SANDPILE GROUP ON THE GRAPH $\mathcal{D}_n$ OF THE DIHEDRAL GROUP

ARNAUD DARTOIS, FRANCESCA FIORENZI, AND PAOLO FRANCINI

ABSTRACT. In this paper we study the structure of the Abelian sandpile group on the Cayley graph $\mathcal{D}_n$ of the dihedral group $D_n =< a, b \mid a^n = b^2 = (ab)^2 = 1 >$. We prove that the Smith normal form of the sandpile group is not cyclic as one can generaly expect but is always the direct product of two or three cyclic groups. We conclude by considering some particular cases.

1. INTRODUCTION

The Abelian sandpile model has been introduced by Bak, Tang and Wiesenfeld in [2]. It has been widely studied as one of the simplest models that shows Self-organized criticality (SOC) [11, 1, 3, 12]. Its underlying Abelian structure was discovered by Dhar [9] and Creutz [8]. In particular, the order of the group is precisely the number of spanning trees of the graph [10], and two principal bijections exist [4].

A sandpile model could be seen as a cellular automaton on a rooted graph $\mathcal{G}$ whose cells are the vertices $\mathcal{V}(\mathcal{G})$ of $\mathcal{G}$ and each cell contains a certain number of grains of sand. The transitions of the automaton are given by the following "toppling rule": each cell i containing at least as many grains as its degree, transfers a grain of sand to each of its neighbors j. After a toppling of the vertex i, the number of grains in this cell decreases by its degree, while the number of those of its neighbors increases by 1. The root r does not topple and could be considered as collecting all the grains leaving the system. The *sandpile group* on $\mathcal{G}$ is the quotient of $\mathbb{Z}^{\mathcal{V}(\mathcal{G})}$ by the subgroup generated by the $|\mathcal{V}(\mathcal{G})| - 1$ elements expressing the toppling rules (that is, if $i \neq r$ is a vertex of degree d_i and $(j_k)_{1 \leq k \leq d_i}$ are the neighbors of i, a generator of this subgroup is $\Delta_i := d_i x_i - \sum_{1 \leq k \leq d_i} x_{j_k}$), and the element x_r.

The aim of this work is to characterize the structure of the Abelian sandpile group on the graph $\mathcal{D}_n$, for any $n \geq 3$. As shown in Figure 1, the graph $\mathcal{D}_n$ is the non oriented Cayley representation for the dihedral group D_n of symmetries for an n-sided regular polygon, with the presentation

$$< a, b \mid a^n = b^2 = (ab)^2 = 1 > .$$

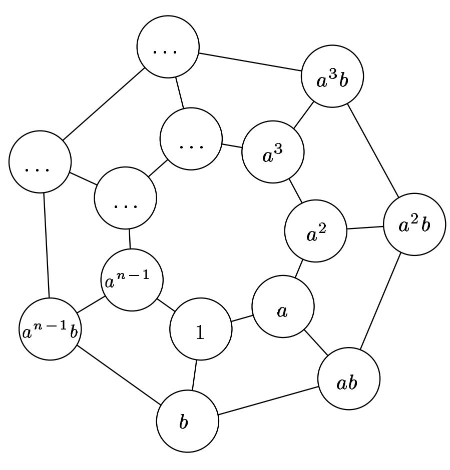

FIGURE 1. Graph $\mathcal{D}_n$

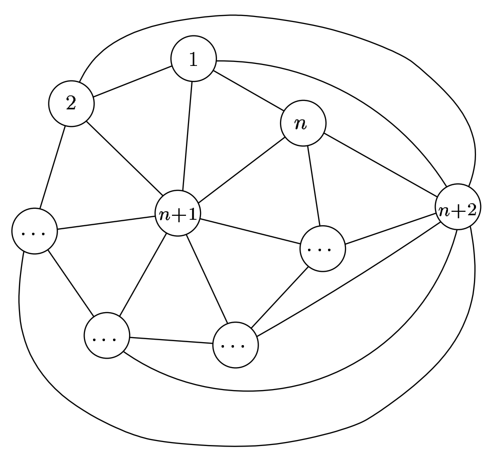

FIGURE 2. Dual graph of $\mathcal{D}_n$

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We will see that the sandpile group on $\mathcal{D}_n$ is not cyclic and it is always the direct product of at most three cyclic groups.

