Orbital profile and orbit algebra of oligomorphic permutation groups

Conjecture of Macpherson

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March 29th of 2017

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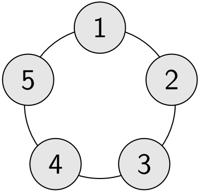
$\to$ induced action on subsets of pearls



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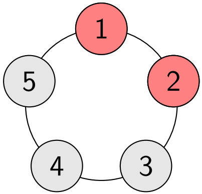

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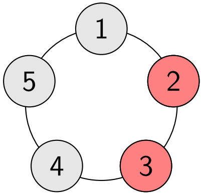

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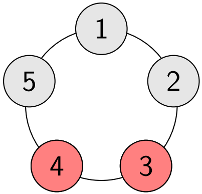

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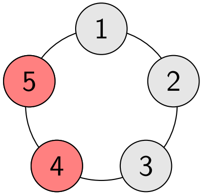

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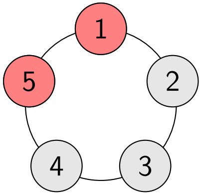

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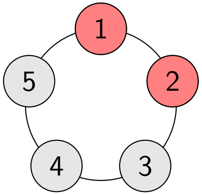

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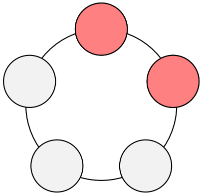

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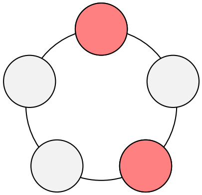

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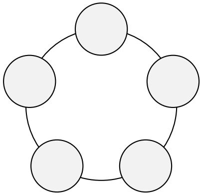

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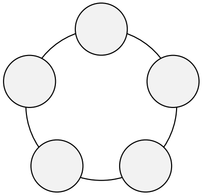

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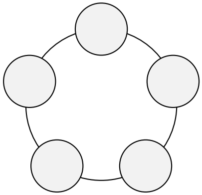

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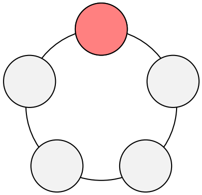

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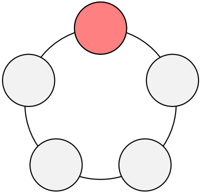

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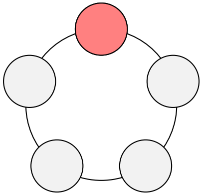

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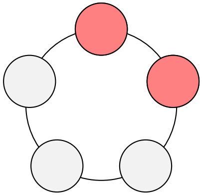

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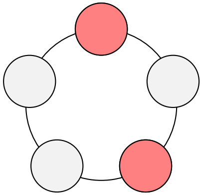

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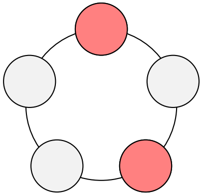

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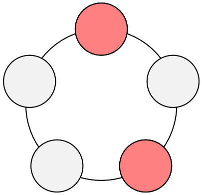

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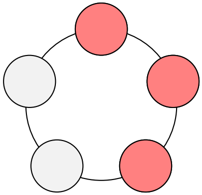

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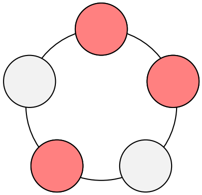

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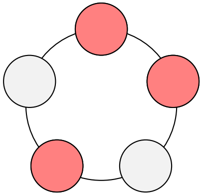

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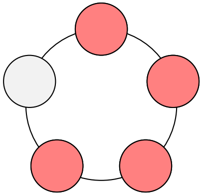

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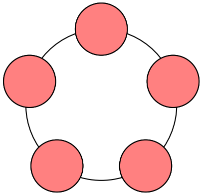

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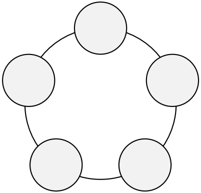
$\varphi_G(n) = 0$ si $n > 5$



Age and profile: example on a finite group (2)

Generating polynomial of the profile:

$$\mathcal{H}_G(z) = \sum_{n \geq 0} \varphi_G(n) z^n = 1 + z + 2z^2 + 2z^3 + z^4 + z^5$$

