Minimal forbidden patterns of multi-dimensional shifts

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Abstract

We study whether the entropy (or growth rate) of minimal forbidden patterns of symbolic dynamical shifts of dimension 2 or more, is a conjugacy invariant. We prove that the entropy of minimal forbidden patterns is a conjugacy invariant for uniformly semi-strongly irreducible shifts. We prove a weaker invariant in the general case.

Key Words: higher dimensional shifts, minimal forbidden patterns, conjugacy invariants. AMS Classification: 37B10 – 28D20.

1 Introduction

Symbolic dynamical systems are often defined by a set of forbidden patterns. In dimension two for instance, a shift is the set of labellings of the square lattice, called configurations, which avoid any forbidden pattern. Shifts of finite type are those which can be defined by a finite set of forbidden patterns. This property is a conjugacy invariant, see for instance [20] or [16]. Many natural examples of two-dimensional shifts of finite type arise from lattice systems in statistical mechanics [1].

The dynamic of multi-dimensional shifts is much more complex than the one of one-dimensional shifts. For instance the entropy of a shift, which is a conjugacy invariant that gives the complexity of the allowed patterns (i.e. patterns contained in some configuration of the shift), is easily computable for a one-dimensional shift of finite type. It leads to remarkable difficult problems even for the simplest examples in dimension two.

In [3] has been introduced the notion of minimal forbidden word for a one-dimensional shift: a word is minimal forbidden if it is forbidden and if all its proper factors are allowed words of the shift. The set of minimal forbidden words is finite for a shift of finite type. The entropy, or complexity, of the set of minimal forbidden words is a conjugacy invariant for one-dimensional shifts. Moreover this invariant is independent of some other known invariants like the entropy of the shift or the zeta function for instance. Note that this invariant is not meaningful for shifts of finite type, or it just says that shifts of finite type are only conjugate to shifts of finite type, which is well known. Minimal forbidden words have applications in several areas like lossless compression (data compression using antidictionaries [7]) and also reconstruction of DNA sequences from its fragments (the fragment assembly problem [4], [18] and [21]).

In this paper, we study the notion of minimal forbidden patterns for multi-dimensional shifts, with the goal to provide some new conjugacy invariants for multi-dimensional shifts. We give two notions of minimal forbidden patterns. The first one is the direct extension of the notion of minimal forbidden words for one-dimensional shifts. It can be considered for patterns with a square or rectangular shape. For instance, a square is minimal forbidden if it is a forbidden pattern such that each strict subsquare is allowed. It turns out that this definition is too weak. Indeed, a shift of finite type can have an infinite number of minimal forbidden patterns of this type. Thus we consider a stronger notion which leads to much less minimal forbidden patterns. In dimension two for instance, a forbidden square of size n is minimal for this stronger notion if it is contained in a configuration c such all squares of c of size $n-1$ are allowed patterns of the shift. These two notions coincide in dimension one. Another main difference between the one-dimensional and higher dimensional case appears with the computational point of view. The computation of the set of minimal forbidden words for one-dimensional shifts of finite type, or one-dimensional sofic shifts, and the computation of its growth rate is easily done in polynomial time (see [6] and [2]), while the problem seems to have at least the same difficulty as the problem of computing the entropy of the shift in dimension two. It is also undecidable to check whether a given pattern is contained in a configuration of a given shift of finite type (see [14] for instance).

For a multi-dimensional shift X, we denote by $h^1(X)$ the complexity of the strong minimal forbidden patterns and by $h^2(X)$ the complexity of the weak minimal forbidden patterns. We prove two partial results of invariance under conjugacy. First, if X and Y are two conjugate shifts, $h^1(X) \leq h^2(Y)$. Second, the strong entropy $h^1(X)$ of minimal forbidden patterns is a conjugacy invariant for shifts which are uniformly semi-strongly irreducible. This latter property is a property of irreducibility of shifts of finite type approximating the shift from the outside. It is always satisfied by one-dimensional irreducible sofic shifts. We prove our main results for square shapes and the results are valid for any dimension. The proofs of these results cannot be generalized in the case of rectangular shapes.

The paper is organized as follows. In Section 2, we recall some basic definitions from symbolic dynamics. The reader is referred to [15] or [13] for more details, see also [14], [20], and [16] for multi-dimensional shifts, [8] for shifts on Cayley graphs. We define here the notions of minimal forbidden patterns, the weaker and the stronger. We prove our main results in Section 3.

2 Definitions and examples

2.1 Background on shifts and conjugacies

We recall here some basic definitions and properties about multi-dimensional shifts and conjugacies. We also fix some notations.

Let A be a finite *alphabet* and d be a positive integer. The *d-dimensional full A-shift* is the set $A^{\mathbb{Z}^d}$ of all functions $c : \mathbb{Z}^d \to A$. In the language of cellular automata we have a space in which the "universe" is the integer lattice $\mathbb{Z}^d$ and a *configuration* is an element of $A^{\mathbb{Z}^d}$, that is a function assigning to each cell of the grid a letter of A. On this set we have a natural metric: if $c_1, c_2 \in A^{\mathbb{Z}^d}$ are two configurations, we define the distance

$$\mathrm{dist}(c_1, c_2) := \frac{1}{n+1},$$

where n is the least natural number such that $c_1 \neq c_2$ in $D_n := [-n, n]^d$. If such an n does not exist, that is if $c_1 = c_2$, we set their distance equal to zero. This metric induces a topology equivalent to the usual product topology, where the topology in A is the discrete one. In the sequel we focus on the case $d = 2$.

The group $\mathbb{Z}^2$ acts on $A^{\mathbb{Z}^2}$ as follows:

$$(c^\gamma)_{|\alpha} := c_{|\gamma+\alpha}$$

for each $c \in A^{\mathbb{Z}^2}$ and each $\gamma, \alpha \in \mathbb{Z}^2$, where $c_{|\alpha}$ is the value of c at α and the addition is the usual operation in the direct sum $\mathbb{Z}^2$.

Now we give a topological definition of a *shift space* (briefly *shift*). As stated in Proposition 2.2, this definition is equivalent to the classical combinatorial one.

Definition 2.1 A subset X of $A^{\mathbb{Z}^2}$ is called a *shift* if it is topologically closed and $\mathbb{Z}^2$-invariant.

Here $\mathbb{Z}^2$-invariance means that X is invariant under the action of $\mathbb{Z}^2$ on $A^{\mathbb{Z}^2}$ (that is $c^\gamma \in X$ for each $c \in X$ and each $\gamma \in \mathbb{Z}^2$). Notice that this is equivalent to have $c^{(1,0)} \in X$ and $c^{(0,1)} \in X$ for each $c \in X$.

A *pattern* is a function $p : E \to A$, where E is a non-empty finite subset of $\mathbb{Z}^2$. The set E is called the *support of* the pattern. In the sequel, we do not distinguish between a pattern p with support E and the pattern obtained by copying p on a translated support of E. A *block* is a pattern with a connected support. A *pattern (resp. block) of* X is the restriction of a configuration of X

to a finite (resp. a finite connected) subset of $\mathbb{Z}^2$. Notice that, being a shift $\mathbb{Z}^2$-invariant, these notions are independent of the position of their supports.

We denote by $\mathcal{B}(X)$ the set of blocks of a shift X and by $\mathcal{B}_n(X)$ the set of square blocks of size n of X. If X is a subshift of $A^{\mathbb{Z}}$, a configuration is a bi-infinite word and a block of X is a finite word appearing in some configuration of X.

Let $\mathcal{F}$ be a set of patterns, we denote by $\mathsf{X}_{\mathcal{F}}$ the set of all configurations of $A^{\mathbb{Z}^2}$ avoiding each pattern of $\mathcal{F}$. It is easy to prove that the topological definition of a shift space is equivalent to the following combinatorial one involving the avoidance of certain *forbidden patterns*.