We want to point out that the sandpile group on the Cayley graph on a finite group, is not independent on the presentation of such a group. For example if we consider the presentation $< a, b \mid a^2 = b^2 = (ab)^n = 1 >$ of D_n, we have that its Cayley graph is circle with $2n$ vertices and the sandpile group on this graph is cyclic of order $2n$.

2. A system of relations for the sandpile group on $\mathcal{D}_n$

Cori and Rossin have proved in [6] that the Smith normal form of the sandiple group on a graph is the same of that on its planar dual. Being simpler the induced system of relations, we work on the dual graph of $\mathcal{D}_n$ (Figure 2). Here we have chosen the vertex $n+2$ as the root, such that $x_{n+2} = 0$. Applying the toppling rule to each of the remaining vertices, we get the following system of equations:

$$(1)\qquad \begin{cases} x_1 &= 4x_n - x_{n-1} - x_{n+1} \\ x_2 &= 4x_1 - x_n - x_{n+1} \\ x_i &= 4x_{i-1} - x_{i-2} - x_{n+1} \quad \text{(for each } 3 \le i \le n) \\ 0 &= x_1 + \ldots + x_n. \end{cases}$$

From this system we can already see that there are at most three generators. Indeed each x_i can be expressed in term of x_1, x_2 and x_{n+1}. For all $3 \le i \le n$, we define a_i, b_i and c_i such that $x_i = a_i x_2 - b_i x_1 - c_i x_{n+1}$.

Proposition 2.1. *If we extend $(c_i)_{i\ge 3}$ by $c_0 = 1$ and $c_1 = 0$, we get:*

$$(2)\qquad c_i = 4c_{i-1} - c_{i-2} + 1,$$

$$i \ge 3 \begin{cases} a_i &= c_{i+1} - c_i \\ b_i &= c_i - c_{i-1}. \end{cases}$$

Proof From Equation (1), we get three recurrence relations for a_i, b_i and c_i with $3 \le i \le n$. From these relations it turns out that we could express the a_i's and b_i's in term of the c_i's. □

This leads us to simplify the matrix $\mathbf{A} = \mathbf{A}_n$ of the system.

Theorem 2.2. *The matrix $\mathbf{A}_n$ of the Abelian sandpile group on $\mathcal{D}_n$ is given by*

$$\mathbf{A}_n = \begin{pmatrix} 0 & c_n - 1 & c_{n+1} \\ 0 & c_{n+1} & c_{n+2} \\ n & \frac{c_{n+2} - 3c_{n+1} - n}{2} & \frac{c_{n+2} - c_{n+1} - n}{2} \end{pmatrix}.$$

Proof From Equation 1 and Proposition 2.1 we can deduce a new system of relations between generators x_1, x_2 and x_{n+1}, that is

$$\begin{cases} x_1 &= (c_{n+2} - c_{n+1})x_2 - (c_{n+1} - c_n)x_1 - c_{n+1}x_{n+1} \\ x_2 &= (c_{n+3} - c_{n+2})x_2 - (c_{n+2} - c_{n+1})x_1 - c_{n+2}x_{n+1} \\ x_i &= (c_{i+1} - c_i)x_2 - (c_i - c_{i-1})x_1 - c_i x_{n+1} \qquad \text{(for each } 3 \le i \le n). \end{cases}$$

By some reductions, one can verify that $\mathbf{A}_n$ is the matrix of such a system. □

We conclude this section giving an explicit formula for the sequence $(c_n)_{n\ge 0}$. Solving the Equation (2) we get

$$(3)\qquad c_n = \frac{9 - 5\sqrt{3}}{12}(2 + \sqrt{3})^n + \frac{9 + 5\sqrt{3}}{12}(2 - \sqrt{3})^n - \frac{1}{2}.$$

3. Analysis of the coefficients of the Smith normal form

In this section we give an explicit expression of the Smith normal form $\mathcal{S} = \mathcal{S}_n$ of the matrix $\mathbf{A}_n$.

