# The Garden of Eden Theorem for Cellular Automata and for Symbolic Dynamical Systems

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Dedicated to Benjamin Weiss on his 60th birthday

**Abstract.** We survey the most recent and general results on Garden of Eden (briefly GOE) type theorems in the setting of Symbolic Dynamical Systems and Cellular Automata. We present a GOE type theorem for (one-dimensional) irreducible sub-shifts of finite type and show that a generalization to sofic shifts does not hold in general. We also present a detailed and self-contained proof of a GOE type theorem of Gromov for maps of bounded propagation, between strongly-irreducible stable spaces of finite type over amenable graphs, admitting a dense pseudogroup of holonomy maps.

## 1. Introduction

The notion of a *cellular automaton* was introduced by Ulam [28] and von Neumann [26]. In the classical situation [24, 25, 26, 28] the *universe* $U$ is the lattice $\mathbf{Z}^2$ of integer points of Euclidean plane. If $S$ is a finite set, the *set of states* or the *alphabet*, a *configuration* is a map $c : U \longrightarrow S$. A *transition* (or *local*) *map* is a function $\tau : \mathcal{C} \longrightarrow \mathcal{C}$ from the set $\mathcal{C}$ of all configurations into itself such that the state $\tau[c](x)$ at a point $x \in U$ only depends on the states $c(y)$ at the *neighbours* $y$'s of $x$. In the literature there are different nighbourhoods (corresponding to different metric distances in the universe $U = \mathbf{Z}^2$ ), for instance the Moore-neighbourhood [24] and that of von Neumann [26]: to fix the ideas we choose the latter. Thus, denoting by $B(x;1) = \{x, y_1 = x + (1,0), y_2 = x + (-1,0), y_3 = x + (0,1), y_4 = x + (0,-1)\}$, the ball of radius 1 centered at $x \in U$, there exists a function, called *local rule*, $f : S^{B((0,0);1)} \to S$ such that

$$\tau[c](x) = f[c(x), c(y_1), c(y_2), c(y_3), c(y_4)]. \tag{1.1}$$

As $U$ is countable, $\mathcal{C}$ is a compact *metrizable* space and one shows (see Proposition 4.4 that $\tau : \mathcal{C} \longmapsto \mathcal{C}$ is a transition map induced by a local rule $f$ (associated with a suitable metric distance on $U$) if and only if it is (uniformly) continuous and $\mathbf{Z}^2-$*equivariant* (or, equivalently, *commutes* with the *shift*): $\tau[c^g] = \tau[c]^g$, where (using the multiplicative notation), $c^g(h) = c(g^{-1}h)$, for all $c \in \mathcal{C}$ and $g, h \in \mathbf{Z}^2$.

One often speaks of $\tau$ as being time: if $c$ is the configuration at time $t$, then $\tau[c]$ is the configuration at time $t+1$. An *initial configuration* is a configuration at time $t = 0$. A configuration $c$ not in the image of $\tau$, namely $c \in \mathcal{C} \setminus \tau[\mathcal{C}]$, is called a *Garden of Eden* (briefly *GOE*) *configuration*, this biblical terminology being motivated by the fact that GOE configurations may only appear as initial configurations.

Given a non-empty finite subset $F \subset U$, a *pattern* of *support* $F$ is a map $p : F \longrightarrow S$. A pattern $p$ is called GOE if any configuration extending $p$ outside its support $F$ is a GOE configuration: $\tau(c)|_F \neq p$ for all $c \in \mathcal{C}$. Using topological methods, namely a compactness argument, one shows (see Proposition 4.1), that the existence of GOE patterns is equivalent to that of GOE configurations.

Two *distinct* patterns $p$ and $p'$ with a common support $F$ are said to be $\tau-$*mutually erasable* if, for all configurations $c$ and $c'$ extending $p$ and $p'$, respectively and that are equal outside $F$, i.e. $c|_{U\setminus F} = c'|_{U\setminus F}$, one has $\tau(c) = \tau(c')$.

Also, $\tau$ is *pre-injective* if for all $c, c' \in \mathcal{C}$ such that $c \neq c'$ but differ only in finitely many points, i.e. $\exists F \subseteq U$ finite s.t. $c|_{U\setminus F} = c'|_{U\setminus F}$, then $\tau[c] \neq \tau[c']$. It is easy to show (see Proposition 4.2) that non-existence of $\tau-$mutually erasable patterns and pre-injectivity of $\tau$ are equivalent notions.

Given a finitely generated group $G =< A >$, with a finite and symmetric system of generators $A = A^{-1}$, one can define, in perfect analogy with the above setting, the notion of a *cellular automaton* over the universe $G$, [4, 9, 23].

One then says that a group $G$ satisfies the *Myhill property* (resp. the *Moore property*) if given any cellular automaton over $G$, $\tau$ pre-injective $\Rightarrow$ $\tau$ surjective ($\tau$ surjective $\Rightarrow$ $\tau$ pre-injective, respectively).

The classical *Garden of Eden Theorem*, due to Moore and Myhill [24, 25], states the equivalence between *surjectivity* and *pre-injectivity* of $\tau$ for a cellular automaton over $G = \mathbf{Z}^2$. In other words both the Moore and the Myhill properties hold for $\mathbf{Z}^2$.

This theorem has been extended to groups of sub-exponential growth in [23] and, more generally, to *amenable* groups [4]. In [4] it is also shown that for groups containing the non-abelian free group $\mathbf{F}_2$ (thus highly non-amenable) both the Myhill property and the Moore property fail to hold in general: it was posed as a problem whether this failure holds also for all (other) non-amenable groups (e.g. for the free Burnside groups $B(m, n)$ or the Ol'shanskii groups, [3]); clearly a positive answer to this question would give a new characterization of amenability.

A group $G$ is *surjunctive*, [18, 20, 30] if, for any finite alphabet $S$, any transition map $\tau : S^G \to S^G$ is either surjective or non-injective; equivalently, $\tau$ injective $\Rightarrow$ $\tau$ surjective, which can be reh""arded as a sort of *co-hopfianity* condition (see [5] for *hopfianity* and this latter notion). Since injectivity implies pre-injectivity, we have that groups satisfying the Myhill property are surjunctive. We recall that *sofic groups* – a notion due to Gromov and Weiss, generalizing both amenable groups and *residually finite* groups (e.g. the free groups) – are surjunctive.

Section 2 of the present paper is devoted to one-dimensional (i.e. with universe $U = \mathbf{Z}$) Garden of Eden type theorems. Basic notions like *irreducibility*, being of *finite type*, are introduced in this setting where, we believe the reader has a gentler approach than in the multi-dimensional case. Thus, although the main result, namely Theorem A, cannot be recovered as a particular case of Theorem B (i.e. Gromov's theorem, which constitutes the main result of Section 3) because different kinds of irreducibility are involved, this section should be interepreted as a preparation thowards the much more articulated following section.

The book by Lind and Marcus [22] is an excellent comprehensive introduction to the theory of symbolic dynamical systems. The theory of *one-dimensional* shifts (or subshifts), i.e. closed shift-invariant subspaces of $S^{\mathbf{Z}}$, is investigated there in full detail focusing the strong connections with other settings (graph theory, theory of formal languages, Perron-Frobenius theory, etc.).

In her PhD dissertation [9], the second named author investigated GOE type theorems for one-dimensional subshifts and for subshifts over amenable groups: clearly the set up is slightly more delicate than in [4] since notions like *irreducibility* and *finiteness* (e.g. requirement for a subshift to be of *finite type* or to be *sofic* [22, 29, 30]: see Section 2, for all the details) play a determinant role. Using the notion of *entropy* (see, e.g., [20, 22]) one can relax the condition for a transition map to be an *endo*morphism of a single sub-shift, by considering a transition map between two distinct subshifts. The Garden of Eden Theorem in this setting then becomes

**Theorem A.** *Let $X$ and $Y$ be irreducible (one-dimensional) sub-shifts of finite type and $\tau : X \to Y$ a transition map. Suppose that* $\mathrm{ent}(X) = \mathrm{ent}(Y)$*, e.g. if $X = Y$. Then $\tau$ is pre-injective if and only if it is surjective.*

Sofic shifts were introduced by B. Weiss [29] as the minimal class containing all the shifts of finite type and closed under *factorizations*: a *factor* is the image under a local map. A natural question then arises: what can one say about GOE type theorems for *sofic* shifts? The answer is known from [9, 10]: the Myhill property holds for irreducible sofic shifts, see Theorem 2.17, but the Moore property in general fails to hold, see Counterexample 2.18.

The last section is devoted to a Garden of Eden type theorem due to Gromov. In [20], using entropic arguments, Gromov generalized the Garden of Eden theorem of [4] in the following way (see our Section 3 for all definitions and notions involved in the statement).

**Theorem B.** *Let $\Delta$ be an amenable graph. Suppose that $X, Y \subseteq S^{\Delta}$ are strongly-irreducible stable spaces of finite type with the same entropy:* $\mathrm{ent}(X) = \mathrm{ent}(Y)$*, e.g. if $X = Y$. Then a map of bounded propagation $\tau : X \to Y$ admitting a dense* *holonomy is surjective if and only if it is pre-injective.*

The remarkable point is that Gromov does not restict himself to graphs $\Delta$ which might occur as Cayley graphs of groups, but to graphs having a sufficient regularity (*dense pseudogroups of isometries* and *dense holonomy*); also he considers, more generally, *subshifts* rather than *full shifts*: the set of configurations $\mathcal{C}$ is now a *closed shift-invariant* subset of $S^{\Delta}$. Thus [20] covers [4] and, in this setting, the generalization can be formulated as follows (Garden of Eden Theorem for strongly-irreducible amenable subshifts):

**Corollary C.** *Let $G$ be an amenable finitely generated group and $X, Y \subseteq S^G$ two strongly-irreducible subshifts of finite type with the same entropy $ent(X) = ent(Y)$ (e.g. if $X = Y$). Then a local map $\tau : X \to Y$ is surjective if and only if it is pre-injective.*

We analyze in detail all definitions and terminolgy considered by Gromov: since these latter slightly differ from the usual ones from the current literature (e.g. from [22]), our pourpose is to provide a "dictionary" between these different points of view: the parallelism is not always complete and we shall point out the differences. Also, although the key-point in Gromov's proof is essentially of the fundamental Lemma 3.18 on strict-monotonicity – all other statements, together with several definitions and notions, being heavily sprouded, even with more general results, between various preceeding sections of the long paper [20] – our proof should be thought of as a completely self-contained and thus more accessible version of the original proof.

# 2. GOE Theorem for one-dimensional sub-shifts

We start this section by reviewing some known facts about directed graphs and their entropy. The fundamental entropic inequality for subgraphs of irreducible graphs (Theorem 2.2) is presented here in a new version from [27] which avoids the Perron-Frobenius theory, by means of which it is usually proved, but is based on the techniques of Gromov [20] further developed in Section 3.

## 2.1. Directed Graphs and their entropy

A *finite, directed graph* $G$ is given by a finite set $V(G)$ of *vertices*, a finite set $E(G)$ of *edges* and two functions $i, t : E(G) \to V(G)$. If $e \in E(G)$ then $i(e)$ and $t(e)$ are the *initial vertex* and the *terminal vertex* of $e$, respectively. We say that $G$ is *simple* if $e, e' \in E(V)$ and $e \neq e'$ implies $(i(e), t(e)) \neq (i(e'), t(e'))$. A *path of lenght* $n$ in $G$ is a finite sequence $\pi = e_1 e_2 \cdots e_n$ of edges such that $t(e_k) = i(e_{k+1})$ for $k = 1, 2, \ldots, n-1$; $\pi$ starts at edge $i(\pi) = e_1$ and terminates at edge $t(\pi) = e_n$. A *bi-infinite path* in $G$ is a sequence $\xi = \{e_k\}_{k \in \mathbf{Z}}$ of edges such that $t(e_k) = i(e_{k+1})$ for any $k \in \mathbf{Z}$.

Let $G$ be a finite, simple, directed graph. A *word* in $G$ is a finite sequence $a_1 a_2 \cdots a_m$ of vertices such that there is an edge starting in $a_i$ and terminating in $a_{i+1}$ for $i = 1, 2, \ldots m-1$. Define $B_m(G)$ as the set of all words in $G$ of lenght $m$. The entropy $\mathrm{ent}(G)$ of $G$ is defined as

$$\mathrm{ent}(G) = \lim_{m \to \infty} \frac{\log |B_m(G)|}{m}. \tag{2.2}$$

Such a limit always exists (use the Fekete-Polya lemma, see, e.g. Lemma 4.17 and Proposition 4.18 in [22]).

A graph $G$ is *irreducible* when, given any ordered pair of vertices $a$ and $b$, there exists a path from $a$ to $b$. A word $a_1 a_2 \cdots a_m$ in $G$ is *simple* when all the vertices are distinct; it is a *cycle* when $a_1 = a_m$ and $a_i \neq a_j$ if $\{i, j\} \neq \{1, m\}$.