Can be calculated straightly by Pólya's theory

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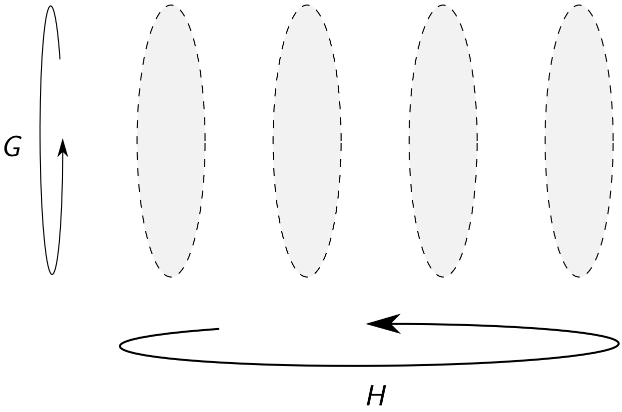
$\rightarrow$ **Oligomorphic permutation groups**:

$$\varphi_G(n) < \infty \quad \forall n \in \mathbb{N}$$

Wreath product of two permutation groups

$G \leq \mathfrak{S}_M$, $H \leq \mathfrak{S}_N$

$G \wr H$ has a natural action on $E = \sqcup_{i=1}^{N} E_i$, with $\mathrm{card} E_i = M$.



Examples

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$ (action on a denumerable set of copies of $\mathbb{N}$)

  An orbit of degree $n \longleftrightarrow$ a partition of n

  $\varphi_G(n) = \mathscr{P}(n)$, the number of partitions of n

$$\mathcal{H}_G = \frac{1}{\prod_{i=1}^{\infty}(1 - z^i)}$$

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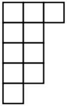
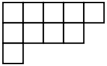
- $G = \mathfrak{S}_m \wr \mathfrak{S}_\infty$
  $\varphi_G(n) = \mathscr{P}_m(n)$, number of partitions into parts of size $\leq m$

  $$\mathcal{H}_G = \frac{1}{\prod_{i=1}^{m}(1-z^i)}$$

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_m$
  $\varphi_G(n) = \mathscr{P}_m(n)$, number of partitions into at most m parts

  $$\mathcal{H}_G = \frac{1}{\prod_{i=1}^{m}(1-z^i)}$$

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If an orbital profile is bounded by a polynomial, it is asymptotically equivalent to a polynomial.

Note

The number $\mathscr{P}(n)$ of partitions of n is neither bounded by a polynomial nor exponential.

Conjecture of Cameron

Conjecture (Cameron)

If a profile is bounded by a polynomial (thus polynomial) it is **quasi-polynomial**:

$$\varphi_G(n) = a_s(n)n^s + \cdots + a_1(n)n + a_0(n),$$

where the a_i's are periodic functions.

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Note

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})} \implies \varphi_G \text{ quasi-polynomial of degree at most } k-1$$

Graded algebras

Definition: Graded algebra

$A = \oplus_n A_n$ such that $A_i A_j \subseteq A_{i+j}$.

Example

$A = \mathbb{K}[x_1, \ldots, x_m]$ is a graded algebra.
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Proposition

A is finitely generated $\implies$ $\text{Hilbert}\,(A) = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$

Example

$\text{Hilbert}\,(\mathbb{Q}[x, y, t^3]) = \frac{1}{(1-z)^2(1-z^3)}$

A strategy to prove Cameron's conjecture?

- G: an oligomorphic permutation group with polynomial profile
- Find a graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n\geq 0} A_n$ such that

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- Try to show that $\mathbb{Q}\mathcal{A}(G)$ is finitely generated
- Deduce:

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$$

and thus the quasi-polynomiality of $\varphi_G(n)$

Cameron, 1980: the orbit algebra $\mathbb{Q}\mathcal{A}(G)$

- a commutative connected graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n\geq 0} A_n$
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Product?

