A Hierarchy of Irreducible Sofic Shifts

Marie-Pierre Béal, Francesca Fiorenzi, and Dominique Perrin

Institut Gaspard-Monge, Université de Marne-la-Vallée
77454 Marne-la-Vallée Cedex 2, France
{beal,perrin,fiorenzi}@univ-mlv.fr

Abstract. We define new subclasses of the class of irreducible sofic shifts. These classes form an infinite hierarchy where the lowest class is the class of almost finite type shifts introduced by B. Marcus. We give effective characterizations of these classes with the syntactic semigroups of the shifts.

Keywords: Automata and formal languages, symbolic dynamics.

1 Introduction

Sofic shifts [18] are sets of bi-infinite labels in a labeled graph. If the graph can be chosen strongly connected, the sofic shift is said to be irreducible. An irreducible sofic shift has a unique (up to isomorphisms of automata) minimal deterministic presentation called its right Fischer cover. A particular subclass of sofic shifts is the class of shifts of finite type which are defined by a finite set of forbidden blocks. Two sofic shifts X and Y are conjugate if there is a bijective block map from X onto Y. It is an open question to decide whether two sofic shifts are conjugate, even in the particular case of irreducible shifts of finite type.

Almost finite type shifts have been introduced in [13] (see also [15]). They constitute a meaningful intermediate class above the class of shifts of finite type for several reasons. For instance, if $\tilde{X}$ is the shift presented by the reversed presentation of a shift X that has almost finite type, then X and $\tilde{X}$ are conjugate [6]. Almost finite type shifts are of practical interest in coding for constrained channels. Sliding block decoding theorems hold in the case of almost finite type constraints while they do not hold beyond this class [10].

In this article, we first give a characterization of almost finite type shifts based on the syntactic semigroup of the shift. This semigroup S is the transition semigroup of the right Fischer cover of the irreducible sofic shift. The structure of a finite semigroup is determined by the Green's relations (denoted $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}, \mathcal{D}, \mathcal{J}$), see for instance [17]. We show that an irreducible sofic shift has almost finite type if and only if for any regular $\mathcal{H}$-class of S with image I and any $\mathcal{R}$-class of the $\mathcal{D}$-class of rank 1 with domain D, the intersection $D \cap I$ has at most one element. In general, the greatest cardinality of $D \cap I$, where I is an image and D is a domain as above, is called the degree of the shift. This enables the definition of a hierarchy of subclasses of irreducible sofic shifts with respect to this degree, where the lowest class (that with degree 1) is the class of almost finite type shifts. In particular, we prove that conjugate irreducible sofic shifts have the same degree. This degree is thus a conjugacy invariant.

The proof of this invariant uses Nasu's Classification Theorem for sofic shifts [16] that extends William's one for shifts of finite type. This theorem says that two irreducible sofic shifts X, Y are conjugate if and only if there is a sequence of symbolic adjacency matrices of right Fischer covers $A = A_0, A_1, \dots, A_{l-1}, A_l = B$, such that A_{i-1} and A_i are elementary strong shift equivalent for $1 \leq i \leq l$, where A and B are the adjacency matrices of the right Fischer covers of X and Y, respectively. This means that, for each i, there are two symbolic matrices U_i and V_i such that, after recoding the alphabets of A_{i-1} and A_i, one has $A_{i-1} = U_iV_i$ and $A_i = V_iU_i$. A bipartite shift is associated in a natural way to a pair of elementary strong shift equivalent and irreducible sofic shifts [16]. Another syntactic conjugacy invariant called the syntactic graph of the sofic shift was defined in [2]. We give an example of two non-almost finite type shifts with different degrees (and therefore not conjugate) that have the same syntactic graph.

Basic definitions related to symbolic dynamics are given in Section 2.1. We refer to [12] or [11] for more details. See also [13], [12, Section 13.1], [15], [10],[5], [6], [19] and [7] about almost finite type shifts. Basic definitions and properties related to finite semigroups and their structure are given in Section 2.2. We refer to [17, Chapter 3] for a more comprehensive expository. Nasu's Classification Theorem is recalled in Section 2.3. In Section 3, we define a hierarchy of irreducible sofic shifts. In Section 4, we consider the problem of characterizing classes of shifts (as the class of almost finite type shifts), by algebraic properties of the syntactic semigroup.

2 Definitions and Background

2.1 Almost Finite Type Shifts and Their Presentations

Let $\mathcal{A}$ be a finite alphabet, i.e. a finite set of symbols. The shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma((a_i)_{i\in\mathbb{Z}}) = (a_{i+1})_{i\in\mathbb{Z}}$, for $(a_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$. If $\mathcal{A}^{\mathbb{Z}}$ is endowed with the product topology of the discrete topology on $\mathcal{A}$, a *shift* is a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$.