Given an arbitrary word $w = a_1 a_2 a_3 \cdots a_n$, we can form its *decomposition into cycles* as follows. Let $i_1$ be the largest index such that the vertices $a_1, a_2 \ldots a_{i_1 - 1}$ are all distinct; then $a_{i_1} = a_{j_1}$ for a suitable $j_1 < i_1$ and $c_1 = a_{j_1} a_{j_1+1} \cdots a_{i_1}$ is the *first cycle* of the word; also set $r_1 = a_1 a_2 \cdots a_{j_1}$. Successively consider the largest index $i_2 > i_1$, such that the vertices $a_{i_1} a_{i_1+1} \cdots a_{i_2-1}$ are all distinct; then $a_{i_2} = a_{j_2}$ for a suitable $j_2 \in \{i_1, i_1+1, \ldots i_2 - 1\}$ and $c_2 = a_{j_2} a_{j_2+1} \cdots a_{i_2}$ is the *second cycle* of the word; set $r_2 = a_{i_1+1} a_{i_1+2} \cdots a_{j_2}$. Continuing this way we obtain a (unique) canonical decomposition $w = r_1 c_1 r_2 c_2 \cdots r_k c_k r_{k+1}$, with some overlappings at the extremities, where $c_1, c_2, \ldots, c_k$ are the cycles and $r_1, r_2, \ldots, r_{k+1}$ are simple (possibly empty) words. With this notation we say that $\overline{w} = a_1 \cdots a_{j_s - 1} a_{j_s} a_{i_s + 1} \cdots a_n$ is obtained from $w$ by *collapsing* the $s-$th cycle $c_s$.

**Lemma 2.1.** *Let $G$ be an irreducible graph and $e \in E(G)$ an edge in $G$. Then there exists $n$ such that if $w = a_1 a_2 \cdots a_n$ is a word in $G$ of lenght $n$ then there exists a word $w' = a_1 a_2' \cdots a_{n-1}' a_n$ containing $e$.*

*Proof.* From the irreducibility it follows the existence of a word $v$ starting at $a_1$, terminating in $a_1$ and containing $e$. Also, if $n$ is large enough, in the canonical decomposition $w = r_1 c_1 r_2 c_2 \cdots r_k c_k r_{k+1}$ of $w$ there exists a cycle $c$ repeated many times. If the lenght of $c$ is $\ell$, the lenght of $v$ is $m$ and the cycle $c$ is repeated at least $m$ times we may collapse the first $m-1$ copies of $c$ and add $\ell - 1$ copies of $v$ at the beginning obtaining the desired word. □

The following fundamental entropic inequality is usually proved by means of the Perron-Frobenius theorem (see, e.g., [22]); we present a new proof from [27] based on Gromov's techniques in [20].

**Theorem 2.2.** *If $G$ is an irreducible graph and $H$ is obtained from $G$ by deleting one edge $e \in E(G)$, then* $\mathrm{ent}(H) < \mathrm{ent}(G)$.

*Proof.* Let $n$ be the integer of Lemma 2.1. We first show that setting

$$\alpha = \frac{1}{|B_n(G)|}$$

one has, for $k = 1, 2, \ldots$

$$|B_{kn}(H)| \leq (1-\alpha)^k |B_{kn}(G)|. \tag{2.3}$$

Clearly we have $|B_{n(k-1)}(G)| \geq \alpha \cdot |B_{kn}(G)|$. In what follows a word $w$ of lenght $kn$ will be represented as the concatenation of words of lenght $n$, i.e. in the form $w = w_1 w_2 \cdots w_k$, where $w_h$ is a word of lenght $n$ for $h = 1, 2, \ldots k$. Define $\pi_h$ as the set of all $w \in B_{kn}(G)$ such that $w_h$ contains the edge $i(e)t(e)$. By Lemma 2.1 for any $w' \in B_{(k-1)n}(G)$ there exists a word $vw' \in B_{kn}(G)$ such that $v$ contains $i(e)t(e)$. Then $|\pi_1| \geq |B_{n(k-1)}(G)|$. Therefore $|B_{nk}(G) \setminus \pi_1| \leq (1-\alpha)|B_{nk}(G)|$. Then define $B^h_{nk}(G) = B_{nk}(G) \setminus \bigcup_{l=1}^{h-1} \pi_l$, $C^h = \{w \in B^h_{nk}(G) : w_h \text{ contains } i(e)t(e)\}$ and $D^h$ as the set of all couples of words $(v_1, v_2)$ such that $v_1 \in B_{n(h-1)}(G)$, $v_2 \in B_{n(k-h)}(G)$ and there exists $v \in B_n(G)$ such that $v_1 v v_2 \in B^h_{kn}(G)$. By Lemma 2.1, for any $(v_1, v_2) \in D^h$ there exists $w = v_1 v' v_2 \in B_{kn}(G)$ such that $v'$ contains $ab$. Thus $w \in C^h$. Therefore we have again: $|C^h| \geq |D^h| \geq \alpha \cdot |B^h_{kn}(G)|$ and so:

$$|B_{kn}(G) \setminus \bigcup_{l=1}^{h} \pi_l| = |(B_{kn}(G) \setminus \bigcup_{l=1}^{h-1} \pi_l) \setminus \pi_h| = |B^h_{kn}(G) \setminus \pi_h| = |B^h_{kn}(G) \setminus C^h| \leq$$
$$\leq (1-\alpha)|B^h_{kn}(G)| \leq (1-\alpha)^h |B_{kn}(G)|,$$

where the last inequality follows by an obvious inductive argument on $h$. Then for $h = k$ we obtain (2.3).

Taking logarithms in (2.3), it follows that

$$\frac{\log|B_{kn}(H)|}{kn} \leq \frac{\log(1-\alpha)}{n} + \frac{\log|B_{kn}(G)|}{kn}$$

and therefore, letting $k \to \infty$, we have

$$\text{ent}(H) \leq \frac{\log(1-\alpha)}{n} + \text{ent}(G) < \text{ent}(G).$$

□

## 2.2. Symbolic Dynamical Systems

We start by recalling from [22] some basic facts on symbolic dynamical systems.

Let $S$ be a finite alphabet. A *word $w$ of lenght $n$* over $S$ is a finite sequence $w = x_1 x_2 \cdots x_n$ of symbols $x_i \in S, i = 1, 2, \ldots, n$. A *bi-infinite word* $x$ is a bi-infinite sequence of symbols of $S$: $x = \cdots x_{-2} x_{-1} x_0 x_1 x_2 \cdots$. We say that a word $w$ of lenght $n$ is contained in the bi-infinite word $x$ if there is an index $i \in \mathbf{Z}$ such that $w = x_i x_{i+1} \cdots x_{i+n-1}$ The set of all bi-infinite words is denoted as usual by $S^{\mathbf{Z}}$ and it is called the *full shift*. It is a compact space if endowed with the product topology ($S$ is a discrete space) and the map $\sigma : S^{\mathbf{Z}} \to S^{\mathbf{Z}}$ defined by $\sigma(x)_i = x_{i+1}$, called the *shift*, is continuous. A *subshift*, or *symbolic dynamical system*, is a subset of $S^{\mathbf{Z}}$ that is closed and shift invariant. If $X$ is a subshift, its *language* $B(X)$ is the set of all words $\{x_i x_{i+1} x_{i+2} \cdots x_{i+n} \mid x \in X,\ i \in \mathbf{Z} \text{ and } n \in \mathbf{N}\}$. A subshift $X \subseteq S^{\mathbf{Z}}$ is always described by means of a set $\mathcal{F}$ of *forbidden words*: there exists a set $\mathcal{F}$ of words over the alphabet $S$ such that $X$ is the set of all $x \in S^{\mathbf{Z}}$ containing no words in $\mathcal{F}$ as subwords; conversely, for any set $\mathcal{F}$, the set of all $x \in S^{\mathbf{Z}}$ not containig the words in $\mathcal{F}$ as subwords is a subshift, denoted $X_{\mathcal{F}}$ (see Proposition 4.3). A subshift $X = X_{\mathcal{F}}$ is *of finite type* if it is possible to describe it in terms of a finite set $\mathcal{F}$ of forbidden words. If this is the case, the maximum $M$ of the lenghts of the world in $\mathcal{F}$ is called the *memory* of $X$. The shifts of finite type are characterized by the following *overlapping property*: $X$ is a shift of memory $M$ if and only if whenever $uv, vw \in B(X)$ and $|v| \geq M-1$ then $uvw \in B(X)$.

A subshift $X$ is *irreducible* if, for every pair of finite words $u, v \in B(X)$, there exists a word $w \in B(X)$ such that $uwv \in B(X)$.

Let $X$ be a subshift. Define $B_n(X)$ as the set of all words in $B(X)$ of lenght $n$. The entropy $\mathrm{ent}(X)$ of $X$ is defined as

$$\mathrm{ent}(X) = \lim_{n\to\infty} \frac{\log |B_n(X)|}{n} \tag{2.4}$$

and, as for the entropy of a graph (2.2), such a limit always exists.

Let now $X$ be a shift of finite type with memory $M$. For $m \geq M$ consider the graph $G = G(X, m)$ whose vertices are the allowed words of lenght $m$, i.e. $V(G) = B_m(X)$ and the edges are the words of lenght $m+1$ in $B(X)$: $E(G) = B_{m+1}(X)$ with $i(t) = a_1 a_2 \cdots a_m$ and $t(e) = a_2 \cdots a_m a_{m+1}$ for an edge $e = a_1 a_2 \cdots a_m a_{m+1}$. This construction comes from [22] where the shift associated to $G(X, m)$ is called the *$m-$higher block shift.* It is easy to show that if $X$ is irreducible then also $G$ is irreducible. Note also that $|B_n(G)| = |B_{n+m-1}(X)|$, so that $\mathrm{ent}(G) = \mathrm{ent}(X)$. We can now prove the following.

**Theorem 2.3.** *If $X$ is an irreducible shift of finite type and $Y$ is a proper subshift of $X$ then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

Let $X = X_{\mathcal{F}}$ and $Y = X_{\mathcal{F}'}$ with $\mathcal{F}$ finite and contained in $\mathcal{F}'$. Choose $W \in \mathcal{F}' \setminus \mathcal{F}$ and set $m = \max\{M, |W|\}$, were $M = \max\{|v| : v \in \mathcal{F}\}$ is the memory of $X$. Then if $w \in B_{m+1}(X)$ contains $W$ as a subword one has

$$\mathrm{ent}(Y) \leq \mathrm{ent}(X_{\mathcal{F}\cup\{W\}}) \leq \mathrm{ent}\ (X_{\mathcal{F}\cup\{w\}}) \leq \mathrm{ent}\ (X). \tag{2.5}$$

Consider the graph $G = G(X, m)$. Forbidding word $w$ corresponds to deleting an edge in $G$, thus Theorem 2.2 applies and the last inequality in (2.5) is strict. □

A map $\tau : S^{\mathbf{Z}} \to S^{\mathbf{Z}}$ is *$k-$local* (compare with the notion of *transition map* for a cellular automaton from the Introduction) if there exists a function $\overline{\tau} : S^{2k+1} \to S$ such that for all $x \in S^{\mathbf{Z}}$ the bi-infinite word $y = \tau(x)$ is defined by:

$$y_n = \overline{\tau}(x_{n-k}, x_{n-k+1}, \ldots, x_n, \ldots, x_{n+k-1}, x_{n+k})$$

for every $n \in \mathbf{Z}$. We also say that $\tau$ has memory $k$. It is a well known fact (see Proposition 4.4) that if $X$ and $Y$ are subshifts then a function $\tau : X \to Y$ is $k-$local for some $k$ if and only if it is continuous and commutes with the shifts.

A local map $\tau$ is called *pre-injective* if whenever $x, y \in X$ and the set $\{i \in \mathbf{Z} : x_i \neq y_i\}$ is finite and non-empty, then $\tau(x) \neq \tau(y)$.

If $Y$ is a subset of $S^{\mathbf{Z}}$ not necessarily a subshift, we can define

$$\mathrm{ent}(Y) = \liminf_{m\to\infty} \frac{\log |B_{[-m+1,m]}(Y)|}{2m}, \tag{2.6}$$

where $B_{[-m+1,m]}(Y) = \{y_{-m+1}y_{-m+2}\cdots y_0 \cdots y_{m-1}y_m : y \in Y\}$. Then we have:

**Proposition 2.4.** *If $Y \subseteq S^{\mathbf{Z}}$ and $\tau$ is a local map, then* $\mathrm{ent}(\tau(Y)) \leq \mathrm{ent}(Y)$.

*Proof.* If $\tau$ is $k$-local, then $|B_{[-m+1,m]}(\tau(Y))| \leq |B_{[-m-k+1,m+k]}(Y)|$; dividing by $2m$ and taking the lim inf we obtain the desired inequality. □

Let $G$ be a finite, directed graph. The *period* of a vertex $v \in V(G)$ is the greatest common divisor of the lenghts of the closed paths starting and terminating at $v$. We recall from Sections 4.4 and 4.5 of [22] the following:

**Lemma 2.5.** *Let $G$ be an irreducible directed graph. Then:*

*(i) All vertices $v \in V(G)$ have the same period, called the* period *of $G$;*

*(ii) if $m$ is relatively prime with the period $p$ of $G$ then, for every $u, v \in V(G)$, there exist $k \in \mathbf{N}$ and a path of lenght $km$ starting at $u$ and terminating at $v$.*

**Corollary 2.6.** *If $X$ is an irreducible shift of memory $M$ and the associated graph $G = G(X, M)$ has period $p$ then, for every $m \geq M$ prime with $p$, and for every $u, v \in B_m(X)$, there exist $k \in \mathbf{N}$ and a word $w \in B_{km}(X)$ such that $uwv \in B(X)$.*

If $X$ is a shift of memory $M$ and $m \geq M$, the *m-power* shift $X^m$ of $X$ is the 2-memory shift defined by taking $B_m(X)$ as the alphabet and forbidding all words $uv$ with $u, v \in B_m(X)$ such that $uv \notin B(X)$. Form the Corollary 2.6 it follows that if $X$ is irreducible and $m$ is prime with the period of the graph associated with $X$, then $X^m$ is irreducible too. There is a canonical bijection $\psi : X \to X^m$ given by

$$\psi(x)_k = x_{km+1}x_{km+2}\cdots x_{(k+1)m}$$

for any $x \in X$. If $\psi(x)_k = u$ for some $k \in \mathbf{Z}$ we say that $x$ contains $u$ in an *$m-$position.*

**Theorem 2.7.** *Let $X$ be an irreducible shift of memory $M$. Suppose that $Y$ is the subset of $X$ obtained from $X$ by forbidding a word in $B_m(X)$ in any $m-$position, with $m$ prime to the period of $G(X, M)$. Then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

*Proof.* First note that $|B_k(X^m)| = |B_{mk}(X)|$ so that $\mathrm{ent}(X^m) = m \cdot \mathrm{ent}(X)$. Moreover we may apply Theorem 2.3: $Y$ corresponds under $\psi$ to a proper subshift $Y'$ of $X^m$ so that $\mathrm{ent}(Y') < \mathrm{ent}(X^m)$. Finally, from $|B_{[-mk+1,mk]}(Y)| = |B_{2k}(Y')|$ and, recalling (2.6), we can deduce that $m \cdot \mathrm{ent}(Y) \leq \mathrm{ent}(Y')$ and the theorem follows. □

We can now prove the following result that stems from the works of Hedlund [16] and of Coven and Paul [7]; see also [20] pag. 268-269:

**Theorem 2.8.** *Let $X$ be an irreducible shift of finite type and $\tau : X \to A^{\mathbf{Z}}$ a local map. Then $\tau$ is pre-injective if and only if* ent($X$)=ent($\tau(X)$).