- Defined on subsets:

$$ef = \begin{cases} e \cup f & \text{if } e \cap f = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

- $o = \{e_1, e_2, \ldots\} \quad \longleftrightarrow \quad e_1 + e_2 + \cdots$

Example of product on a finite case

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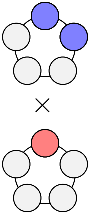

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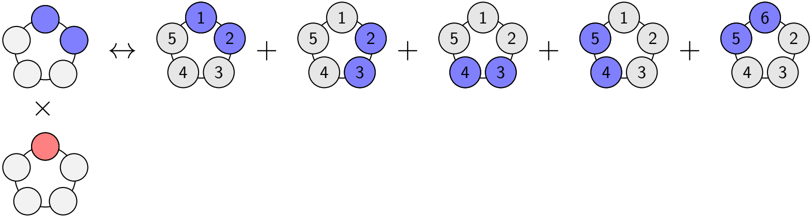

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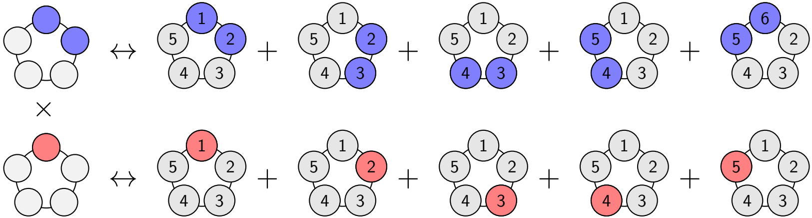

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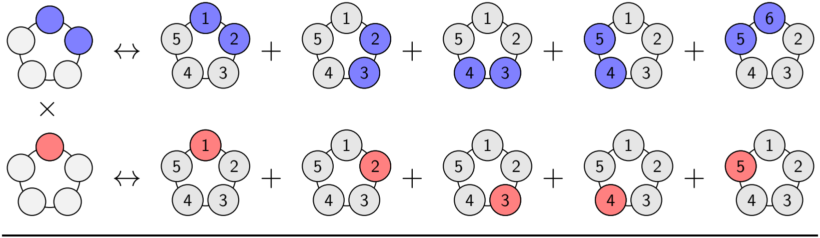

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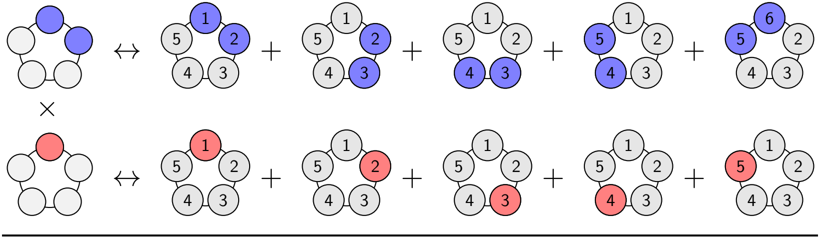

$= \quad 0$

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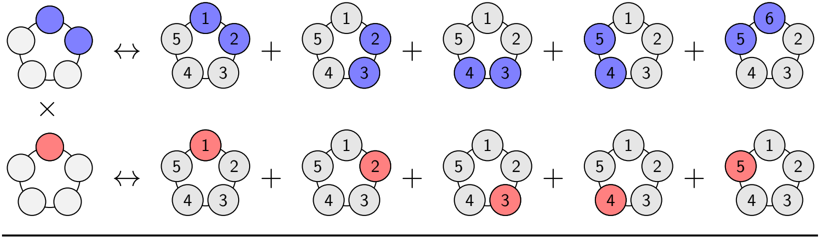

$$= \quad 0 \quad + \quad 0$$

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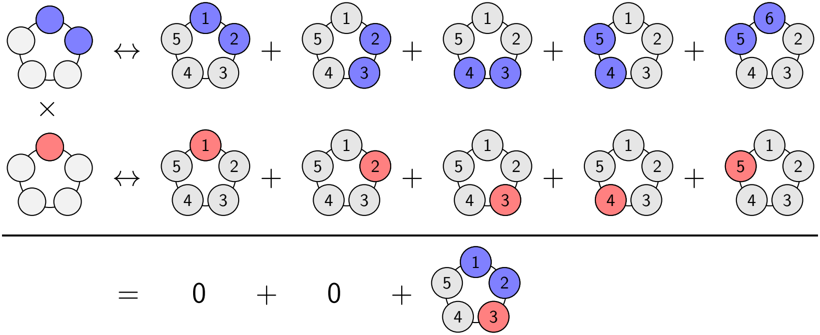

