PERIODIC CONFIGURATIONS OF SUBSHIFTS ON GROUPS

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Abstract. We study the density of periodic configurations for shift spaces defined on (the Cayley graph of) a finitely generated group. We prove that in the case of a full shift on a residually finite group and in that of a group shift space on an abelian group, the periodic configurations are dense. In the one–dimensional case we prove the density for irreducible sofic shifts. In connection with this we study the surjunctivity of cellular automata and local selfmappings. Some related decision problems for shift spaces of finite type are also investigated.

1. Introduction

In symbolic dynamics, a shift space is a set of bi–infinite words over a finite alphabet which avoid a fixed set of forbidden factors. It is so called because of its invariance under the shift map. A shift is an example of a discrete–time dynamical system, i.e. a compact metric space equipped with a continuous selfmapping that describes one step of the evolution. In this framework, it is interesting to study the behavior of points and sets under iteration. In particular, some typical questions concern the density of periodic points and the topological transitivity – the latter corresponding to the so–called irreducibility of the shift space.

A continuous map between two shifts that commutes with the shift map is called a local function. A bijective local function is called a conjugacy. An important open question in symbolic dynamics is to decide whether two shifts are conjugate, even when they are of finite type (i.e. described by a finite set of forbidden factors).

In this work we consider a more general class of shift spaces, in which instead of bi–infinite words we consider tilings of a suitable regular graph avoiding some forbidden patterns. More precisely, let Γ be a finitely generated group represented by its Cayley graph. A configuration is an element of the space $\mathcal{A}^\Gamma$, i.e. a tiling of Γ by means of letters in a finite alphabet $\mathcal{A}$. A subset X of $\mathcal{A}^\Gamma$ whose configurations avoid a fixed set of forbidden patterns is called subshift, shift space or simply shift. An n–dimensional shift is a subshift defined on the group $\mathbb{Z}^n$. It is clear that one–dimensional shifts are suitable subsets of bi–infinite words over a finite alphabet. Also in this case a shift is naturally endowed with a compact metric and shift maps in direction of each neighbor are as many continuous selfmappings. On these shift spaces is still possible to give the notion of local function. A cellular automaton is a local function defined on the whole $\mathcal{A}^\Gamma$ (the full $\mathcal{A}$–shift).

In this setting we prove that the density of the periodic configurations is a conjugacy invariant, as is the number of periodic configurations of a fixed period. Moreover, we show that a group Γ is residually finite if and only if periodic configurations are dense in $\mathcal{A}^\Gamma$. If the alphabet $\mathcal{A}$ is a finite group and Γ is abelian, then periodic configurations of a subshift which is also a subgroup of $\mathcal{A}^\Gamma$ (namely a group shift) are dense. In the one–dimensional case, we prove the density of the periodic configurations for an irreducible subshift of finite type of $\mathcal{A}^\mathbb{Z}$. A sofic shift being the image under a local map of a shift of finite type, this implies also the density of the periodic configurations for an irreducible sofic subshift of $\mathcal{A}^\mathbb{Z}$. We see that these results cannot be generalized to higher dimensions.

We also investigate the surjunctivity of local selfmappings (a selfmapping is surjunctive if it is either non–injective or surjective). Richardson has proved in [22] that an n–dimensional cellular automaton is surjunctive. In fact, the surjunctivity problem is related to that of the density of periodic configurations: we prove that if the periodic configurations of a shift are dense, then its local selfmappings are surjunctive. As a consequence we have that a cellular automaton on $\mathcal{A}^\Gamma$ is surjunctive if Γ is a residually finite group.

Cellular automata have mainly been investigated in the n–dimensional case. There is a deep difference between the one–dimensional cellular automata and the higher dimensional ones. For example, Amoroso and Patt have shown in [1] that surjectivity and injectivity of one–dimensional cellular automata are decidable.

On the other hand Kari has shown in [13] and [14] that both the injectivity and the surjectivity problems are undecidable for cellular automata of higher dimension. In this work we extend the Amoroso–Patt's results to local functions defined on shifts of finite type. Some other well–known decision problems for n–dimensional shifts of finite type are listed, proving that in the one–dimensional case they can be solved. More generally they can be solved for the class of group shifts using some results due to Wang [25] and Kitchens and Schmidt [15].

The paper is organized as follows. In Sections 2 and 3 the notions of shift space and local function are formally defined, also proving that many basic results for the subshifts of $\mathcal{A}^{\mathbb{Z}}$ given in the book of Lind and Marcus [17], can be generalized to the subshifts of $\mathcal{A}^{\Gamma}$.

In Section 4, we recall some relevant classes of one–dimensional shifts and we give our extension of the Amoroso–Patt's decidability results.

In Section 5, after some generalities about periodic configurations, we prove that their density is a conjugacy invariant.

In Section 6 we recall the class of residually finite groups and we prove that a group Γ is residually finite if and only if periodic configurations are dense in $\mathcal{A}^{\Gamma}$.

In Section 7 we prove the density of periodic configurations for group shifts.

In Section 8, we study the surjunctivity of a general cellular automaton on a group Γ. We also mention the related "Garden of Eden" theorem which, whenever holds, guarantees the surjunctivity of local selfmappings.

Section 9 is devoted to establish for which classes of n–dimensional shifts the periodic configurations are dense. We conclude by listing some other well–known decision problems for this class of shifts.

2. Cayley graphs of finitely generated groups

Let Γ be a finitely generated group for which we fix a finite set of generators. Without loss of generality we suppose that it is symmetric (the inverse of a generator is still a generator).

, we fix Each $\gamma \in \Gamma$ can be written as

$$\gamma = x_{i_1}^{\delta_1} x_{i_2}^{\delta_2} \dots x_{i_n}^{\delta_n} \tag{1}$$

where the x_{i_j}'s are generators and $\delta_j \in \mathbb{Z}$. We define the *length of* γ as the natural number

$$\|\gamma\| = \min\{|\delta_1| + |\delta_2| + \cdots + |\delta_n| \mid \gamma \text{ is written as in (1)}\}.$$

Hence Γ is naturally endowed with a metric space structure, with the distance given by

$$\operatorname{dist}(\alpha, \beta) = \|\alpha^{-1}\beta\|. \tag{2}$$

For $\gamma \in \Gamma$, we denote by $D_n(\gamma)$ the disk of radius n centered at γ and by D_n the disk $D_n(1)$. Notice that D_1 is the set of generators of Γ. The set D_n provides, by left translation, a *neighborhood of* γ, that is $\gamma D_n = D_n(\gamma)$. Indeed, if $\alpha \in \gamma D_n$ then $\alpha = \gamma\beta$ with $\|\beta\| \leq n$. Hence $\operatorname{dist}(\alpha, \gamma) = \|\alpha^{-1}\gamma\| = \|\beta^{-1}\| \leq n$. Conversely, if $\alpha \in D_n(\gamma)$ then $\|\gamma^{-1}\alpha\| \leq n$, that is $\gamma^{-1}\alpha \in D_n$). Hence $\alpha = \gamma\gamma^{-1}\alpha \in \gamma D_n$.