A finite *automaton* is a finite multigraph labeled by $\mathcal{A}$. It is denoted $\mathsf{A} = (Q, E)$, where Q is a finite set of states, and E a finite set of edges labeled by $\mathcal{A}$. It is equivalent to a *symbolic adjacency* $(Q \times Q)$*-matrix* A, where A_{pq} is the finite formal sum of the labels of all the edges from p to q. A *sofic shift* is the set of the labels of all the bi-infinite paths on a finite automaton. If A is a finite automaton, we denote by X_{A} the sofic shift defined by the automaton A. Several automata can define the same sofic shift. They are also called *presentations* or *covers* of the sofic shift. We will assume that all presentations are *essential*: all states have at least one outgoing edge and one incoming edge. An automaton is *deterministic* if for any given state and any given symbol, there is at most one outgoing edge labeled by this given symbol. An automaton is *left closing with delay D* if whenever two paths of length $D+1$ end at the same state and have the same label, then they have the same final edge. An automaton is *left closing* if it is left-closing with some delay $D \geq 0$. A sofic shift is *irreducible* if it has a presentation with a strongly connected graph. Irreducible sofic shifts have a unique (up to isomorphisms of automata) *minimal deterministic presentation*, that is a deterministic presentation having the fewest states among all deterministic presentations of the shift. This presentation is called the *right Fischer cover* of the shift.

An irreducible sofic shift has *almost finite type* (AFT) if it has a deterministic and left-closing presentation. The class of almost finite type shifts was introduced by B. Marcus in [13], see also [15] and [12, Section 13.1].

Let $\mathsf{A} = (Q, E)$ be a deterministic automaton labeled by $\mathcal{A}$. The *square* of A is the deterministic automaton $(Q \times Q, F)$ where $(p,q) \xrightarrow{a} (p', q') \in F$ if and only if $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q' \in E$. A *diagonal* state of the square of A is a state (p,p), with $p \in Q$.

An almost finite type shift is an irreducible shift whose right Fischer cover is left-closing. Thus the square of its right Fischer cover has no strongly connected component with at least one edge containing a non-diagonal state and admitting a path going from this component to a diagonal state (see for instance [13], [1]). Checking whether an irreducible sofic shift has almost finite type can thus be done in a quadratic time in the number of states of the right Fischer cover of the shift.

2.2 The Syntactic Semigroup of an Irreducible Sofic Shift

In this section, we recall the definition and the structure of the syntactic semigroup of an irreducible sofic shift [2].

Let $\mathsf{A} = (Q, E)$ be a finite deterministic (essential) automaton on the alphabet $\mathcal{A}$. Each finite word w of $\mathcal{A}^*$ defines a partial function from Q to Q. This function sends the state p to the state q, if w is the label of a path from p to q. The semigroup generated by all these functions is called the *transition semigroup* of the automaton. When X_A is not the full shift, the semigroup has a null element, denoted 0, which corresponds to words which are not factors of any bi-infinite word of X_A. The *syntactic semigroup* of an irreducible sofic shift is defined as the transition semigroup of its right Fischer cover.

Given a semigroup S, we denote by S^1 the following monoid: if S is a monoid, $S^1 = S$. If S is not a monoid, $S^1 = S \cup \{1\}$ together with the law $*$ defined by $x * y = xy$ if $x, y \in S$ and $1 * x = x * 1 = x$ for every $x \in S^1$.

We recall the *Green's relations* $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{J}$, which are fundamental equivalence relations defined in a semigroup S. They are defined as follows. Let $x, y \in S$,

$$\begin{aligned}
x\mathcal{R}y &\Leftrightarrow xS^1 = yS^1,\\
x\mathcal{L}y &\Leftrightarrow S^1x = S^1y,\\
x\mathcal{J}y &\Leftrightarrow S^1xS^1 = S^1yS^1,\\
x\mathcal{H}y &\Leftrightarrow x\mathcal{R}y \text{ and } x\mathcal{L}y.
\end{aligned}$$

Another relation $\mathcal{D}$ is defined by:

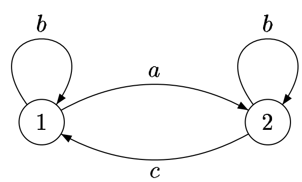
$$x\mathcal{D}y \Leftrightarrow \exists z \in S \;\; x\mathcal{R}z \text{ and } z\mathcal{L}y.$$

In a finite semigroup $\mathcal{J} = \mathcal{D}$.