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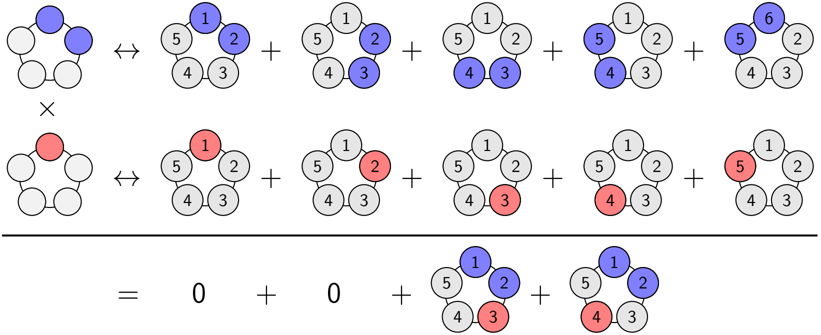

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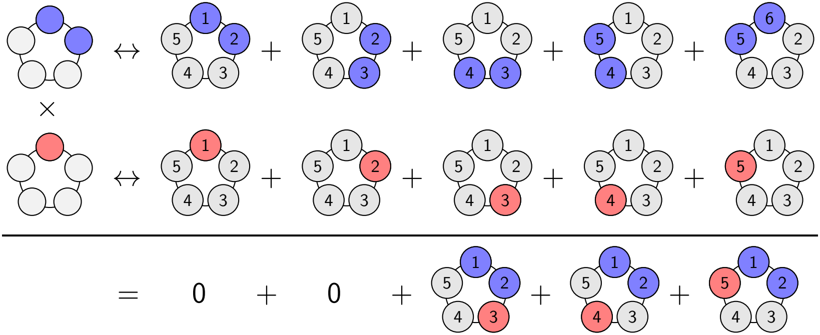

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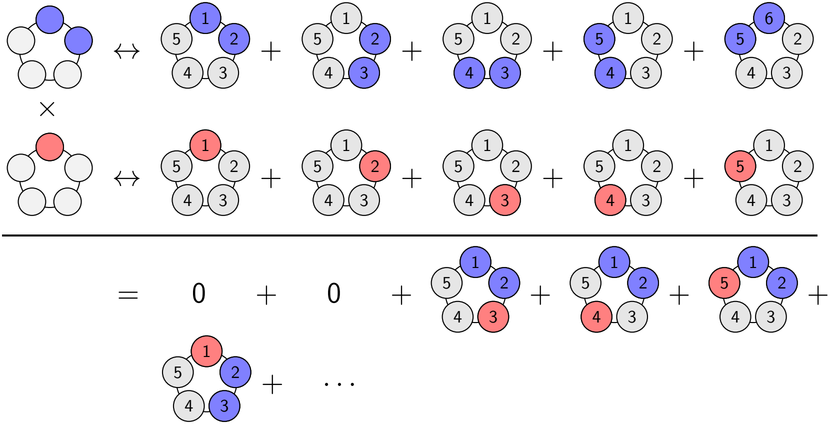

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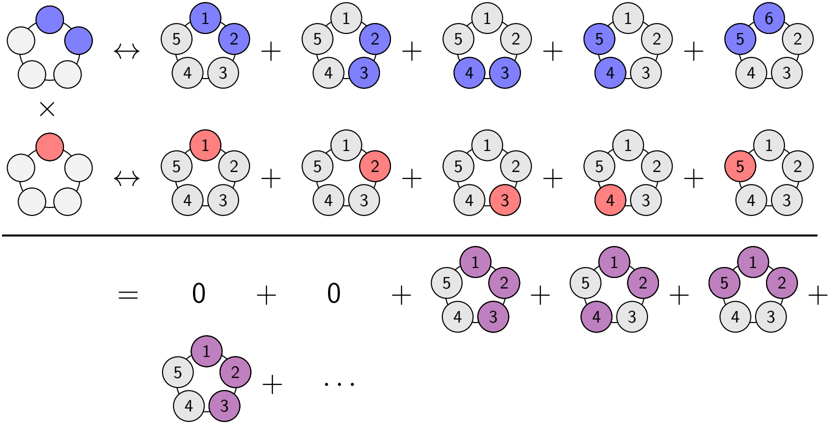