*Proof.* Let $M$ denote the maximum between the memory of $X$ and that of $\tau$. Firstly suppose that $\tau$ is pre-injective. Let $n \geq 2M$; for each pair of $u, v \in B_M(X)$ and $w \in B_n(\tau(X))$ there is at most one $p \in B_n(X)$ from $u$ to $v$ such that $\tau(upv) = w$. Thus

$$|B_{n-2M}(\tau(X))| \leq |B_n(X)| \leq |B_n(\tau(X))| \cdot |B_M(X)|^2$$

and from the definition of entropy it follows that $\text{ent}(X) = \text{ent}(\tau(X))$.

Now suppose that $\tau$ is not pre-injective. Then there exists $x, y \in X$, $x \neq y$ but $x_i = y_i$ for $i \notin \{1, 2, \ldots, k\}$ for some $k \in \mathbf{N}$, such that $\tau(x) = \tau(y)$. Define $u = x_{-M+1} \cdots x_0 x_1 \cdots x_k \cdots x_{k+t}$ and $v = y_{-M+1} \cdots y_0 y_1 \cdots y_k \cdots y_{k+t}$; where $t \geq M$ is choosen in such a way that the common lenght $m = k + t + M$ of $u$ and $v$ is prime with the period of the associated graph $G(X, M)$. If now $z'$ is obtained from $z \in X$ by replacing a given occurrence of $u$ by $v$, then $z' \in X$ and $\tau(z') = \tau(z)$. Therefore if $Y$ is obtained from $X$ by forbidding $u$ in any $m$-position then $\tau(X) = \tau(Y)$ and combining Theorem 2.7 with Proposition 2.4 we get $\text{ent}(\tau(X)) \leq \text{ent}(Y) < \text{ent}(X)$. □

Now we can prove Theorem A from the Introduction, namely the Garden of Eden theorem for irreducible shifts of finite type.

*Proof of Theorem A.* If $\tau$ is pre-injective then, by Theorem 2.8, $\text{ent}(\tau(X)) = \text{ent}(X)$ and Theorem 2.3 applied to $\tau(X) \subseteq Y$ ensures the surjectivity of $\tau$. Conversely, if $\tau$ is surjective, then $\text{ent}(\tau(X)) = \text{ent}(Y) = \text{ent}(X)$ so that, again from Theorem 2.8, it follows that $\tau$ is pre-injective. □

**Counterexample 2.9.** Myhill property no longer holds for a 1-dimensional subshift of finite type but not irreducible.

*Proof.* Set $X = \{0, 1\}^{\mathbf{Z}}$ and $\overline{X} = X \cup \{\overline{2}\}$, where $\overline{2}$ is the bi-infinite word with constant value 2. Then $\overline{X}$ is a subshift of finite type over the alphabet $\{0, 1, 2\}$ with set of forbidden words $\{02, 20, 12, 21\}$.

Also, $\overline{X}$ is not irreducible; indeed $1, 2 \in B(\overline{X})$ but for no word $w \in B(\overline{X})$ the word $1w2$ belongs to $B(\overline{X})$. Consider the transition map $\tau : \overline{X} \to \overline{X}$ defined by

$$\tau(c) = \begin{cases} c & \text{if } c \in X \\ \overline{0} & \text{if } c = \overline{2} \end{cases}$$

Then it is easy to show that $\tau$ is pre-injective but not surjective. □

**Counterexample 2.10.** Moore-property no longer holds for a shift of finite type but not irreducible.

*Proof.* Let X be the shift over the alphabet $\{0,1,2\}$ with set of forbidden words $\{01,02\}$. Thus

$$X = \{1,2\}^{\mathbf{Z}} \cup \{\overline{0}\} \cup \{w\overline{0}^r : w = \cdots w_i w_{i+1} \cdots w_k;\ w_t \in \{1,2\}\}$$

where, $\{1,2\}^{\mathbf{Z}}$ denotes, as usual, the full shift on the letters 1 and 2; $\overline{0}$ is, as above, the bi-infinite sequence of zeroes and, finally $w$ is a *left*-infinite sequence in $\{1,2\}^{\mathbf{N}}$ and $\overline{0}^r = 00\cdots \in \{0\}^{\mathbf{N}}$ is a *right*-infinite sequence of zeroes. $X$ is not irreducible, since for no word $u \in B(X)$ the word $0u1$ belongs to $B(X)$. Consider the transition map $\tau : X \to X$ defined by the local rule:

$$f(a_{-1}, a_0, a_1) = \begin{cases} a_0 & \text{if } a_1 \neq 0 \\ 0 & a_1 = 0 \end{cases}$$

The function $\tau$ is surjective because we have

$$\begin{aligned} &\tau[\overline{0}] = \overline{0} \\ &\tau[w1\overline{0}^r] = w0\overline{0}^r \text{ for all } w \in \{1,2\}^{\mathbf{N}} \\ &\tau[v] = v \text{ for all } v \in \{1,2\}^{\mathbf{Z}}. \end{aligned}$$

This also shows thay $\tau$ is not pre-injective; indeed $\tau[w\mathbf{1}\overline{0}^r] = w0\overline{0}^r = \tau[w\mathbf{2}\overline{0}^r]$. □

In two-dimensions or, more generally for subshifts over finitely generated groups there is a notion of irreducibility which extends naturally that for one-dimenional sub-shifts: a shift $X \subseteq S^G$ is irreducible if for all pair of patterns $p$ and $p'$ with supports $F$ and $F'$, respectively, there exist $g \in G$ and a configuration $c \in X$ such that $F \cap gF' = \emptyset$, $c|_F = p$ and $c'|_{gF'} = p'$ (here $gF' = \{gf' : f' \in F'\}$ is the (left-)translation of $F'$ by the element $g$). In topological terms, $X$ is irreducible if and only if for all pairs of open subsets $U, V \subseteq X$, there exists $g \in G$ such that $U \cap V^g \neq \emptyset$ (where $V^g = \{c^g : c \in V\}$ and, as usual, $c^g[h] = c[g^{-1}h]$); indeed for any pattern $p$ with support $F$ consider the set $U_p = \{c \in X : c|_F = p\}$ consisting of all configurations extending $p$ outside its support: this is and open set in the topology of $X$.

It turns out – see the next counterexample – that this is too weak a notion of irreducibility to guarantee a multi-dimensional GOE type theorem. There is a notion of *strong irreducibility* (from [9]: see Definition 3.8 in our Section 3) which ensures a GOE theorem (see, e.g. Theorem B and Corollary C).

**Counterexample 2.11.** The Garden of Eden theorem no longer holds, in general, for two-dimensional irreducible shifts of finite type.

*Proof.* If $X$ is a subshift of $S^{\mathbf{Z}}$, the subshift

$$X^2 = \{(s_{ij})_{i,j\in\mathbf{Z}} : (s_{\bar{\imath}j})_{j\in\mathbf{Z}} \in X, \text{ for all } \bar{\imath} \in \mathbf{Z}\}$$

consisting of (independent) horizontal copies of $X$, is always irreducible and it is of finite type if $X$ is so. A transition map $\tau : X \to X$ may be extended to a transition map $\tau_2 : X^2 \to X^2$ acting independently on each horizontal line. Now, to obtain counterexamples to the Garden of Eden Theorem it suffices to consider the previous one-dimensional counterexamples and apply the above construction.□

## 2.3. Sofic Shifts

Let $G$ be a finite, directed graph. If $S$ is a finite alphabet, a *labelling* of $G$ is a map $L : E(G) \to S$. The label of a path $\pi = e_1 e_2 \cdots e_n$ is the word $L(\pi) = L(e_1)L(e_2)\cdots L(e_n)$. If $\xi = \{e_k\}_{k\in\mathbf{Z}}$ is a bi-infinite path in $G$, its label is given by

$$L(\xi) = \cdots L(e_{-1})L(e_0)L(e_1)\cdots \in S^{\mathbf{Z}}$$

The *sofic shift* $X$ *presented* by $(G, L)$ is the set of the labels of the infinite paths in $G$:

$$X = X(G, L) = \{L(\xi) \mid \xi \text{ is a bi-infinite path in } G\}$$

If $S = E(G)$ and $L$ is the identity map, then $X$ is called the *edge shift* associated with $G$; it is of finite type with memory $M = 2$. Every sofic shift is a shift space and every shift of finite type is sofic. A labelled graph $(G, L)$ is *deterministic* or *right resolving* if for each vertex $v \in V(G)$ the edges starting at $v$ carry different labels, i.e. $e, e' \in E(G)$, $e \neq e'$ and $i(e) = i(e')$ implies $L(e) \neq L(e')$. We recall a basic fact on irreducible sofic shifts (see Section 3.3 in [22] and, in particular, Theorem 3.3.2):

**Theorem 2.12.** *An irreducible sofic shift may be presented by an irreducible deterministic labelled graph.*

**Proposition 2.13.** *Let $(G, L)$ be a deterministic labelled graph, $Y$ the edge shift associated to $G$ and $X$ the sofic shift presented by $(G, L)$. Then* $\mathrm{ent}(Y) = \mathrm{ent}(X)$.

*Proof.* Clearly $L$ is a 1-local map from $Y$ onto $X$. Then, by Proposition 2.4, $\mathrm{ent}(X) \leq \mathrm{ent}(Y)$. Moreover since $(G, L)$ is deterministic every word in $B_n(X)$ has at most $|V(G)|$ preimages under $L$. Therefore $|B_n(X)| \geq \frac{1}{|V(G)|}|B_n(Y)|$. Taking logarithms, dividing by $n$ and letting $n \to \infty$ we obtain the reverse inequality. □

**Theorem 2.14.** *If $X$ is an irreducible sofic shift and $Y$ is a proper subshift of $X$ then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

*Proof.* From Theorem 2.12 there exists an irreducible, deterministic labelled graph $(G, L)$ that presents $X$. Let $X'$ be the edge shift of $G$. Then, by Proposition 2.13, $\mathrm{ent}(X) = \mathrm{ent}(X')$. Using the 1-local map $L$ from $X'$ onto $X$, define $Y' = L^{-1}(Y)$, which is a proper subshift of $X'$. Then, from Theorem 2.3,

$$\mathrm{ent}(Y) \leq \mathrm{ent}(Y') < \mathrm{ent}(X') = \mathrm{ent}(X)$$

□

**Remark 2.15.** A *finite-state-automaton* is a labeled graph with a distinguished *initial state* and a distinguished subset of *terminal states*. A *language* $L$ is a set of words over a finite alphabeth. The language associated with a finite-state-automaton is the set of all labels of paths that begin at the initial state and end at a terminal state, and a language is called a *regular language* if it is of this form; there are other equivalent constructive definitions of such languages in terms of *right-* (or equivalently *left-*) *linear grammars*, see [15, 17]. As remarked by W.Krieger [21], there is a connection between sofic shifts and regular languages: the languages of sofic shifts are precisely the *factorial* (i.e. closed under subwords) and *prolongable* (i.e. every word in the language can be extended to the left and the right to obtain a longer word still in the language) regular languages. For further reading on the connections between automata theory (= theory of formal languages) and symbolic dynamics, see the nice survey of Béal and Perrin [1].

The above Theorem 2.14 is well-known in the setting of the theory of formal languages and in that of geometric group theory [8, 19], where irreduciblility is often called *ergodicity*, the logarithm of the entropy is usually called the *(exponential) growth rate* and "sofic shift" is replaced by "regular language". This result has been recently extended to *irreducible unumbiguous non-linear context-free languages* (a class of languages generalizing the regular languages) in [6]; "unumbiguous" corresponds to "deterministic".

**Lemma 2.16.** *Let $(G, L)$ be an irreducible deterministic labelled graph and denote by $X = X(G, L)$ the corresponding irreducible sofic shift. Let $Y$ be another shift and $\tau : X \to Y$ a local map. Then $\tau \circ L$ is pre-injective if and only if* $\mathrm{ent}(X) = \mathrm{ent}(\tau(X))$.

