

Orbital profile and orbit algebra of oligomorphic permutation groups

Conjecture of Macpherson

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joint work with Nicolas M. Thiéry

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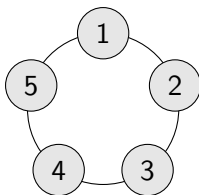
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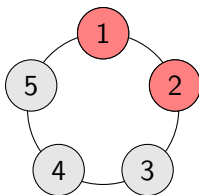
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Action of the cyclic group $G = C_5$ on the five pearl necklace
→ induced action on subsets of pearls



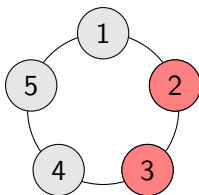
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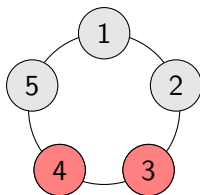
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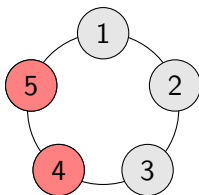
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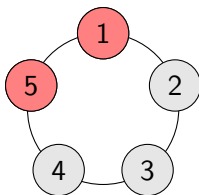
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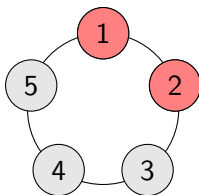
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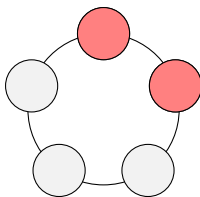
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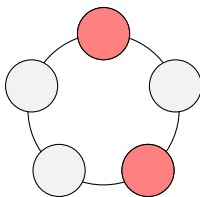
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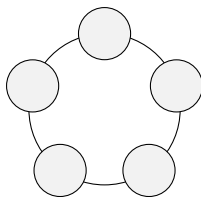
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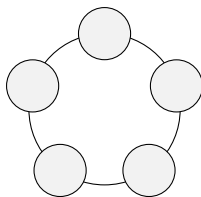
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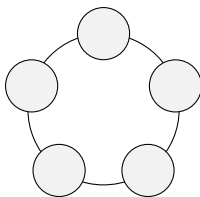
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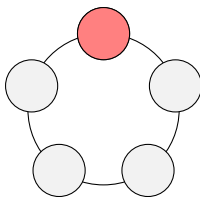
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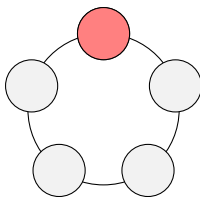
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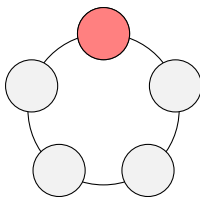
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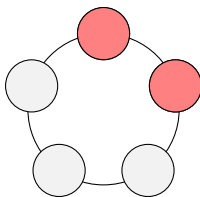
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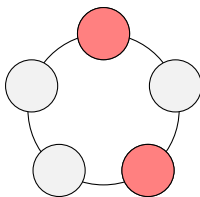
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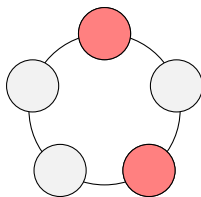
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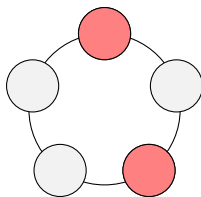
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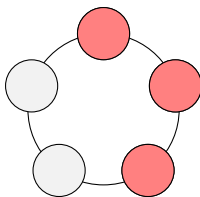
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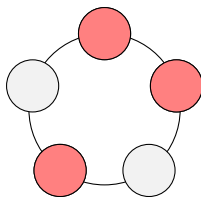
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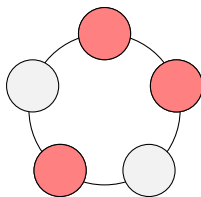
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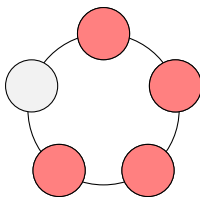
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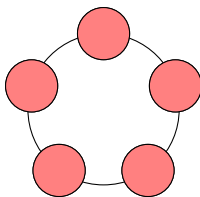
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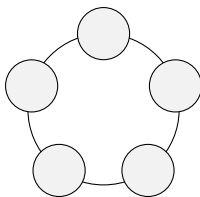
$$\varphi_G(2) = 2$$

$$\varphi_G(3) = 2$$

$$\varphi_G(4) = 1$$

$$\varphi_G(5) = 1$$

$$\varphi_G(n) = 0 \text{ si } n > 5$$



Age and profile: example on a finite group (2)