An $\mathcal{R}$-*class* is an equivalence class for a relation $\mathcal{R}$ (similar notations hold for the other Green's relations). An *idempotent* is an element $e \in S$ such that $ee = e$. A *regular* class is a class containing an idempotent. In a regular $\mathcal{D}$-class, any $\mathcal{H}$-class containing an idempotent is a maximal subgroup of the semigroup. Moreover, two regular $\mathcal{H}$-classes contained in a same $\mathcal{D}$-class are isomorphic (as groups), see for instance [17, Chapter 3 Proposition 1.8].

We say that two elements $x, y \in S$ are *conjugate* if there are elements $u, v \in S^1$ such that $x = uv$ and $y = vu$.

Let S be a transition semigroup of an automaton $\mathsf{A} = (Q, E)$ and $x \in S$. The *rank* of x is the cardinal of the image of x as a partial function from Q to Q. The *kernel* of x is the partition induced by the equivalence relation $\sim$ over the domain of x where $p \sim q$ if and only p, q have the same image under x. We describe the so called "egg-box" pictures with the sofic shifts of Figure 1 and Figure 2 which have almost finite type and not almost finite type, respectively.



	12
1/2	*b

	1	2
1	*ac	a
2	c	*ca

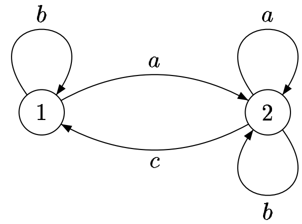
	−
−	*0

Fig. 1. An irreducible sofic shift which has almost finite type. Its syntactic semigroup is represented on the right part of the figure. It is composed of three $\mathcal{D}$-classes of rank 2, 1 and 0, respectively, represented by the above tables from left to right. Each square in a table represents an $\mathcal{H}$-class. Each row represents an $\mathcal{R}$-class and each column an $\mathcal{L}$-class. The common kernel of the elements in each row is written on the left of each row. The common image of the elements in each column is written above each column. Idempotents are marked with the symbol $*$. Each $\mathcal{D}$-class of this semigroup is regular.

The syntactic semigroup of an irreducible sofic shift has a unique $\mathcal{D}$-class of rank 1 which is regular (see for instance [3] or [4], and also [9]). Moreover, if u is a non null element of this semigroup, there is a word w such that uw has rank 1.

2.3 Nasu's Classification Theorem for Sofic Shifts

In this section, we recall Nasu's Classification Theorem for sofic shifts [16] (see also [12, Theorem 7.2.12]), which extends William's Classification Theorem for shifts of finite type (see [12, Theorem 7.2.7]).



$$\begin{array}{c|c|} & 12 \\ \hline 1/2 & {}^{*}b \\ \hline \end{array} \qquad \begin{array}{c|c|c|} & 1 & 2 \\ \hline 12 & {}^{*}ac & a \\ \hline 2 & c & {}^{*}ca \\ \hline \end{array} \qquad \begin{array}{c|c|} & - \\ \hline - & {}^{*}0 \\ \hline \end{array}$$

Fig. 2. An irreducible sofic shift which has not almost finite type. Indeed, there are two distinct left-infinite paths labelled ...*bbbbbba* ending at state 2. Also in this case, each $\mathcal{D}$-class is regular.

Let $X \subset \mathcal{A}^{\mathbb{Z}}$, $Y \subset \mathcal{B}^{\mathbb{Z}}$ be two shifts and m, a be nonnegative integers. A map $\phi : X \to Y$ is a (m, a)-*block map* (or (m, a)-*factor map*) if there is a map $\delta : \mathcal{A}^{m+a+1} \to \mathcal{B}$ such that $\phi((a_i)_{i\in\mathbb{Z}}) = (b_i)_{i\in\mathbb{Z}}$ where $\delta(a_{i-m} \ldots a_{i-1}\ a_i\ a_{i+1} \ldots a_{i+a}) = b_i$. A *block map* is a (m, a)-*block map* for some nonnegative integers m, a (respectively called its *memory* and *anticipation*). The well known theorem of Curtis, Hedlund, and Lyndon [8] asserts that continuous maps commuting with the shift map σ, are exactly block maps. A *conjugacy* is a one-to-one and onto block map (then, being a shift compact, also its inverse is a block map).

Having almost finite type is a property of shifts which is invariant under conjugacy [13].

Let A be the symbolic adjacency $(Q \times Q)$-matrix of an automaton A with entries in a finite alphabet $\mathcal{A}$. Let $\mathcal{B}$ be a finite alphabet and f a one-to-one map from $\mathcal{A}$ to $\mathcal{B}$. The map f is extended to a morphism from finite formal sums of elements of $\mathcal{A}$ to finite formal sums of elements of $\mathcal{B}$. We say that f *transforms* A into a symbolic $(Q \times Q)$-matrix B if $B_{pq} = f(A_{pq})$ for each $p, q \in Q$.