*Proof.* Let $X'$ be the edge shift of $G$. Then $\tau \circ L : X' \to Y$ is a local map; thus by Theorem 2.8 applied to the irreducible shift $X'$ of finite type we have that $\tau \circ L$ is pre-injective if and only if $\mathrm{ent}(X') = \mathrm{ent}\ (\tau(L(X'))) = \mathrm{ent}\ (\tau(X))$. By Proposition 2.13, $\mathrm{ent}(X') = \mathrm{ent}(X)$ and the assertion follows. □

Now we can prove the Myhill property for irreducible sofic shifts.

**Theorem 2.17.** *Let $X, Y$ be irreducible sofic shifts with the same entropy:* $\mathrm{ent}(X) = \mathrm{ent}(Y)$ *and let $\tau : X \to Y$ be a local map. Then $\tau$ pre-injective implies $\tau$ surjective.*

*Proof.* Let $(G, L)$ be an irreducible deterministic labelled graph presenting $X$ and denote by $X'$ the corresponding edge-shift on $G$. We first show that if $\tau$ is pre-injective then $\tau \circ L$ is pre-injective. Indeed if $\tau \circ L$ is not pre-injective then there exist two bi-infinite paths $\xi_1 = \cdots e_{-1}e_0e_1 \cdots e_ne_{n+1} \cdots$ and $\xi_2 = \cdots e_{-1}f_0f_1 \cdots f_ne_{n+1} \cdots$ in $G$ which differ only for finitely many edges (in particular, say $e_0 \neq f_0$) such that $\tau(L(\xi_1)) = \tau(L(\xi_2))$. Setting $a_i := L(e_i), i \in \mathbf{Z}$ and $b_i := L(f_i), i = 0, \ldots, n$, the labelld graph beeing deterministic we have $a_0 \neq b_0$ and hence

$$L(\xi_1) = \cdots a_{-2}a_{-1}a_0a_1 \cdots a_{n-1}a_na_{n+1} \cdots$$

and

$$L(\xi_2) = \cdots a_{-2}a_{-1}b_0b_1 \cdots b_{n-1}b_na_{n+1} \cdots$$

are two configurations in $X$ which differ only on a finite (non empty) set and whose image under $\tau$ are equal. Therefore $\tau$ is not pre-injective.

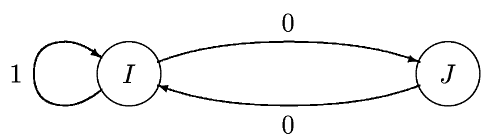
Thus, if $\tau$ is pre-injective, the same is for $\tau \circ L$; by Lemma 2.16 we have $\mathrm{ent}(Y) = \mathrm{ent}(X) = \mathrm{ent}(\tau(X))$. By Theorem 2.14, $\tau(X)$ cannot be a proper subshift of the irreducible shift $Y$. Hence $\tau$ is surjective. □

**Counterexample 2.18.** There exists an irreducible sofic shift (not of finite type) for which the transition map is surjective but not pre-injective; this yields a counterexample to the Moore property.

*Proof.* Let $X_e$ denote the *even shift*, that is the subshift of $\{0,1\}^{\mathbf{Z}}$ with forbidden words:

$$\{10^{2n+1}1 \mid n \geq 0\}.$$

The shift $X_e$ is sofic, indeed it is accepted by the following labeled graph:




Consider the function $f : S^5 \to S$ defined by

$$f(a_1a_2a_3a_4a_5) = \begin{cases} 1 & \text{if } a_1a_2a_3 = 000 \text{ or } a_1a_2a_3 = 111 \text{ or } a_2a_3a_4 = 010, \\ 0 & \text{otherwise.} \end{cases}$$

and denote by $\tau : X_e \to S^{\mathbf{Z}}$ the induced local map, i.e.

$$\tau[c](x) = f[c(x-2), c(x-1), c(x), c(x+1), c(x+2)].$$

We want to show that $\tau : X_e \to X_e$ is well-defined and that it is surjecive but not pre-injective. We devide the proof in a few steps. By abuse of notation, for a word (often called also *block*) $w = a_1a_2\cdots a_k$, with $k \geq 5$ we set

$$\tau(w) = f(a_1a_2a_3a_4a_5)f(a_2a_3a_4a_5a_6)\cdots f(a_{k-4}a_{k-3}a_{k-2}a_{k-1}a_k).$$

**Step 1.** *If a block $0^n1$ with $n \geq 3$ has a pre-image under $\tau$ of lenght $n+5$ in the language of $X_e$, say*

| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $\dots$ | $a_{n+1}$ | $a_{n+2}$ | $a_{n+3}$ | $a_{n+4}$ | $a_{n+5}$ |
|---|---|---|---|---|---|---|---|---|---|
| | | 0 | 0 | $\dots$ | 0 | 0 | 1 | | |

,

*then this pre–image is necessarily of one of the forms*

1. (*i*) $a_1a_2\ xx\ (1-x)(1-x)\dots 11\ 00\ 11\ 000a_{n+4}a_{n+5}$,

   (*ii*) $a_1a_2\ xx\ (1-x)(1-x)\dots 11\ 00\ 11\ 00100$, *(iii)* $a_1a_2\ (1-x)(1-x)\ xx\ldots 00\ 11\ 00\ 111a_{n+4}a_{n+5}$,

*when n is even and for a suitable* $x \in \{0,1\}$*;*

2. *(i)* $a_1a_2\ (1-x)\ xx\ldots 11\ 00\ 11\ 000a_{n+4}a_{n+5}$,

   *(ii)* $a_1a_2\ (1-x)\ xx\ldots 11\ 00\ 11\ 00100$

   *(iii)* $a_1a_2\ x\ (1-x)(1-x)\ldots 00\ 11\ 00\ 111a_{n+4}a_{n+5}$

   *when n is odd and for a suitable* $x \in \{0,1\}$.

*Proof of Step* 1. The statement may be proved by induction on $n \geq 3$. Suppose $n = 3$. When $\tau(a_1a_2a_3a_4a_5a_6a_7a_8) = 0001$ we have three cases: if $a_4a_5a_6 = 000$ then $a_3 = 1$, if $a_4a_5a_6a_7a_8 = 00100$ then again $a_3 = 1$ and finally if $a_4a_5a_6 = 111$ then necessarily $a_3 = 0$.

Now suppose that the statement is true for $n$ and that $\tau(a_1\ldots a_{n+6}) = 0^{n+1}1$: if $n$ is even, by the inductive hypothesis one has either

$$a_4\ldots a_{n+4} = xx\ (1-x)(1-x)\ldots 11\ 000$$

or

$$a_4\ldots a_{n+6} = xx\ (1-x)(1-x)\ldots 11\ 00100$$

or

$$a_4\ldots a_{n+4} = (1-x)(1-x)\ xx\ldots 00\ 111$$

for a suitable $x \in \{0,1\}$.

In any case we have $a_4 = a_5$. If $a_3 = a_4$, then $f(a_3a_4a_5a_6a_7) = f(a_4a_4a_4a_6a_7) = 1 \neq 0$. Thus $a_3 \neq a_4$. Then the theorem follows by the inductive hypotesis. The case $n$ odd may be proved in a similar way. □

**Step 2.** *The map* $\tau$ *is an* endo*morphism of* $X_e$*, that is* $\tau(X_e) \subseteq X_e$.

*Proof of Step* 2. It suffices to prove that no forbidden word $10^n1$ with $n$ odd, has a pre-image of lenght $n+6$ in the language of $X_e$. First of all one has to check that there is no block $a_1a_2a_3a_4a_5a_6a_7$ of lenght 7 such that $\tau(a_1a_2a_3a_4a_5a_6a_7) = 101$, distinguishing two cases: $a_3a_4 = 00$ and $a_3a_4a_5 = 111$.

We now prove that no block $a_1\ldots a_{n+6}$ of length $n+6$ has $10^n1$ as image under $\tau$, where $n \in \mathbf{N}$ is odd and strictly greater than 1. If $\tau(a_1\ldots a_{n+6}) = 10^n1$ then by previous step we have $a_4a_5a_6\ldots = x(1-x)(1-x)\ldots$, and being $f(a_1a_2a_3a_4a_5) = 1$, we distinguish two cases:

- $x = 0$. Then $a_3 = 0$ (otherwise we had a forbidden block) and $a_2 = 1$ because $f(a_2a_3a_4a_5a_6) = f(a_20011) = 0$. It follows that $f(a_1a_2a_3a_4a_5) = f(a_11001) = 0 \neq 1$.

- $x = 1$. If $a_3 = 0$ then $a_2 = 0$ and $f(a_2a_3a_4a_5a_6) = f(00100) = 1 \neq 0$. Thus $a_3 = 1$. Then $f(a_2a_3100) = f(a_21100)$ and $f(a_21100) = 0$ implies $a_2 = 0$. Thus $f(a_1a_2a_310) = f(a_10110) = 0 \neq 1$. Hence $10^n1$ has no pre–image under $\tau$.

□

**Step 3.** *The transition map $\tau : X_e \to X_e$ is surjective.*

*Proof of Step* 3. By the compactness argument we alluded to before (see also Proposition 4.1) it suffices to prove the non-existence of Garden of Eden words (=patterns), and in our setting it suffices to prove that each block of the form $10^{n_1}10^{n_2}\dots10^{n_k}1$ (where $n_1, \dots, n_k$ are even integers), has a preimage-block. Suppose first that $k = 1$:

| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $\dots$ | $a_{n+2}$ | $a_{n+3}$ | $a_{n+4}$ | $a_{n+5}$ | $a_{n+6}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 0 | 0 | $\dots$ | 0 | 0 | 1 | | |

,

we distinguish the following three cases in which $a_{n+4} \mapsto 1$.

- $n = 0$. Then either $a_1a_2a_3a_4 = 0000$, $a_1a_2a_3a_4a_5a_6 = 000100$ or $a_1a_2a_3a_4 = 1111$.
- $n = 2$. Then either $a_1a_2a_3a_4a_5a_6 = a_1a_1 1000$, $a_1a_2a_3a_4a_5a_6a_7a_8 = a_1a_1 100100$ or $a_1a_2a_3a_4a_5a_6 = 000111$.
- $n \geq 4$. Then, for a suitable $x \in \{0, 1\}$, $\tau[(1-x)(1-x)(1-x)xx\dots000a_{n+5}a_{n+6}] = 10^n1$. Similarly $\tau[(1-x)(1-x)(1-x)xx\dots00100] = 10^n1$, and finally $\tau[xxx(1-x)(1-x)\dots111a_{n+5}a_{n+6}] = 10^n1$.

Now, given any word of the form $10^{n_1}10^{n_2}\dots10^{n_k}1$ we can construct a pre-image starting from the most right block $10n_k1$: over the first on the right 1 we can write arbitrarily $000**$, $111**$ or $00100$. In this way we get a word $a_1a_2a_3a_4a_5$ over the second on the left 1 and we can start from this word over 1 to construct a pre-image for the second on the right block $10^{n_{k-1}}$, and so on. In each of the possible choices we can find a block whose image under $\tau$ is our fixed word. □

**Step 4.** *The transition map is not pre-injective.*

*Proof of Step* 4. Consider the configuration $c_1$:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\dots$ | 0 | 0 | 0 | 0 | 0 | **1** | **0** | **0** | 1 | 0 | 0 | 0 | 0 | 0 | $\dots$ |

and the configuration $c_2$:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\dots$ | 0 | 0 | 0 | 0 | 0 | **0** | **1** | **1** | 1 | 0 | 0 | 0 | 0 | 0 | $\dots$ |

.

These configurations differ only on a finite subset of $\mathbf{Z}$, but they have the same image under $\tau$, namely the configuration

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\dots$ | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | $\dots$ |

,

so that $\tau$ is not pre–injective. □

# 3. Gromov's Theorem

This section is devoted to the GOE type theorem of Gromov [20]: all new definitions (with the exception of *strong-irreducibility* and the *finite type* condition) are contained there. We analyze all these concepts and illustrate them with several examples, counterexamples and remarks (in particular, Lemma 3.11 and the related Counterexample 3.12 are from [9, 11]).

## 3.1. Pseudogroups of partial isometries of a graph.

Let $\Delta$ be a simple, infinite countable connected (undirected) graph of bounded valency; i.e. $\Delta$ has no loops or multiple edges, each pair of vertices is connected by a path and there is a positive integer $d$ such that $\Delta$ has at most $d$ edges at each vertex. We will no distiguish between the graph and the set of its vertices, that will be denoted $\Delta$. As usual, define a metric distance on $\Delta$ by setting $\text{dist}(\delta, \delta')$ as the minimal lenght of a path of edges joining $\delta$ and $\delta'$; the ball of radious $r$ and center $\delta$ will be denoted $D(\delta, r)$, that is $D(\delta, r) = \{\delta' \in \Delta \mid \text{dist}(\delta, \delta') \leq r\}$. In general $\Delta$ does not have isometries; thus we put our attention on the *partial isometries* of the graph. A partial isometry $\gamma$ is a bijective map between two subsets $\Omega$ and $\Omega'$ that preserves the metric dist. $\Delta$ has many partial isometries: since it is simple and of bounded valency, for a positive integer $r$ there are at most finitely many isometry classes of balls of radious $r$. A set $\Gamma$ of partial isometries of $\Delta$ will be called *a pseudogroup of partial isometries acting on* $\Delta$ if it satisfies the following four axioms:

(A) $\Gamma$ contains the identity map $\text{Id}_\Delta : \Delta \to \Delta$,

(B) $\gamma \in \Gamma \Rightarrow \gamma^{-1} \in \Gamma$,

(C) If $\gamma : \Omega \to \Omega'$ and $\gamma' : \Omega' \to \Omega''$ are in $\Gamma$ then $\gamma' \circ \gamma : \Omega \to \Omega''$ is in $\Gamma$,

(D) For every $\gamma \in \Gamma$, $\gamma : \Omega \to \Omega'$, its restriction $\gamma : \Omega_0 \to \gamma(\Omega_0)$ is also in $\Gamma$ for all $\Omega_0 \subseteq \Omega$.