Proposition 2.2 *A subset $X \subseteq A^{\mathbb{Z}^2}$ is a shift if and only if there exists a set of patterns $\mathcal{F}$ such that $X = \mathsf{X}_{\mathcal{F}}$. In this case, $\mathcal{F}$ is a* set of forbidden patterns *of X.*

Definition 2.3 Let X be a subshift of $A^{\mathbb{Z}^2}$. A map $\tau : X \to A^{\mathbb{Z}^2}$ is *k-local* if there exists $\delta : \mathcal{B}_{2k+1}(X) \to A$ such that for every $c \in X$ and $\gamma \in \mathbb{Z}^2$

$$(\tau(c))_{|\gamma} = \delta((c^{\gamma})_{|D_k}) = \delta(c_{|\gamma+\alpha_1}, c_{|\gamma+\alpha_2}, \ldots, c_{|\gamma+\alpha_m}),$$

where $\alpha_1, \ldots, \alpha_m$ denote the elements of $D_k = [-k, k]^2$.

In this definition, we have assumed that the alphabet of the shift X is the same as the alphabet of its image $\tau(X)$. In this assumption there is no loss of generality because if $\tau : X \subseteq A^{\mathbb{Z}^2} \to B^{\mathbb{Z}^2}$, one can always consider X as a shift over the alphabet $A \cup B$.

It is known from the Curtis-Lyndon-Hedlund theorem that local maps are exactly the functions which are continuous and commute with the $\mathbb{Z}^2$-action (i.e. for each $c \in X$ and each $\gamma \in \mathbb{Z}^2$, one has $\tau(c^{\gamma}) = \tau(c)^{\gamma}$). Hence if a local map is one-to-one and onto, its inverse is also a local map.

This result leads us to give the following definition.

Definition 2.4 Two subshifts $X, Y \subseteq A^{\mathbb{Z}^2}$ are *conjugate* if there exists a local bijective map between them (namely a *conjugacy*). The *invariants* are the properties of a shift invariant under conjugacy.

The basic definition of a shift of finite type is in terms of forbidden patterns. In a sense we may say that a shift is of finite type if we can decide whether or not a configuration belongs to the shift only by checking its blocks of a fixed (and only depending on the shift) size.

Definition 2.5 A shift is *of finite type* if it admits a finite set of forbidden blocks.

We will see later some examples of shifts of finite type.

Definition 2.6 If $X \subseteq A^{\mathbb{Z}^2}$ is a shift, the *entropy of* X is defined as

$$h(X) := \lim_{n\to\infty} \frac{\log |\mathcal{B}_n(X)|}{n^2}. \qquad (1)$$

We will always use the base 2 for logarithms.

The existence of the limit in (1) is proved for instance in [15, Proposition 4.1.8] for the one-dimensional case and in [12] for the multi-dimensional one.

For each $\gamma \in \mathbb{Z}^2$, the set D_n provides, by translation, a *neighborhood of* γ, that is the set $D(\gamma, n) := \gamma + D_n = [\gamma - n, \gamma + n]^2$. Given a subset $E \subseteq \mathbb{Z}^2$ and for each $k \in \mathbb{N}$ we denote by

$$E^{+k} := \bigcup_{\alpha \in E} D(\alpha, k) \text{ and } E^{-k} := \{\alpha \in E \mid D(\alpha, k) \subseteq E\}$$

the k-*closure of* E and the k-*interior of* E, respectively.

Let $\tau : X \to Y$ be a k-local map. If p is a pattern of X with support E, the map τ is defined on p and gives a pattern of Y with support E^{-k}. Indeed one can define

$$\tau(p) := \tau(c)_{|E^{-k}},$$

where c is any configuration of X extending p.

The following well-known result guarantees that *the entropy is invariant under conjugacy.*

Proposition 2.7 *Let X be a shift and let $\tau : X \to A^{\mathbb{Z}^2}$ be a local map. Then $h(\tau(X)) \leq h(X)$.*

PROOF Let τ be k-local and let $Y := \tau(X)$. The map $\tau : \mathcal{B}_{n+2k}(X) \to \mathcal{B}_n(Y)$ is surjective and hence

$$|\mathcal{B}_n(Y)| \leq |\mathcal{B}_{n+2k}(X)| \leq |\mathcal{B}_n(X)||A^{[1,n+2k]^2 \setminus [1,n]^2}|.$$

From the previous inequalities we have

$$\frac{\log |\mathcal{B}_n(Y)|}{n^2} \leq \frac{\log |\mathcal{B}_n(X)|}{n^2} + \frac{((n+2k)^2 - n^2)\log |A|}{n^2}$$

and hence, taking the limit, $h(Y) \leq h(X)$. $\square$

In the case of Cayley graphs the entropy is defined as a maximum limit and, as proved in [8, Theorem 2.12], it is an invariant if the group is amenable.

2.2 Minimal forbidden patterns

We define below several notions of minimal forbidden patterns. In the one-dimensional case, a word is minimal forbidden if it is forbidden and if each strict factor is allowed. The natural extension of this property leads to a forbidden block whose proper subblocks are allowed. This corresponds to our second definition below. But, as we will see later, this definition is too weak. For instance, a shift of finite type does not necessarily have a finite set of minimal forbidden patterns with respect to this definition. For this reason, our first definition below corresponds to a stronger property which is equivalent to the other one in the one-dimensional case. We also make a distinction between the cases in which these blocks are squares or rectangles. We will see below that this distinction is relevant.

Let m, n be two nonnegative integers. We denote by $\mathcal{F}_n(X)$ the set of forbidden squares of X of size n, and by $\mathcal{F}_{m,n}(X)$ the set of forbidden rectangles of X of size $m \times n$. If it is not specified a particular set of forbidden patterns for X, with forbidden pattern we mean a pattern which is not allowed.

Now we give four different possible definitions of *minimal forbidden patterns* of a shift X. Let m, n be two integers greater than or equal to 1.

- $\mathcal{M}^1_n(X) := \mathcal{F}_n(X) \cap \mathcal{B}(\mathsf{X}_{\mathcal{F}_{n-1}(X)})$. That is a square of size n is minimal forbidden if it is forbidden and if it is contained in a configuration in which each square of size $n-1$ is allowed;
- $\mathcal{M}^2_n(X)$ is the set of squares of $\mathcal{F}_n(X)$ such that each subsquare of size $n-1$ is an element of $\mathcal{B}(X)$;
- $\mathcal{M}^1_{m,n}(X) := \mathcal{F}_{m,n}(X) \cap \mathcal{B}(\mathsf{X}_{\mathcal{F}_{m-1,n}(X)}) \cap \mathcal{B}(\mathsf{X}_{\mathcal{F}_{m,n-1}(X)})$;
- $\mathcal{M}^2_{m,n}(X)$ is the set of rectangles of $\mathcal{F}_{m,n}(X)$ such that each proper subrectangle is an element of $\mathcal{B}(X)$.

It is straightforward that, for any integers m, n, we have the inclusions $\mathcal{M}^1_{n,n}(X) \subseteq \mathcal{M}^1_n(X) \subseteq \mathcal{M}^2_n(X)$ and $\mathcal{M}^1_{m,n}(X) \subseteq \mathcal{M}^2_{m,n}(X)$. We will see that there are examples in which these inclusions are strict.

We denote with $\mathcal{M}^i(X)$ the set $\bigcup_n \mathcal{M}^i_n(X)$ (for $i = 1, 2$). We prove below that the sets $\bigcup_{m,n} \mathcal{M}^1_{m,n}(X)$ and $\mathcal{M}^1(X)$ are sets of forbidden patterns for X.

Proposition 2.8 *The set $\bigcup_{m,n} \mathcal{M}^1_{m,n}(X)$ is a set of forbidden patterns for X, that is $X = \mathsf{X}_{\bigcup_{m,n} \mathcal{M}^1_{m,n}(X)}$.*

PROOF If $c \in X$ and p is a rectangle of c, we have $p \notin \mathcal{F}_{m,n}(X)$ for every $m, n \geq 1$ and then $p \notin \bigcup_{m,n} \mathcal{M}^1_{m,n}(X)$. Hence $X \subseteq \mathsf{X}_{\bigcup_{m,n} \mathcal{M}^1_{m,n}(X)}$.