We now define the notion of strong shift equivalence between two symbolic adjacency matrices.

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets. We denote by $\mathcal{AB}$ the set of words ab with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Two symbolic matrices A and B with entries in $\mathcal{A}$ and $\mathcal{B}$ respectively, are *elementary strong shift equivalent* if there is a pair of symbolic matrices (U, V) with entries in disjoint alphabets $\mathcal{U}$ and $\mathcal{V}$ respectively, such that there is a one-to-one map from $\mathcal{A}$ to $\mathcal{UV}$ which transforms A into UV, and there is a one-to-one map from $\mathcal{B}$ to $\mathcal{VU}$ which transforms B into VU.

Two symbolic adjacency matrices A and B are *strong shift equivalent within right Fischer covers* if there is a sequence of symbolic adjacency matrices of right Fischer covers

$$A = A_0, A_1, \ldots, A_{l-1}, A_l = B$$

such that for $1 \leq i \leq l$ the matrices A_{i-1} and A_i are elementary strong shift equivalent.

Theorem 1 (Nasu). *Let X and Y be irreducible sofic shifts and let A and B be the symbolic adjacency matrices of the right Fischer covers of X and Y,* *respectively. Then X and Y are conjugate if and only if A and B are strong shift equivalent within right Fischer covers.*

Let us consider the two irreducible sofic shifts X and Y defined by the right Fischer covers in Figure 3. The symbolic adjacency matrices of these automata

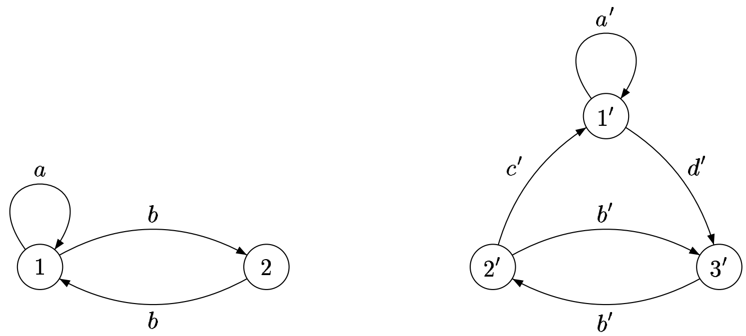


Fig. 3. Two conjugate shifts X and Y.

are respectively

$$A = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}, \quad B = \begin{bmatrix} a' & 0 & d' \\ c' & 0 & b' \\ 0 & b' & 0 \end{bmatrix}.$$

Then A and B are elementary strong shift equivalent with

$$U = \begin{bmatrix} u_1 & 0 & u_2 \\ 0 & u_2 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & 0 \\ v_2 & 0 \\ 0 & v_2 \end{bmatrix}.$$

Indeed,

$$UV = \begin{bmatrix} u_1v_1 & u_2v_2 \\ u_2v_2 & 0 \end{bmatrix}, \quad VU = \begin{bmatrix} v_1u_1 & 0 & v_1u_2 \\ v_2u_1 & 0 & v_2u_2 \\ 0 & v_2u_2 & 0 \end{bmatrix}.$$

The one-to-one maps from $\mathcal{A} = \{a, b\}$ to $\mathcal{UV}$ and from $\mathcal{B} = \{a', b', c', d'\}$ to $\mathcal{VU}$ are described in the tables below.

a	u_1v_1
b	u_2v_2

,

a'	v_1u_1
b'	v_2u_2
c'	v_2u_1
d'	v_1u_2

.

An elementary strong shift equivalence between $\mathsf{A} = (Q, E)$ and $\mathsf{B} = (Q', E')$, enables the construction of an irreducible sofic shift Z on the alphabet $\mathcal{U} \cup \mathcal{V}$ as follows. The sofic shift Z is defined by the automaton $\mathsf{C} = (Q \cup Q', F)$, where the symbolic adjacency matrix C of C is

$$\begin{array}{cc} & \begin{array}{cc} Q & Q' \end{array} \\ \begin{array}{c} Q \\ Q' \end{array} & \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} \end{array}.$$

The shift Z is called the *bipartite shift* defined by U, V (see Figure 4). An edge of C labeled by $\mathcal{U}$ goes from a state in Q to a state in Q'. An edge of C labeled by $\mathcal{V}$ goes from a state in Q' to a state in Q. Remark that C is a right Fischer cover (i.e. is minimal).

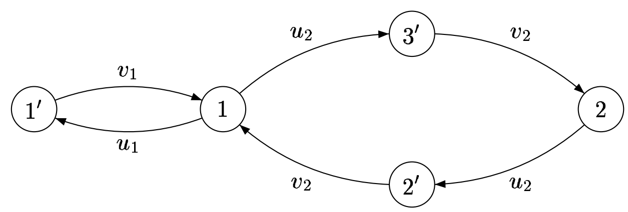


Fig. 4. The bipartite shift Z of the shifts X and Y in Figure 3.