Clearly, the set of all partial isometries is a pseuodogroup. A pseuodogroup $\Gamma$ is *cofinite* on $\Delta$ if, for every $r = 0, 1, 2, \dots$, there are at most finitely many mutually non-$\Gamma$-isometric balls of radiuos $r$. Since $\Delta$ is simple and of bounded valency, the pseudogroup of all partial isometries is cofinite. We will also say that two points $\delta$ and $\delta'$ are *$r$-equivalent* with respect to $\Gamma$ if the $r$-balls centered at these points are $\Gamma$-isometric. $\Gamma$ is *dense on* $\Delta$ if, for every $r = 0, 1, 2, \dots$, each *non-empty* class $\Delta'$ of $r$-equivalent points forms a net in $\Delta$, that is there exists an $R = R(\Delta')$ such that $\Delta'$ meets every ball of radious $R$ in $\Delta$ (equivalently, $\cup_{\delta' \in \Delta'} D(\delta', R) = \Delta$).

## 3.2. Subproduct systems on $\Delta$.

Let now $S$ be a finite or countable alphabet. An *$S$-valued subproduct system on* $\Delta$, denoted by $\{X_\Omega\}$, consists of an assignement to each *finite* subset $\Omega$ of $\Delta$ of a *finite* set $X_\Omega$ of $S$-valued functions defined on $\Omega$ yelding a *projective system* with respect to inclusion of finite subsets; this means that, if $\Sigma \subseteq \Omega$ is an arbitrary finite subset of $\Omega$ and $x \in X_\Omega$, then the restriction $x|_\Sigma$ of $x$ to $\Sigma$ belongs to $X_\Sigma$. The term "subproduct" comes form the fact that one might regard each $X_\Omega$ as a subset of the cartesian product $\times_{\delta\in\Omega} X_{\{\delta\}}$. *The projective limit* $X = \varprojlim X_\Omega$ of a subproduct system $\{X_\Omega\}$ is the set of all functions $x : \Delta \to S$ such that the restriction $x|_\Omega$ of $x$ to $\Omega$ belongs to $X_\Omega$ for every finite $\Omega \subset \Delta$. An *exhaustion of* $\Delta$ *by finite subsets* is a sequence $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \ldots \subset \Omega_n \subset \ldots$ of finite subsets such that $\bigcup_{n=1}^\infty \Omega_n = \Delta$. It is not difficult to prove that $x$ belongs to the projective limit of $\{X_\Omega\}$ if and only if the restriction of $x$ to $\Omega_n$ belongs to $X_{\Omega_n}$ for all $n = 1, 2, \ldots$ The projective limits just defined are always non-empty:

**Proposition 3.1.** *The projective limit of a subproduct system $\{X_\Omega\}$ is non empty.*

*Proof.* Let $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \ldots \subset \Omega_n \subset \ldots$ be an exhaustion of $\Delta$ by finite subsets. For $j \geq i$, let $X_i^j$ be the set of the restrictions to $\Omega_i$ of the functions in $X_{\Omega_j}$. Then the intersection $X_i^\infty = \bigcap_{j=i}^\infty X_i^j$ is non empty. In fact, this is the intersection of a decreasing ($X_i^j \geq X_i^{j+1}$) sequence of non empty finite subsets. Now denote by $\pi_{i+1,i}$ the projection (restriction of functions) from $X_{\Omega_{i+1}}$ to $X_{\Omega_i}$. We claim that $\pi_{i+1,i}$ is *onto* from $X_{i+1}^\infty$ to $X_i^\infty$. In fact, if $x \in X_i^\infty$ and $j \geq i+1$ then $x \in X_i^j$, that is there exists $x_j \in X_{\Omega_j}$ such that $x = x_j|_{\Omega_i}$. Setting $x' = x_j|_{\Omega_{i+1}}$ we have $x = x'|_{\Omega_i}$ so that $x' \in \pi_{i+1,i}^{-1}(x) \bigcap X_{i+1}^j \neq \emptyset$. Then, as before,

$$\pi_{i+1,i}^{-1}(x) \bigcap X_{i+1}^\infty = \bigcap_{j=i}^\infty \left(\pi_{i+1,i}^{-1}(x) \bigcap X_{i+1}^j\right) \neq \emptyset.$$

Now we can construct an element of the projective limit of $\{X_\Omega\}$: choose a function of $X_1^\infty$, it may be extended to a function of $X_2^\infty$, that may be extended to a function of $X_3^\infty$ and so on, obtaining a function on all $\Delta$. □

**Remark 3.2.** Observe that the local finiteness of the projective system $\{X_\Omega\}$ is essential. For instance if $S = \{0, 1, 2, \ldots\}$ and $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \ldots \subset \Omega_n \subset \ldots$ is an exhaustion of $\Delta$ by finite subsets, setting

$$X_{\Omega_i} = \{x : \Omega_i \to \{i, i+1, i+2, \ldots\}\}$$

we clearly have that $\varprojlim X_{\Omega_i}$ is empty.

Now let $X$ be a set of $S$-valued functions defined on $\Delta$. $X$ is *locally-finite* if the set $X_{\{\delta\}} = \{x(\delta) : x \in X\}$ is finite for every $\delta \in \Delta$. If, in addition, $\sup\{|X_{\{\delta\}}| : \delta \in \Delta\} < \infty$, then $X$ is *uniformly*-locally-finite. Clearly, if $S$ is finite, every $X \subseteq S^\Delta$ is (uniformly) locally-finite. Now consider a locally-finite set of

$S$-valued functions defined on $\Delta$. For every finite subset $\Omega$ of $\Delta$ let $X_\Omega$ be the space of the restrictions of the functions of $X$ to $\Omega$. Clearly $\{X_\Omega\}$ is a subproduct system; we call it *the subproduct system generated by* $X$.

We now present two examples that show what may happen.

**Example 3.3.** Let $S = \{0,1\}$ and denote by $X$ be the set of $S$-valued functions on $\Delta$ defined by the following condition (finite support): $x : \Delta \to S$ is in $X$ if and only if there exists a finite subset $\Sigma = \Sigma(x) \subset \Delta$ such that $x(\delta) = 0$ for every $\delta \notin \Sigma$. Then if $\{X_\Omega\}$ is the subproduct generated by $X$, for any $\Omega$, $X_\Omega$ is equal to the set of all $S$-valued functions defined on $\Omega$; consequently the projective limit $X'$ of $\{X_\Omega\}$ is equal to the space of *all* $S$-valued functions defined on $\Delta$. In particular $X \subsetneq X'$.

**Example 3.4.** Let $\Delta$ be the usual Cayley graph of the integers (the bi-infinite line graph). If $\Omega$ is a (finite) subset of $\Delta$ we say that three consecutive integers $i-1, i, i+1$ contained in $\Omega$ are in the interior of $\Omega$ if $i-2$ and $i+2$ are in $\Omega$ as well. Let $S = \{0,1\}$ and define the subproduct system $\{X_\Omega\}$ by the following condition: $x : \Omega \to S$ is in $X_\Omega$ if and only if $x(i-1) = x(i+1)$ and $x(i) \neq x(i-1)$ whenever $i-1, i, i+1$ are in the interior of $\Omega$. This means that $x : \Omega \to S$ is in $X_\Omega$ if and only it has period 2 except (possibly) at the boundary of $\Omega$. Then the projective limit $X'$ of $\{X_\Omega\}$ consists of two functions: those that have period 2. Now, denoting by $\{X'_\Omega\}$ the subproduct system generated by $X'$, one has that $x : \Omega \to S$ is in $X'_\Omega$ if and only if it has period 2 on the *whole* of $\Omega$; this shows that, for all finite $\Omega \subset \Delta$ with non-empty interior, the strict inequality $X'_\Omega \subsetneq X_\Omega$ holds.

These examples suggest the following definition, which is not considered, at least in this form, in [20]:

**Definition 3.5.** A subproduct system $\{X_\Omega\}$ is *stable* when it is equal to the subproduct system generated by its projective limit. Analogously, we will say that a locally-finite set $X$ of $S$-valued functions is a *stable space* if it is the projective limit of the subproduct system generated by it.

**Remark 3.6.** Let $X$ be a set of $S$–valued functions on $\Delta$ and denote by $\{X_\Omega\}$ the projective system induced by the restrictions. If $X' = \varprojlim X_\Omega$ then $X \subseteq X'$. Thus $\{X_\Omega\}$ is stable. On the other side, starting from a projective system $\{X_\Omega\}$, denoting by $\{X'_\Omega\}$ the projective system induced by the projective limit $X' = \varprojlim X_\Omega$, we have $X'_\Omega \subseteq X_\Omega$ and therefore $X'$ is stable. In other words, the projective limit of any projective system or any induced subproduct system are stable.

Our definition has a simple topological interpretation. Endowing the space $S^\Delta$ of all $S$-valued functions with the product topology ($S$ is a discrete countable space), a subspace $X$ of $S^\Delta$ is closed (in fact compact, by local finiteness) if and only if it is stable.

Gromov gives local conditions of stability [20], slightly different from ours; our definition seems more natural since it also ensures the following fundamental properties: if $\{X_\Omega\}$ is stable and $X$ is its projective limit, then the projection (restriction) from $X$ to each $X_\Omega$ is onto, and, if $\Omega_0 \subseteq \Omega$ then the projection $X_\Omega \to X_{\Omega_0}$ is also onto.

## 3.3. Irreducibility conditions for subproduct systems

**Definition 3.7.** A subproduct system $\{X_\Omega\}$ has *propagation* $\leq \ell$ if it satisfies the following condition: $x : \Omega \to S$ belongs to $X_\Omega$ if (and only if) the restrictions of $x$ to the intersections $\Omega \cap D(\delta, \ell)$ are contained in $X_{\Omega \cap D(\delta,\ell)}$ for all $\delta \in \Omega$.

For two subsets $\Omega, \Omega' \subseteq \Delta$ set $\mathrm{dist}(\Omega, \Omega') = \min\{\mathrm{dist}(\delta, \delta') : \delta \in \Omega, \delta' \in \Omega'\}$.

**Definition 3.8.** A subproduct system $\{X_\Omega\}$ and its projective limit $X$ are *$M$-irreducible* if, for each pair of finite sets $\Omega, \Omega' \subset \Delta$ such that $\mathrm{dist}(\Omega, \Omega') > M$ and for each $x \in X_\Omega$ and $y \in X_{\Omega'}$, the function $z : \Omega \cup \Omega' \to S$ that equals $x$ on $\Omega$ and $y$ on $\Omega'$ belongs to $X_{\Omega \cup \Omega'}$, equivalently $X_{\Omega \cup \Omega'} = X_\Omega \times X_{\Omega'}$. $X$ is *strongly-irreducible* if it is $M$-irreducible for some $M > 0$.

**Definition 3.9.** A stable subproduct system $\{X_\Omega\}$ has *memory $M$* if it satisfies the following condition: $x : \Delta \to S$ is in the projective limit $X = \lim\limits_{\leftarrow} X_\Omega$ if (and only if) the restriction of $x$ to $D(\delta, M)$ is in $X_{D(\delta,M)}$ for any $\delta \in \Delta$. We say that it is *of finite type* if it has memory $M$ for some $M > 0$.

**Remark 3.10.** Let $X$ be a (stable) subproduct system. It is obvious that if it has propagation $\leq \ell$ then it also has propagation $\leq \ell + 1$, etc.: one says simply that it has *bounded propagation.* The same holds for strong-irreducibility and for the memory. Thus, in the remaining of this section, when thinking of a strongly-irreducible space $X$ of bounded propagation and of finite type, it is not restrictive to suppose that, for a *common* $\ell$, the space $X$ is $\ell-$irreducible, with propagation $\leq \ell$ and memory $\ell$.

**Lemma 3.11.** *A stable space of bounded propagation is strongly irreducibile and of finite type.*

*Proof.* Let $X$ be a system of propagation $\leq \ell$. Suppose that $\Omega$ and $\Omega'$ are finite subsets of $\Delta$ such that $\mathrm{dist}(\Omega, \Omega') > \ell$. A ball of radious $\ell$ centered in a point in $\Omega \cup \Omega'$ cannot intersect both $\Omega$ and $\Omega'$. From this fact it follows immediately the $\ell-$irreducibility. Now suppose that the restrictions of $x : \Delta \to S$ to $D(\delta, \ell)$ is in $X_{D(\delta,\ell)}$ for any $\delta \in \Delta$. Then for any finite set $\Omega \subseteq \Delta$ and for any $\delta \in \Omega$ the restriction of $x$ to $\Omega \cap D(\delta, \ell)$ is in $X_{\Omega \cap D(\delta,\ell)}$, so that, by the bounded propagation property, the restriction of $x$ to $\Omega$ belongs to $X_\Omega$ and, in virtue of the stability condition, $x \in X$. This latter shows that $X$ is of finite type with memory $\ell$. □

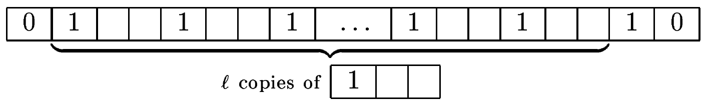
For a partial converse of the above statement in the 1-dimensional setting see Proposition 3.5.11 of [9]. In general such a converse does not hold as the following counterexample shows.