Generating polynomial of the profile:

$$\mathcal{H}_G(z) = \sum_{n \geq 0} \varphi_G(n) z^n = 1 + z + 2z^2 + 2z^3 + z^4 + z^5$$

Can be calculated straightly by Pólya's theory

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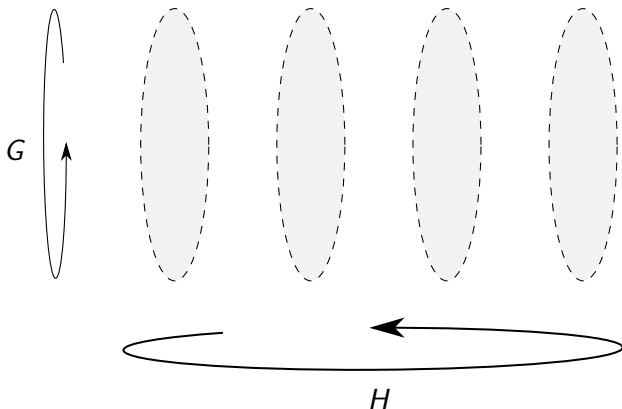
→ **Oligomorphic permutation groups:**

$$\varphi_G(n) < \infty \quad \forall n \in \mathbb{N}$$

Wreath product of two permutation groups

$$G \leq \mathfrak{S}_M, H \leq \mathfrak{S}_N$$

$G \wr H$ has a natural action on $E = \sqcup_{i=1}^N E_i$, with $\text{card} E_i = M$.



Examples

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$ (action on a denumerable set of copies of \mathbb{N})

An orbit of degree $n \longleftrightarrow$ a partition of n

$\varphi_G(n) = \mathcal{P}(n)$, the number of partitions of n

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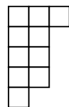
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- $G = \mathfrak{S}_m \wr \mathfrak{S}_\infty$

$\varphi_G(n) = \mathcal{P}_m(n)$, number of partitions into parts of size $\leq m$

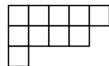
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- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_m$

$\varphi_G(n) = \mathcal{P}_m(n)$, number of partitions into at most m parts

$$\mathcal{H}_G = \frac{1}{\prod_{i=1}^m (1 - z^i)}$$



Growth of the profile

Proposition

Orbital profiles are non decreasing.

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Theorem (Pouzet)

If an orbital profile is bounded by a polynomial, it is asymptotically equivalent to a polynomial.

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Theorem (Pouzet)

If an orbital profile is bounded by a polynomial, it is asymptotically equivalent to a polynomial.

Note

The number $\mathcal{P}(n)$ of partitions of n is neither bounded by a polynomial nor exponential.

Conjecture of Cameron

Conjecture (Cameron)

If a profile is bounded by a polynomial (thus polynomial) it is **quasi-polynomial**:

$$\varphi_G(n) = a_s(n)n^s + \cdots + a_1(n)n + a_0(n),$$

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Note

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})} \implies \varphi_G \text{ quasi-polynomial of degree at most } k - 1$$

Graded algebras

Definition: Graded algebra

$A = \bigoplus_n A_n$ such that $A_i A_j \subseteq A_{i+j}$.

Example

$A = \mathbb{K}[x_1, \dots, x_m]$ is a graded algebra.

A_n : homogeneous polynomials of degree n

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Proposition

A is finitely generated $\implies \text{Hilbert}(A) = \frac{P(z)}{(1-z^{d_1}) \dots (1-z^{d_k})}$

Example

$\text{Hilbert}(\mathbb{Q}[x, y, t^3]) = \frac{1}{(1-z)^2(1-z^3)}$

A strategy to prove Cameron's conjecture?