Suppose that $c \notin X$. Since $X = \bigcap_{m,n} \mathsf{X}_{\mathcal{F}_{m,n}(X)}$ we have $c \notin \mathsf{X}_{\mathcal{F}_{m,n}(X)}$ for some $m, n \geq 0$. Notice that the shifts $\mathsf{X}_{\mathcal{F}_{m,n}(X)}$ have the property that $c \notin \mathsf{X}_{\mathcal{F}_{m,n}(X)}$ implies $c \notin \mathsf{X}_{\mathcal{F}_{m+1,n}(X)}$ and $c \notin \mathsf{X}_{\mathcal{F}_{m,n+1}(X)}$. This means that in the grid of the natural numbers in which a pair (m, n) is "marked" if and only if $c \notin \mathsf{X}_{\mathcal{F}_{m,n}(X)}$, there are some *extremal* pairs, that is, pairs which are marked but such that the pair on the left and the pair below are not marked. In Figure 1 the left corners of the dashed line show the extremal pairs for c. Notice that since $\mathcal{F}_{m,0} = \emptyset = \mathcal{F}_{0,n}$, the pairs $(m, 0)$ and $(0, n)$ are always unmarked. Hence if (m, n) is an extremal pair for c we have $m, n \geq 1$, $c \in \mathsf{X}_{\mathcal{F}_{m-1,n}(X)}$ and

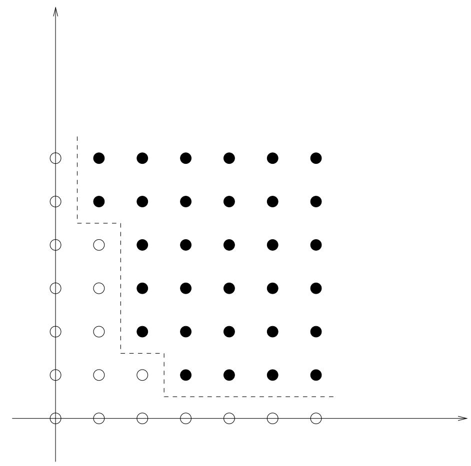


Figure 1: The extremal pairs for c.

$c \in \mathsf{X}_{\mathcal{F}_{m,n-1}(X)}$. Since $c \notin \mathsf{X}_{\mathcal{F}_{m,n}(X)}$ there exists a forbidden rectangle p of size $m \times n$ in c and this is also a pattern of $\mathsf{X}_{\mathcal{F}_{m-1,n}(X)}$ and a pattern of $\mathsf{X}_{\mathcal{F}_{m,n-1}(X)}$. This means that $p \in \mathcal{M}^1_{m,n}(X)$ and hence $c \notin X_{\bigcup_{m,n} \mathcal{M}^1_{m,n}(X)}$. □

Remark. One can see that Proposition 2.8 holds in dimension d. Indeed the shifts $\mathsf{X}_{\mathcal{F}_{n_1,\ldots,n_d}(X)}$ are such that $c \notin \mathsf{X}_{\mathcal{F}_{n_1,\ldots,n_d}(X)}$ implies $c \notin \mathsf{X}_{\mathcal{F}_{n_1,\ldots,n_k+1,\ldots,n_d}(X)}$ for each $k = 1, \ldots, d$.

Proposition 2.9 *The set $\mathcal{M}^1(X)$ is a set of forbidden patterns for X, that is $X = \mathsf{X}_{\mathcal{M}^1(X)}$.*

Proof If $c \in X$ and p is a square of c, we have $p \notin \mathcal{F}_n(X)$ for every $n \geq 1$ and hence $p \notin \mathcal{M}^1_n(X)$. Thus $X \subseteq \mathsf{X}_{\mathcal{M}^1(X)}$.

Suppose that $c \notin X$. If $c \notin \mathsf{X}_{\mathcal{F}_1(X)}$ we have $c \notin \mathsf{X}_{\mathcal{M}^1_1(X)}$ and hence $c \notin \mathsf{X}_{\mathcal{M}^1(X)}$ (notice that $\mathcal{M}^1_1(X) = \mathcal{F}_1(X)$ since $\mathcal{F}_0(X) = \emptyset$). Otherwise, since

$$\mathsf{X}_{\mathcal{F}_1(X)} \supseteq \mathsf{X}_{\mathcal{F}_2(X)} \supseteq \cdots \supseteq \mathsf{X}_{\mathcal{F}_n(X)} \cdots \supseteq X,$$

there exists an integer i such that $c \in \mathsf{X}_{\mathcal{F}_i(X)}$ and $c \notin \mathsf{X}_{\mathcal{F}_{i+1}(X)}$. Hence there is a pattern p of c of size $i + 1$ which is forbidden in X (that is $p \in \mathcal{F}_{i+1}(X)$). Moreover we have $p \in \mathcal{B}(\mathsf{X}_{\mathcal{F}_i(X)})$ and hence $p \in \mathcal{M}^1_{i+1}(X)$ which implies $c \notin \mathsf{X}_{\mathcal{M}^1(X)}$. □

Notice that, as an easy consequence of Propositions 2.8 and 2.9, the sets $\bigcup_{m,n} \mathcal{M}^2_{m,n}(X)$ and $\mathcal{M}^2(X)$ also are possible sets of forbidden patterns for X.

Proposition 2.10 *A shift X is of finite type if and only if $\mathcal{M}^1(X)$ is finite.*

PROOF If $\mathcal{M}^1(X)$ is finite, the shift X is of finite type by Proposition 2.9. Conversely, suppose that X is of finite type and hence that $X = \mathsf{X}_{\mathcal{F}}$, where $\mathcal{F}$ is a finite set of forbidden squares. Let n be such that there are no squares in $\mathcal{F}$ of size greater than or equal to n. If $h \geq n$ and p is a square of $\mathcal{M}^1_h(X)$, there exists a configuration $c \in \mathsf{X}_{\mathcal{F}_{h-1}(X)}$ which contains p. Thus $c \notin X$. Hence there is a square of $\mathcal{F}$ contained in c, and the size of this square must be greater than or equal to $h \geq n$, which is excluded. Hence $\mathcal{M}^1_h(X) = \emptyset$ for each $h \geq n$. $\square$

By Proposition 2.8, if $\bigcup_{m,n} \mathcal{M}^1_{m,n}(X)$ is finite then X is of finite type. We will see with an example that the converse is not true. Nevertheless we have the following result.

Proposition 2.11 *A shift X is of finite type if and only if there is a positive integer n_0 such that $\mathcal{M}^1_{m,n}(X) = \emptyset$ for $m, n \geq n_0$.*

PROOF Suppose that X is of finite type and hence that $X = \mathsf{X}_{\mathcal{F}}$ where $\mathcal{F}$ is a finite set of forbidden rectangles. Let n_0 be an integer such that there are no rectangles in $\mathcal{F}$ of size $m \times n$, when $m \geq n_0$ or $n \geq n_0$. Let m and n be two integers such that $m \geq n_0$ and $n \geq n_0$. If p is a rectangle of $\mathcal{M}^1_{m,n}(X)$, there exists a configuration c containing p which belongs to $\mathsf{X}_{\mathcal{F}_{m,n-1}(X)}$. This configuration is not in X. Then c contains a rectangle of $\mathcal{F}$ of size $\bar{m} \times \bar{n}$ with $\bar{m} > m \geq n_0$ or $\bar{n} \geq n \geq n_0$. This contradicts the fact that there are no rectangles in $\mathcal{F}$ of size $m \times n$, when $m \geq n_0$ or $n \geq n_0$. Hence $\mathcal{M}^1_{m,n}(X) = \emptyset$ for each $m, n \geq n_0$. $\square$

In the following proposition, we prove that the possible notions of minimal forbidden patterns coincide in the one-dimensional case.

Proposition 2.12 *Let X be a one-dimensional shift. Then $\mathcal{M}^1_n(X) = \mathcal{M}^2_n(X)$.*

PROOF Let w be a minimal forbidden word in $\mathcal{M}^2_n(X)$, where w_1 is its left prefix of length $n-1$, and w_2 its right suffix of length $n-1$. Then w_1, w_2 are allowed words of X. Let l be a left-infinite word and r be a right-infinite word such that $lw_1r \in X$. Similarly let $\bar{l}$ be a left-infinite word and $\bar{r}$ be a right-infinite word with $\bar{l}w_2\bar{r} \in X$. Then $lw\bar{r}$ belongs to $\mathsf{X}_{\mathcal{F}_{n-1}(X)}$. Thus $w \in \mathcal{M}^1_n(X)$. $\square$

Now we give two examples of shifts of finite type for which the sets of minimal forbidden squares $\mathcal{M}^1$ and $\mathcal{M}^2$ are both finite. Other examples can be found in [16] and [20].