**Counterexample 3.12.** Strong irreducibility and finite type conditions do not imply, in general, bounded propagation.

*Proof.* Let $X$ be the subshift of $\{0,1\}^{\mathbf{Z}}$ determined by the set of forbidden blocks:

$$\{010, 111\}.$$

It is easy to show that $X$ is a strongly-irreducible (in fact 3-irreducible) shift of finite type; for any $\ell \geq 1$ consider the following pattern $p$ whose support $F$ consists only of the boxes containing a digit.




We have $p|_{F \cap D(\alpha,\ell)} \in X_{F \cap D(\alpha,\ell)}$ but $p \notin X_F$: indeed one can *locally* insert zeroes in the empty boxes yielding admissible words (or patterns); this is no more possible *globally*, i.e. for all the $3\ell + 3$ boxes.

Hence $X$ is not of bounded propagation $\leq \ell$ for each $\ell \geq 1$. □

## 3.4. Maps of bounded propagation.

If $\Omega$ is a finite subset of $\Delta$ define $\Omega^{-\ell}=\{\delta \in \Delta : D(\delta, \ell) \text{ is contained in } \Omega\}$ as the $\ell$*−interior* of $\Omega$. Let $\{X_\Omega\}$ and $\{Y_\Omega\}$ be $S$-valued subproduct systems on $\Delta$ (if $\{X_\Omega\}$ and $\{Y_\Omega\}$ have different alphabets, say $S_X$ and $S_Y$ we may take the union $S = S_X \cup S_Y$); a *morphism of bounded propagation* $\leq \ell$ between the two subproduct systems consists of a set of functions $\tau_\Omega : X_\Omega \to Y_{\Omega^{-\ell}}$, denoted by $\{\tau_\Omega\}$, commuting with the (respective) restrictions. Clearly $\{\tau_\Omega\}$ gives rise to a function $\tau$ between the projective limits $X$ of $X_\Omega$ and $Y$ of $Y_\Omega$: if $x \in X$ and $\delta \in \Delta$ then $y = \tau(x)$ is defined on $\delta$ by: $y(\delta) = \tau_{D(\delta,\ell)}[x|_{D(\delta,\ell)}](\delta)$, where $x|_{D(\delta,\ell)}$ denotes, as usual, the restriction of $x$ to $D(\delta, \ell)$. In other words, for every $\delta \in \Delta$ we have a map from $X_{D(\delta,\ell)}$ to $Y_{\{\delta\}}$ and the set of all these maps determines both $\{\tau_\Omega\}$ and $\tau$.

## 3.5. Holonomy maps.

Let $\{X_\Omega\}$ be a subproduct system. Fix two balls $D$ and $D'$ in $\Delta$. A *holonomy map* $h$ between the projective systems $\{X_\Omega\}_{\Omega \subseteq D}$ and $\{X_{\Omega'}\}_{\Omega' \subseteq D'}$ consists of the following data:

(i) an isometry $\gamma = \gamma_h : D \to D'$ sending the center of $D$ to that of $D'$,

(ii) a set of bijective maps $h_\Omega : X_\Omega \to X_{\gamma(\Omega)}$ for all $\Omega \subset D$ which commute with the restriction maps.

A *holonomy over* $\Delta$ is defined as a set $H$ of holonomy maps defined between certain pairs of balls $D$ and $D'$. The balls admitting holonomies betweeen them are called *equivalent.* According to Gromov [20], we say that $H$ is a *pseudogroup of holonomies* if it satisfies the following axioms:

(1) The identity $\mathrm{Id}_D$, given by the identity map $D \to D$ and the identity map $X_D \to X_D$ is in $H$.

(2) $h \in H \Rightarrow h^{-1} \in H$.

(3) If $h$ and $h'$ are in $H$, $h$ is defined between $D$ and $D'$ and $h'$ is between $D'$ and $D''$, then their composition $h' \circ h$ between $D$ and $D''$ is also in $H$.

(4) If a ball $D_0$ is contained in $D$ then the obvious restriction of each holonomy from $D$ to $D_0$ belongs to the holonomy over $D_0$ (where $D_0$ is not necessarily a concentric ball).

If $H$ is a pseudogroup of holonomies we denote by $\Gamma(H)$ the associated pseudogroup of partial isometries: $\Gamma(H) = \{\gamma_h : h \in H\}$. $H$ is called *cofinite* or *dense* whenever $\Gamma(H)$ is such. Given a morphism $f = \{f_\Omega\}$ of bounded propagation $\leq \ell$ between two projective systems over $\Delta$ we shall consider holonomies that *commute* with $\{f_\Omega\}$. This corresponds to the notion of $G-$equivariance we alluded to in the Introduction when $\Delta$ is the Cayley graph of a finitely generated group.

## 3.6. Entropy.

Let $X$ be a set of $S$-valued functions defined on $\Delta$ and $\Omega_n \subset \Delta, n = 1, 2, \ldots,$ be a sequence of finite subsets. Denote as usual by $X_{\Omega_n}$ the set of the restrictions of the functions of $X$ to $\Omega_n$ and by $|A|$ the cardinality of a set $A$. Then we define *the entropy of $X$ with respect to* $\{\Omega_n\}$ by setting:

$$\mathrm{ent}(X) = \mathrm{ent}(X : \{\Omega_n\}) = \liminf_{n\to\infty} \frac{\log |X_{\Omega_n}|}{|\Omega_n|}. \tag{3.7}$$

Clearly, the entropy is monotone for inclusion: if $X' \subset X$ then $\mathrm{ent}(X') \leq \mathrm{ent}(X)$. Note that if $S$ is finite and $X$ is the space of all $S$-valued functions on $\Delta$ then $|X_\Omega| = |S|^{|\Omega|}$ and therefore $\mathrm{ent}(X) = \log |S|$.

## 3.7. Amenability.

If $\Omega$ is a subset of $\Delta$, its *$\ell$-boundary* $\partial_\ell\Omega$ is the set of all $\delta \in \Delta$ such that the ball $D(\delta, \ell)$ intersects both $\Omega$ and $\Delta \setminus \Omega$. The 1-boundary will be denoted simply by $\partial\Omega$ and called *the boundary.* We also set $\Omega^{+\ell}=\Omega \cup \partial_\ell\Omega$. The proof of the following proposition is trivial:

**Proposition 3.13.** *Let $\Delta$ a graph and $\Omega \subseteq \Delta$ a (not necessarily finite) subset.*
*(i)* $\partial_\ell\Omega \subseteq \bigcup_{\delta\in\partial\Omega} D(\delta, \ell)$;
*(ii)* $\Omega^{+\ell} = \bigcup_{\delta\in\Omega} D(\delta, \ell)$;
*(iii) if* $a = \max\{|(D(\delta, \ell)| : \delta \in \Delta\}$ *and $\Omega$ is finite then* $|\partial_\ell\Omega| \leq a \cdot |\Omega|$.

□

A sequence $\Omega_n \subset \Delta, n = 1, 2, \ldots$ of finite subsets is called *amenable* if

$$\lim_{n\to\infty} \frac{|\partial\Omega_n|}{|\Omega_n|} = 0.$$

Clearly, by Proposition 3.13, if $\{\Omega_n\}$ is amenable we have also $\lim_{n\to\infty} \frac{|\partial_\ell\Omega_n|}{|\Omega_n|} = 0$ for every positive integer $\ell$. (The converse is also true since $\partial_\ell\Omega \supseteq \partial\Omega$).

The graph $\Delta$ is *amenable* if it admits an exhaustion by finite substs $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \ldots \subset \Omega_n \subset \ldots$ that is amenable.

From now on, as the graphs $\Delta$ we shall deal with, are always amenable, when referring to the entropy of a stable space $X \subseteq S^\Delta$, we shall assume that an exhausting amenable sequence $\{\Omega_n\} \subseteq \Delta$ has been fixed once for all. Thus we shall simply denote the entropy by $\mathrm{ent}(X)$ instead of $\mathrm{ent}(X : \{\Omega_n\})$.

The following is an analogue of Proposition 2.4.

**Proposition 3.14.** *Suppose that $\Delta$ is an amenable graph and that $X$ and $Y$ are uniformly-locally-finite sets of $S$-valued functions defined on $\Delta$. If a map of bounded propagation $\tau : X \to Y$ is surjective then* $\mathrm{ent}(Y) \leq \mathrm{ent}(X)$. *Equivalently, one has* $\mathrm{ent}(\tau(X)) \leq \mathrm{ent}(X)$ *for all maps $\tau$ of bounded propagation (not necessarily surjective).*

*Proof.* Since $\Omega_n \subseteq (\Omega_n^{+\ell})^{-\ell}$, by the surjectivity of $\tau$, the cardinality of $Y_{\Omega_n}$ does not exceed that of $X_{\Omega_n^{+\ell}}$. Setting $b=\sup\{|X_{\{\delta\}}| : \delta \in \Delta\}$, one has $|X_{\Omega_n^{+\ell}}| \leq |X_{\Omega_n}| \cdot b^{|\partial_\ell\Omega_n|}$; thus

$$\frac{\log|Y_{\Omega_n}|}{|\Omega_n|} \leq \frac{\log|X_{\Omega_n^{+\ell}}|}{|\Omega_n|} \leq \frac{\log|X_{\Omega_n}| + \log(b)\cdot|\partial_\ell\Omega_n|}{|\Omega_n|}$$

and taking the lim inf we get the desired inequality. □

## 3.8. Splicable spaces.

Given $\Omega \subset \Delta$ and two $S$-valued functions $x$ and $x'$ on $\Delta$, define their *splice over* $\Omega$ as the function $x$ which is equal to $x$ on $\Omega$ and to $x'$ outside $\Omega$. A space $X$ of $S$-valued functions on $\Delta$ is *$\ell$-splicable* if the conditions $x, x' \in X$ and $x = x'$ on $\partial_\ell \Omega$ imply that their splice $x$ over $\Omega$ belongs to $X$ whenever $\Omega$ is a finite subset of $\Delta$.

**Example 3.15.** Let $S$ be any countable set and let $\Delta$ be the bi-infinite line graph. Let $X(\ell)$ denote the set of all $\ell$-periodic functions. Then $X(\ell)$ is $[\frac{\ell+1}{2}]$-splicable.

**Proposition 3.16.** *A stable space $X$ of finite type with memory $\ell$ is $2\ell-$splicable.*

*Proof.* Let $\Omega$ be a finite subset of $\Delta$, $x, x' \in X$ such that $x = x'$ on $\partial_{2\ell}\Omega$ and $x$ their splice over $\Omega$. Then for every ball $D$ of radious $\ell$, the restriction of $x$ to $D$ is equal either to the restriction of $x$ (when $D \subseteq \Omega^{+2\ell}$) or to the restriction of $x'$ (when $D \cap \Omega = \emptyset$), thus in any case it belongs to $X_D$ and since $X$ has memory $\ell$ we have that $x \in X$. □

## 3.9. Preinjectivity.

A map $\tau : X \to Y$ between two spaces $X$ and $Y$ of $S$-valued functions on $\Delta$ is *pre-injective* if whenever $x, x' \in X$ are such that $x \neq x'$ in a finite non-empty subset $\Omega \subset \Delta$ and $x = x'$ outside $\Omega$, then $\tau(x) \neq \tau(x')$.

The following corresponds to one implication in the statement of Theorem 2.8 (the Hedlund - Coven - Paul theorem):

**Proposition 3.17.** *Let $\Delta$ be an amenable graph and let $X$ be a uniformly-locally-finite irreducible stable space of finite type. If $\tau : X \to S^\Delta$ is a pre-injective map of bounded propagation, then* $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$.

*Proof.* Denote by $Y = \tau(X)$ the image of $X$ and by $\{Y_\Omega\}$ the corresponding subproduct system. Set $b = \sup\{|X_{\{\delta\}}| : \delta \in \Delta\}$ and $c = \sup\{|Y_{\{\delta\}}| : \delta \in \Delta\}$. Since $|Y_{\Omega_n^{+2\ell}}| \leq |Y_{\Omega_n}| \cdot c^{|\partial_{2\ell}\Omega_n|}$, $|X_{\Omega_n}| \leq |X_{\Omega_n^{-2\ell}}| \cdot b^{|\Omega_n \setminus \Omega_n^{-2\ell}|}$, $|\partial_{2\ell}\Omega_n|/|\Omega_n| \to 0$ and $|\Omega_n \setminus \Omega_n^{-2\ell}|/|\Omega_n| \leq (|\partial\Omega_n| \cdot \max\{|D(\delta, 2\ell)| : \delta \in \Delta\})/|\Omega_n| \to 0$ as $n \to \infty$, we have

$$\frac{\log |Y_{\Omega_n^{+2\ell}}|}{\Omega_n} \leq \frac{\log |Y_{\Omega_n}|}{\Omega_n} + \frac{\partial_{2\ell}\Omega_n|}{|\Omega_n|} \log c$$

and

$$\frac{\log |X_{\Omega_n}|}{\Omega_n} \leq \frac{\log |X_{\Omega_n^{-2\ell}}|}{\Omega_n} + \frac{|\Omega_n \setminus \Omega_n^{-2\ell}|}{|\Omega_n|} \log b.$$