It can be easily seen that, for each n, we have $c_{n-1}c_{n+1} = c_n^2 - c_n$. Hence the determinant of the minor $\begin{pmatrix} c_n - 1 & c_{n+1} \\ c_{n+1} & c_{n+2} \end{pmatrix}$ of $\mathbf{A}_n$ is $-w_n := -(c_{n+1} + c_{n+2})$.

Proposition 3.1. *For $n = 2m+1$ odd, we have that $w_n = c_{2m+2} + c_{2m+3} = h_m^2$ where the sequence h_m is defined as:*

$$\begin{cases} h_0 = 1 \\ h_1 = 5 \\ h_m = 4h_{m-1} - h_{m-2}. \end{cases}$$

For $n = 2m$ even, we have that $w_n = c_{2m+1} + c_{2m+2} = 6\,k_m^2$, where the sequence k_m is defined as:

$$\begin{cases} k_0 = 0 \\ k_1 = 1 \\ k_m = 4k_{m-1} - k_{m-2}. \end{cases}$$

PROOF It can be seen by induction that, for every $m \geq 1$, we have $h_m^2 = w_{2m+1}$, $h_{m-1}h_m = w_{2m} - 1$, $6k_m^2 = w_{2m}$ and $6k_{m-1}k_m = w_{2m+1} - 1$. □

Proposition 3.2. *For each $m, n \geq 1$ we have*

$$k_{m+n} = k_{m+1}k_n - k_m k_{n-1} \quad \textit{and} \quad h_{m+n} = k_{m+1}h_n - k_m h_{n-1}.$$

PROOF Set $\mathcal{K} := \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}$ we have that

$$\mathcal{K}^m = \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}^m = \begin{pmatrix} k_{m+1} & -k_m \\ k_m & -k_{m-1} \end{pmatrix}$$

Since $\mathcal{K}^{m+n-1} = \mathcal{K}^m\,\mathcal{K}^{n-1}$, we have

$$\begin{pmatrix} k_{m+n} & -k_{m+n-1} \\ k_{m+n-1} & -k_{m+n-2} \end{pmatrix} = \begin{pmatrix} k_{m+1} & -k_m \\ k_m & -k_{m-1} \end{pmatrix} \cdot \begin{pmatrix} k_n & -k_{n-1} \\ k_{n-1} & -k_{n-2} \end{pmatrix}.$$

Comparing the top left entry in the left hand side with the corresponding in the right side gives the first equality. For the second identity, we use the first one and the identity $h_m = k_m + k_{m+1}$. □

Now we list some other relationships linking the sequences h_n, k_n and c_n. They can be easily prooved by induction.

- $h_m\,k_m = c_{2m+2}$
- $h_{m-1}\,k_m = c_{2m+1}$
- $h_m\,k_{m+1} = c_{2m+3}$

With this, one can prove the following theorem.

Theorem 3.3. *For a dividing b, we have that k_a divides k_b. Moreover, we also have that* $\det(\mathbf{A}_a)$ *divides* $\det(\mathbf{A}_b)$ *(these are the orders of the sandpile groups).*

PROOF For the first statement, we prove by induction on t that k_a divides k_{at}. This is true if $t = 0$. If k_a divides k_{at}, hence we have $k_{a(t+1)} = k_{at+1}k_a - k_{at}k_{a-1}$. The inductive hypothesis implies that k_a divides the second term, hence it also divides $k_{a(t+1)}$.

Now we prove that if $2a+1$ divides $2b+1$, then h_a divides h_b. First notice that, by Proposition 3.2, we have $k_{2m+1} = k_{m+1}^2 - k_m^2$. Let $2b+1 = (2a+1)(2t+1)$. We prove by induction on t that h_a divides $h_b = h_{2at+a+t}$. This is true if $t = 0$. If h_a divides $h_{2at+a+t}$, we have $h_{2a(t+1)+a+(t+1)} = h_{(2a+1)+(2at+a+t)} = k_{2a+2}h_{2at+a+t} - k_{2a+1}h_{2at+a+t-1}$. The first term is a multiple of h_a by inductive hypothesis. Moreover we have $k_{2a+1} = k_{a+1}^2 - k_a^2 = (k_{a+1} + k_a)(k_{a+1} - k_a) = h_a(k_{a+1} - k_a)$. Hence also the second term is a multiple of h_a.