The *Cayley graph* of Γ, has state set Γ and there is an edge from γ to $\bar{\gamma}$ if there exists a generator x such that $\gamma x = \bar{\gamma}$. Hence the distance defined in (2) coincides with the graph distance (that is, the minimal length of a path between to states) on the Cayley graph of Γ. Notice that, by the symmetry of the set of generators, this graph is non–oriented. For example, we may look at the classical cellular decomposition of Euclidean space $\mathbb{R}^n$ as the Cayley graph of the group $\mathbb{Z}^n$ with the presentation $\langle x_1, x_1^{-1}, \dots, x_n, x_n^{-1} \mid x_i x_j = x_j x_i \rangle$.

We recall that the function $g : \mathbb{N} \to \mathbb{N}$ defined by

$$g(n) = |D_n|$$

which counts the elements of the disk D_n, is called *growth function* of Γ. One can prove that the limit

$$\lambda = \lim_{n\to\infty} g(n)^{\frac{1}{n}}$$

always exists. If $\lambda > 1$ then, for all sufficiently large n, we have $g(n) \geq \lambda^n$, and the group Γ has *exponential growth*. If $\lambda = 1$, we distinguish two cases. Either there exists a polynomial $p(n)$ such that, for all sufficiently large n, we have $g(n) \leq p(n)$, in which case Γ has *polynomial growth*. Otherwise Γ has *intermediate growth* (i.e. $g(n)$ grows faster than any polynomial in n and slower then any exponential function x^n with $x > 1$). Moreover, it is possible to prove that the type of growth is a property of the group Γ (i.e. it does not depend on the choice of a set of generators). For this reason we deal with the *growth of a group*. This notion has been independently introduced by Milnor [19], Efremovič [7] and Švarc [24] and it is very useful in the theory of cellular automata.

A group Γ is *amenable* if it admits a Γ–invariant probability measure, that is a function $\mu : 2^\Gamma \longrightarrow [0,1]$ defined on the subsets of Γ such that for $A, B \subseteq \Gamma$ and for every $\gamma \in \Gamma$

- $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$ (*finite additivity*)
- $\mu(\gamma A) = \mu(A)$ (Γ*–invariance*)
- $\mu(\Gamma) = 1$ (*normalization*).

Finite groups, abelian groups, solvable groups, subgroups of amenable groups are all examples of amenable groups. Moreover, a finitely generated group of non–exponential growth is amenable. The free group $\mathbb{F}_2$ of rank 2 has exponential growth and is non–amenable, but there exist examples of amenable groups of exponential growth [5].

3. Shift spaces and cellular automata

Let $\mathcal{A}$ be a finite set (with at least two elements) called *alphabet*. Let Γ a finitely generated group as in the previous section. An element of the set $\mathcal{A}^\Gamma$ (i.e. the set of all functions $c : \Gamma \to \mathcal{A}$), is called a *configuration*. If $\Gamma = \mathbb{Z}$, a configuration is clearly a bi–infinite word over the alphabet $\mathcal{A}$.

For every $X \subseteq \mathcal{A}^\Gamma$ and $E \subseteq \Gamma$, we denote by X_E the restrictions to E of the configurations in X, that is

$$X_E = \{c_{|E} \mid c \in X\}.$$

A *pattern of* X is an element of X_E where E is a non–empty finite subset of Γ. The set E is called the *support* of the pattern. A *block* of X is a pattern of X whose support is a disk. If $X \subseteq \mathcal{A}^\mathbb{Z}$ (i.e. $\Gamma = \mathbb{Z}$), a block of X (also called a *factor*) is a finite word appearing in some bi–infinite word of X. In this case, the *language of* X is the set $\mathcal{L}(X)$ of all factors of X.

A subset $X \subseteq \mathcal{A}^\Gamma$ is *subshift* if there exists a set of blocks $\mathcal{F} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{A}^{D_n}$ such that $X = \mathsf{X}_\mathcal{F}$, where

$$\mathsf{X}_\mathcal{F} = \{c \in \mathcal{A}^\Gamma \mid {c^\alpha}_{|D_n} \notin \mathcal{F} \text{ for every } \alpha \in \Gamma, n \in \mathbb{N}\}.$$

In this case, $\mathcal{F}$ is a *set of forbidden blocks* of X. A subshift is also indifferently called *shift space* or simply *shift*. An n*–dimensional* shift is a subshift of $\mathcal{A}^{\mathbb{Z}^n}$.

On the set $\mathcal{A}^\Gamma$ is defined the usual product topology, where the topology in $\mathcal{A}$ is the discrete one. By Tychonoff's theorem, $\mathcal{A}^\Gamma$ is also compact. If $c_1, c_2 \in \mathcal{A}^\Gamma$ are two configurations, we define the distance

$$\mathrm{dist}(c_1, c_2) = \frac{1}{n+1}$$

where n is the least natural number such that $c_1 \neq c_2$ in D_n. If such an n does not exist, that is if $c_1 = c_2$, we set their distance equal to zero. Notice that the topology induced by this metric is equivalent to the product topology.

The group Γ acts on $\mathcal{A}^\Gamma$ on the right as follows:

$$(c^\gamma)_{|\alpha} = c_{|\gamma\alpha}$$

for each $c \in \mathcal{A}^\Gamma$ and each $\gamma, \alpha \in \Gamma$ (where $c_{|\alpha}$ denotes the value of c at α).

In [4] we prove that the combinatorial definition of a shift space is equivalent to a topological one. This fact is well known in the n–dimensional case.

Proposition 3.1. [4, Proposition 4.3] *A subset $X \subseteq \mathcal{A}^\Gamma$ is a shift if and only if it is topologically closed and Γ–invariant (i.e. $X^\Gamma = X$).*

Remark. A pattern with support E is a function $p : E \to \mathcal{A}$. If $\gamma \in \Gamma$, we have that the function $\bar{p} : \gamma E \to \mathcal{A}$ defined as $\bar{p}_{|\gamma\alpha} = p_{|\alpha}$ (for each $\alpha \in E$), is the pattern obtained copying p on the translated support γE. Moreover, if X is a shift, we have that $\bar{p} \in X_{\gamma E}$ if and only if $p \in X_E$. For this reason, in the sequel we do not make distinction between p and $\bar{p}$ (when the context makes it possible). For example, a word $a_1 \dots a_n$ is simply a finite sequence of symbols for which we do not specify (if it is not necessary), if the support is the interval $[1, n]$ or the interval $[2, n+1]$.

Let X be a subshift of $\mathcal{A}^\Gamma$. A function $\tau : X \to \mathcal{A}^\Gamma$ is *M–local* if there exists $\delta : X_{D_M} \to \mathcal{A}$ such that for every $c \in X$ and $\gamma \in \Gamma$

$$(\tau(c))_{|\gamma} = \delta(c^\gamma{}_{|D_M}) = \delta(c_{|\gamma\alpha_1}, c_{|\gamma\alpha_2}, \ldots, c_{|\gamma\alpha_m}),$$

where $D_M = \{\alpha_1, \ldots, \alpha_m\}$. Hence locality means that the value of $\tau(c)$ at γ only depends on the values of c at the elements of a fixed neighborhood of γ.

A local function defined on the whole $\mathcal{A}^\Gamma$ is called a *cellular automaton*.

