On the orbit algebra of P-oligomorphic permutation groups

Justine Falque
(LRI, Univ. Paris-Sud, France)

Nicolas Thiéry
(LRI, Univ. Paris-Sud, France)

Proving two conjectures of Cameron and Macpherson

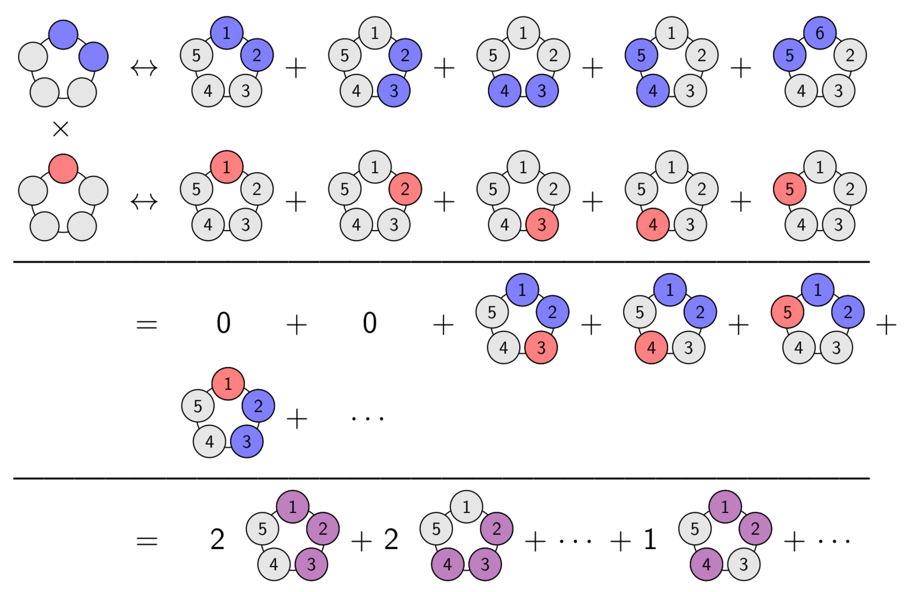
A conjecture of Cameron (70's)

- G : infinite permutation group, acting on a denumerable set E
- The *age* of G is the set of orbits of G for its induced action on the finite subsets of E.
  The orbit of a subset of size n is of *degree* n.
- The *profile* $\varphi_G(n)$ counts the number of orbits of degree n.
- G is *P-oligomorphic* when $\varphi_G(n)$ is bounded by a polynomial.
- **Conjecture** (Cameron, 70's): If G is P-oligomorphic, the generating series of its profile is of shape $\frac{P(z)}{\prod_i (1-z^{d_i})}$ $(P(z) \in \mathbb{Z}[z])$.
  The analogue was proved in 2003 for permutation classes and in 2009 for undirected graphs.

Algebraization and a stronger conjecture

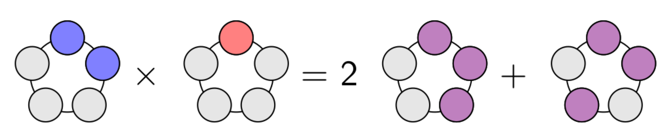
- The *set algebra* of E: a graded connected commutative algebra on (possibly infinite) formal linear combinations of finite subsets of E. Product: induced by the disjoint union of sets.
- The *orbit algebra* of G: the subalgebra $\mathscr{A}_G$ whose basis is indexed by the orbits of subsets, each basis element being the formal sum of the subsets in that orbit.
  Fact: The generating series of the profile $\sum_n \varphi_G(n) z^n$ coincides with the *Hilbert series* of the age algebra.
- **Conjecture** (Macpherson, 1985): If G is P-oligomorphic, then its orbit algebra is finitely generated.

Example: detailed product in a finite case

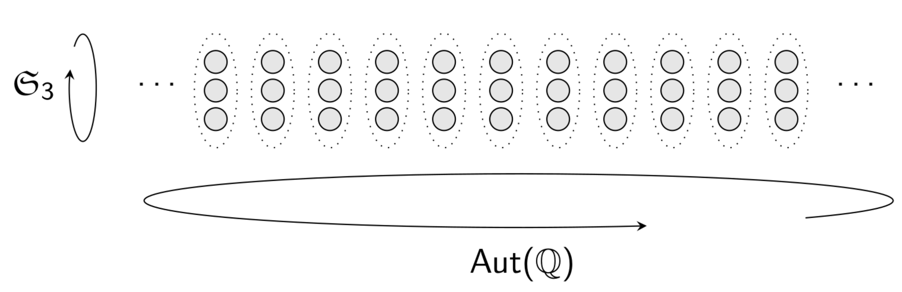


Product between two orbits (blue and red, resp.) of the cyclic group C_5

Result of the product:

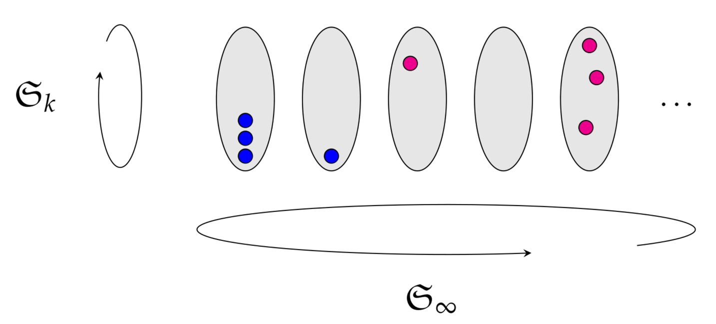


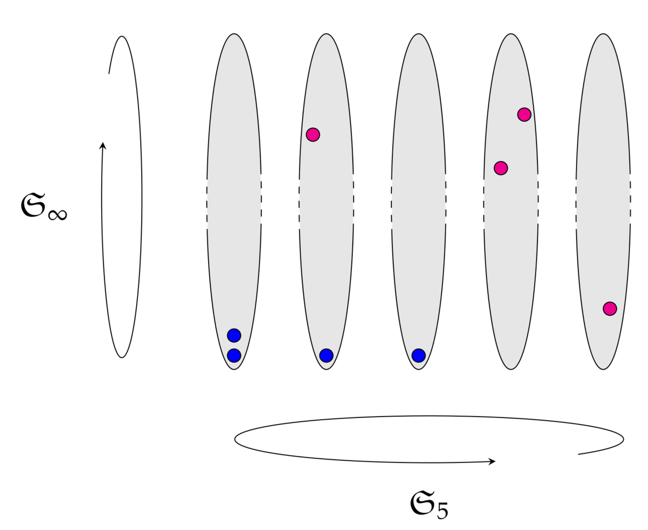
Infinite example: a wreath product



$G = \mathfrak{S}_3 \wr \mathrm{Aut}(\mathbb{Q})$
Independant copies of $\mathfrak{S}_3$ act within each block of size 3.
Age: all integer compositions