Suppose, by contradiction, that $\mathrm{ent}(Y) < \mathrm{ent}(X)$. Then taking the lim inf in the above formulas we get:

$$\liminf_{n\to\infty} \frac{\log |Y_{\Omega_n^{+2\ell}}|}{|\Omega_n|} < \liminf_{n\to\infty} \frac{\log |X_{\Omega_n^{-2\ell}}|}{|\Omega_n|}$$

and we can find $n$ such that:

$$|Y_{\Omega_n^{+2\ell}}| < |X_{\Omega_n^{-2\ell}}|.$$

Now take $z \in X_{\Omega_n^{+2\ell}\setminus\Omega_n}$; then for every $y \in X_{\Omega_n^{-2\ell}}$ by the strong irrducibility there exists $x' \in X_{\Omega_n^{-2\ell}\cup(\Omega_n^{+2\ell}\setminus\Omega_n)}$ that is equal to $z$ on $\Omega_n^{+2\ell} \setminus \Omega_n$ and to $y$ on $\Omega_n^{-2\ell}$. Using the stability of $X$, $x'$ may be extended to a function $x \in X_{\Omega_n^{+2\ell}}$. Thus $|\{x \in X_{\Omega_n^{+2\ell}} : x = z \text{ on } \Omega_n^{+2\ell} \setminus \Omega_n\}| \geq |Y_{(\Omega_n^{+2\ell})^{-l}}|$. It follows that there exist $x_0, x_1 \in X_{\Omega_n^{+2\ell}}$ such that: $x_0 \neq x_1$, $x_0 = x_1 = z$ on $\Omega_n^{+2\ell} \setminus \Omega_n$ and $\tau_{\Omega_n^{+2\ell}}(x_0) = \tau_{\Omega_n^{+2\ell}}(x_1)$. Then, using the stability of $X$, we can extend $x_0$ and $x_1$ to functions $\tilde{x}_0, \tilde{x}_1$ on all $\Delta$. By Proposition 3.16 $X$ is $2\ell$-splicable; thus the splice $\tilde{x}$ of $\tilde{x}_0$ and $\tilde{x}_1$ over $\Omega_n$ belongs to $X$. But now $\tilde{x} \neq \tilde{x}_1$ on $\Omega_n$, $\tilde{x} = \tilde{x}_1$ outside $\Omega_n$ and $\tau(\tilde{x}) = \tau(\tilde{x}_1)$, since $\tau$ has bounded propagation $\leq \ell$ and a ball of radious $\ell$ cannot intersect both $\Omega_n$ and $\Delta \setminus \Omega_n^{+2\ell}$. This makes $\tau$ *not* pre-injective and proves the proposition. □

## 3.10. Strict monotonicity.

Now we prove the following fundamental analogue of Theorems 2.2, 2.7 and 2.14.

**Lemma 3.18.** *Let $\Delta$ be an amenable graph and let $X \subseteq S^\Delta$ be a uniformly locally-finite irreducible stable space. Suppose a set $\{D_j : j = 1, 2, \dots\}$ of balls of radious $\rho$ constitutes a net in $\Delta$ (i.e. some $R$-neighbourhood of their union equals the whole of $\Delta$). If $X'$ is a subspace of $X$ such that $X'_{D_j}$ is strictly smaller than $X_{D_j}$ on every ball $D_j$: $|X'_{D_j}| < |X_{D_j}|$, then*

$$\mathrm{ent}(X') < \mathrm{ent}(X).$$

*Proof.* We may assume (throwing away some balls if necessary) that the mutual distances between the balls are large say $\geq 10\ell$. Consider a large $\Omega_n$ and suppose that $D_{j_1}, D_{j_2}, \dots, D_{j_N}$, where $N = N(n)$, are *all* the balls that are contained in $\Omega_n^{-2\ell}$. Set

$$\alpha = \left(\max\{|X_{D(\delta,\rho+2\ell)}| : \delta \in \Delta\}\right)^{-1}.$$

We prove by induction that

$$|X'_{\Omega_n}| \leq (1-\alpha)^{N(n)} |X_{\Omega_n}|. \tag{3.8}$$

For $i = 1, 2, \dots N$ choose $x_i \in X_{D_{j_i}} \setminus X'_{D_{j_i}}$ and denote by $\pi_i$ the projection from $X_{\Omega_n}$ to $X_{D_{j_i}}$. Setting $Y = X_{\Omega_n \setminus D_{j_1}^{+2\ell}}$, we have $|Y| \geq \alpha \cdot |X_{\Omega_n}|$ and $|\pi_1^{-1}(x_1)| \geq |Y|$;

the last inequality follows from the strong irreducibility of $X$: the distance between $D_{j_1}$ and $\Omega_n \setminus D_{j_1}^{+2\ell}$ is $> 2\ell$, thus every $y \in Y$ may be extended to a function on $\Omega_n$ which is equal to $x_1$ on $D_{j_1}$. It follows that $|X_{\Omega_n} \setminus \pi_1^{-1}(x_1)| \leq (1-\alpha)|X_{\Omega_n}|$. Now suppose (inductive hypothesis) that

$$|X_{\Omega_n} \setminus \bigcup_{k=1}^{N-1} \pi_k^{-1}(x_k)| \leq (1-\alpha)^{N-1}|X_{\Omega_n}|.$$

Define $\overline{X}_{\Omega_n} = X_{\Omega_n} \setminus \bigcup_{k=1}^{N-1} \pi_k^{-1}(x_k)$, $B = \{z \in \overline{X}_{\Omega_n} : z = x_N \text{ on } D_{j_N}\}$ and denote by $\overline{Y}$ the set of the restrictions of the functions of $\overline{X}_{\Omega_n}$ to $\Omega_n \setminus D_{j_N}^{+2\ell}$. By strong irreducibility, given any $y \in \overline{Y}$ we can find $z \in X_{\Omega_n}$ such that $z = y$ on $\Omega_n \setminus D_{j_N}^{+2\ell}$ and $z = x_N$ on $D_{j_N}$. But for $k = 1, 2, \ldots, N-1$, $z = y \neq x_{j_k}$ on $D_{j_k}$: thus $z \in \overline{X}_{\Omega_n}$ and so $z \in B$. Therefore we have again: $|B| \geq |\overline{Y}| \geq \alpha \cdot |\overline{X}_{\Omega_n}|$ so that, being $B \subset \pi_N^{-1}(x_N)$, one has

$$\begin{aligned}|X_{\Omega_n} \setminus \textstyle\bigcup_{k=1}^{N} \pi_k^{-1}(x_k)| = |(X_{\Omega_n} \setminus \bigcup_{k=1}^{N-1} \pi_k^{-1}(x_k)) \setminus \pi_N^{-1}(x_N)| = |\overline{X}_{\Omega_n} \setminus \pi_N^{-1}(x_N)| \leq \\ \leq |\overline{X}_{\Omega_n} \setminus B| \leq (1-\alpha)|\overline{X}_{\Omega_n}| \leq (1-\alpha)^N |X_{\Omega_n}|.\end{aligned}$$

Thus (3.8) is proved:

$$|X'_{\Omega_n}| \leq |X_{\Omega_n} \setminus \bigcup_{k=1}^{N} \pi_k^{-1}(x_k)| \leq (1-\alpha)^N |X_{\Omega_n}|.$$

Set now $\beta = \max\{|D(\delta, R+\rho+\ell)| : \delta \in \Delta\}$. Since all the balls $D_{j_1}, \ldots, D_{j_N}$ are contained in $\Omega_n^{-2\ell}$, we have:

$$\Omega_n \subset \bigcup_{k=1}^{N(n)} D_{j_k}^{+R} \cup (\Omega_n \setminus \Omega_n^{-(R+2\rho+2\ell)});$$

thus

$$|\Omega_n| \leq N(n) \cdot \beta + |\Omega_n \setminus \Omega_n^{-(R+2\rho+2\ell)}|$$

and, from the amenability of $\{\Omega_n\}$, it follows that

$$\liminf_{n \to \infty} \frac{N(n)}{|\Omega_n|} \geq \frac{1}{\beta} > 0$$

Then from (3.8) we have:

$$\frac{\log |X'_{\Omega_n}|}{|\Omega_n|} - \frac{N(n)}{|\Omega_n|} \cdot \log(1-\alpha) \leq \frac{\log |X_{\Omega_n}|}{|\Omega_n|}$$

and taking the $\liminf$ we obtain:

$$\mathrm{ent}(X') < \mathrm{ent}(X') - \tfrac{1}{\beta} \log(1-\alpha) \leq \mathrm{ent}(X).$$

□

## 3.11. Preinjectivity corollary.

The following corresponds to the other implication (see Proposition 3.17 for the first one) of the Hedlund-Coven-Paul theorem (Theorem 2.8).

**Proposition 3.19.** *Let $X$ be a strongly-irreducible stable space of finite type. Suppose that $\tau : X \to S^\Delta$ is a map of bounded propagation admitting a dense pseudogroup of holonomies. Then the equality* $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$ *implies that $\tau$ is pre-injective.*

*Proof.* If $\tau$ is not pre-injective, there exist $x$ and $x'$ in $X$ and a ball $D$ such that: $x \neq x'$ on $D$, $x = x'$ outside $D$ and $\tau(x) = \tau(x')$. Using the density of the holonomy pseudogroup we can form a net of balls $\{D_j^{+2\ell}\}$ with corresponding functions $x_j$ and $x'_j$ in $X_{D_j^{+2\ell}}$ such that: $x_j \neq x'_j$ on $D_j$, $x_j = x'_j$ on $D_j^{+2\ell} \setminus D_j$ and $\tau_{D_j^{+2\ell}}(x_j) = \tau_{D_j^{+2\ell}}(x'_j)$. Then define $X'$ as the set of those functions in $X$ such that no restriction to $D_j^{+2\ell}$ equals $x_j$.

We claim that $\tau(X') = \tau(X)$. In fact, if $z \in X \setminus X'$ then there exists a non-empty set of integers $J$ such that $z = x_j$ on $D_j^{+2\ell}$ for all $j \in J$. Then define $z'$ as follows: $z' = z$ outside $\bigcup_{j \in J} D_j^{+2\ell}$, and $z'|_{D_j^{+2\ell}} = x'_j$ for all $j \in J$. Clearly, $z' \in X'$ since $X$ is a stable projective limit of memory $\ell$; since $\tau$ is of bounded propagation $\leq \ell$ we also have $\tau(z') = \tau(z)$.

From Proposition 3.14 and Lemma 3.18 it then follows that $\mathrm{ent}(\tau(X)) = \mathrm{ent}(\tau(X')) \leq \mathrm{ent}(X') < \mathrm{ent}(X)$. □

## 3.12. Surjectivity corollary.

**Proposition 3.20.** *Let $\tau : X \to Y$ be a map of bounded propagation admitting a dense holonomy and suppose that $Y$ is stable and strongly-irreducible. Then the equality* $\mathrm{ent}(Y) = \mathrm{ent}(\tau(X))$ *implies that $\tau$ is surjective.*

*Proof.* If $Y' = \tau(X) \subset Y$ misses some $y \in Y$, there exists a ball $D$ such that the restriction of $y$ to $D$ does not belong to $Y'_D$. Then the dense holonomy carries $D$ densely over $\Delta$ and Lemma 3.18 (with $X$ and $X'$ substituted by $Y$ and $Y'$, respectively) applies, yielding $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y') < \mathrm{ent}(Y)$. □

## 3.13. Garden of Eden theorem for stable spaces and for amenable shifts

*Proof of Theorem B.* If $\tau$ is surjective then $\mathrm{ent}\tau(X) = \mathrm{ent}(X)$ and Proposition 3.19 implies that $\tau$ is pre-injective.

On the other hand if $\tau$ is pre-injective, then, by Proposition 3.17, $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$ and, by Proposition 3.20, $\tau$ is surjective. □

**Remark 3.21.** The original statement of Gromov is slightly weaker: instead of *strong-irreducibility* and *finite type* conditions, he assumes, in the hypotesis the *bounded propagation* property for the stable space $X$, which, as shown in Lemma 3.11 and the relative Counterexample 3.12, is a stronger condition.

Also, one could further generalize the statement by assuming the holonomy to be *big* in the sense of Furstenberg (see [14] and the nice survey on Ergodic Ramsey Theory of Bergelson [2]), rather than dense, to obtain a more general statement.

Finally observe that Gromov statement holds for *uniformly-locally-finite* spaces. If the alphabet $S$ is finite (as we assumed in the Introduction) all subsets $X \subseteq S^\Delta$ are clearly uniformly-locally-finite.

We can now show why Theorem B generalizes the Garden of Eden theorem for amenable subshifts [4, 9].

*Proof of Corollary C.* Let $\Delta$ denote the Cayley graph of $G$ with respect to a suitable finite symmetric generating system $A$. Then $\Delta$ is simple, with the same cardinality of $G$, thus at most countable, and of bounded valency: the graph is indeed regual of degree $|A|$.

Because of the strong-regularity of $\Delta$ we might consider as holonomy the pseudogroup $H_G$ of partial isometries generated by all the translations $t_g : \Delta \to \Delta$, $g \in G$, where $t_g(\delta) = g\delta$, $\delta \in \Delta$. In other words we set

$$H_G = \{t_g|_\Omega : \Omega \to g\Omega; \quad g \in G, \quad \Omega \subset \Delta\}.$$

The holonomy $H_G$ is clearly cofinite (indeed all $r-$balls are $H_G-$equivalent) and dense.

A subspace $X \subseteq S^G$ is always *uniformly*-locally-finite and, as we observed in Remark 3.6, it is closed if and only if it is stable.

On the other hand, a map $\tau : X \to Y$ is of bounded propagation ($\leq \ell$) if and only if (tautologically) it is ($\ell-$)local. Also, $\tau$ admits $H_G$ as a (dense) pseudogroup of holonomies if and only if it is $G-$equivariant.

Finally it is well-known, see e.g. [3], that $G$ is amenable (as a group) if and only if $\Delta$ is amenable (as a graph). □

**Remark 3.22.** As for Theorem A, in the statement of Corollary C we can relax the finite type condition for the subshift $Y$ by requiring $Y$ to be sofic (see Lemma 3.5.4 in [9]).