Remark. In the definition of locality, we assume that the alphabet of the shift X is the same as the alphabet of its image $\tau(X)$. In this assumption there is no loss of generality because if $\tau : X \subseteq \mathcal{A}^\Gamma \to \mathcal{B}^\Gamma$, one can always consider X as a shift over the alphabet $\mathcal{A} \cup \mathcal{B}$.

The following characterization of local functions is, in the one–dimensional case, known as the Curtis–Lyndon–Hedlund theorem. In [4] it has been generalized as follows to any local function.

Proposition 3.2. [4, Proposition 4.4] *A function $\tau : X \to \mathcal{A}^\Gamma$ is local if and only if it is continuous and commutes with the Γ–action (i.e. for each $c \in X$ and each $\gamma \in \Gamma$, one has $\tau(c^\gamma) = \tau(c)^\gamma$).*

As a consequence of this fact, we have that *the composition of two local functions is still local*.

Proposition 3.3. *Let X be a shift. For each $\gamma \in \Gamma$ the function $X \to X$ that associates with each $c \in X$ its translated configuration c^γ, is continuous.*

Proof. Let $n \geq 0$ and let $m \geq 0$ such that $\gamma D_n \subseteq D_m$. If $\mathrm{dist}(c, \bar{c}) < \frac{1}{m+1}$, then c and $\bar{c}$ agree on D_m and therefore on γD_n. Hence $\alpha \in D_n \Rightarrow c_{|\gamma\alpha} = \bar{c}_{|\gamma\alpha} \Rightarrow c^\gamma{}_{|\alpha} = \bar{c}^\gamma{}_{|\alpha}$. That is c^γ and $\bar{c}^\gamma$ agree on D_n so that $\mathrm{dist}(c^\gamma, \bar{c}^\gamma) < \frac{1}{n+1}$. $\square$

Remark. Notice that, in general, this function does not commute with the Γ–action (and therefore it is not local). Indeed, if Γ is not abelian and $\gamma\alpha \neq \alpha\gamma$, we may have $(c^\gamma)^\alpha \neq (c^\alpha)^\gamma$.

If X is a subshift of $\mathcal{A}^\Gamma$ and $\tau : X \to \mathcal{A}^\Gamma$ is a local function, Proposition 3.2 guarantees that the image $Y = \tau(X)$ is still a subshift of $\mathcal{A}^\Gamma$. Indeed Y is closed (or, equivalently, compact) and it is also Γ–invariant. In fact we have that $Y^\Gamma = (\tau(X))^\Gamma = \tau(X^\Gamma) = \tau(X) = Y$. Moreover, if τ is injective then $\tau : X \to Y$ is a homeomorphism and τ^{-1} commutes with the Γ–action. Indeed, if $c \in Y$ then $c = \tau(\bar{c})$ for a unique $\bar{c} \in X$ and we have

$$\tau^{-1}(c^\gamma) = \tau^{-1}(\tau(\bar{c})^\gamma) = \tau^{-1}(\tau(\bar{c}^\gamma)) = \bar{c}^\gamma = (\tau^{-1}(c))^\gamma.$$

By Proposition 3.2, we have that τ^{-1} is local and the well–known Richardson's theorem [22], stating that the inverse of an invertible n–dimensional cellular automaton is a cellular automaton, holds also in this more general setting. In the one–dimensional case, Lind and Marcus [17, Theorem 1.5.14] give a direct proof of this fact. This result leads us to say that two subshifts are *conjugate* if there exists a local bijective function between them (namely a *conjugacy*). The *invariants* are the properties of a shift which are invariant under conjugacy.

3.1. **Irreducibility.** A one–dimensional shift $X \subseteq \mathcal{A}^\mathbb{Z}$ is *irreducible* if for each pair of words $u, v \in \mathcal{L}(X)$, there exists a word $w \in \mathcal{L}(X)$ such that the concatenated word $uwv \in \mathcal{L}(X)$. This corresponds to the transitivity of the related discrete–time dynamical system. The natural generalization of this property to any subshift $X \subseteq \mathcal{A}^\Gamma$ is that for each pair of patterns $p_1 \in X_E$ and $p_2 \in X_F$, there exists an element $\gamma \in \Gamma$ such that $E \cap \gamma F = \emptyset$ and a configuration $c \in X$ such that $c_{|E} = p_1$ and $c_{|\gamma F} = p_2$. In other words, a shift is irreducible if whenever we have two patterns appearing each one in some configuration of X, there exists a configuration $c \in X$ in which these two patterns appear simultaneously on disjoint supports. In the one–dimensional case, these two definitions are equivalent, as proved in [9, Section 2].

A stronger notion is that of *mixing* shift: for each pair of patterns $p_1 \in X_E$ and $p_2 \in X_F$, there exists $M > 0$ such that for each $\gamma \notin D_M$ there is a configuration $c \in X$ such that $c_{|E} = p_1$ and $c_{|\gamma F} = p_2$ (notice that if M is big enough, then $E \cap \gamma F = \emptyset$). In other words, a shift X is mixing if and only if for each pair of open sets $U, V \subseteq X$ there exists $M > 0$ such that $U \cap V^\gamma \neq \emptyset$ for all $\gamma \notin D_M$. Indeed, given a pattern p with support E, consider the set $U = \{c \in X \mid c_{|E} = p\}$. If $E = \{\gamma_1, \ldots, \gamma_n\}$ then $U = \bigcap_{i=1}^n \{c \in X \mid c_{|\gamma_i} = p_{|\gamma_i}\}$ is a finite intersection of open sets and hence is open.

Further forms of irreducibility has been introduced in [9] and [10]. First, *strong irreducibility* states that if the supports of the patterns are far enough, than it is not necessary to translate them in order to find a configuration in which both the patterns appear. Hereafter, *semi–strong irreducibility* states that if the supports of the patterns are far enough, than translating them "a little" (the length of this difference being bounded and only depending on the shift), we can find a common extension.

3.2. **Shifts of finite type.** A shift is *of finite type* if it admits a finite set of forbidden blocks. Hence we can decide whether or not a configuration belongs to such a shift only checking its blocks of a fixed (and only depending on the shift) radius.

More precisely, if X is a shift of finite type, since a finite set $\mathcal{F}$ of forbidden blocks of X has a maximal support, we can always suppose that each block of $\mathcal{F}$ has the disk D_M as support (indeed each block that contains a forbidden block is forbidden). In this case the shift X is called *M–step* and the number M is called the *memory of X*.

If X is a one–dimensional shift, we define the memory of X as the number M, where $M+1$ is the maximal length of a forbidden word. For these shifts we have the following useful property:

Proposition 3.4. [17, Theorem 2.1.8] *A shift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is an M–step shift of finite type if and only if whenever $uv, vw \in \mathcal{L}(X)$ and $|v| \geq M$, then $uvw \in \mathcal{L}(X)$ (where $|v|$ denotes the length of the word v).*

As proved in [9, Corollary 2.11], this "overlapping" property holds more generally for subshifts of finite type of $\mathcal{A}^{\Gamma}$.

4. One–dimensional shifts spaces

4.1. **Edge shifts.** A relevant class of one–dimensional subshifts of finite type, is that of edge shifts. This class is strictly tied up with that of finite graphs. This relation allows us to study the properties of an edge shift (possible quite complex) by studying the properties of its underling graph.