3 A Hierarchy of Irreducible Sofic Shifts

In this section, we first give a syntactic characterization of almost finite type shifts. Next, we define a hierarchy of irreducible sofic shifts.

Proposition 1. *Let X be an irreducible sofic shift and S its syntactic semigroup. Then X has almost finite type if and only if for any regular $\mathcal{H}$-class of S with image I and any $\mathcal{R}$-class of the $\mathcal{D}$-class of rank 1 with domain D, the intersection $D \cap I$ has at most one element.*

Proof Let us assume that X has not almost finite type. Let A be the right Fischer cover of X. Then there are two states $p \neq q$ and a state r, two words u, v and two paths on A labeled uv as follows.

$$\begin{array}{c} p \xrightarrow{u} p \xrightarrow{v} r \\ q \xrightarrow{u} q \xrightarrow{v} r \end{array}$$

There is a positive integer n that u^n is a non null idempotent in S. Let I be the image of u^n. It contains p and q. Let w be a word of rank 1 in S. Since A has a strongly connected graph, there is a word w' such that the domain of $w'w$ contains the state r. The word $vw'w$ has rank 1 and its domain contains p

and q. Thus the intersection of the domain of $vw'w$ and I contains at least two elements.

Conversely, let us assume that there is in S a regular $\mathcal{H}$-class H with image I and an $\mathcal{R}$-class R of the $\mathcal{D}$-class of rank 1 with domain D, such that $D \cap I$ has at least two elements. Let e be an idempotent of H. Then e induces the identity map on its image I. Let $p \neq q \in D \cap I$, and $v \in R$. Then there is a state r and two paths on A as follows

$$p \xrightarrow{e} p \xrightarrow{v} r$$
$$q \xrightarrow{e} q \xrightarrow{v} r$$

It follows that X has not almost finite type. $\square$

For instance, the shift presented in Figure 2 has not almost finite type since it has a regular $\mathcal{H}$-class of rank 2 (containing the idempotent b) whose image is $\{1, 2\}$. This image intersects the domain of $\{ac, a\}$ (which is a $\mathcal{R}$-class contained in the $\mathcal{D}$-class of rank 1), with $\{1, 2\}$ as intersection.

We now introduce the following classification of irreducible sofic shifts. An irreducible sofic shift is *d-non-left closing* if its syntactic semigroup has a regular $\mathcal{H}$-class with image I and an $\mathcal{R}$-class of the $\mathcal{D}$-class of rank 1 with domain D, such that $D \cap I$ has d elements. An irreducible sofic shift has *degree d* if it is d-non-left closing with $d \geq 0$ and not d'-non-left closing for any $d' > d$. Notice that the degree of an irreducible sofic shift is always non null.

The following proposition allows us to prove that the class of irreducible sofic shifts with degree d is a class of shifts invariant under conjugacy. In this classification, the class of almost finite type shifts is the class of irreducible sofic shifts with degree 1. This enables to recover that the property of having almost finite type is a conjugacy invariant.

Proposition 2. *Let X and Y be two conjugate irreducible sofic shifts and let d be a positive integer. If X is d-non-left closing, then Y is d-non-left closing.*

Before proving Proposition 2, we recall some results from [2] about the syntactic semigroup of a bipartite shift. Let X (respectively Y) be an irreducible sofic shift whose symbolic adjacency matrices of its right Fischer cover is a $(Q \times Q)$-matrix (respectively $(Q' \times Q')$-matrix) denoted by A (respectively by B). We assume that A and B are elementary strong shift equivalent through a pair of matrices (U, V). The corresponding alphabets are denoted $\mathcal{A}$, $\mathcal{B}$, $\mathcal{U}$, and $\mathcal{V}$ as before. We denote by f a one-to-one map from $\mathcal{A}$ to $\mathcal{UV}$ which transforms A into UV and by g a one-to-one map from $\mathcal{B}$ to $\mathcal{VU}$ which transforms B into VU. Let Z be the bipartite irreducible sofic shift associated to U, V. We denote by S (respectively T, R) the syntactic semigroup of X (respectively Y, Z).

Let $w \in R$. If w is non null, the bipartite nature of Z implies that w is a function from $Q \cup Q'$ to $Q \cup Q'$ whose domain is included either in Q or in Q', and whose image is included either in Q or in Q'. If $w \neq 0$ with a domain included in P and an image included in P', we say that w has the *type* (P, P'). Remark that w has type (Q, Q) if and only if $w \neq 0$ and $w \in (f(\mathcal{A}))^*$, and w has type (Q', Q') if and only if $w \neq 0$ and $w \in (g(\mathcal{B}))^*$. Elements of R in a same non null $\mathcal{H}$-class have the same type.