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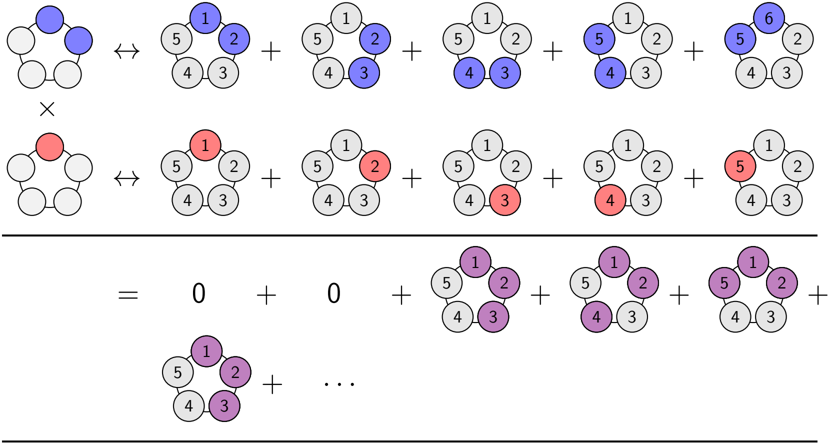

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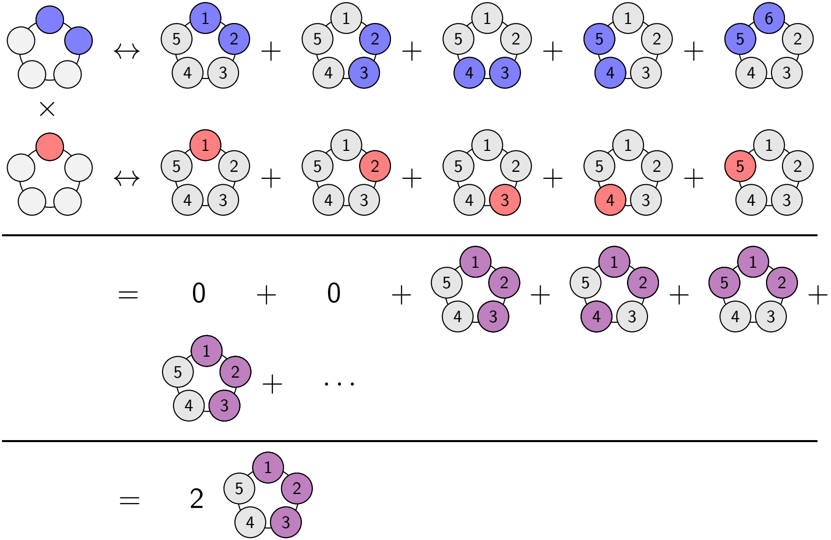

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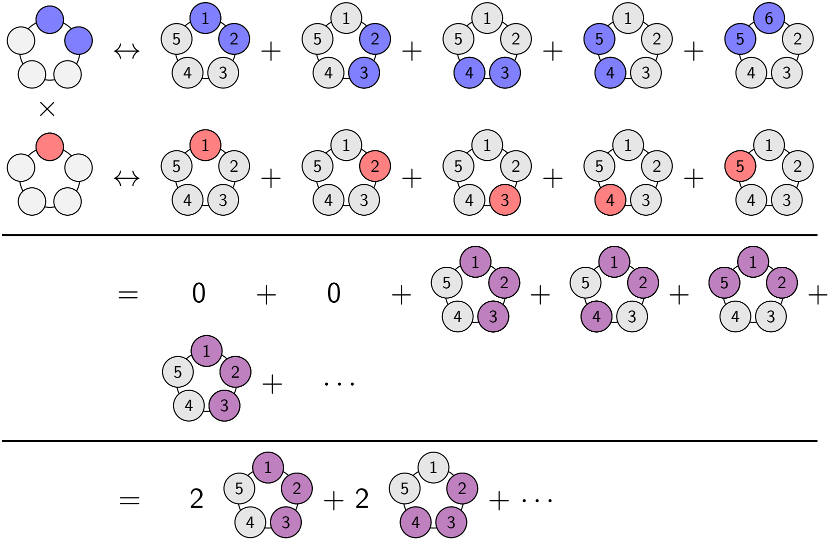