More precisely, let $\mathsf{G} = (Q, \mathcal{E})$ be a finite directed multigraph with state set Q and edge set $\mathcal{E}$. The *edge shift* X_{G} is the subshift of $\mathcal{E}^{\mathbb{Z}}$ defined by

$$X_{\mathsf{G}} = \{(e_z)_{z\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} \mid \mathsf{t}(e_z) = \mathsf{i}(e_{z+1}) \text{ for all } z \in \mathbb{Z}\},$$

where the edge $e \in \mathcal{E}$ has initial state $\mathsf{i}(e)$ and terminal state $\mathsf{t}(e)$.

Each one–dimensional shift of finite type is conjugate to an edge shift and hence they have the same invariants. Thus, also in this case, the properties of the shift depend on the structure of a suitable graph. For this, it is easy to see that every edge shift is a 1–step shift of finite type with set of forbidden blocks $\{ef \mid e, f \in \mathcal{E} \text{ and } \mathsf{t}(e) \neq \mathsf{i}(f)\}$. Conversely, given a M–step shift of finite type X, we give an effective procedure to construct a suitable graph G such that X is conjugate to X_{G}. The states of G are the words of $\mathcal{L}(X)$ of length M and there is an edge from state $a_1 \dots a_M$ to state $a_2 \dots a_{M+1}$ if the word $a_1 \dots a_M a_{M+1}$ still belongs to $\mathcal{L}(X)$.

$$a_1 \dots a_M \longrightarrow a_2 \dots a_{M+1}$$

$$\text{if } a_1 \dots a_M a_{M+1} \in \mathcal{L}(X)$$

The edge shift accepted by this graph is the $(M+1)$th *higher block shift* of X and is denoted by $X^{[M+1]}$. The shifts X and $X^{[M+1]}$ are conjugate by the function $\tau : X^{[M+1]} \to X$ defined by setting $\tau(c)_{|z}$ equal to the first letter of the word $c_{|z}$, for each $c \in X^{[M+1]}$ and each $z \in \mathbb{Z}$. This function is bijective and local. The table below points out its behavior.

$\dots$	$a_{-1}a_0 \dots a_M$	$a_0 a_1 \dots a_{M+1}$	$a_1 a_2 \dots a_{M+2}$	$\dots$
$\dots$	a_{-1}	a_0	a_1	$\dots$

Notice that in a graph G, there can be a state from which no edges start or at which no edges end. Such a state is called *stranded*. Clearly no bi–infinite paths in X_{G} involve a stranded state, hence the stranded states and the edges starting or ending at them are inessential for the edge shift X_{G}. Following Lind and Marcus [17, Definition 2.2.9], a graph is *essential* if no state is stranded. Removing step by step the stranded states of G, we get an essential graph $\bar{\mathsf{G}}$ that recognizes the same edge shift. This procedure is effective, because G has a finite number of states. Moreover, this “essential form” of G is unique.

4.2. **Sofic shifts.** The class of *sofic* shifts has been introduced by Weiss in [26] as the smallest class of shifts containing the shifts of finite type and closed under *factorization* (i.e. the image under a local map). Equivalently, one can see that a sofic shift is the set of labels of bi–infinite paths in a finite automaton.

More precisely, a *finite automaton* A is a finite directed multigraph labeled by a finite alphabet. A subshift of $\mathcal{A}^{\mathbb{Z}}$ is sofic if and only if it is the set of the labels of all the bi–infinite paths on a finite automaton A labeled by $\mathcal{A}$. In this case we say that the shift is *accepted* by A and it is denoted X_{A}. The automaton A is called *presentation* of the shift.

Obviously each edge shift is sofic, where the label of an edge is the edge itself. Hence, by conjugation, each shift of finite type is sofic. An accepting automaton is given below by considering the graph G with labeling τ introduced in the previous section.

$$a_1 \dots a_M \xrightarrow{\quad a_1 \quad} a_2 \dots a_{M+1} \xrightarrow{\quad a_2 \quad} a_3 \dots a_{M+2}$$

Remarks. (1) If we deal only with essential graphs, the language of a sofic shift is regular. (2) A sofic shift is irreducible if and only if it has a strongly connected presentation.

An automaton is *deterministic* if for any state and any symbol, there is at most one outgoing edge labeled by this symbol. An irreducible sofic shift has a unique (up to isomorphisms of automata) *minimal deterministic presentation*, that is a deterministic presentation having the fewest states among all deterministic presentations of the shift [17, Theorem 3.3.2]. Lind and Marcus also proved in [17, Lemma 3.3.10] that the minimal deterministic presentation of an irreducible sofic shift is strongly connected and, in [17, Proposition 2.2.14], that if G is a strongly connected graph, then the edge shift X_{G} is irreducible. As a consequence of this two facts, we have the following corollary.

Corollary 4.1. *A subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is an irreducible sofic shift if and only if it is the image under a local function of an irreducible shift of finite type.*

Proof. Let X be an irreducible sofic shift and let G be the underlying graph of the minimal deterministic presentation of X. Then the edge shift X_{G} is irreducible. Conversely, the image under a local function of an irreducible shift is also irreducible. $\square$

4.3. **Decision problems.** A natural decision problem arising in the theory of cellular automata concerns the existence of effective procedures to establish the surjectivity and the injectivity of the transition function. Amoroso and Patt have shown in [1] that there are algorithms to decide surjectivity and injectivity of one–dimensional cellular automata. On the other hand Kari has shown in [13] and [14] that both the injectivity and the surjectivity problems are undecidable for n–dimensional cellular automata with $n > 1$.

In this section we extend the problem to local function over subshifts of finite type of $\mathcal{A}^{\mathbb{Z}}$, giving in both cases a positive answer to the existence of decision procedures. More decision problems will be stated in Section 9.1.

4.3.1. *A decision procedure for surjectivity.* If X is a shift of finite type and Y is sofic, the problem of deciding whether or not a function $\tau : X \to Y$ given in terms of local map is surjective, is decidable.

Let $\tau : X \to \mathcal{A}^{\mathbb{Z}}$ be a function defined by a local rule δ. Without loss of generality, we can assume that X has memory $2M$ and that τ is M–local. The function τ can be represented in this way: consider the presentation of the edge shift $X^{[2M+1]}$ constructed in Section 4.1. The label of the edge between $u_1 \dots u_M\ a\ v_1 \dots v_{M-1}$ and $u_2 \dots u_M\ a\ v_1 \dots v_M$ (both in $\mathcal{L}(X)$), is the letter $\delta(u_1 \dots u_M\ a\ v_1 \dots v_M)$, that is the letter to write in place of a in the image block:
In this way we get a finite automaton A which is the presentation of the (sofic) shift $\tau(X)$. To see whether or not the function τ is surjective, Lind and Marcus give in [17, Section 3.4] an effective procedure to decide whether two finite automata accept the same shifts.

$$u_1 \dots u_M \; a \; v_1 \dots v_{M-1} \xrightarrow{\delta(u_1 \dots u_M \; a \; v_1 \dots v_M)} u_2 \dots u_M \; a \; v_1 \dots v_M$$

4.3.2. *A decision procedure for injectivity.* If X is a shift of finite type, the problem of deciding whether or not a function $\tau : X \to \mathcal{A}^{\mathbb{Z}}$ given in terms of local map is injective, is decidable.