Example 2.13 The following example is a two-dimensional shift of finite type X for which the value of its entropy of allowed blocks $h(X)$ is known. Let A be the alphabet $\{0, 1, 2\}$. We define the shift of finite type $X = \mathsf{X}_{\mathcal{F}}$ where $\mathcal{F}$ is the following set of patterns:

$$\begin{array}{|c|c|}\hline x & x \\ \hline \end{array} \qquad \begin{array}{|c|}\hline x \\ \hline x \\ \hline \end{array}$$

with $x \in A$. The configurations of this bidimensional shift are the three colorings of a square lattice. Two adjacent cells have a different color. It turns out that the exact value of the entropy of this shift is known (see [1]) and equal to

$$h(X) = \frac{3}{2} \log \frac{4}{3}. \quad \square$$

Example 2.14 We now give an example of a two-dimensional shift of finite type X for which the exact value of its entropy of allowed blocks is not known. Let A be the alphabet $\{0, 1\}$. We define the shift of finite type $X = \mathsf{X}_{\mathcal{F}}$ where $\mathcal{F}$ is the following set of patterns:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \end{array}$$

These constraints are known as the hard square constraints. They correspond to some lattice gas models [1]. $\square$

For the two shifts of Examples 2.13 and 2.14, both sets $\mathcal{M}^1$ and $\mathcal{M}^2$ are finite. Indeed, $\mathcal{M}^2$ only contains the 2×2 squares containing a forbidden rectangle of $\mathcal{F}$. We now give an example of a bidimensional shift of finite type for which the sets $\mathcal{M}^2$ and $\bigcup_{m,n} \mathcal{M}^2_{m,n}$ are not finite.

Example 2.15 Let A be an alphabet and $\bar{A} := A \cup \{a, b\}$, where a and b do not belong to A. We define the shift of finite type $X = \mathsf{X}_{\mathcal{F}}$ where $\mathcal{F}$ is the following set of patterns:

$$\begin{array}{|c|c|}\hline & x \\ \hline a & \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline y & \\ \hline & b \\ \hline \end{array}$$

where $x \neq a$ and $y \neq b$. For n big enough we have $\mathcal{M}^1_n(X) = \emptyset$, but $\mathcal{M}^2_n(X)$ contains, if n is odd, the following squares of size n:

$$\begin{array}{|c|c|c|c|c|}\hline a & * & \dots & * & b \\ \hline * & * & \dots & * & * \\ \hline * & * & \dots & * & * \\ \hline \vdots & \vdots & \dots & \vdots & \vdots \\ \hline * & * & \dots & * & * \\ \hline \end{array}$$

where each $*$ can be replaced by any letter in A. Thus $|\mathcal{M}^2_n(X)| \geq |A|^{n^2-2}$. Moreover, for m, n big enough, we have $\mathcal{M}^1_{m,n}(X) = \emptyset$. In this example we also have that $\bigcup_{m,n} \mathcal{M}^1_{m,n}(X)$ is not finite. Indeed $\mathcal{M}^1_{1,n}(X)$ contains, if n is odd, the following rectangle:

$$\begin{array}{|c|c|c|c|c|}\hline a & * & \dots & * & b \\ \hline \end{array}$$

where each $*$ can be replaced by any letter in A. This can be easily seen observing that the configuration filled of a's outside the rectangle is contained in $\mathsf{X}_{\mathcal{F}_{0,n}(X)} \cap \mathsf{X}_{\mathcal{F}_{1,n-1}(X)}$. Hence $|\mathcal{M}^1_{1,n}(X)| \geq |A|^{n-2}$. $\square$

Example 2.16 Let A be the alphabet $\{a, b, c\}$ and $\mathcal{F}$ be the set of squares bordered by b's as follows,

$$\begin{array}{|c|c|c|c|c|}\hline b & b & \dots & b & b \\ \hline b & * & \dots & * & b \\ \hline \vdots & \vdots & \dots & \vdots & \vdots \\ \hline b & * & \dots & * & b \\ \hline b & b & \dots & b & b \\ \hline \end{array} \tag{2}$$

where each $*$ can be replaced with an a or a c. Let $X = \mathsf{X}_{\mathcal{F}}$. We have $\mathcal{M}_n^1(X) = \mathcal{M}_n^2(X) = \mathcal{M}_{n,n}^1(X)$, and a minimal forbidden square of size n in X is a square $n \times n$ bordered by b's and contained in $\mathcal{F}$. Hence $|\mathcal{M}_n^1(X)| = |\mathcal{M}_n^2(X)| = |\mathcal{M}_{n,n}^1(X)| = 2^{(n-2)^2}$. Moreover we have $\mathcal{M}_{m,n}^1(X) = \emptyset$ if $m \neq n$. $\square$

Example 2.17 With a slight modification of Example 2.16, one can see that in general $\mathcal{M}_n^1 \neq \mathcal{M}_{n,n}^1$. Let A be the alphabet $\{a, b, c\}$ and $\mathcal{F}$ be the set of *rectangles* $n \times (n+1)$ bordered by b's as in (2), where each $*$ can be replaced with an a or a c. The square $(n+1) \times (n+1)$

$$\underbrace{\begin{array}{|c|c|c|c|c|}\hline b & b & \dots & b & b \\ \hline b & a & \dots & a & b \\ \hline \vdots & \vdots & \dots & \vdots & \vdots \\ \hline b & a & \dots & a & b \\ \hline b & b & \dots & b & b \\ \hline a & a & \dots & a & a \\ \hline \end{array}}_{(n+1)\times(n+1)}$$

is contained in $\mathcal{M}_{n+1}^1(X)$, but $\mathcal{M}_{n+1,n+1}^1(X) = \emptyset$. In this example one can also see that in general $X \neq \mathsf{X}_{\bigcup_n \mathcal{M}_{n,n}^1(X)}$. $\square$

3 Entropy of minimal forbidden patterns

In this section, we state and prove our main invariance results on the entropy of minimal forbidden patterns in the case of square blocks. We will explain later why these results cannot be extended to the case of rectangular shapes.

Definition 3.1 For $i = 1, 2$, we denote by $h^i(X)$ the *entropy of the sequence* $(\mathcal{M}_n^i(X))$ *of minimal forbidden patterns* of X, that is:

$$h^i(X) := \limsup_{n\to\infty} \frac{1}{n^2} \log |\mathcal{M}_n^i(X)|.$$

Notice that h^1 is always $-\infty$ for shifts of finite type. In the Example 2.15 we have $h^1(X) = -\infty$ and $h^2(X) \geq \log(|A|)$.

Let τ be a k-local map defined on X. The map τ_n is well defined on $\mathsf{X}_{\mathcal{F}_n(X)}$ if $n \geq 2k+1$ and

$$\tau_n(c)_{|\alpha} := \tau(c_{|D(\alpha,k)}) \tag{3}$$

(indeed $c_{|D(\alpha,k)}$ is a pattern of X).