Finally, we prove that if $2a+1$ divides $2b$, then h_a divides k_b. Let $2b = (2a+1)2t$. We have to prove that h_a divides $k_{(2a+1)t}$. As we have already seen, h_a divides k_{2a+1}. Moreover, we have that k_{2a+1} is a divisor of $k_{(2a+1)t} = k_b$.

By these facts and being $\det(\mathbf{A}_a) = -aw_a$, we have the second statement. □

Corollary 3.4. *If 2^t divides n with $t \geq 1$, then 2^{t+1} divides k_n. Moreover, if 3^t divides n, then 3^t also divides k_n.*

PROOF We obtain an explicit formula for k_n, solving the recurrence:

$$k_n = \frac{1}{2\sqrt{3}}((2+\sqrt{3})^n - (2-\sqrt{3})^n) =$$

$$= \frac{1}{\sqrt{3}} \sum_{1 \leq 2j+1 \leq n} \binom{n}{2j+1} \sqrt{3}^{2j+1} 2^{n-(2j+1)} = \sum_{1 \leq 2j+1 \leq n} \binom{n}{2j+1} 3^j 2^{n-(2j+1)}.$$

From this we have that 2^{t+1} divides k_{2^t} for $t \geq 1$. Moreover, 3^t divides k_{3^t}. □

3.1. **Computation of $\mathcal{S}_{11}$.** Notice that for each m we have $(k_m, k_{m+1}) = (h_m, h_{m+1}) = 1$. This implies that

- If $n = 2m+1$ is odd $(c_{n+1}, c_{n+2}) = (h_m k_m, h_m k_{m+1}) = h_m$.
- If $n = 2m$ is even then $(c_{n+1}, c_{n+2}) = (h_{m-1} k_m, h_m k_m) = k_m$.

In general

$$\mathcal{S}_{11} = (n, c_{n+1}, c_{n+2}, \frac{c_{n+2} - c_{n+1} - n}{2}).$$

- If $n = 2m+1$ is odd then $(n, c_{n+1}, c_{n+2}, \frac{c_{n+2}-c_{n+1}-n}{2}) = (n, c_{n+1}, c_{n+2})$. Indeed, if d divides n, c_{n+1} and c_{n+2} hence d is odd because n is odd and d divides also $\frac{c_{n+2}-c_{n+1}-n}{2}$. Hence $\mathcal{S}_{11} = (n, c_{n+1}, c_{n+2}) = (n, h_m)$.
- If $n = 2m$ is even then
$$\mathcal{S}_{11} = (n, k_m, \frac{h_m k_m - h_{m-1} k_m - n}{2}) = (n, k_m, k_m \frac{h_m - h_{m-1}}{2} - m) = (n, k_m, m) = (m, k_m).$$
  - If m is odd hence k_m is odd and then $\mathcal{S}_{11} = (n, k_m)$.
  - If m is even we have $\mathcal{S}_{11} = (m, k_m) = \frac{(n,k_m)}{2}$. This latter equality comes from Corollary 3.4, indeed k_m contains in its factorization the maximal power of 2 contained in $2m$. With this, notice that if n_1 and n_2 are two integers then $2(n_1, n_2) = (2n_1, n_2)$ if the power of 2 contained in n_1 is strictly smaller than the power of 2 contained in n_2.