- G : an oligomorphic permutation group with polynomial profile
- Find a graded algebra $\mathbb{Q}\mathcal{A}(G) = \bigoplus_{n \geq 0} A_n$ such that

$$\mathcal{H}_G = \text{Hilbert}(\mathbb{Q}\mathcal{A}(G))$$

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- Try to show that $\mathbb{Q}\mathcal{A}(G)$ is finitely generated
- Deduce:

$$\mathcal{H}_G = \frac{P(z)}{(1 - z^{d_1}) \cdots (1 - z^{d_k})}$$

and thus the quasi-polynomiality of $\varphi_G(n)$

Cameron, 1980: the orbit algebra $\mathbb{Q}\mathcal{A}(G)$

- a commutative connected graded algebra $\mathbb{Q}\mathcal{A}(G) = \bigoplus_{n \geq 0} A_n$
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Vector space structure

- finite formal linear combinations of orbits (ex: $2o_1 + 5o_2 - o_3$)
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Product?

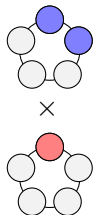
- Defined on subsets:

$$ef = \begin{cases} e \cup f & \text{if } e \cap f = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

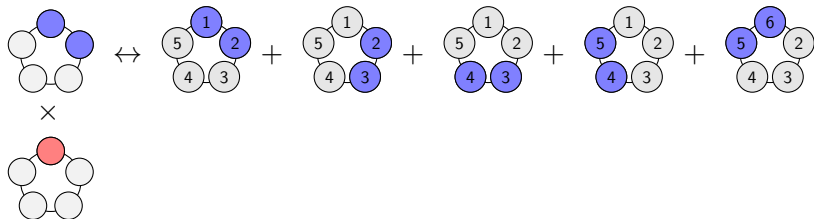
- $o = \{e_1, e_2, \dots\} \longleftrightarrow e_1 + e_2 + \dots$

Example of product on a finite case

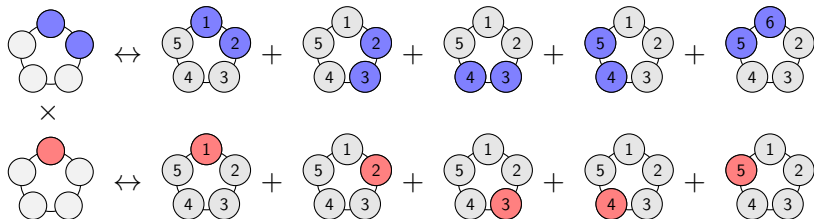
Example of product on a finite case



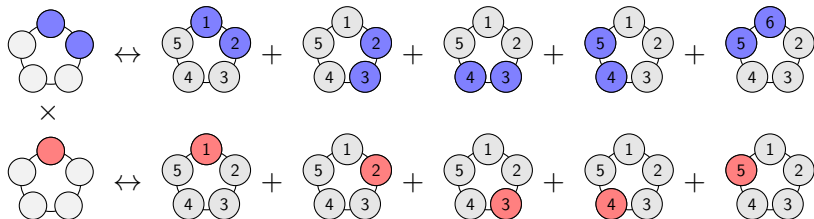
Example of product on a finite case



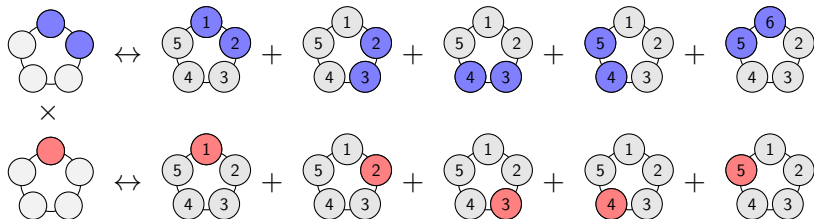
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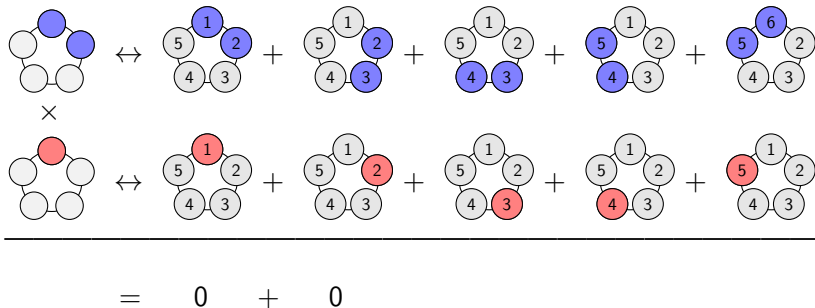