Lemma 3.2 *Let τ be a k-local map defined on X. If $c \in X$ then $c \in \mathsf{X}_{\mathcal{F}_n(X)}$ and $\tau_n(c) = \tau(c)$.*

PROOF We have $\tau_n(c)_{|\alpha} = \tau(c_{|D(\alpha,k)}) = \tau(c)_{|D(\alpha,k)^{-k}} = \tau(c)_{|\alpha}$. □

Lemma 3.3 *Let τ be a k-local map defined on X. If p is a pattern of X then it is also a pattern of $\mathsf{X}_{\mathcal{F}_n(X)}$ and $\tau_n(p) = \tau(p)$.*

PROOF Let E be the support of p and let $c \in X$ be a configuration extending p. One has $\tau_n(p) = \tau_n(c)_{|E^{-k}}$. By Lemma 3.2, we have $\tau_n(c)_{|E^{-k}} = \tau(c)_{|E^{-k}} = \tau(p)$. □

Proposition 3.4 *Let $\tau : X \to Y$ be a k-local map. If $n \geq 2k+1$, we have $\tau_n : \mathsf{X}_{\mathcal{F}_n(X)} \to \mathsf{X}_{\mathcal{F}_{n-2k}(Y)}$.*

PROOF Let $c \in \mathsf{X}_{\mathcal{F}_n(X)}$ and let p a square of size $n-2k$ of $\tau_n(c)$. There exists a square $\bar{p}$ of c with size n such that $\tau_n(\bar{p}) = p$. We have $\bar{p} \in \mathcal{B}(X)$ and hence there exists a $\bar{c} \in X$ such that $\bar{p}$ is a square of $\bar{c}$. This means that p is a square of $\tau(\bar{c}) \in \tau(X) \subseteq Y$. □

Proposition 3.5 *Let $\tau : X \to Y$ be an injective k-local map. Then there exists an n such that τ_n is injective on $\mathsf{X}_{\mathcal{F}_n(X)}$ (and hence it is injective on each $\mathsf{X}_{\mathcal{F}_m(X)}$ with $m \geq n$).*

PROOF Suppose that for each n the map τ_n is not injective on $\mathsf{X}_{\mathcal{F}_n(X)}$. Then there exist two sequences $(c_n)_n$ and $(\bar{c}_n)_n$ with $c_n, \bar{c}_n \in \mathsf{X}_{\mathcal{F}_n(X)}$ such that for each n we have $c_n \neq \bar{c}_n$ and $\tau_n(c_n) = \tau_n(\bar{c}_n)$. We can always suppose that $c_n \neq \bar{c}_n$ at the center $(0,0)$ of $A^{\mathbb{Z}^2}$. Being $A^{\mathbb{Z}^2}$ compact, there exist $c, \bar{c} \in A^{\mathbb{Z}^2}$ and two subsequences $(c_{n_k})_k$ and $(\bar{c}_{n_k})_k$ such that $\lim_k c_{n_k} = c$ and $\lim_k \bar{c}_{n_k} = \bar{c}$. For each h, the sequence $(c_{n_k})_{k \geq h}$ is contained in $\mathsf{X}_{\mathcal{F}_{n_h}(X)}$ and it being closed, $c \in \mathsf{X}_{\mathcal{F}_{n_h}(X)}$. This implies that $c \in X$ and analogously $\bar{c} \in X$. Moreover the continuity of τ_{n_h} implies that $\tau(c) = \tau(\bar{c})$. But being $\mathrm{dist}(c_n, \bar{c}_n) = 1$, we have $c \neq \bar{c}$, which contradicts the injectivity of τ. □

Lemma 3.6 *Let $\tau : X \to Y$ be a bijective k-local map, let τ^{-1} be $\bar{k}$-local and let $c \in \mathsf{X}_{\mathcal{F}_n(X)}$ with $n \geq 2k + 2\bar{k} + 1$. Hence $(\tau^{-1})_{n-2k}(\tau_n(c)) = c$.*

PROOF Let α be an element of $\mathbb{Z}^2$. We have

$$(\tau^{-1})_{n-2k}(\tau_n(c))_{|\alpha} = \tau^{-1}(\tau_n(c)_{|D(\alpha,\bar{k})}) = \tau^{-1}(\tau_n(c_{|D(\alpha,\bar{k}+k)})).$$

By Lemma 3.3, we have

$$\tau^{-1}(\tau_n(c_{|D(\alpha,\bar{k}+k)})) = \tau^{-1}(\tau(c_{|D(\alpha,\bar{k}+k)})) = \tau^{-1}(\tau(\bar{c})_{|D(\alpha,\bar{k})}),$$

where $\bar{c} \in X$ extends $c_{|D(\alpha,\bar{k}+k)}$. Now $\tau^{-1}(\tau(\bar{c})_{|D(\alpha,\bar{k})}) = \tau^{-1}(\tau(\bar{c}))_{|\alpha} = \bar{c}_{|\alpha} = c_{|\alpha}$. $\square$

Notice that in Proposition 3.5 we have proved that, if τ is one-to-one, the map $\tau_n : X_{\mathcal{F}_n(X)} \to \tau_n(X_{\mathcal{F}_n(X)})$ is invertible. In Lemma 3.6, we have proved that the inverse of this map is $((\tau^{-1})_{n-2k})_{|\tau_n(X_{\mathcal{F}_n(X)})}$. In particular the constant of locality of $(\tau_n)^{-1}$ is the same as that of τ^{-1} (that is $\bar{k}$).

Lemma 3.7 *Let $\tau : X \to Y$ be a bijective k-local map, let τ^{-1} be $\bar{k}$-local and let p be a pattern of $\mathsf{X}_{\mathcal{F}_n(X)}$ with $n \geq 2k + 2\bar{k} + 1$. Hence, if E is the support of p, we have $(\tau^{-1})_{n-2k}(\tau_n(p)) = p_{|E^{-k-\bar{k}}}$.*

Proof Let c be a configuration of $\mathsf{X}_{\mathcal{F}_n(X)}$ extending p. We have

$$(\tau^{-1})_{n-2k}(\tau_n(p)) = (\tau^{-1})_{n-2k}(\tau_n(c)_{|E^{-k}}) = (\tau^{-1})_{n-2k}(\tau_n(c))_{|E^{-k-\bar{k}}}.$$

By Lemma 3.6, we have $(\tau^{-1})_{n-2k}(\tau_n(c))_{|E^{-k-\bar{k}}} = c_{|E^{-k-\bar{k}}} = p_{|E^{-k-\bar{k}}}$. $\square$

Observe that a shift of finite type can also be defined using the notion of *allowed patterns*. More precisely a shift X is of finite type if and only if there exists a finite set $\mathcal{C}$ of patterns such that $X = \mathsf{X}(\mathcal{C})$, where

$$\mathsf{X}(\mathcal{C}) := \{c \in A^{\mathbb{Z}^2} \mid \text{each pattern of } c \text{ belongs to } \mathcal{C}\}.$$

Indeed if $\mathcal{F}$ is a finite set of forbidden patterns, we can always suppose that each of them has the same support F and we can define $\mathcal{C} := A^F \backslash \mathcal{F}$. Observe that in this case it is not necessary that each pattern of $\mathcal{C}$ is a pattern of X, but we can always suppose that $\mathcal{C} = \mathcal{C} \cap \mathcal{B}(X)$.

With this equivalent characterization it is possible to prove the invariance of the notion of being of finite type. For this we can define $\bar{\mathcal{C}} := \{\tau(c)_{|F^{+\bar{k}}} \mid c \in X \text{ and } c \text{ extends } p \in \mathcal{C}\}$ and prove, using Lemma 3.7, that $Y = \mathsf{X}(\bar{\mathcal{C}})$. In the following theorem we prove this invariance as a consequence of Proposition 3.5.

Proposition 3.8 *The notion of being of finite type for multi-dimensional shifts is invariant under conjugacy.*

Proof Suppose that $\tau : X \to Y$ is a k-local conjugacy and suppose that Y is of finite type. Hence there exists n such that $Y = \mathsf{X}_{\mathcal{F}_n(Y)}$ (obviously this condition is also sufficient). Suppose that $c \in \mathsf{X}_{\mathcal{F}_{n+2k}(X)}$. The configuration $\tau_{n+2k}(c)$ belongs to $\mathsf{X}_{\mathcal{F}_n(Y)} = Y$ and then $\tau_{n+2k}(c) = \tau(\bar{c})$ with $\bar{c} \in X$. By Lemma 3.2 we have $\tau_{n+2k}(c) = \tau_{n+2k}(\bar{c})$ and being τ_{n+2k} one-to-one we have $c = \bar{c}$. This implies $X = \mathsf{X}_{\mathcal{F}_{n+2k}(X)}$. $\square$

The following theorem extends the one-dimensional construction of [2] to multi-dimensional shifts.

Theorem 3.9 *Let $\tau : X \to Y$ be a conjugacy, let τ be k-local and let τ^{-1} be $\bar{k}$-local. Hence for $n \geq 2k + 2\bar{k} + 1$*

$$|\mathcal{M}_n^1(X)| \leq C(n) \sum_{r=-2k}^{2\bar{k}} |\mathcal{M}_{n+r}^2(Y)|,$$

where $C(n) = (2\bar{k} + 2k + 1)^2 |A|^{4(\bar{k}+k)(n+\bar{k}-k)}$.