3.2. **Computation of $\mathcal{S}_{22}$.** In general

$$\mathcal{S}_{11}\mathcal{S}_{22} = (c_{n+1} + c_{n+2}, nc_{n+1}, nc_{n+2}, \frac{(c_{n+1} + c_{n+2})^2 + nc_{n+1} - nc_{n+2}}{2} - 3c_{n+1}c_{n+2}).$$

- If $n = 2m+1$ is odd then $\mathcal{S}_{11}\mathcal{S}_{22} = (c_{n+1} + c_{n+2}, nc_{n+1}, nc_{n+2}, 3c_{n+1}c_{n+2})$. Indeed it can be easily seen that either c_{n+1} or c_{n+2} is odd and if d divides $c_{n+1} + c_{n+2}$, nc_{n+1} and nc_{n+2} hence d is odd and divides also $\frac{(c_{n+1}+c_{n+2})^2+nc_{n+1}-nc_{n+2}}{2}$. Thus
$$\mathcal{S}_{11}\mathcal{S}_{22} = (h_m^2, nh_m, 3h_m^2 k_m k_{m+1}) = h_m(h_m, n, 3h_m k_m k_{m+1}) = h_m(n, h_m).$$
Hence $\mathcal{S}_{22} = h_m$.
- If $n = 2m$ is even, we have
$$\mathcal{S}_{11}\mathcal{S}_{22} = (6k_m^2, nk_m, \frac{36k_m^2 + nk_m(h_m - h_{m-1})}{2} - 3h_{m-1}k_m h_m k_m) =$$
$$= k_m(6k_m, n, 18k_m + n\frac{(h_m - h_{m-1})}{2} - 3h_{m-1}h_m k_m) =$$
$$= k_m(6k_m, n, 3h_{m-1}h_m k_m) = k_m(n, 3k_m(2, h_{m-1}h_m)) = k_m(n, 3k_m).$$
Hence
  - If m is odd then $\mathcal{S}_{22} = k_m \alpha$, with $\alpha := \frac{(n,3k_m)}{(n,k_m)}$.
  - If m is even then $\mathcal{S}_{22} = 2k_m \alpha$.

From Corollary 3.4 we have that the numbers α which could be 1 or 3, is actually 1. Moreover, we have that $\det(\mathcal{S}_n) = |\det(\mathbf{A}_n)|$. Thus $\mathcal{S}_{33} = \frac{nw_n}{\mathcal{S}_{11}\mathcal{S}_{22}}$. Hence the Smith form $\mathcal{S}_n$ of the matrix $\mathbf{A}_n$ is

- For $n = 2m + 1$ odd
$$\begin{pmatrix} (n, h_m) & 0 & 0 \\ 0 & h_m & 0 \\ 0 & 0 & \frac{n\, h_m}{(n,h_m)} \end{pmatrix},$$
- for $n = 2m$ with m odd
$$\begin{pmatrix} (n, k_m) & 0 & 0 \\ 0 & k_m & 0 \\ 0 & 0 & \frac{6n k_m}{(n,k_m)} \end{pmatrix},$$
- for $n = 2m$ with m even
$$\begin{pmatrix} \frac{(n,k_m)}{2} & 0 & 0 \\ 0 & 2k_m & 0 \\ 0 & 0 & \frac{6n k_m}{(n,k_m)} \end{pmatrix}.$$

By the statements verified during the proof of Theorem 3.3, one can also see that, for a dividing b, each entry of the matrix $\mathcal{S}_a$ divides the corresponding one in the matrix $\mathcal{S}_b$. This leads to the following theorem.

Theorem 3.5. *For a dividing b, the sandpile group on $\mathcal{D}_a$ is a subgroup of the sandpile group on $\mathcal{D}_b$.*

4. Some particular cases

It is interesting to investigate the cases in which the sandpile group on $\mathcal{D}_n$ is the direct product of exactly three cyclic groups, that is in which $\mathcal{S}_{11} \neq 1$. We have already seen that, if $t \geq 2$ and 2^t divides n, hence 2^{t-1} divides $\mathcal{S}_{11}$. Moreover, if $t \geq 1$ and $2 \cdot 3^t$ divides n, hence 3^t divides $\mathcal{S}_{11}$.