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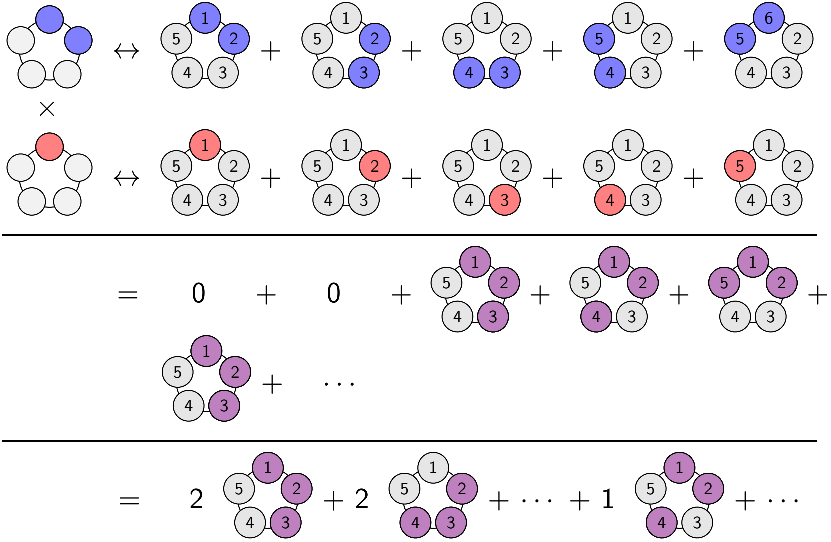
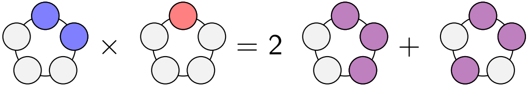


In the end:



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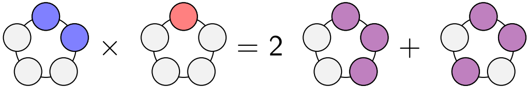

Non trivial fact

Product well defined (and graded) on the space of orbits.

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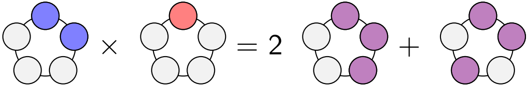

Non trivial fact

Product well defined (and graded) on the space of orbits.

$\longrightarrow$ **The orbit algebra of a permutation group**

Examples of orbit algebras (1)

Example 1

If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n, and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

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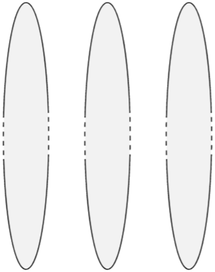
$G = \mathfrak{S}_\infty \wr \mathfrak{S}_3$, recall that $\varphi_G(n) = \mathscr{P}_3(n)$.

$A_n =$ homogeneous symmetric polynomials of degree n in x_1, x_2, x_3



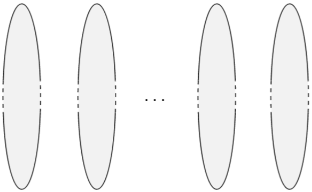
$\rightarrow \mathbb{Q}\mathcal{A}(\mathfrak{S}_\infty \wr \mathfrak{S}_3) = \mathbb{K}[x_1, x_2, x_3]^{\mathfrak{S}_3}$

Examples of orbit algebras (2)

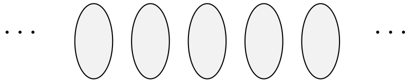
More generally, for H subgroup of $\mathfrak{S}_m$:

- $G = \mathfrak{S}_\infty \wr H$:
  $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x_1, \dots, x_m]^H$, the algebra of invariants of H

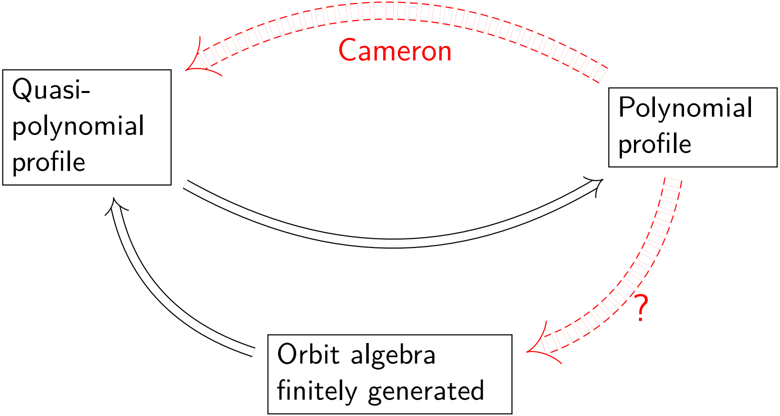
  $\mathbb{Q}\mathcal{A}(G)$ is finitely generated by Hilbert's theorem.