Proof If $p \in \mathcal{M}_n^1(X)$ we have $p \in \mathcal{B}(\mathsf{X}_{\mathcal{F}_{n-1}(X)})$, and hence there exists $c \in \mathsf{X}_{\mathcal{F}_{n-1}(X)}$ such that c contains p in $E := [1, n]^2$. Consider the pattern $\bar{p} := \tau_{n-1}(c)_{|E^{+\bar{k}}}$ of $\tau_{n-1}(c)$. If $\bar{p}$ is a pattern of Y there exists $\bar{c} \in Y$ such that $\bar{p}$ is a pattern of $\bar{c}$. Now $\tau^{-1}(\bar{c}) \in X$ and $\tau^{-1}(\bar{c})_{|E} = \tau^{-1}(\bar{c}_{|E^{+\bar{k}}}) = \tau^{-1}(\bar{p})$. Being $\bar{p}$ a pattern of $\mathsf{X}_{\mathcal{F}_{n-1-2k}(Y)}$, we can apply Lemma 3.3 and then $\tau^{-1}(\bar{p}) = (\tau^{-1})_{n-1-2k}(\bar{p}) = (\tau^{-1})_{n-1-2k}(\tau_{n-1}(c)_{|E^{+\bar{k}}}) = (\tau^{-1})_{n-1-2k}(\tau_{n-1}(c_{|E^{+\bar{k}+k}}))$. Now notice that $c_{|E^{+\bar{k}+k}}$ is a pattern in $\mathsf{X}_{\mathcal{F}_{n-1}(X)}$ and we can apply Lemma 3.7. Hence we have $\tau^{-1}(\bar{c})_{|E} = (\tau^{-1})_{n-1-2k}(\tau_{n-1}(c_{|E^{+\bar{k}+k}})) = c_{|E} = p$, which contradicts the fact that $p \notin \mathcal{B}(X)$.

Hence to each $p \in \mathcal{M}_n^1(X)$ one can associate a pattern $\bar{p} \in \mathcal{F}_{n+2\bar{k}}(Y) \cap \mathcal{B}(\mathsf{X}_{\mathcal{F}_{n-1-2k}(Y)})$ and being $(\tau^{-1})_{n-1-2k}(\bar{p}) = p$ this association is one-to-one. Notice that in $\bar{p}$ there is a pattern of $\mathcal{M}_{n+r}^2(Y)$, where $-2k \leq r \leq 2\bar{k}$. Hence the maximal number of such patterns contained in $\bar{p}$ is

$$(2\bar{k} - r + 1)^2 |A|^{(n+2\bar{k})^2 - (n+r)^2} |\mathcal{M}_{n+r}^2(Y)|,$$

where $(2\bar{k} - r + 1)^2$ is the number of positions in which we can insert the left bottom vertex of a square of size $n + r$ in a square of size $n + 2\bar{k}$ (see Figure 2), and $(n + 2\bar{k})^2 - (n + r)^2$ is the number of free positions which we can fill with letters in the alphabet A. Hence

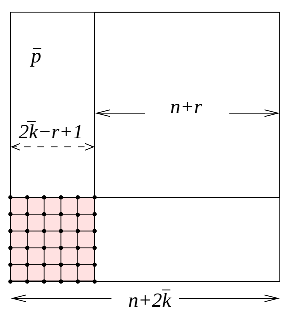


Figure 2: How a pattern in $\mathcal{M}_{n+r}^2(Y)$ can appear in $\bar{p}$.

$$|\mathcal{M}_n^1(X)| \leq \sum_{r=-2k}^{2\bar{k}} (2\bar{k} - r + 1)^2 |A|^{(n+2\bar{k})^2 - (n+r)^2} |\mathcal{M}_{n+r}^2(Y)|$$

$$\leq (2\bar{k}+2k+1)^2|A|^{4(\bar{k}+k)(n+\bar{k}-k)} \sum_{r=-2k}^{2\bar{k}} |\mathcal{M}^2_{n+r}(Y)|. \quad \square$$

Remark. In the d-dimensional case we have

$$|\mathcal{M}^1_n(X)| \leq C(n) \sum_{r=-2k}^{2\bar{k}} |\mathcal{M}^2_{n+r}(Y)|, \tag{4}$$

where $C(n) = (2\bar{k}+2k+1)^d|A|^{(n+2\bar{k})^d-(n-2k)^d}$.

Corollary 3.10 *Let X and Y be two conjugate multi-dimensional shifts. Then $h^1(X) \leq h^2(Y)$.*

Proof By Equation (4) we have that

$$|\mathcal{M}^1_n(X)| \leq (2\bar{k}+2k+1)^d|A|^{(n+2\bar{k})^d-(n-2k)^d} \sum_{r=-2k}^{2\bar{k}} |\mathcal{M}^2_{n+r}(Y)|.$$

Thus

$$|\mathcal{M}^1_n(X)| \leq (2\bar{k}+2k+1)^{d+1}|A|^{(n+2\bar{k})^d-(n-2k)^d} \max_{r=-2k}^{2\bar{k}} |\mathcal{M}^2_{n+r}(Y)|.$$

Hence we have

$$\frac{\log(|\mathcal{M}^1_n(X)|)}{n^d} \leq \frac{(d+1)\log(2\bar{k}+2k+1)}{n^d} +$$

$$+\frac{((n+2\bar{k})^d-(n-2k)^d)\log|A|}{n^d} + \frac{\log|\mathcal{M}^2_{n+\bar{r}}(Y)|}{n^d},$$

where $-2k \leq \bar{r} = \bar{r}(n) \leq 2\bar{k}$. By taking the maximum limits, we have the conclusion. $\square$

From the previous result and by Proposition 2.12, we recover the known result for one-dimensional shifts (see [2]).

Corollary 3.11 *Let X and Y be two conjugate one-dimensional shifts. Then $h^1(X) = h^1(Y)$.*

3.1 Semi-strongly irreducible shifts

In this section we prove that h^1 is an invariant for a suitable class of shifts.

A one-dimensional shift X is *irreducible* if for every $u, v \in \mathcal{B}(X)$ there exists a word w such that the concatenation uwv belongs to $\mathcal{B}(X)$.

This concept can be generalized in the multi-dimensional case: a shift $X \subseteq A^{\mathbb{Z}^d}$ is called *irreducible* if for each pair of blocks $p, q \in \mathcal{B}(X)$ with supports E

and F, there exists a configuration $c \in X$ such that $c = p$ in E and $c = q$ in $\bar{F}$, where $\bar{F}$ is a translation of F contained in $\complement E$.

We now give the definition of semi-strong irreducibility for a shift. This definition is strictly weaker than the one given in [9] needed to prove a *Garden of Eden* theorem for shifts of finite type defined on the Cayley graph of a finitely generated group.

Definition 3.12 A shift X is called (M,h)-*irreducible* (where M,h are natural numbers such that $M \geq h$) if for each pair of blocks $p,q \in \mathcal{B}(X)$ whose supports E and F have distance greater than M, there exists a configuration $c \in X$ such that $c = p$ in E and $c = q$ in $\bar{F}$, where $\bar{F}$ is a translation of F contained in F^{+h}. The shift X is called *semi-strongly irreducible* if it is (M,h)-irreducible for some $M,h \in \mathbb{N}$.

A shift X is *uniformly* (M,h)-*irreducible* if the sequence $(X, \mathsf{X}_{\mathcal{F}_n(X)})$ is uniformly (M,h)-irreducible, i.e. X and $\mathsf{X}_{\mathcal{F}_n(X)}$ are (M,h)-irreducible for any non-negative integer n. The shift X is *uniformly semi-strongly irreducible* if it is uniformly (M,h)-irreducible for some $M,h \in \mathbb{N}$.

Recall (see for instance [15]), that a one-dimensional shift is *sofic* if it is the set of labels of all bi-infinite paths on a finite labeled graph. It is irreducible if and only if one of these graphs is strongly connected.