Example of product on a finite case

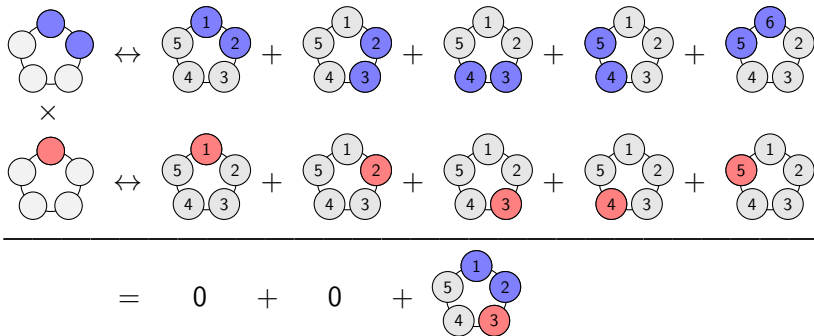


= 0

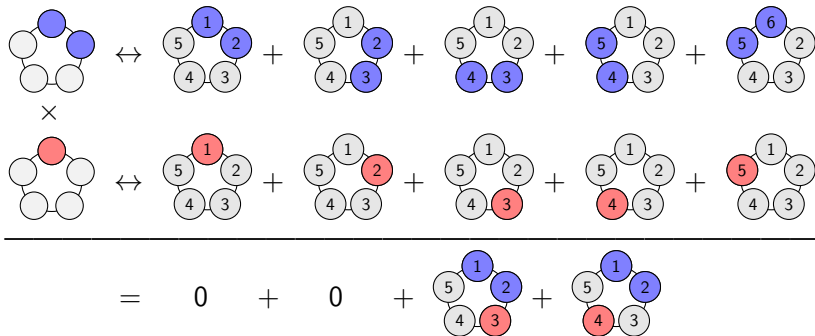
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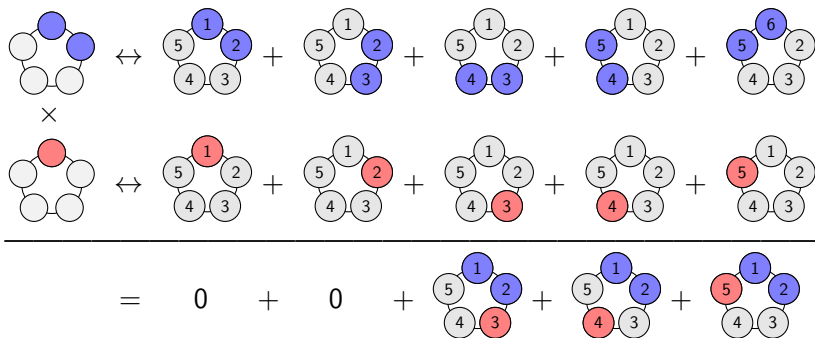
Example of product on a finite case