As we have seen above, we can construct a finite automaton A which is a presentation of the sofic shift $\tau(X)$. From A, we construct another automaton $\mathsf{A} * \mathsf{A}$. Its states are couples (p, q), where p and q are states of A. There is an edge $(p, q) \xrightarrow{a} (r, s)$ labeled a, if in A there are two edges labeled a such that:

$$p \xrightarrow{a} r \quad \text{and} \quad q \xrightarrow{a} s.$$

Notice that, in general, $X_{\mathsf{A}} = X_{\mathsf{A}*\mathsf{A}}$ and hence $\mathsf{A} * \mathsf{A}$ is another presentation of $\tau(X)$.

A state (p, q) of $\mathsf{A} * \mathsf{A}$ is *diagonal* if $p = q$. Notice that the function τ is non–injective if and only if on the graph A there are two different bi–infinite paths with the same label. This fact is equivalent to the existence of a bi–infinite path on $\mathsf{A} * \mathsf{A}$ that involves a non–diagonal state. Hence, starting from the graph $\mathsf{A} * \mathsf{A}$ we construct an essential graph that accepts the same bi–infinite paths. It suffices to check, on this latter graph, if some non–diagonal state is left.

5. Density of periodic configurations

In the n–dimensional case, a periodic configuration is obtained "repeating" in each direction the same finite block. Hence, translating such a configuration, we get only a finite number of new configurations. This property leads us to define *periodic* a configuration whose Γ–orbit is finite.

In this section we establish some generalities about periodic configurations. We also prove that the density of periodic configurations is an invariant of the shifts, as is the number of the periodic configuration with a fixed period.

Definition 5.1. A configuration $c \in \mathcal{A}^{\Gamma}$ is *n–periodic* if its orbit $c^{\Gamma} = \{c^{\gamma} \mid \gamma \in \Gamma\}$ consists of n elements. In this case n is the *period* of c. A configuration is *periodic* if it is n–periodic for some $n \in \mathbb{N}$.

From now on, $\mathcal{Q}_n$ (resp. $\mathcal{P}_n$) denotes the set of the periodic configurations with period (resp. dividing) n and $\mathcal{P}$ is the set $\bigcup_{n \geq 1} \mathcal{P}_n = \bigcup_{n \geq 1} \mathcal{Q}_n$ of all periodic configurations.

In general, a configuration $c \in \mathcal{A}^{\Gamma}$ is constant on the right cosets of its own stabilizer H_c (i.e. the subgroup of all $\gamma \in \Gamma$ such that $c^{\gamma} = c$). Indeed, if $\gamma \in H_c$, we have

$$c_{|\gamma\alpha} = (c^{\gamma})_{|\alpha} = c_{|\alpha}.$$

Hence, if c is periodic it is constant on the right cosets of a subgroup of finite index. Now we prove that this property characterizes periodic configurations.

Proposition 5.2. *A configuration $c \in \mathcal{A}^{\Gamma}$ belongs to $\mathcal{P}_n$ if and only if there exists a subgroup $H \leq \Gamma$ with finite index dividing n, such that c is constant on the right cosets of H.*

Proof. Let c be m–periodic with $m|n$. By definition, the stabilizer H_c has finite index m and, as we seen, c is constant on the right cosets of H_c. Conversely, if H has finite index dividing n and c is constant on the right cosets of H, we have that $H \subseteq H_c$. Indeed, if $\gamma \in H$ and $\alpha \in \Gamma$ we have $(c^{\gamma})_{|\alpha} = c_{|\gamma\alpha} = c_{|\alpha}$ and hence $c^{\gamma} = c$ so that $\gamma \in H_c$. Since H is of finite index, H_c has finite index as well and the index of H_c divides that of H so that it divides n. Hence $c \in \mathcal{P}_n$. □

Corollary 5.3. *The set $\mathcal{P}_n$ is finite.*

Proof. By Proposition 5.2, a configuration $c \in \mathcal{A}^{\Gamma}$ belongs to $\mathcal{P}_n$ if and only if it is constant on the right cosets of a subgroup H with finite index dividing n. Being Γ a finitely generated group, there are finitely many subgroups of Γ of a fixed finite index (see, for example, [23]). Thus, these subgroups H are in finite number. For a fixed H among them, there are finitely many functions from the right cosets of H to $\mathcal{A}$, that is $|\mathcal{A}|^{[\Gamma:H]}$. □

For a shift space X, we denote by $\mathcal{P}(X)$ the set of all periodic configuration of X, that is $\mathcal{P}(X) = \mathcal{P} \cap X$. Similarly, we denote by $\mathcal{Q}_n(X)$ (resp. $\mathcal{P}_n(X)$), the intersection $\mathcal{Q}_n \cap X$ (resp. $\mathcal{P}_n \cap X$), and $q_n(X)$ (resp. $p_n(X)$), denotes its cardinality.

Proposition 5.4. *If $X \subseteq \mathcal{A}^\Gamma$ is a shift such that $\mathcal{P}(X)$ is dense in X and $\tau : X \to \mathcal{A}^\Gamma$ is a local function, then $\mathcal{P}(\tau(X))$ is dense in $\tau(X)$.*

Proof. Set $Y = \tau(X)$. First we prove that $\tau(\mathcal{P}(X)) \subseteq \mathcal{P}(Y)$. Indeed if $c \in X$ the stabilizer H_c is contained in $H_{\tau(c)}$ (we have $c^\gamma = c \Rightarrow (\tau(c))^\gamma = \tau(c^\gamma) = \tau(c)$), and if H_c has finite index, then $H_{\tau(c)}$ has finite index as well. Then $\overline{\mathcal{P}(Y)} \supseteq \overline{\tau(\mathcal{P}(X))} \supseteq \tau(\overline{\mathcal{P}(X)}) = \tau(X) = Y$. □

Corollary 5.5. *The density of its periodic configurations is an invariant of the shift.*

Proposition 5.6. *Let $X \subseteq \mathcal{A}^\Gamma$, then the numbers $q_n(X)$ and $p_n(X)$ are invariants of X.*

Proof. Let $\tau : X \to Y$ be a conjugacy. We prove that $H_c = H_{\tau(c)}$. As proved in Proposition 5.4, we always have that $H_c \subseteq H_{\tau(c)}$. Conversely, if $\gamma \in H_{\tau(c)}$, we have $\tau(c^\gamma) = \tau(c)^\gamma = \tau(c)$. The function τ being injective, we have $c^\gamma = c$. Thus, $c \in \mathcal{Q}_n(X)$ if and only if $\tau(c) \in \mathcal{Q}_n(Y)$. The same holds for configurations whose period divides n. □

6. Residually finite groups

A group Γ is *residually finite* if for every $\gamma \in \Gamma \setminus \{1\}$, there exists a subgroup of finite index $H \leq \Gamma$ such that $\gamma \notin H$. In other words, a group if residually finite if the intersection of all its subgroup of finite index is trivial. Examples of residually finite groups are the groups $\mathbb{Z}^n$ and, in general, all finitely generated abelian groups. The free group $\mathbb{F}_n$ of rank n is an example of residually finite, non–abelian group. The additive group of rational numbers $\mathbb{Q}$ is an example of abelian, non–finitely generated and non–residually finite group.