Proposition 3.13 *Every one-dimensional irreducible sofic shift is semi-strongly irreducible.*

Proof Let X be the set of labels of an M-state labeled strongly connected graph. We prove that X is $(M-1, M-1)$-irreducible. Let $n \geq M-1$ and let $u,v \in \mathcal{B}(X)$. Being u contained in a configuration of X, there exists $\bar{u} \in \mathcal{B}_n(X)$ such that $u\bar{u} \in \mathcal{B}(X)$. Since the graph has M states and is strongly connected, there is word w of length at most $M-1$ such that $u\bar{u}wv \in \mathcal{B}(X)$. Being $n \leq |\bar{u}w| \leq n + (M-1)$, we have the conclusion. $\Box$

Theorem 3.14 *Let X be a uniformly semi-strongly irreducible shift. Let $\tau : X \to Y$ be a conjugacy, let τ be k-local and let τ^{-1} be $\bar{k}$-local. Hence if $n \geq M + 4h + 4k + 2\bar{k}$*

$$|\mathcal{M}^1_n(X)| \leq C(n) \sum_{r=-2k}^{2\bar{k}} |\mathcal{M}^1_{n+r}(Y)|,$$

where $C(n) = (a+n)^2 |A|^{bn+c}$, a,b,c are constants depending only on $k, \bar{k}, M, h$ and the shift X is uniformly (M,h)-irreducible.

Proof Consider $p \in \mathcal{M}^1_n(X)$. Being $\mathsf{X}_{\mathcal{F}_{n-1}(X)}$ an (M,h)-irreducible shift and being p a pattern of it, there exists a configuration $c \in \mathsf{X}_{\mathcal{F}_{n-1}(X)}$ such that c contains p in $E := [1,n]^2$ and a copy of p in each translation $\bar{E}$ of E contained in squares of size $n + 2h$ at mutual distance $M+1$ and positioned as in Figure 3. Hence $\bar{c} := \tau_{n-1}(c)$ is a configuration in $\mathsf{X}_{\mathcal{F}_{n-1-2k}(Y)}$ and, as

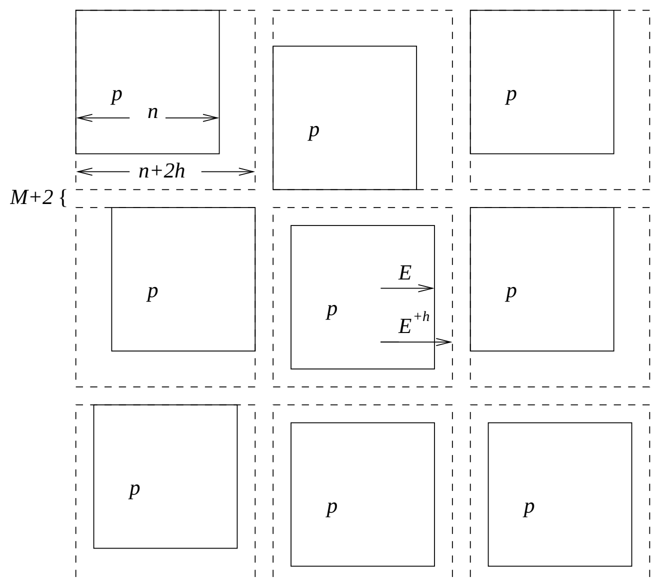


Figure 3: The configuration $c \in \mathsf{X}_{\mathcal{F}_{n-1}(X)}$.

proved in Theorem 3.9, we have $\bar{c} \notin \mathsf{X}_{\mathcal{F}_{n+2\bar{k}}(Y)}$. Then there exists an integer r, with $-2k \leq r \leq 2\bar{k}$ such that $\bar{c} \in \mathsf{X}_{\mathcal{F}_{n+r-1}(Y)}$ and $\bar{c} \notin \mathsf{X}_{\mathcal{F}_{n+r}(Y)}$. This means that $\bar{c}$ contains a pattern $\bar{p} \in \mathcal{M}^1_{n+r}(Y)$ with a support F whose left bottom corner belongs to some square of size $n + 2h + M$ obtained by covering the plane with disjoint copies of $[1 - h, n + h + M]^2$ (recall that $[1, n]^2 = E$ and $[1 - h, n + h]^2 = E^{+h}$). The number of possible positions of this left bottom corner of F inside this square is then $(n + 2h + M)^2$. Let q be the pattern of $\bar{c}$ of size $n - 2k$ defined by $q := \tau_{n-1}(p)$. As one can see in Figure 4, the pattern $\bar{p}$ determines (at most) four rectangles in the copies of q intersecting $\bar{p}$, and hence it determines (at most) four rectangles of q. We are going to count the maximal number of points in q which are not contained in one of these four rectangles. First notice that the maximal distance between two copies of q in $\bar{c}$ is $M+4h+2k+1$ and hence the minimal number of points in the four rectangles is $(n+r)^2-(2(M+4h+2k)(n+r)-(M+4h+2k)^2) = (n+r-(M+4h+2k))^2$ (notice that our condition on n garanties that at least one of these four rectangles is not empty). It turns out that the maximal number of points in q which are not in the rectangles is $(n-2k)^2-(n+r-(M+4h+2k))^2$. By the restrictions on r this number is less than or equal to $e(n) := 2(M+4h+2k)(n-2k)-(M+4h+2k)^2$.

Now $\bar{p}$ determines q excepting for at most $e(n)$ points and hence we can complete q in at most $|A|^{e(n)}$ ways. On the other side, q determines p excepting for at most $f(n) := n^2 - (n - 2k - 2\bar{k})^2$ points and hence we can complete p in at most $|A|^{f(n)}$ ways. Indeed q determines, by $(\tau^{-1})_{n-1-2k}$, a square contained

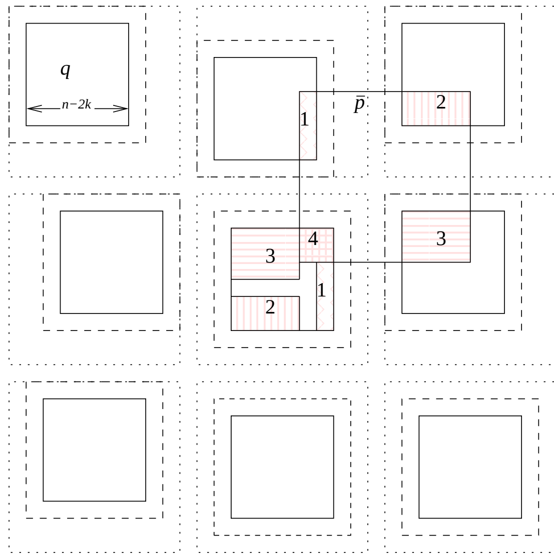


Figure 4: The configuration $\bar{c} := \tau(c)$ containing the pattern $\bar{p} \in \mathcal{M}^1_{n+r}(Y)$.

in p of size $n - 2k - 2\bar{k}$. Thus we have:

$$|\mathcal{M}^1_n(X)| \leq (n + 2h + M)^2 |A|^{e(n)+f(n)} \sum_{r=-2k}^{2\bar{k}} |\mathcal{M}^1_{n+r}(Y)|. \quad \square$$

Remark. In the d-dimensional case we have

$$|\mathcal{M}^1_n(X)| \leq C(n) \sum_{r=-2k}^{2\bar{k}} |\mathcal{M}^2_{n+r}(Y)|,$$

where $C(n) = (n + 2h + M)^d |A|^{e(n)+f(n)}$ with $e(n) := d(M + 4h + 2k)(n + 2\bar{k})^{d-1} - (M + 4h + 2k)^d$, $f(n) := n^d - (n - 2k - 2\bar{k})^d$ and the shift X is uniformly (M, h)-irreducible.

Corollary 3.15 *Let X and Y be two conjugate uniformly semi-strongly irreducible shifts. Then $h^1(X) = h^1(Y)$.*

Example 3.16 Here is an example of two shifts $X, Y \subseteq A^{\mathbb{Z}^2}$ such that $h^1(X) \not\leq h^2(Y)$ and hence which are not conjugate. Consider the shift X of Example 2.16 and let Y be the shift in which is forbidden to replace each $*$ with an a in the block (2). As we have seen, $|\mathcal{M}^1_n(X)| = 2^{(n-1)^2}$ and then $h^1(X) = \log(2)$. On the other side we have $\mathcal{M}^1_n(Y) = \mathcal{M}^2_n(Y)$ and a minimal forbidden square of size n in Y is a square $n \times n$ bordered by b's and only with a's inside:

$$\underbrace{\begin{array}{|c|c|c|c|c|}\hline b & b & \ldots & b & b \\ \hline b & a & \ldots & a & b \\ \hline \vdots & \vdots & \ldots & \vdots & \vdots \\ \hline b & a & \ldots & a & b \\ \hline b & b & \ldots & b & b \\ \hline \end{array}}_{n\times n}$$

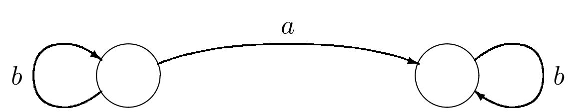
Hence $|\mathcal{M}_n^1(Y)| = 1 = |\mathcal{M}_n^2(Y)|$ and $h^1(Y) = 0 = h^2(Y)$. This implies that X and Y are not conjugate.