Example of product on a finite case



Example of product on a finite case

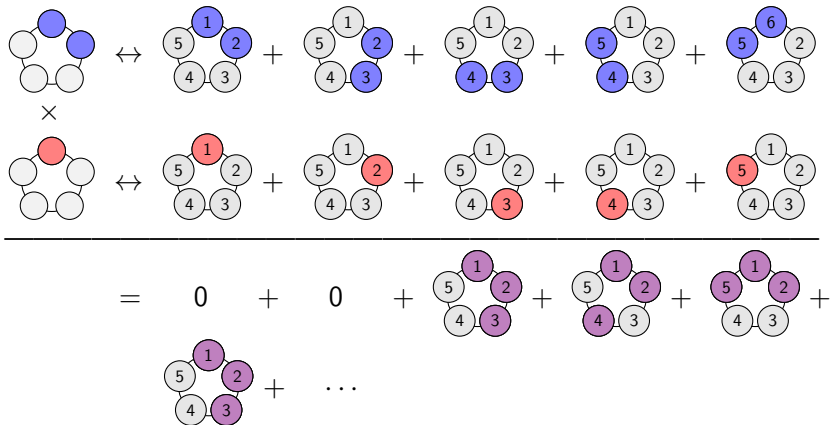


Example of product on a finite case

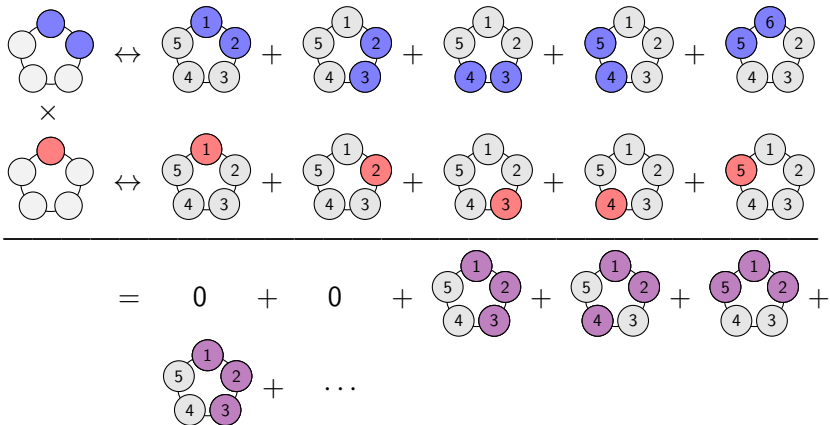
$$\begin{array}{c}
 \begin{array}{c}
 \text{Diagram 1} \\
 \times \\
 \text{Diagram 2}
 \end{array} \\
 \Leftrightarrow \\
 \begin{array}{c}
 \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 + \\
 \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12}
 \end{array} \\
 \hline
 = \\
 \begin{array}{c}
 0 + 0 + \text{Diagram 13} + \text{Diagram 14} + \text{Diagram 15} \\
 + \\
 \text{Diagram 16} + \dots
 \end{array}
 \end{array}$$

The diagrams are 5-cycles with nodes labeled 1 to 5. The top-left diagram has nodes 1 and 2 colored blue. The bottom-left diagram has node 1 colored red. The top row of diagrams shows the expansion of the product of the top-left and bottom-left diagrams. The bottom row shows the simplification of this product, with the first two terms being zero.

Example of product on a finite case



Example of product on a finite case



Example of product on a finite case

$$\begin{array}{c}
 \begin{array}{c}
 \text{Diagram 1} \\
 \times \\
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 \end{array} \\
 \Leftrightarrow \begin{array}{c}
 \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 + \\
 \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12}
 \end{array} \\
 \hline
 = \begin{array}{c}
 0 + 0 + \text{Diagram 13} + \text{Diagram 14} + \text{Diagram 15} \\
 + \text{Diagram 16} + \dots
 \end{array} \\
 \hline
 = 2 \begin{array}{c}
 \text{Diagram 17}
 \end{array}
 \end{array}$$

The diagrams are 5-cycles with nodes labeled 1, 2, 3, 4, 5. In the first row, the top diagram has nodes 1 and 2 colored blue, and the bottom diagram has node 1 colored red. The second row shows the expansion of the product into 12 terms, where nodes are colored blue, red, or grey. The third row shows the simplification of these terms, with some terms being zero and others being purple. The final row shows the result as 2 times a purple 5-cycle with nodes 1, 2, 3, 4, 5.

Example of product on a finite case

$$\begin{array}{c}
 \begin{array}{c}
 \text{Diagram 1} \\
 \times \\
 \text{Diagram 2}
 \end{array} \\
 \Leftrightarrow \begin{array}{c}
 \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 + \\
 \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12}
 \end{array} \\
 \hline
 = \begin{array}{c}
 0 + 0 + \text{Diagram 13} + \text{Diagram 14} + \text{Diagram 15} + \\
 \text{Diagram 16} + \dots
 \end{array} \\
 \hline
 = 2 \begin{array}{c}
 \text{Diagram 17} \\
 + \\
 \text{Diagram 18} \\
 + \dots
 \end{array}
 \end{array}$$

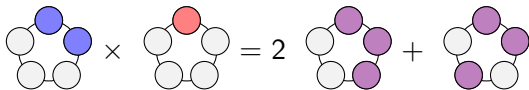
The diagrams are pentagons with nodes labeled 1, 2, 3, 4, 5. In the first row, the top node is 1, the right node is 2, the bottom node is 3, the left node is 4, and the top-left node is 5.