4.1. **The case $n = p^t$, where p is a prime.** If $p \neq 2$, we have the following equalities mod p:
$$w_{p^t} = c_{p^t+1} + c_{p^t+2} = \frac{(2+\sqrt{3})^{p^t} + (2-\sqrt{3})^{p^t}}{2} - 1 = \frac{(2^{p^t} + \sqrt{3}^{p^t}) + (2^{p^t} - \sqrt{3}^{p^t})}{2} - 1 = 1.$$
This argument shows that $(p, w_{p^t}) = 1$ for p odd prime, thus the Smith form is in this case
$$\mathcal{S}_{p^t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & h_{\frac{p^t-1}{2}} & 0 \\ 0 & 0 & p^t\, h_{\frac{p^t-1}{2}} \end{pmatrix}.$$
Otherwise, for $p = 2$, we have
$$\mathcal{S}_{2^t} = \begin{pmatrix} 2^{t-1} & 0 & 0 \\ 0 & 2k_{2^{t-1}} & 0 \\ 0 & 0 & 12k_{2^{t-1}} \end{pmatrix}.$$

4.2. **The case $n = 2p^t$, where p is a prime.** If $p > 3$ we have the following calculation mod p:
$$w_{2p^t} = c_{2p^t+1} + c_{2p^t+2} = \frac{(2+\sqrt{3})^{2p^t} + (2-\sqrt{3})^{2p^t}}{2} - 1 = \frac{(7+4\sqrt{3})^{p^t} + (7-4\sqrt{3})^{p^t}}{2} - 1 = 6.$$
This means that $2p^t$ and k_{p^t} (which is odd) are coprime. Hence, for $p > 3$ we have the Smith form
$$\mathcal{S}_{2p^t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{p^t} & 0 \\ 0 & 0 & 12\, p^t\, k_{p^t} \end{pmatrix}.$$
Otherwise, for $p = 3$, we have
$$\mathcal{S}_{2\cdot 3^t} = \begin{pmatrix} 3^t & 0 & 0 \\ 0 & k_{3^t} & 0 \\ 0 & 0 & 12k_{3^t} \end{pmatrix}.$$

4.3. **Building a sandpile group with $\mathcal{S}_{11}$ divided by a given $m \in \mathbb{N}$.** We conclude this section proving that for any integer m, there exists an integer n such that the sandpile group on $\mathcal{D}_n$ is the product of 3 cyclic groups with m dividing their order. As conjectured in [7], the sandpile groups which are at least the product of three cyclic groups are rare. Some classes are well known, as complete graphs K_n or bipartite complete graphs $K_{n,p}$ (see [5]).

Let $m \geq 2$, we denote by $\pi(m)$ the period of the values of $(c_n)_{n\geq 0}$ mod m. Hence $c_{n+1} = c_{n+2} = 0$ mod m if and only if $\pi(m)$ divides n. For $p > 3$ prime we have two cases:

- if $\left(\frac{3}{p}\right) = 1$, then Equation (3) is valid on $\mathbb{F}_p$. By Fermat's little theorem, we have that $\pi(p)$ divides $p-1$;
- if $\left(\frac{3}{p}\right) = -1$, we have $c_{p+2} = \frac{2^p+2^{p+1}+3^{\frac{p+1}{2}}-3}{6} = 0$ mod p. Analogously, $c_{p+3} = 0$ mod p and this implies that $\pi(p)$ divides $p+1$.

We denote by $\phi(m)$ the smallest positive period of the values of $(c_n)_{n\geq 0}$ mod m which is also a multiple of m, that is $\phi(m) := \text{lcm}(\pi(m), m)$. Notice that if m and n are coprime, then $\phi(mn) = \text{lcm}(\phi(n), \phi(m))$. Moreover, we have $\phi(p) = p(p - \left(\frac{3}{p}\right))$.

By these facts, we can prove the following proposition:

Proposition 4.1. *Let m be an integer and $\mathcal{P}$ be the set of prime numbers greater than 3 dividing m. Then there exists n smaller than $4m^3 \cdot \prod_{p\in\mathcal{P}}(p - \left(\frac{3}{p}\right))$ such that the Smith normal form of the sandpile group on $\mathcal{D}_n$ is the product of 3 cyclic groups and m divides the order of each of them.*

PROOF From the equation

$$6k_m^2 = c_{2m+1}^2 + c_{2m+2}^2 - 4c_{2m+1}c_{2m+2}$$

we deduce that $2m$ divides $k_{\frac{\phi(2m)}{2}}$. Set $n := \phi(2m)$. Then m divides $\frac{(n,k_{\frac{n}{2}})}{2}$ and, since n is even, m divides $(\mathcal{S}_n)_{11}$ whatever is the parity of $\frac{n}{2}$.