Let $w = a_1 \dots a_n$ be an element of S, we define the element $f(w)$ as $f(a_1)\dots f(a_n)$. Note that this definition is consistent since if $a_1 \dots a_n = a'_1 \dots a'_m$ in S, then $f(a_1)\dots f(a_n) = f(a'_1)\dots f(a'_m)$ in R. Similarly we define an element $g(w)$ for any element w of T.

Conversely, let w be an element of R belonging to $f(\mathcal{A})^*$ $(\subset (\mathcal{U}\mathcal{V})^*)$. Then $w = f(a_1)\dots f(a_n)$, with $a_i \in \mathcal{A}$. We define $f^{-1}(w)$ as $a_1 \dots a_n$. Similarly we define $g^{-1}(w)$. Again these definitions and notations are consistent. Thus f is a semigroup isomorphism from S to the subsemigroup of R of transition functions defined by the words in $(f(\mathcal{A}))^*$. Notice that $f(0) = 0$ if $0 \in S$. Analogously, g is a semigroup isomorphism from T to the subsemigroup of R of transition functions defined by the words in $(g(\mathcal{B}))^*$.

We now prove Proposition 2.

Proof[of Proposition 2] By Nasu's Theorem [16] we can assume, without loss of generality, that the symbolic adjacency matrices of the right Fischer covers of X and Y are elementary strong shift equivalent. We define the bipartite shift Z as above. We denote by S, T and R the syntactic semigroups of X, Y and Z respectively.

Let us assume that X is d-non-left closing. Thus S has a regular $\mathcal{H}$-class H with image I and an $\mathcal{R}$-class of the $\mathcal{D}$-class of rank 1 with domain D, such that $D \cap I$ has d elements. Let e be an idempotent of H. It induces the identity map on its image I.

The element $f(e)$ is an idempotent element of type (Q, Q) in R. Let $u_1v_1 \dots u_nv_n \in (\mathcal{U}\mathcal{V})^*$ such that $f(e) = u_1v_1 \dots u_nv_n$. We define an element $\bar{e}$ as $\bar{e} = v_1 \dots u_nv_nu_1$. Thus $f(e)u_1 = u_1\bar{e}$ in R. Remark that $\bar{e}$ depends on the choice of the word $u_1v_1 \dots u_nv_n$ representing $f(e)$ in R. Notice that $\bar{e}$ and $f(e)$ are conjugate. Indeed, if $w = v_1 \dots u_nv_n$, then $f(e) = u_1w$ and $\bar{e} = wu_1$. Hence, $\bar{e}^3 = wu_1wu_1wu_1 = wf(e)^2u_1 = wf(e)u_1 = wu_1wu_1 = \bar{e}^2$. We have $\bar{e}^2 \neq 0$ since $f(e) \neq 0$ and $f(e) = f(e)^2 = f(e)^3 = u_1\bar{e}^2w$. Thus $\bar{e}^2$ is an idempotent of R of type (Q', Q').

Let $I \cap D = \{x_1, \dots, x_d\}$. Then there is a word $z \in \mathcal{A}^*$ of rank 1, a state $x \in Q$ and a path on the right Fischer cover of X labeled z from any state in $I \cap D$ to x. Moreover, there are a letter $u \in \mathcal{U}$, a state $y \in Q'$ and an edge $x \xrightarrow{u} y$ in the right Fischer cover of Z. It follows that there are paths as follows in the right Fischer cover of Z.

$$\begin{array}{l}
x_1 \xrightarrow{u_1} y_1 \xrightarrow{w} x_1 \xrightarrow{u_1} y_1 \xrightarrow{w} x_1 \xrightarrow{f(z)} x \xrightarrow{u} y \\
x_2 \xrightarrow{u_1} y_2 \xrightarrow{w} x_2 \xrightarrow{u_1} y_2 \xrightarrow{w} x_2 \xrightarrow{f(z)} x \xrightarrow{u} y \\
\vdots \\
x_d \xrightarrow{u_1} y_d \xrightarrow{w} x_d \xrightarrow{u_1} y_d \xrightarrow{w} x_d \xrightarrow{f(z)} x \xrightarrow{u} y
\end{array}$$

The states y_i, for $1 \le i \le d$, belong to Q'. Since the states x_i are distinct, also the states y_i are distinct. Indeed, let us assume for instance that $y_1 = y_2$. Then $x_1 = x_2$ by considering the paths labeled w from y_i to x_i for $i = 1, 2$. Thus, in the right Fischer cover of Z there are the following paths, for $1 \le i \le d$.