Example of product on a finite case

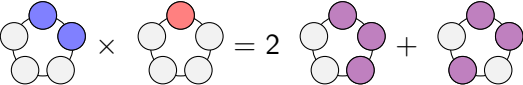
$$\begin{array}{c}
 \begin{array}{c}
 \text{Diagram 1} \\
 \times \\
 \text{Diagram 2}
 \end{array} \\
 \Leftrightarrow \begin{array}{c}
 \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 + \\
 \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12}
 \end{array} \\
 \hline
 = \begin{array}{c}
 0 + 0 + \text{Diagram 13} + \text{Diagram 14} + \text{Diagram 15} + \\
 \text{Diagram 16} + \dots
 \end{array} \\
 \hline
 = \begin{array}{c}
 2 \text{Diagram 17} + 2 \text{Diagram 18} + \dots + 1 \text{Diagram 19} + \dots
 \end{array}
 \end{array}$$

The diagrams are 5-cycles with nodes labeled 1, 2, 3, 4, 5. The top row shows the product of a cycle with nodes 1 and 2 highlighted in blue and a cycle with node 1 highlighted in red. The middle row shows the resulting sum of cycles with various nodes highlighted in blue, red, or purple. The bottom row shows the final simplified sum with coefficients.

In the end:



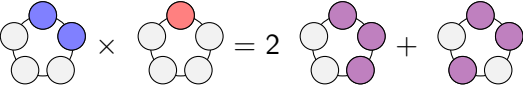
In the end:

$$\begin{array}{c} \bullet \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \end{array} \times \begin{array}{c} \bullet \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \end{array} = 2 \begin{array}{c} \bullet \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \end{array} + \begin{array}{c} \bullet \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \end{array}$$


Non trivial fact

Product well defined (and graded) on the space of orbits.

In the end:

$$\begin{array}{c} \bullet \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \end{array} \times \begin{array}{c} \bullet \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \end{array} = 2 \begin{array}{c} \bullet \\ \bullet \\ \circ \quad \bullet \\ \circ \quad \bullet \\ \circ \end{array} + \begin{array}{c} \bullet \\ \bullet \\ \circ \quad \bullet \\ \bullet \quad \circ \\ \circ \end{array}$$


Non trivial fact

Product well defined (and graded) on the space of orbits.

→ **The orbit algebra of a permutation group**

Examples of orbit algebras (1)

Example 1

If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n , and $\mathcal{QA}(G) = \mathbb{K}[x]$.

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Examples of orbit algebras (1)

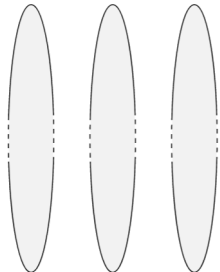
Example 1

If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n , and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

Example 2

$G = \mathfrak{S}_\infty \wr \mathfrak{S}_3$, recall that $\varphi_G(n) = \mathcal{P}_3(n)$.

$A_n =$ homogeneous symmetric polynomials of degree n in x_1, x_2, x_3

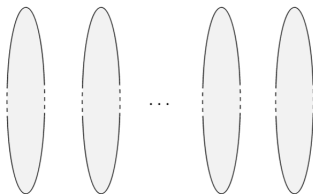


$$\rightarrow \mathbb{Q}\mathcal{A}(\mathfrak{S}_\infty \wr \mathfrak{S}_3) = \mathbb{K}[x_1, x_2, x_3]^{\mathfrak{S}_3}$$

Examples of orbit algebras (2)

More generally, for H subgroup of \mathfrak{S}_m :

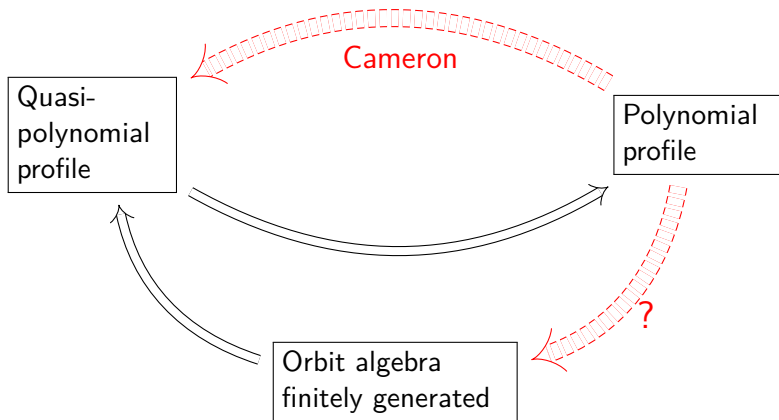
- $G = \mathfrak{S}_\infty \wr H$:
 $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x_1, \dots, x_m]^H$, the algebra of invariants of H
 $\mathbb{Q}\mathcal{A}(G)$ is finitely generated by Hilbert's theorem.