In this section we see that (finitely generated) residually finite groups are precisely those groups such that for each finite set $\mathcal{A}$, the set $\mathcal{P}$ of periodic configurations is dense in $\mathcal{A}^\Gamma$.

The first part of the following lemma is well known. Its extension is due to T. Ceccherini–Silberstein and A. Machì.

Lemma 6.1. *Let Γ be a residually finite group and let $F = \{\gamma_1, \dots, \gamma_n\}$ be a finite subset of $\Gamma \setminus \{1\}$. There exists a subgroup $H \leq \Gamma$ of finite index such that $F \subseteq \Gamma \setminus H$ and $H\gamma_i \neq H\gamma_j$ for each $i \neq j$.*

Proof. For every $i = 1, \dots, n$ let H_i be a subgroup of finite index such that $\gamma_i \notin H_i$ and let H_{ij} be a subgroup of finite index such that $\gamma_i\gamma_j^{-1} \notin H_{ij}$ (where $i \neq j$). The intersection H of all these subgroups has finite index as well. Moreover $\gamma_i \notin H$ (for each i) and $\gamma_i\gamma_j^{-1} \notin H$ $(i \neq j)$. □

Remark. In particular, the set F in previous lemma can be extended to a set of right coset representatives of the subgroup H.

Theorem 6.2. *Let Γ be a finitely generated group and $\mathcal{A}$ a finite alphabet. If Γ is residually finite, then the set $\mathcal{P}$ of periodic configurations is dense in $\mathcal{A}^\Gamma$.*

Proof. Suppose that Γ is residually finite. We have to prove that $\mathcal{A}^\Gamma = \overline{\mathcal{P}}$. Fix $c \in \mathcal{A}^\Gamma$ and let H be the subgroup of finite index whose existence is guaranteed by Lemma 6.1 with $F = D_n \setminus \{1\}$, and let D be a set of right coset representatives of H containing D_n. If $\gamma \in \Gamma$ and $\gamma = hd$ with $h \in H$ and $d \in D$, define a configuration c_n such that $(c_n)_{|\gamma} = c_{|d}$. This configuration being constant on the right cosets of H, it is periodic. Moreover c and c_n agree on D_n and hence $\mathrm{dist}(c, c_n) < \frac{1}{n+1}$. Then the sequence of periodic configurations $(c_n)_n$ converges to c. □

The same result is also given by Yukita [28]. The converse of this theorem also holds.

Theorem 6.3. *Let Γ be a finitely generated group and $\mathcal{A}$ a finite alphabet. Then Γ is residually finite if and only if the set $\mathcal{P}$ of periodic configurations is dense in $\mathcal{A}^\Gamma$.*

Proof. If Γ is not residually finite, let $\gamma \neq 1$ be an element belonging to all the subgroups of Γ of finite index. In particular $\gamma \in \bigcap_{c \in \mathcal{P}} H_c$ so that, for each $c \in \mathcal{P}$, we have $c^\gamma = c$ and hence $c_{|\gamma} = c_{|1}$. Let $\bar{c} \in \mathcal{A}^\Gamma$ such that $\bar{c}_{|\gamma} \neq \bar{c}_{|1}$, then for each n such that $\gamma \in D_n$ and each $c \in \mathcal{P}$ we have $\bar{c}_{|D_n} \neq c_{|D_n}$. Hence $\mathrm{dist}(\bar{c}, c) \geq \frac{1}{n+1}$ and $\bar{c} \notin \overline{\mathcal{P}}$. □

7. Group shifts

If the alphabet $\mathcal{A}$ is a finite group, the full shift $\mathcal{A}^\Gamma$ is also a group with product defined as in the direct product of infinitely many copies of $\mathcal{A}$. Endowed with this operation the space $\mathcal{A}^\Gamma$ is a compact metric topological group. A subshift $X \subseteq \mathcal{A}^\Gamma$ which is also a subgroup is called *group shift*.

In this section we prove (as a consequence of a more general theorem in [16]), that for this class of shifts the periodic configurations are dense. Moreover, as we will see in Section 9.1, some well–known decision problems can be solved for the class of group shifts.

Clearly a group shift is also a compact (metric) group. Hence it is an example of dynamical system (X, Γ), where X is a compact group and Γ is a subgroup of the group $\mathrm{Aut}(X)$ of the automorphisms of X which are also continuous. Indeed the action of Γ defines a subgroup of $\mathrm{Aut}(X)$: for a fixed $\gamma \in \Gamma$, the bijective function $c \mapsto c^\gamma$ from X to X is obviously a group homomorphism and, as proved in Lemma 3.3, it is also continuous.

If (X, Γ) is such a dynamical system, the group Γ *acts expansively* on X if there exists a neighborhood U of the identity 1 in X such that $\bigcap_{\gamma\in\Gamma} \gamma(U) = \{1\}$. The set of Γ*–periodic* points is the set of points $x \in X$ such that $\{\gamma(x) \mid \gamma \in \Gamma\}$ is finite. Clearly it coincides with the set $\mathcal{P}(X)$ if X is a group shift.

If X is metrizable and Γ is an infinite and finitely generated abelian group, Kitchens and Schmidt prove in [16, Theorem 3.2] that if Γ acts expansively on X then (X, Γ) satisfies the descending chain condition (i.e. each nested decreasing sequence of closed Γ–invariant subgroups is finite), if and only if (X, Γ) is conjugate to a dynamical system (Y, Γ), where Y is a group subshift of $\mathcal{A}^\Gamma$ and $\mathcal{A}$ is a compact Lie group. Notice that, in this context, a conjugation is a continuous groups isomorphism that commutes with the Γ–action.

A consequence of this fact is the following theorem.

Theorem 7.1. [16, Corollary 7.4] *Let X be a compact group and $\Gamma \le \mathrm{Aut}(X)$ a finitely generated, abelian group. If Γ acts expansively on X then the set of Γ–periodic points is dense in X.*

Hence we can prove the following result for group shifts.

Corollary 7.2. *Let $\mathcal{A}$ be a finite group and let Γ be a finitely generated, abelian group. If $X \le \mathcal{A}^\Gamma$ is a group shift, then the set $\mathcal{P}(X)$ of periodic configurations of X is dense in X.*

Proof. We prove that the group Γ acts expansively on X. Indeed the identity in X is the configuration c assuming the constant value $1_\mathcal{A}$, where $1_\mathcal{A}$ is the identity of $\mathcal{A}$. Consider the neighborhood U of c consisting of all those configurations of X assuming the value $1_\mathcal{A}$ at 1_Γ. Obviously $\bigcap_{\gamma\in\Gamma}\{c^\gamma \mid c \in U\} = \bigcap_{\gamma\in\Gamma}\{c \in X \mid c_{|\gamma} = 1_\mathcal{A}\} = \{c\}$. □

In [16] is also proved that if X is a group shift, then X is of finite type. Indeed the following theorem is proved.

Theorem 7.3. [16, Corollary 3.9] *Let $\mathcal{A}$ be a compact Lie group. If $X \le \mathcal{A}^\Gamma$ is a closed Γ–invariant subgroup there exists a finite set $D \subseteq \Gamma$ such that*

$$X = \{c \in \mathcal{A}^\Gamma \mid c^\gamma{}_{|D} \in H \text{ for every } \gamma \in \Gamma\},$$

where H is a closed subgroup of $\mathcal{A}^D$.

