as an R -module and is equinumerous in each degree with the family $(v_k)_k$. By dimension count, it is an R -basis of M . \square

Lemma 6.4. *Let F be a finite permutation group of a set $X \sqcup Y$ of variables. Assume that F stabilizes X and Y . Then, $\mathbb{Q}[X \sqcup Y]^F / (y^2, y \in Y)$ is Cohen-Macaulay.*

Proof. Since F stabilizes X and Y , we may take $\text{Sym}(X) \text{Sym}(Y)$ as ring of primary invariants, over which $A = \mathbb{Q}[X \sqcup Y]^F$ is a finite dimensional free module; in other words, there exists a finite collection of homogeneous invariants v_k such that we have:

$$A = \bigoplus_{k \in K} v_k \text{Sym}(X) \text{Sym}(Y).$$

It follows that A is an (infinite dimensional) $\text{Sym}(X)$ -free module. For instance, we have:

$$A = \bigoplus_{k, \lambda} v_k p_\lambda(Y) \text{Sym}(X),$$

where $(p_\lambda(Y))_\lambda$ is the *power sum multiplicative basis* of $\text{Sym}(Y)$: for λ an integer partition, $p_\lambda(Y) = \prod_{d \in \lambda} p_d(Y)$, where $p_d(Y) := \sum_{y \in Y} y^d$ is the d -th *symmetric power sum*. A first consequence is that the quotient $A / (y^2, y \in Y)$ is a finitely generated $\text{Sym}(X)$ -module:

$$A / (y^2, y \in Y) = \sum_{k \in K, i=1, \dots, |Y|} v_k p_1^i(Y) \text{Sym}(X).$$

There remains to show that it is a free module. To this end, let us come back to A and consider its monomial basis, that is the collection of all F -orbit sums of monomials in $\mathbb{Q}[X \sqcup Y]$. By Fact 6.3, we may extract from the monomial basis a family $(\mu_\ell)_{\ell \in L}$ which forms a $\text{Sym}(X)$ -basis of A :

$$A = \bigoplus_{\ell \in L} \mu_\ell \text{Sym}(X).$$

Remark that, if a monomial is divisible by y^2 for some $y \in Y$, then the same holds for any other monomial in its orbit sum. Therefore, in the quotient, each monomial μ_ℓ either vanishes completely or is left unaffected, and the same holds accordingly for the principal ideal $\mu_\ell \text{Sym}(X)$. Therefore, we have indeed:

$$A/(y^2, y \in Y) = \bigoplus_{\ell \in L'} \mu_\ell \text{Sym}(X),$$

for some subset L' of L , so that $A/(y^2)$ is a finite dimensional free $\text{Sym}(X)$ -module, as desired. \square

Corollary 6.5. *The orbit algebra $\mathbb{Q}\mathcal{A}(G)$ of a P -oligomorphic permutation group G is Cohen-Macaulay.*

References

- [1] J. Balogh et al. “The unlabelled speed of a hereditary graph property”. In: *J. Combin. Theory Ser. B* 99.1 (2009), pp. 9–19. doi: [10.1016/j.jctb.2008.03.004](https://doi.org/10.1016/j.jctb.2008.03.004).
- [2] B. Bollobás. “Hereditary properties of graphs: asymptotic enumeration, global structure, and colouring”. In: *Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998)*. Extra Vol. III. 1998, 333–342 (electronic).
- [3] P. J. Cameron. *Oligomorphic permutation groups*. Vol. 152. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1990, pp. viii+160.
- [4] P. J. Cameron. “Oligomorphic permutation groups”. In: *Perspectives in Mathematical Sciences II: Pure Mathematics*. Ed. by Mohan Delampady N. S. Narasimha Sastry T. S. S. R. K. Rao and B. Rajeev. World Scientific, Singapore, 2009, pp. 37–61.
- [5] P. J. Cameron. “The algebra of an age”. In: *Model theory of groups and automorphism groups (Blaubeuren, 1995)*. Vol. 244. London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, 1997, pp. 126–133.
- [6] Justine Falque. “Classification of P-oligomorphic groups; Conjectures of Cameron and Macpherson”. 136 pages. PhD thesis. Université Paris-Sud, Nov. 2019.
- [7] Justine Falque and Nicolas M. Thiéry. “Macpherson’s conjecture holds: The orbit algebra of a permutation group with polynomial profile is finitely generated”. In: *30th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2018, Hanover)*. 2018.
- [8] R. Fraïssé. *Theory of relations*. Revised. Vol. 145. Studies in Logic and the Foundations of Mathematics. With an appendix by Norbert Sauer. Amsterdam: North-Holland Publishing Co., 2000, pp. ii+451.
- [9] T. Kaiser and M. Klazar. “On growth rates of closed permutation classes”. In: *Electron. J. Combin.* 9.2 (2002/03). Permutation patterns (Otago, 2003), Research paper 10, 20 pp. (electronic). URL: http://www.combinatorics.org/Volume_9/Abstracts/v9i2r10.html.
- [10] M. Klazar. “Overview of some general results in combinatorial enumeration”. In: *Arxiv preprint arXiv:0803.4292* (2008).
- [11] H. D. Macpherson. “Growth rates in infinite graphs and permutation groups”. In: *Proceedings of the London Mathematical Society* 3.2 (1985), pp. 285–294.
- [12] H. D. Macpherson. “Orbits of infinite permutation groups”. In: *Proc. London Math. Soc.* (3) 51.2 (1985), pp. 246–284. ISSN: 0024-6115. doi: [10.1112/plms/s3-51.2.246](https://doi.org/10.1112/plms/s3-51.2.246). URL: <https://doi.org/10.1112/plms/s3-51.2.246>.
- [13] M. Pouzet. “The profile of relations”. In: *Glob. J. Pure Appl. Math.* 2.3 (2006), pp. 237–272.

- [14] M. Pouzet. “When is the orbit algebra of a group an integral domain? Proof of a conjecture of P. J. Cameron”. In: *Theor. Inform. Appl.* 42.1 (2008), pp. 83–103.
- [15] M. Pouzet and N. M. Thiéry. “Some relational structures with polynomial growth and their associated algebras I: Quasi-polynomiality of the profile”. In: *Electron. J. Combin.* 20.2 (2013), Paper 1, 35.
- [16] M. Pouzet and N. M. Thiéry. *Some relational structures with polynomial growth and their associated algebras II: Finite generation*. In preparation, 26 pages. 2018.
- [17] C. Ryll-Nardzewski. “On the categoricity in power $\leq \aleph_0$ ”. In: *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 7 (1959), 545–548. (unbound insert). issn: 0001-4117.
- [18] R. P. Stanley. “Invariants of finite groups and their applications to combinatorics”. In: *Bull. Amer. Math. Soc. (N.S.)* 1.3 (1979), pp. 475–511.