The problem is now the computing of n. Let $m = 2^{r_1} \cdot 3^{r_2} \cdot \prod_{p\in\mathcal{P}} p^r$ be the prime decomposition of m (with $r_1, r_2 \geq 0$). By the multiplicative property of ϕ, we need to compute ϕ at p^r for each $p \in \mathcal{P}$. By Corollary 3.4, we have $\phi(2^t) = 2^t$ for $t \geq 2$ and $\phi(3^t) = 2 \cdot 3^t$.

For $p > 3$ prime, we can easily compute a good upper bound of $\phi(p^r)$. First notice that $\phi(p^t) \leq p^3\phi(p^{t-1})$ for $t \geq 2$. This implies $\phi(p^t) \leq p^{3t}(p - \left(\frac{3}{p}\right))$, which still holds for $t = 1$. Hence $4m^3 \cdot \prod_{p\in\mathcal{P}}(p - \left(\frac{3}{p}\right))$ is greater than n. □

It is clear that there are infinitely many numbers n with the properties stated in Proposition 4.1. The following conjecture gives the explicit form of one of them.

Conjecture 1. *Let m be an integer and $\mathcal{P}$ be the set of prime numbers greater than 3 dividing m. If $n := 4m \cdot \text{lcm}_{p\in\mathcal{P}}(p - \left(\frac{3}{p}\right))$, then the the Smith normal form of the sandpile group on $\mathcal{D}_n$ is the product of 3 cyclic groups and m divides the order of each of them.*

This conjecture arise from the fact that the value of $\phi(p^t)$ is naturally conjectured to be $p^t\pi(p)$ for $p > 3$ prime. Let $m = 2^{r_1} \cdot 3^{r_2} \cdot \prod_{p\in\mathcal{P}} p^r$ be the prime decomposition of m (with $r_1, r_2 \geq 0$). We have $\phi(2m) = \text{lcm}_{p\in\mathcal{P}}(\phi(2^{r_1+1}), \phi(3^{r_2}), \phi(p^r))$, with $\phi(1) := 1$. This value divides $4m \cdot \text{lcm}_{p\in\mathcal{P}}(\pi(p))$ and hence it divides n.

5. Conclusion

In this paper, we have studied the structure of the Abelian sandpile group on the Cayley graph $\mathcal{D}_n$ of the dihedral group $D_n =< a, b \mid a^n = b^2 = (ab)^2 = 1 >$.

First, we have given the explicit Smith normal form of this group, pointing out that the group is never cyclic, but it is always the product of 2 or 3 cyclic groups. As a by-product this implies that this sandpile group is strongly dependent on the Cayley representation of D_n.

After that, we have detailled some particular cases. We have seen that the group is always the product of 2 cyclic groups for $n = p^t$ with $p > 2$ prime, or $n = 2 \cdot p^t$ with $p > 3$ prime. Moreover, we have proved that for any integer m there exists n such that the sandpile group is the product of 3 cyclic groups and m divides the order of each of them. This fact implies the existence of an infinite family of graphs on which the sandpile group is particular. Indeed, Cori and Rossin conjecture in [7] that the Smith normal form of the sandpile group on a graph is often cyclic (that is, asymptotically on the number $n \geq 2$, there are more than 75% graphs with n vertices, for which the sandpile group is cyclic), sometimes the product of two cyclic groups (more than 15%), and very rarely the product of more (less than 5% for any remaining case).

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ARNAUD DARTOIS, LIX - Ecole Polytechnique, 91128 Palaiseau Cedex - France
E-mail address: dartois@lix.polytechnique.fr

FRANCESCA FIORENZI, Institut Gaspard Monge, Université de Marne la Vallée, 5 Bd Descartes - Champs-sur-Marne, F-77454 Marne la Vallée - France
E-mail address: fiorenzi@mat.uniroma1.it

PAOLO FRANCINI, Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università degli Studi di Roma "La Sapienza", Via A. Scarpa 16, 00161 Roma - Italy
E-mail address: paolo.francini@uniroma1.it