Two ages isomorphic to integer partitions





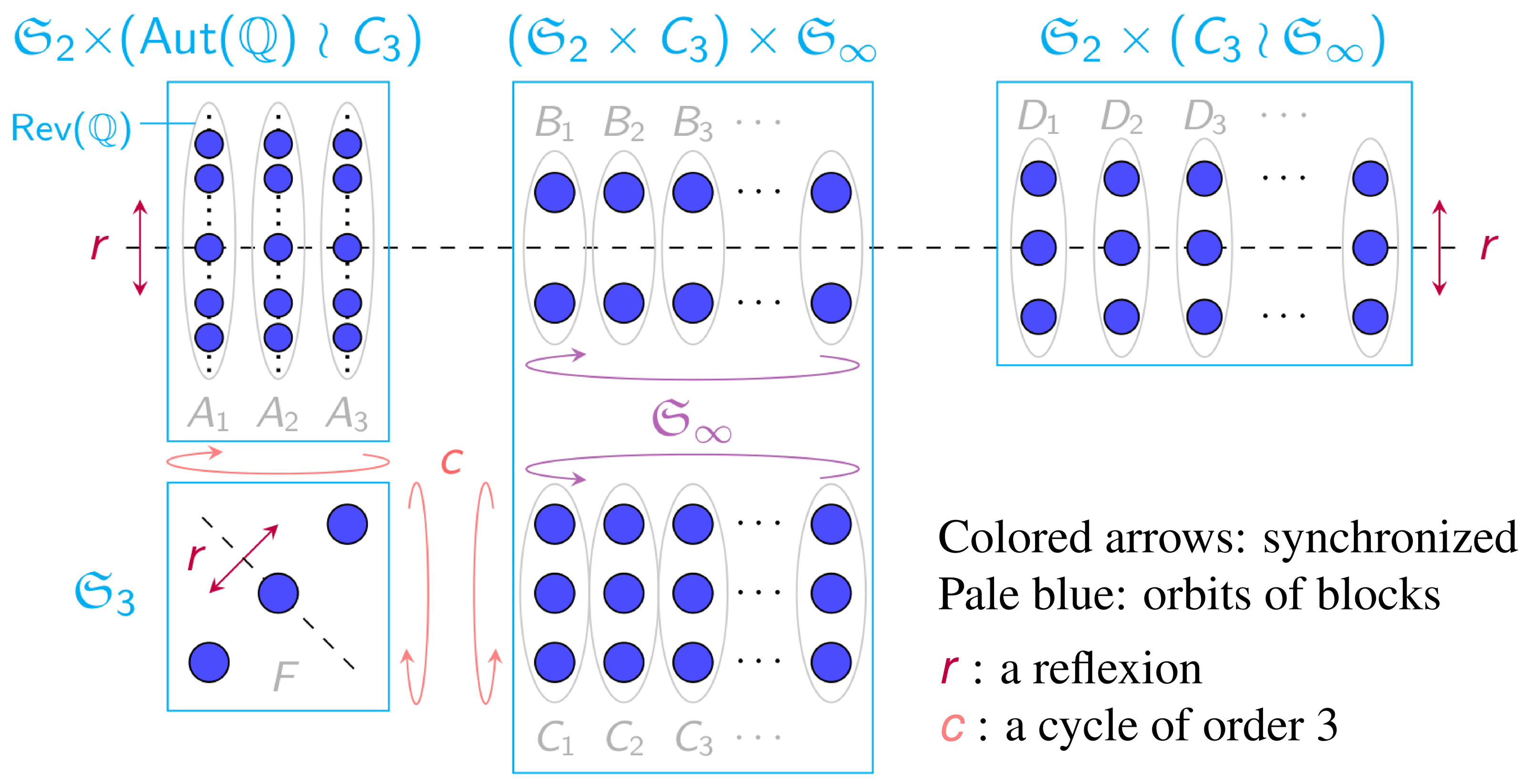
- $G = \mathfrak{S}_k \wr \mathfrak{S}_\infty$
  Age: the integer partitions of n with parts at most k, for every $n \in \mathbb{N}$
- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_5$
  Age: the integer partitions of n with at most k parts, $n \in \mathbb{N}$
  $\mathscr{A}_G \sim \mathbb{K}[x_1, \ldots, x_5]^{\mathfrak{S}_5}$

The blue and red subsets are in the same orbit in each case.

Main Theorem (J.F. & N.T. , 2017)

The orbit algebra of a P-oligomorphic group is finitely generated. Furthermore, it is a Cohen-Macauley algebra, leading the series of the profile to be of shape $\frac{P(z)}{\prod_i (1-z^{d_i})}$ with $P(z) \in \mathbb{N}[z]$.

Example: a typical P-oligomorphic group



Colored arrows: synchronized actions
Pale blue: orbits of blocks
r : a reflexion
c : a cycle of order 3

Tools and ideas of the proof

- Strategy: make use of the fact that if the action of G on two stable parts E_1 and E_2 are independent, then $\mathscr{A}_G = \mathscr{A}_{G_{|E_1}} \otimes \mathscr{A}_{G_{|E_2}}$
- Pb: synchronizations between orbits of elements (ex: $\mathfrak{S}_2$, $\mathfrak{S}_\infty$)
- Idea: split E into *blocks* of elements and consider the G-orbits of blocks (when well defined)
  Only two types if their support is infinite:
  → finitely many infinite blocks (ex: A on figure)
  → infinitely many finite ones (ex: D)
- *Canonical block system*: chosen such that no synchronization between orbits of blocks is infinite (like between B and C).
  On the example, take $(A_i)_i$, $(BC_i)_i$ (joined), $(D_i)_i$ and F.
- This property is obtained using some infinite permutation group theory, such as the study of primitive groups by Cameron and Macpherson and the notion of *subdirect product*.
- The remaining synchronizations are isomorphic to finite groups ($\mathfrak{S}_2$ and C_3 on the example).
  Take a finite index normal subgroup H of G that erases them.
- Apply the strategy on the orbits of blocks of H, which proves that Macpherson's conjecture holds for H.
- Invariant theory ⇒ the result can be lifted from H to G

Special thanks

We warmly thank Maurice Pouzet for suggesting to work on this conjecture and supporting us, and Peter Cameron for enlightening discussions while walking by a misty seashore or enjoying some winter ice cream !

References

P.J. Cameron, *Oligomorphic Permutation Groups* (1990) and "The algebra of an age" (1997) In: *Model theory of groups and auto- morphism groups*
H.D. Macpherson, "Growth rates in infinite graphs and permutation groups", In: *Proceedings of the London Mathematical Society* 3.2