Hence if $\mathcal{A}$ is finite and X is a group shift, the set $\mathcal{A}^D \setminus H$ is finite and is a set of forbidden blocks for X. Although this fact, X is not necessarily irreducible. For example, consider the group shift $\{\bar{0}, \bar{1}, \overline{01}, \overline{10}\}$ in $(\mathbb{Z}/2\mathbb{Z})^\mathbb{Z}$ (where, for each finite word w, we denote by $\bar{w}$ the bi–infinite word $\ldots www\ldots$).

Notice that an abelian, finitely generated group Γ is also residually finite. We have another proof of this fact by fixing a finite group $\mathcal{A}$ and applying Corollary 7.2 to the full group shift $\mathcal{A}^\Gamma$. By Theorem 6.3 we have that Γ is residually finite.

8. Surjunctivity

A selfmapping $\tau : X \to X$ on a set X is *surjunctive* if it is either non–injective or surjective. In other words a function is surjunctive if it is not a strict embedding and hence the implication injective ⇒ surjective holds. This notion is due to Gottschalk [11].

The simplest example is that of a *finite* set X and a selfmapping $\tau : X \to X$. Clearly each function of this kind is surjunctive. Another example is given by an endomorphism of a finite–dimensional vector space and by a regular selfmapping of a complex algebraic variety (see [2]). Many others examples of surjunctive functions are given by Gromov in [12]. Moreover, Richardson proves in [22] that each n–dimensional cellular automaton is surjunctive.

In this section, we consider the surjunctivity of a general cellular automaton over a group Γ and that of a local function on a subshift. In fact, we prove that if the periodic configurations of a subshift $X \subseteq \mathcal{A}^\Gamma$ are dense, then a local function on X is surjunctive.

The following is a sufficient condition for a selfmapping of a topological space to be surjunctive. Similar conditions are stated in [12].

Lemma 8.1. *Let X be a topological space, let $\tau : X \to X$ be a closed function and let $(X_i)_{i\in I}$ be a family of subsets of X such that*

- $X = \overline{\bigcup_{i\in I} X_i}$
- $\tau(X_i) \subseteq X_i$
- $\tau_{|X_i} : X_i \to X_i$ *is surjunctive*

then τ is surjunctive.

Proof. If τ is injective then, for every $i \in I$, the restriction $\tau_{|X_i}$ is injective as well. By the hypotheses we have $\tau(X_i) = X_i$ and hence $\bigcup_{i\in I} X_i = \bigcup_{i\in I} \tau(X_i) = \tau(\bigcup_{i\in I} X_i) \subseteq \tau(\overline{\bigcup_{i\in I} X_i}) = \tau(X)$. Then $X = \overline{\bigcup_{i\in I} X_i} \subseteq \overline{\tau(X)}$, and τ being closed we have $X \subseteq \tau(X)$. □

In the following theorem we prove that the density of the periodic configuration is a sufficient condition for the surjunctivity of a local function defined on a subshift. By Theorem 6.3, we have that from the residual finiteness of Γ it follows that a cellular automaton $\tau : \mathcal{A}^\Gamma \to \mathcal{A}^\Gamma$ is surjunctive. The groups $\mathbb{Z}^n$ being residually finite, we have that this result generalizes Richardson's theorem.

Theorem 8.2. *Let $X \subseteq \mathcal{A}^\Gamma$ be a shift whose set $\mathcal{P}(X)$ of periodic configurations is dense in X. Then every local function $\tau : X \to X$ is surjunctive.*

Proof. By Corollary 5.3 we have that the set $\mathcal{P}_n(X) = \mathcal{P}_n \cap X$ is finite. As proved for Proposition 5.4, we have that if τ is local then $\tau(\mathcal{P}_n(X)) \subseteq \mathcal{P}_n(X)$. Hence τ is surjunctive by Lemma 8.1. □

Corollary 8.3. *If Γ is a residually finite group and $\tau : \mathcal{A}^\Gamma \to \mathcal{A}^\Gamma$ is a cellular automaton, then τ is surjunctive.*

Remark. The implication injective $\Rightarrow$ surjective in Theorem 8.2 is not invertible. An example is the following: let $\mathcal{A} = \{0, 1\}$ and $\Gamma = \mathbb{Z}$. Let τ be the cellular automaton given by the local rule $\delta : \mathcal{A}^3 \to \mathcal{A}$ such that

$$\delta(a_1, a_2, a_3) = a_1 + a_3 \mod 2.$$

The function τ is surjective and not injective. Indeed if $(a_z)_{z\in\mathbb{Z}}$ is a configuration in $\mathcal{A}^\mathbb{Z}$, a pre–image is given by:

$$\begin{cases} b_0 = 0 \\ b_1 = 0 \\ b_{n+1} = a_n - b_{n-2} \mod 2 \text{ if } n \geq 2 \\ b_{-n} = a_{-n+1} - b_{-n+2} \mod 2 \text{ if } n \leq 0 \end{cases}$$

that is

$\dots$	$a_{-2} - a_0$	a_{-1}	a_0	0	0	a_1	a_2	$a_3 - a_1$	$a_4 - a_2$	$\dots$
$\dots$	a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3	a_4	a_5	$\dots$

By taking $b_0 = 1 = b_1$ we get a different pre–image.

8.1. **Garden of Eden theorem.** For n–dimensional cellular automata, Moore [20] has given a sufficient condition for the existence of a a pattern without pre–image. Moore's condition (that is the existence of two different patterns - called *mutually erasable* - for which each pair of extending configurations that coincide outside their supports, have the same image) was also proved to be necessary by Myhill [21]. This equivalence between "local surjectivity" and "local injectivity" of a cellular automaton is the classical well–known *Garden of Eden (GOE) theorem*.

The GOE theorem has been generalized by Machì and Mignosi [18] to any cellular automata defined on a group of non–exponential growth. Later it has been proved by Ceccherini *et al.* [6] for the wider class amenable groups.

It can be proved (see [18, Theorem 5]) that a cellular automaton is surjective if and only if it do not admit patterns without pre–image. The same holds for local functions on subshifts (see [4, Proposition 4.1]). Moreover, we proved [4, Proposition 4.2] that the existence of two mutually erasable patterns is equivalent the *pre–injectivity* of the cellular automaton. This latter property has been introduced by Gromov [12] and corresponds to the injectivity of the automaton on the configuration differing only on a finite set. Hence, the GOE theorem could be restated as the equivalence between pre–injectivity and surjectivity of a cellular automaton.

GOE–like theorems could be investigated even more generally for local functions defined on shift spaces. Obviously, whenever the GOE theorem (or at least Myhill's implication) holds, we have the surjunctivity of the local function. In this connection, the GOE theorem has been proved for one–dimensional irreducible shifts of finite type in [8]. Moreover, we prove Myhill's implication for irreducible sofic shift. In [9] we proved that the GOE theorem for strongly irreducible shifts of finite type on an amenable group. Finally, we proved in [10] that Myhill's implication holds for semi–strongly irreducible shift of finite type on a group of nonexponential growth. In all these cases the surjunctivity of a local function holds.