$$y_i \xrightarrow{(wu_1)^2} y_i \xrightarrow{wf(z)u} y$$

Since $\bar{e} = wu_1$ and $wf(z)u$ are contained in $(g(\mathcal{B}))^*$, the elements $e' = g^{-1}(\bar{e}^2) = g^{-1}((wu_1)^2)$ and $w' = g^{-1}(wf(z)u)$ are in T. Hence the following paths are in the right Fischer cover of Y, for $1 \le i \le d$.

$$y_i \xrightarrow{e'} y_i \xrightarrow{w'} y$$

Notice that e' is an idempotent of T. Since the graph of the right Fischer cover of Y is strongly connected, there is a word w'' of rank 1 whose domain contains y. This allows us to assume that the word w' is an element of rank 1 of T. Hence the domain D' of the $\mathcal{R}$-class of w' and the image I' of the idempotent e', contain $\{y_1, \dots, y_d\}$. We now prove that $D' \cap I'$ is exactly the set $\{y_1, \dots, y_d\}$.

Suppose that there is a state $\bar{y} \in D' \cap I'$ such that $\bar{y} \neq y_i$ for each i. Hence the following path is in the right Fischer cover of Y.

$$\bar{y} \xrightarrow{e'} \bar{y} \xrightarrow{w'} y$$

Thus, in the right Fischer cover of Z there is the following path.

$$\bar{y} \xrightarrow{(wu_1)^2} \bar{y} \xrightarrow{wf(z)u} y$$

Let $\bar{x}$ be the final state of the path labelled by w and starting at $\bar{y}$. It follows that a path of the kind

$$\bar{y} \xrightarrow{w} \bar{x} \xrightarrow{u_1} . \xrightarrow{w} . \xrightarrow{u_1} \bar{y} \xrightarrow{w} \bar{x} \xrightarrow{f(z)} x$$

is in the right Fischer cover of Z (recall that $f(z)$ has rank 1). Being $\bar{x}$ in the image of $u_1wu_1w = f(e)^2 = f(e)$, we have that $\bar{x}$ in the image I of e. Moreover, $\bar{x}$ is in the domain of $f(z)$ and hence it is also in the domain D of z. This implies that $\bar{x}$ is one of the elements x_i and hence $\bar{y}$ is the corresponding y_i, which is a contradiction. Thus the cardinality of $D' \cap I'$ is d. $\square$

We get the following corollary.

Corollary 1. *Let X be an irreducible sofic shift. Then its degree is invariant under conjugacy. Moreover, the increasing sequence $(d_1, d_2, \dots, d_n)$ of positive integers such that X is d_i-non-left closing (where d_n is the degree of the shift), is invariant under conjugacy.*

We show in Figure 5 an example of two sofic shifts X and Y, where X has not almost finite type with degree 3, and Y has not almost finite type with degree 2.

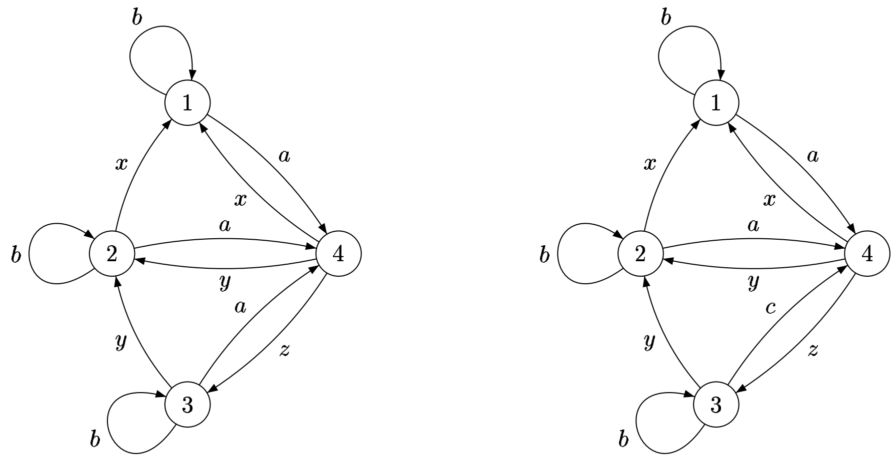


Fig. 5. The right Fischer covers of two non conjugate sofic shifts X (on the left) and Y (on the right), with $\mathcal{A} = \{a, b, x, y, z\}$ and $\mathcal{B} = \{a, b, c, x, y, z\}$. The shift X has degree 3 while the shift Y has degree 2.

Thus these two shifts are not conjugate since their degrees are different. Remark that they have the same syntactic graph, which is another conjugacy invariant defined and described in [2].