- $G = H \wr \mathfrak{S}_\infty$:
  $\mathbb{Q}\mathcal{A}(G) =$ the free algebra generated by the age of H



Overview and conjecture of Macpherson



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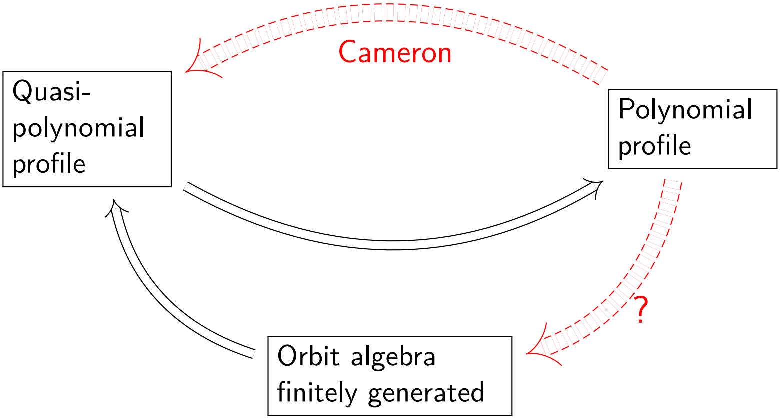

Conjecture (Macpherson, 1985)

Profile of G polynomial $\Longleftrightarrow$ $\mathbb{Q}\mathcal{A}(G)$ finitely generated

Block systems

Definition: Block system

Partition of E such that each part is globally mapped to another one (or itself) by every element of G

(see previous examples)

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Theorem (Cameron)

If G is **primitive** (i.e. admits no non trivial block system) then G has its profile equal to 1 or exponential.

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(see previous examples)

Relevant notion?

Theorem (Cameron)

If G is **primitive** (i.e. admits no non trivial block system) then G has its profile equal to 1 or exponential.

$\rightarrow$ The groups we are interested in have a constanly equal to 1 profile or have a block system.

The complete primitive groups

Theorem (Classification, Cameron)

There are exactly 5 complete groups of constantly equal to 1 profile.

The complete primitive groups

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There are exactly 5 complete groups of constantly equal to 1 profile.

- $\mathrm{Aut}(\mathbb{Q})$: automorphisms of the rational chain (increasing functions)
- $\mathrm{Rev}(\mathbb{Q})$: generated by $\mathrm{Aut}(\mathbb{Q})$ and one reflection
- $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$, preserving the circular order
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- $\mathfrak{S}_\infty$: the symmetric group

Tools

- Block structure of each orbit
- Knowledge of algebras of wreath products
- Embedding

$\Longrightarrow$ lower bound on the profile

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- Block structure of each orbit
- Knowledge of algebras of wreath products
- Embedding

$\Longrightarrow$ lower bound on the profile

- Invariant theory for finite groups (Hilbert's theorem)

$\Longrightarrow$ reduction of the conjecture to essential cases

- Classification of primitive groups
- Goursat's lemma (subdirect product)

$\Longrightarrow$ information on the age

Macpherson for bounded profiles

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- Without reduction (constructive proof):
  $\rightarrow$ same age as $\mathfrak{S}_\infty \times G'$, G' a finite group determined by G
  $\rightarrow$ generating series: $\frac{P(z)}{(1-z)}$,
  where $P(z)$ is the generating polynomial of G'

Macpherson for linear profiles

Two essential cases

- 2 infinite orbits without blocks
- an infinity of blocks of size 2

Macpherson for linear profiles

Two essential cases

- 2 infinite orbits without blocks
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$\rightarrow$ The conjectures of Macpherson and Cameron hold.

Context

- G: permutation group of a countably infinite set E
- Profile φ_G: counts the orbits of finite subsets of E
- **Hypothesis**: $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron): quasi-polynomiality of φ_G
- Conjecture (Macpherson): finite generation of the orbit algebra

Results

- Block structure of $G \Longrightarrow$ minoration of φ_G
- Lemmas and reductions $\Longrightarrow$ bounded and linear cases

Conjectures / intuition

- The orbit algebra is of Cohen-Macaulay
- The growth of the profile is determined by the block structure
- Very rigid: very few groups; classification?