9. The n–dimensional case

In this section we focus on the density of periodic configurations for an n–dimensional shift. In the one–dimensional case we prove this density for an irreducible shift of finite type of and hence, a sofic shift being the image under a local map of a shift of finite type, for an irreducible sofic shift. The situation in the two–dimensional case is deeply different: there are counterexamples of mixing shifts of finite type X for which the set $\mathcal{P}(X)$ is not dense.

We conclude this section by listing some well–known decision problems for n–dimensional shifts proving that in the special case of a one–dimensional shift they can be solved. More generally they can be solved for the class of group shifts using some results due to Wang [25] and Kitchens and Schmidt [15].

Proposition 9.1. *If $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is an irreducible shift of finite type, then $\mathcal{P}(X)$ is dense in X.*

Proof. Suppose that X has memory M. Let $c \in X$ and let $u_n = c_{|[-n,n]}$. Fix $w \in \mathcal{L}(X)$ with $|w| = M$, the shift X being irreducible, there exist two words $v_n, w_n \in \mathcal{L}(X)$ such that

$$w\ v_n u_n w_n\ w \in \mathcal{L}(X).$$

Let c_n be the periodic configuration

$$\ldots w\ v_n u_n w_n\ w\ v_n u_n w_n\ w \cdots = \overline{w\ v_n u_n w_n}\ .$$

By Proposition 3.4, we have that $c_n \in X$. Moreover $c_{n|[-n,n]} = c_{|[-n,n]}$ and hence $\lim_{n\to\infty} c_n = c$. □

Corollary 9.2. *If $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is an irreducible sofic shift, then $\mathcal{P}(X)$ is dense in X.*

Proof. By Corollary 4.1, we have that every irreducible sofic shift is the image under a local map of an irreducible shift of finite type. Hence propositions 9.1 and 5.4 apply. □

Counterexample 9.3. The finite type condition does not imply, in general, the density of the periodic configurations of a shift.

Proof. Let $\mathcal{A} = \{0, 1\}$ and let X be the shift of finite type with set of forbidden blocks $\mathcal{F} = \{01\}$. Then the elements of X are the configurations $\bar{0}$, $\bar{1}$ and the configurations of the type $\ldots$ 111111000000 $\ldots$. Clearly X is not irreducible because there are no words $u \in \mathcal{L}(X)$ such that $0u1 \in \mathcal{L}(X)$. In this shift we have $\mathcal{P}(X) = \{\bar{0}, \bar{1}\}$ which is closed (and so not dense) in X. □

Notice that, for this shift, a local selfmapping is injective if and only if it is surjective and hence surjunctivity holds even if the set of periodic configurations is not dense.

Remark. If X is a subshift of $\mathcal{A}^{\mathbb{Z}}$, it is always possible to define an irreducible subshift $\bar{X}$ of $\mathcal{A}^{\mathbb{Z}^2}$ consisting of copies of X. More precisely, a configuration c belongs to $\bar{X}$ if and only if each horizontal line of c (i.e. the bi–infinite word $(c_{|(z,t)})_{z\in\mathbb{Z}}$, for each fixed $t \in \mathbb{Z}$), belongs to X. Hence X and $\bar{X}$ have the same set of forbidden blocks and it is obvious that the shift $\bar{X}$ is of finite type if X is. The irreducibility of $\bar{X}$ can be easily seen: given two blocks of the shift, it suffices to translate one of them in the vertical direction in such a way that the supports are far enough.

Counterexample 9.4. The density of periodic configurations does not hold, in general, for two–dimensional irreducible shifts of finite type.

Proof. Let $\bar{X}$ be the shift over the alphabet $\mathcal{A} = \{0, 1\}$ generated by the shift X of the previous counterexample. Then $\bar{X}$ is irreducible and of finite type. The set $\mathcal{P}(\bar{X})$ is in this case contained in the set of all those configurations assuming constant value at each horizontal line. It is then clear that a configuration assuming for example the value 1 at $(0, 0)$ and 0 at $(1, 0)$, cannot be approximated with any sequence of periodic configurations. □

Corollary 7.2 gives an answer to the problems arising from this counterexample.

Even if we strengthen the irreducibility hypothesis by assuming that the shift is mixing, there are examples of two–dimensional mixing shifts of finite type and local selfmappings which are injective and not surjective (see [27]).

9.1. **Further decision problems.** We conclude this section with some other decision problems arising in the case of n–dimensional subshifts of finite type.

- The *tiling problem*: given a finite list $\mathcal{F}$ of forbidden blocks is $\mathsf{X}_{\mathcal{F}}$ empty or non–empty? In fact the tiling problem is an equivalent formulation of the *domino problem*, proposed by Wang [25].
- A problem strictly related with this latter is the following: given a finite list $\mathcal{F}$ of forbidden blocks, is there a periodic configuration in $\mathsf{X}_{\mathcal{F}}$?
- Given a finite list $\mathcal{F}$ of forbidden blocks, are the periodic configurations dense in $\mathsf{X}_{\mathcal{F}}$?
- The *extension problem*: given a finite list $\mathcal{F}$ of forbidden blocks and given an *allowable* block (that is a block in which does not appear any forbidden block), is there a configuration in $\mathsf{X}_{\mathcal{F}}$ in which it appears? Clearly a positive answer to the extension problem would imply a positive answer to the tiling problem.

Now we prove that the answers for subshifts of finite type of $\mathcal{A}^{\mathbb{Z}}$ are all positive: there are algorithms to decide, the tiling and the extension problems and there is an algorithm to decide whether or not the periodic configurations are dense in X. In order to see the first two algorithms, consider, more generally, a sofic shift. If A is a finite automaton accepting X (and we may assume that the underlying graph is essential), it can be easily seen that X is non–empty if and only if it exists a cycle on the graph. Hence the shift is non–empty if and only if it contains a periodic configuration. On the other hand, the language of X is the language accepted by A. Hence an allowable word is a word of the language if and only if it is accepted by A.

To establish the density of the periodic configurations, suppose that X is of finite type with memory M. One has that $\mathcal{P}(X)$ is dense in X if and only if $\mathcal{P}(X^{[M+1]})$ is dense in $X^{[M+1]}$. The shift $X^{[M+1]}$ is an edge shift accepted by the graph G constructed in Section 4.2 and hence the set $\mathcal{P}(X^{[M+1]})$ is dense in $X^{[M]}$ if and only if each edge of G is contained in a strictly connected component of G, that is if the graph G has no edges connecting two different connected components.

For the subshifts of finite type of $\mathcal{A}^{\mathbb{Z}^2}$ the answers are quite different. In this setting Berger proved in [3] the existence of a non–empty shift of finite type containing no periodic configurations and the undecidability of the tiling problem. Sufficient conditions to the decidability of tiling and extension problems are the following.

Theorem 9.5 (Wang [25])**.** *If every non–empty subshift of finite type of $\mathcal{A}^{\mathbb{Z}^2}$ contains a periodic configuration then there is an algorithm to decide the tiling problem.*

Theorem 9.6 (Kitchens and Schmidt [15])**.** *If every subshift of finite type of $\mathcal{A}^{\mathbb{Z}^2}$ has dense periodic configurations then there is an algorithm to decide the extension problem.*

The following result is a consequence of these facts and of Corollary 7.2.

Corollary 9.7. *If $X \leq \mathcal{A}^{\mathbb{Z}^2}$ is a group shift, then the tiling and extension problems are decidable for X.*

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