There are irreducible sofic shifts with degree d for every $d > 1$. For instance, consider the right Fischer cover $\mathsf{A} = (\{1, 2, \ldots, d, d+1\}, E)$ on the alphabet $\mathcal{A} = \{a, b, c\}$, where the set of edges is $E = \{i \xrightarrow{b} i,\ i \xrightarrow{a} d+1 \mid i \neq d+1\} \bigcup \{i \xrightarrow{c} i-1 \mid 2 \leq i \leq d\} \bigcup \{d+1 \xrightarrow{c} d\}$. This right Fischer cover has degree d.

4 Links with Semigroup Theory

The above propositions have links with previous results in the theory of varieties of semigroups.

A finite biprefix code (see for instance [4]) defines an almost finite type shift in a natural way. It is known from [14] that, if X is a finite biprefix code and S the syntactic semigroup of X^+, eSe defines a semigroup of partial injective transformations for any idempotent e. Margolis [14] also showed that every semigroup of partial injective transformations divides a semigroup of partial injective transformations which is the syntactic semigroup of a finite biprefix code.

An equivalent formulation of Proposition 1 is the following.

Proposition 3. *Let X an irreducible sofic shift and S its syntactic semigroup. Then X has almost finite type if and only if for any idempotent $e \in S$, the semigroup eSe is a semigroup of partial one-to-one transformations.*

Thus, when S has almost finite type, the semigroup eSe is, for any idempotent e, a subsemigroup of an inverse semigroup. This implies that the semigroup S belongs to the variety of semigroups T such that for each idempotent e, the semigroup eTe is in the variety generated by inverse semigroups. We do not know whether this condition is sufficient to guarantee that X has almost finite type.

ACKNOWLEDGMENT The authors wish to thank the reviewer who suggested the second statement of Corollary 1.

References

1. M.-P. BÉAL, *Codage Symbolique*, Masson, 1993.
2. M.-P. BÉAL, F. FIORENZI, AND D. PERRIN, *The syntactic graph of a sofic shift*, in STACS (Montpellier, 2004), vol. 2996 of Lecture Notes in Comput. Sci., Springer, Berlin, 2004, pp. 282–293.
3. D. BEAUQUIER, *Minimal automaton for a factorial transitive rational language*, Theoret. Comput. Sci., 67 (1989), pp. 65–73.
4. J. BERSTEL AND D. PERRIN, *Theory of Codes*, Academic Press, New York, 1985.
5. F. BLANCHARD, *Codes engendrant certains systèmes sofiques*, Theoret. Comput. Sci., 68 (1989), pp. 253–265.
6. M. BOYLE, B. P. KITCHENS, AND B. H. MARCUS, *A note on minimal covers for sofic shifts*, Proc. Amer. Math. Soc., 95 (1985), pp. 403–411.
7. D. FIEBIG, U.-R. FIEBIG, AND N. JONOSKA, *Multiplicities of covers for sofic shifts*, Theoret. Comput. Sci., 1-2 (2001), pp. 349–375.
8. G. A. HEDLUND, *Endomorphisms and automorphisms of the shift dynamical system*, Math. Systems Theory, 3 (1969), pp. 320–337.
9. N. JONOSKA, *A conjugacy invariant for reducible sofic shifts and its semigroup characterizations*, Israel J. Math., 106 (1998), pp. 221–249.
10. R. KARABED AND B. H. MARCUS, *Sliding-block coding for input-restricted channels*, IEEE Trans. Inform. Theory, IT-34 (1988), pp. 2–26.
11. B. P. KITCHENS, *Symbolic Dynamics: one-sided, two-sided and countable state Markov shifts*, Springer-Verlag, 1997.
12. D. A. LIND AND B. H. MARCUS, *An Introduction to Symbolic Dynamics and Coding*, Cambridge, 1995.
13. B. H. MARCUS, *Sofic systems and encoding data*, IEEE Trans. Inform. Theory, IT-31 (1985), pp. 366–377.
14. S. W. MARGOLIS, *On the syntactic transformation semigroup of a language generated by a finite biprefix code*, Theoret. Comput. Sci., 21 (1982), pp. 225–230.
15. M. NASU, *An invariant for bounded-to-one factor maps between transitive sofic subshifts*, Ergodic Theory Dynam. Systems, 5 (1985), pp. 89–105.
16. ——, *Topological conjugacy for sofic systems*, Ergodic Theory Dynam. Systems, 6 (1986), pp. 265–280.
17. J.-É. PIN, *Varieties of Formal Languages*, Plenum Publishing Corporation, New York, 1986.
18. B. WEISS, *Subshifts of finite type and sofic systems*, Monats. für Math., 77 (1973), pp. 462–474.
19. S. WILLIAMS, *Covers of non-almost-finite-type systems*, Proc. Amer. Math. Soc., 104 (1988), pp. 245–252.