In this example, the shifts have also different entropies. More precisely we have $h(X) < h(Y)$ because X is a proper subshift of Y and this latter is a *strongly irreducible* subshift of $A^{\mathbb{Z}^2}$ (see [8, Lemma 4.4]). With a slight modification of this example (for instance if $\bar{X}$ is the shift in which is forbidden to replace the $*$'s in the block (2) with a's and an *odd number* of c's), it is still easy to prove $h^1(\bar{X}) \not\leq h^2(Y)$. But we get the inequality $h(\bar{X}) < h(Y)$ only by successive approximations and not by previous argument because $\bar{X} \not\subseteq Y$. In any case, for these shifts, the computation of the entropies h^i is quite simpler than that of h. $\square$

Proposition 3.17 *If there exists an integer $\bar{n}$ such that for each $n \geq \bar{n}$ the sequence $(\mathsf{X}_{\mathcal{F}_n(X)})$ is uniformly (M,h)-irreducible then X is (M,h)-irreducible.*

Proof Let p, q be two patterns of X whose supports are at distance $> M$. We have $p, q \in \mathcal{B}(\mathsf{X}_{\mathcal{F}_n(X)})$ for each $n \geq \bar{n}$ and hence there exists $c_n \in \mathsf{X}_{\mathcal{F}_n(X)}$ in which p and q simultaneously appear in a suitable position; we can always suppose that these positions are the same for each c_n. By the compactness of $A^{\mathbb{Z}^2}$, a subsequence of (c_n) converges to $c \in X$ in which p and q appear as in the c_n's. $\square$

Counterexample 3.18 Now we show that there is an example of a reducible shift X such that each $\mathsf{X}_{\mathcal{F}_n(X)}$ is semi-strongly irreducible but the sequence $(\mathsf{X}_{\mathcal{F}_n(X)})$ is not uniformly semi-strongly irreducible. Let X be the sofic shift accepted by the labeled graph in the figure below.



Hence a set of forbidden words for X is given by

$$\{ab^n a \mid n \geq 0\}.$$

In the configurations of $\mathsf{X}_{\mathcal{F}_n(X)}$ there are no words awa such that $0 \leq |w| \leq n-2$. Suppose that $\mathsf{X}_{\mathcal{F}_n(X)}$ is (M,h)-irreducible. Being $a \in \mathcal{B}(\mathsf{X}_{\mathcal{F}_n(X)})$, there exists a word w such that $M-h \leq |w| \leq M+h$ and $awa \in \mathcal{B}(\mathsf{X}_{\mathcal{F}_n(X)})$. It must be $|w| \geq n-1$ and hence $M+h \geq n-1$. This shows that $\mathsf{X}_{\mathcal{F}_n(X)}$ cannot be (M,h)-irreducible for each n (however $\mathsf{X}_{\mathcal{F}_n(X)}$ is $(n-1,0)$-irreducible for each n). $\square$

Proposition 3.19 *A one-dimensional semi-strongly irreducible shift is uniformly semi-strongly irreducible.*

Proof Let X be an (M,h)-irreducible subshift of $A^{\mathbb{Z}}$. Let $p,q \in \mathcal{B}(\mathsf{X}_{\mathcal{F}_n(X)})$ and let $m \geq M$. We can always suppose that the lengths of p and q are both greater than n. Hence $p = \bar{p}u$ with $u \in \mathcal{B}_n(X)$ and $q = v\bar{q}$ with $v \in \mathcal{B}_n(X)$. Since X is (M,h)-irreducible, there exists $w \in \mathcal{B}(X)$ such that $uwv \in \mathcal{B}(X)$ and $m-h \leq |w| \leq m+h$. Moreover, being $p,q \in \mathcal{B}(\mathsf{X}_{\mathcal{F}_n(X)})$, there exist $c, \bar{c} \in \mathsf{X}_{\mathcal{F}_n(X)}$ such that c contains p and $\bar{c}$ contains q. Consider the following configuration.

$$\boxed{\;\ldots \qquad c \qquad \underbrace{\bar{p} \mid u}_{p} \qquad w \qquad \underbrace{v \mid \bar{q}}_{q} \qquad \bar{c} \qquad \ldots\;}$$

As one can see, in this configuration does not appear any forbidden word of length n and hence it must be in $\mathsf{X}_{\mathcal{F}_n(X)}$. Therefore $pwq \in \mathcal{B}(\mathsf{X}_{\mathcal{F}_n(X)})$. $\square$

Corollary 3.20 *An irreducible sofic subshift of $A^{\mathbb{Z}}$ is uniformly semi-strongly irreducible.*

3.2 The case of rectangles

We now discuss the case of minimal forbidden patterns with a rectangular shape.

The entropy of the sequence $(\mathcal{M}^i_{m,n}(X))$ of minimal forbidden rectangles of X is defined as

$$\bar{h}^i(X) := \limsup_{m,n\to\infty} \frac{1}{mn} \log |\mathcal{M}^i_{m,n}(X)|.$$

Let τ be a k-local map defined on X. In this case also the map $\tau_{m,n}$ is well defined on $\mathsf{X}_{\mathcal{F}_{m,n}(X)}$ if $m,n \geq 2k+1$ and its definition coincides with the definition (3) given in the case of squares (indeed if $c \in \mathsf{X}_{\mathcal{F}_{m,n}(X)}$ one has that $c_{|D(\alpha,k)}$ is a pattern of X). With this notation Lemma 3.2 and Lemma 3.3 still hold. Moreover we have $\tau_{m,n} : \mathsf{X}_{\mathcal{F}_{m,n}(X)} \to \mathsf{X}_{\mathcal{F}_{m-2k,n-2k}(Y)}$ and for m,n big enough, $\tau_{m,n}$ is one-to-one if τ is one-to-one.

Let $\tau : X \to Y$ be a bijective k-local map, let τ^{-1} be $\bar{k}$-local, let $c \in \mathsf{X}_{\mathcal{F}_{m,n}(X)}$ and let p be a pattern of $\mathsf{X}_{\mathcal{F}_{m,n}(X)}$. As a generalization of Lemmas 3.6 and 3.7 we have $(\tau^{-1})_{m-2k,n-2k}(\tau_{m,n}(c)) = c$ and $(\tau^{-1})_{m-2k,n-2k}(\tau_{m,n}(p)) = p$.

In spite of these results, we cannot generalize the proof of Theorem 3.9 to get Corollary 3.10. Indeed to each pattern p of $\mathcal{M}^1_{m,n}(X)$ one can associate a pattern $\bar{p} \in \mathcal{F}_{m+2\bar{k},n+2\bar{k}}(Y) \cap \mathcal{B}(\mathsf{X}_{\mathcal{F}_{m-2k,n-1-2k}(Y)}) \cap \mathcal{B}(\mathsf{X}_{\mathcal{F}_{m-1-2k,n-2k}(Y)})$. This means that in $\bar{p}$ there is a rectangle of $\mathcal{M}^2_{\bar{m},\bar{n}}(Y)$, where $\bar{m} \leq m+2\bar{k}$, $\bar{n} \leq n+2\bar{k}$ and $\bar{m} > m-2k$ or $\bar{n} > n-2k$ or both $\bar{m} = m-2k$ and $\bar{n} = n-2k$. Hence it could also happen that the size of this rectangle is $1 \times n$ and this does not allow us to have a constant range into the sum appearing in the statement of Theorem 3.9. For the same reasons as above we cannot generalize either the proof of Theorem 3.14.

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