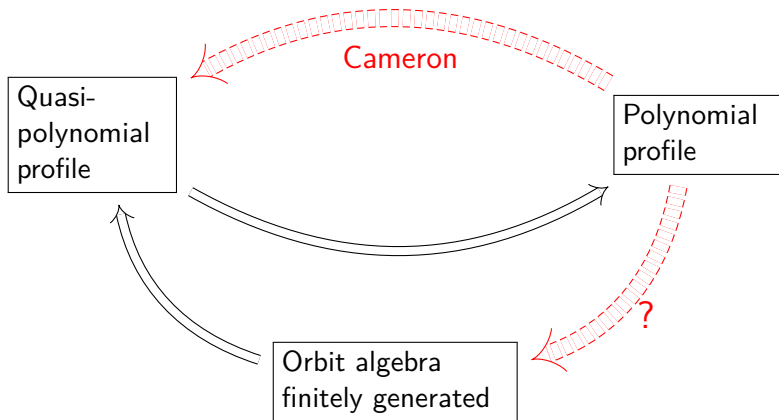
- $G = H \wr \mathfrak{S}_\infty$:
 $\mathbb{Q}\mathcal{A}(G) =$ the free algebra generated by the age of H



Overview and conjecture of Macpherson



Overview and conjecture of Macpherson



Conjecture (Macpherson, 1985)

Profile of G polynomial $\iff \mathcal{QA}(G)$ finitely generated

Block systems

Definition: Block system

Partition of E such that each part is globally mapped to another one (or itself) by every element of G

(see previous examples)

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Theorem (Cameron)

If G is **primitive** (i.e. admits no non trivial block system) then G has its profile equal to 1 or exponential.

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Relevant notion?

Theorem (Cameron)

If G is **primitive** (i.e. admits no non trivial block system) then G has its profile equal to 1 or exponential.

→ The groups we are interested in have a constanly equal to 1 profile or have a block system.

The complete primitive groups

Theorem (Classification, Cameron)

There are exactly 5 complete groups of constantly equal to 1 profile.

The complete primitive groups

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There are exactly 5 complete groups of constantly equal to 1 profile.

- $\text{Aut}(\mathbb{Q})$: automorphisms of the rational chain (increasing functions)
- $\text{Rev}(\mathbb{Q})$: generated by $\text{Aut}(\mathbb{Q})$ and one reflection
- $\text{Aut}(\mathbb{Q}/\mathbb{Z})$, preserving the circular order
- $\text{Rev}(\mathbb{Q}/\mathbb{Z})$: generated by $\text{Aut}(\mathbb{Q}/\mathbb{Z})$ and one reflection
- \mathfrak{S}_∞ : the symmetric group

Tools

- Block structure of each orbit
 - Knowledge of algebras of wreath products
 - Embedding
- ⇒ lower bound on the profile

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Tools

- Block structure of each orbit
- Knowledge of algebras of wreath products
- Embedding
 - \implies lower bound on the profile
- Invariant theory for finite groups (Hilbert's theorem)
 - \implies reduction of the conjecture to essential cases
- Classification of primitive groups
- Goursat's lemma (subdirect product)
 - \implies information on the age

Macpherson for bounded profiles

- First proof by Pouzet

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→ orbit algebra = $\mathbb{K}[x]$

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- By reduction, one can assume G is one of the five primitive groups (with polynomial profile)

→ orbit algebra = $\mathbb{K}[x]$

- Without reduction (constructive proof):
 - same age as $\mathfrak{S}_\infty \times G'$, G' a finite group determined by G
 - generating series: $\frac{P(z)}{(1-z)}$,
where $P(z)$ is the generating polynomial of G'

Macpherson for linear profiles

Two essential cases

- 2 infinite orbits without blocks
- an infinity of blocks of size 2

Macpherson for linear profiles

Two essential cases

- 2 infinite orbits without blocks
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→ The conjectures of Macpherson and Cameron hold.

Context

- G : permutation group of a countably infinite set E
- Profile φ_G : counts the orbits of finite subsets of E
- **Hypothesis**: $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron): quasi-polynomiality of φ_G
- Conjecture (Macpherson): finite generation of the orbit algebra

Results

- Block structure of $G \implies$ minoration of φ_G
- Lemmas and reductions \implies bounded and linear cases

Conjectures / intuition

- The orbit algebra is of Cohen-Macaulay
- The growth of the profile is determined by the block structure
- Very rigid: very few groups; classification?