# 4. Appendix

**Proposition 4.1.** *Let $X, Y \subseteq S^G$ be two subshifts over a finitely generated group $G$ and $\tau : X \to Y$ a transition map. Then the following are equivalent:*
(i) *there exist GOE patterns;*
(ii) *there exist GOE configurations (i.e. $\tau$ is not surjective).*

*Proof.* By a *pattern* $p$ in $Y$, with support say $F$, we clearly mean a $p = c|_F \in Y_F$ with $c \in Y$.

(i) $\Rightarrow$ (ii). It is clear that the surjectivity of $\tau$ implies the non–existence of GOE patterns (in $Y$).

(ii) $\Rightarrow$ (i). Suppose that for each finite set $F \subseteq G$ and each pattern $p \in Y_F$ there is a configuration $c \in X$ such that $\tau(c)|_F = p$; we prove that $\tau$ is surjective. Let $\{\Omega_n\}_n \subseteq G$ be an exhausting increasing sequence of finite subsets of $G$. If $c' \in Y$, let $c_n \in X$ be such that $\tau(c_n)|_{\Omega_n} = c'|_{\Omega_n}$; hence $\lim_{n\to\infty} \tau(c_n) = c'$. $X$ being compact, there is a subsequence $(c_{n_k})_k$ that converges to a configuration $c \in X$ and $\tau$ being continuous, we have that $c' = \lim_{k\to\infty} \tau(c_{n_k}) = \tau(c)$. □

**Proposition 4.2.** *For a transition map $\tau : S^G \to S^G$ the following are equivalent:*
(i) *$\tau$ is pre-injective;*
(ii) *there exist no $\tau-$mutually erasable patterns.*

*Proof.* (i) $\Rightarrow$ (ii). Let $p_1$ and $p_2$ be two $\tau-$mutually erasable patterns with support $F$. Fix $s \in S$ and define $c_1, c_2 \in S^G$ that coincide, respectively, with $p_1$ and $p_2$ on $F$ and that are constant, with value $s$, elsewere, i.e. $c_1|_{G\setminus F} \equiv s \equiv c_2|_{G\setminus F}$.

Then $c_1$ and $c_2$ differ only on a non–empty finite set (since this set is contained in $F$), and $\tau(c_1) = \tau(c_2)$, so that $\tau$ is not pre-injective.

(ii) $\Rightarrow$ (i). Suppose conversely, that $\tau$ is not pre–injective; there exist two configurations $c_1$ and $c_2$ such that, for some non empty finite set $F$, $c_1|_F \neq c_2|_F$, $c_1|_{G\setminus F} = c_2|_{G\setminus F}$, and $\tau(c_1) = \tau(c_2)$.

Set $p_1 := c_1|_{F^{+2M}}$ and $p_2 := c_2|_{F^{+2M}}$, where $M$ is such that $\tau$ is $M$–local. Observe that $p_1 \neq p_2$ and if $\overline{c_1}$, $\overline{c_2}$ are two configurations such that $\overline{c_1}_{|F^{+2M}} = p_1$, $\overline{c_2}_{|F^{+2M}} = p_2$ and $\overline{c_1} = \overline{c_2}$ out of $F^{+2M}$, then $\tau(\overline{c_1}) = \tau(\overline{c_2})$, as it immediately follows from the $M-$locality of $\tau$. Thus $p_1$ and $p_2$ are $\tau-$mutually erasable. □

The following is a generalization of a classical result for 1-dimensional subshifts (see e.g. [1] or Thm. 6.1.21 of [22]). We include both the statement and the proof, in this generalized version, for the sake of completeness and for the convenience of the reader.

**Proposition 4.3.** *A subset $X \subseteq S^G$ is a subshift (i.e. it is a closed and $G-$invariant subspace) if and only if there exists a subset $\mathcal{F}$ of patterns such that $X = X_{\mathcal{F}}$ where, denoting by $F(p)$ the support of a pattern $p$,*

$$X_{\mathcal{F}} = \{c \in S^G : c^g|_{F(p)} \notin \mathcal{F},\ \textit{for all } g \in G \textit{ and } p \in \mathcal{F}\}.$$

*Proof.* Suppose that $X \subseteq S^G$ is a shift. $X$ being closed, for each $c \notin X$ there exists an integer $n_c > 0$ such that

$$B_{S^G}(c, \frac{1}{n_c}) \subseteq S^G \setminus X.$$

Let $\mathcal{F}$ be the set

$$\mathcal{F} := \{c_{|D_{n_c}} : c \notin X\};$$

we prove that $X = X_{\mathcal{F}}$. If $c \notin X_{\mathcal{F}}$, there exists $\overline{c} \notin X$ such that ${c^h}_{|D_{n_{\overline{c}}}} = \overline{c}_{|D_{n_{\overline{c}}}}$ for some $h \in G$. Then $\operatorname{dist}(c^h, \overline{c}) < \frac{1}{n_{\overline{c}}}$ and hence $c^h \in B_{S^G}(\overline{c}, \frac{1}{n_{\overline{c}}}) \subseteq S^G \setminus X$ which implies $c \notin X$, by the $G-$invariance of $X$. This proves that $X \subseteq X_{\mathcal{F}}$. For the other inclusion, if $c \notin X$ then, by definition of $\mathcal{F}$, $c_{|D_{n_c}} \in \mathcal{F}$ and hence $c \notin X_{\mathcal{F}}$.

Now, for the converse, we have to prove that a set of type $X_{\mathcal{F}}$ is a shift. Observe that

$$X_{\mathcal{F}} = \bigcap_{p \in \mathcal{F}} X_{\{p\}}$$

and, if the support of $p$ is $F(p) = \{h_1, \ldots, h_N\}$,

$$X_{\{p\}} = \bigcap_{h \in G} \{c \in S^G : {c^h}_{|F(p)} \neq p\} = \bigcap_{h \in G} \Big( \bigcup_{i=1}^{N} \{c \in S^G : c^h(h_i) \neq p(h_i)\} \Big).$$

Thus, in order to prove that $X_{\mathcal{F}}$ is closed, it suffices to prove that for any $i = 1, \ldots, N$ the set

$$\{c \in S^G : c^h(h_i) \neq p(h_i)\} \tag{4.9}$$

is closed. We have

$$(\{c \in S^G : c^h(h_i) \neq p(h_i)\})^h = \{c \in S^G : c(h_i) \neq p(h_i)\}$$

and then the set in (4.9) is closed being the pre-image of a closed set under a continuous function. Finally we have to prove that $X_{\mathcal{F}}$ is $G-$invariant. If $g \in G$ and $c \in X_{\mathcal{F}}$, for every $h \in G$ and every $p \in \mathcal{F}$ we have ${c^{gh}}_{|F(p)} \notin \mathcal{F} \Rightarrow {(c^g)^h}_{|F(p)} \notin \mathcal{F}$ and hence $c^g \in X_{\mathcal{F}}$. □

The following generalizes the well-known Curtis-Lyndon-Hedlund theorem (see [22], Thm. 6.2.9).

**Proposition 4.4.** *Let $X \subseteq S^G$ be a subshift. A function $\tau : X \to S^G$ is a local map (i.e. induced by a local rule $f$) if and only if it is $G-$equivariant (i.e. $\tau[c^g] = \tau[c]^g$ for all $c \in X$ and $g \in G$) and continuous.*

*Proof.* Suppose that $\tau$ is $M$–local, induced, say by a local rule $f$. Then, denoting by $D_M = \{h_1, \ldots, h_m\}$ the ball of radius $M$ centered at the neutral element $1 \in G$ we have that for $g \in G$ and $c \in X$,

$$\tau[c^g](h) = f[c^g(hh_1), c^g(hh_2), \ldots, c^g(hh_m)] = f[c(ghh_1), \ldots, c(ghh_m)]$$

and

$$\tau[c]^g(h) = \tau[c](gh) = f[c(ghh_1), \dots, c(ghh_m)],$$

so that $\tau$ commutes with the G–action.

We prove that $\tau$ is continuous. A generic element of a sub–basis of $S^G$ is

$$\xi = \xi(h; s) = \{c \in S^G : c(h) = s\}$$

with $h \in G$ and $s \in S$. It suffices to prove that the set

$$\overline{\xi} := \tau^{-1}(\xi) = \{c \in X \ : \tau[c](h) = s\}$$

is open in $X$. Actually, if $c \in \overline{\xi}$ and

$$r := \min\{n \in \mathbf{N} : hh_1, \dots , hh_m \in D_n\}, \tag{4.10}$$

we claim that the ball $B_X(c, \frac{1}{r+1})$ is contained in $\overline{\xi}$. Indeed, if $\overline{c} \in B_X(c, \frac{1}{r+1})$ then

$$\text{dist}(c, \overline{c}) < \frac{1}{r+1},$$

i.e. $c_{|D_r} = \overline{c}_{|D_r}$. As by (4.10), $hh_i \in D_r$, we have $\tau[\overline{c}](h) = f[\overline{c}(hh_1), \dots, \overline{c}(hh_m)] = f[c(hh_1), \dots, c(hh_m)] = \tau[c](h) = s$, so that $\overline{c} \in \overline{\xi}$.

Conversely, suppose that $\tau$ is continuous and commutes with the action of $G$. Since $X$ is compact, $\tau$ is uniformly continuous; fix $M$ in $\mathbf{N}$ such that for every $c, \overline{c} \in X$,

$$\text{dist}(c, \overline{c}) < \frac{1}{M+1} \Rightarrow \text{dist}(\tau[c], \tau[\overline{c}]) < 1.$$

Thus, if $c$ and $\overline{c}$ agree on $D_M$,$\tau(c)$ and $\tau(\overline{c})$ agree at the neutral element $1 \in G$: $\tau[c](1) = \tau[\overline{c}](1)$ so that, the function $f : S^{D_M} \to S$ defined by $f(s_{h_1}, \dots , s_{h_M}) = \tau[c](1)$, where $c$ is any configuration s.t. $c(h_i) = s_{h_i}$, $i = 1, \dots , M$, is well defined. In addition $f$ serves as a *local rule* for $\tau$; indeed, since $\tau$ commutes with the action of $G$, we have

$$\tau[c](h) = \tau[c]^h(1) = \tau[c^h](1) = f[c(hh_1), \dots, c(hh_m)];$$

this shows that $\tau$ is $M$–local and ends the proof. □

From the above Proposition it is clear that *the composition of two local functions is still local.* In any case, this can be easily seen by a direct proof. Also, one gets immediately the fact that the inverse of an invertible transition map is still local; in the terminology of cellular automata this can be rephrased as "the inverse of an (invertible) cellular automaton is a cellular automaton" : this is usually proved in a combinatorial way combined with a compactness argument (see, e.g. [1, 22]):

**Corollary 4.5.** *Suppose that a transition map $\tau : X \to Y$ is invertible (i.e. surjective and injective). Then its inverse $\tau^{-1} : Y \to X$ is also a local map.*

## References

[1] M.-P. Béal and D. Perrin, Symbolic dynamics and finite automata. Handbook of formal languages, Vol. 2, 463–505, Springer, Berlin, 1997.

[2] V. Bergelson, Ergodic Ramsey theory—an update. Ergodic theory of $Z^d$ actions (Warwick, 1993–1994), 1–61, London Math. Soc. Lecture Note Ser. **228**, Cambridge Univ. Press, Cambridge, 1996.

[3] T. Ceccherini-Silberstein, R.I. Grigorchuk and P. de la Harpe, Amenability and paradoxical decompositions for pseudogroups and for discrete metric spaces. Proc. Steklov Inst. Math. **224** (1999), 57-97.

[4] T.G. Ceccherini-Silberstein, A.Machì and F. Scarabotti, Amenable groups and cellular automata, Ann. Inst. Fourier (Grenoble) **49**, 2 (1999), 673-685.

[5] T. Ceccherini-Silberstein and F. Scarabotti, Random walks, entropy and hopfianity of free groups. These Proceedings.

[6] T. Ceccherini-Silberstein and W. Woess, Growth-sensitivity of context-free languages. Preprint 2001.

[7] E. Coven and M. Paul, Endomorphisms of Irreducible Shifts of Finite Type, Math. Systems Theory **8** (1974), 167-175.

[8] D. Epstein, J.Cannon, D. Holt, S. Levy, M. Paterson and W. Thurston, Word Processing in Groups, Jones and Bartlett Publishers, Boston MA, 1992.

[9] F. Fiorenzi, Cellular Automata and Finitely Generated Groups, PhD thesis, Università di Roma "La Sapienza", 2000.

[10] F. Fiorenzi, The Garden of Eden theorem for sofic shifts. Pure Math. Appl. **11** (2000), 471–484.

[11] F. Fiorenzi, Cellular automata and strongly-irreducible shifts of finite type. Preprint 2000.

[12] F. Fiorenzi, Semi-strong irreducibility of shifts and Moore-Myhill property. Preprint 2001.

[13] E. Følner, On groups with full Banach mean value. Math. Scand. **3** (1955), 243-254.

[14] H. Furstenberg, Recurrence in ergodic theory and combinatorial number theory. M. B. Porter Lectures. Princeton University Press, Princeton, N.J., 1981.

[15] M. Harrison, Introduction to formal language theory. Addison-Wesley Publishing Co., Reading, Mass, 1978.

[16] G.A. Hedlund, Endomorphisms and Automorphisms of the Shift Dynamical Systems, Math. Systems Theory **3** (1969), 320-375.

[17] J.E. Hopcroft and J.D. Ullman, Introduction to Automata Theory, Languages and Computation. Publishing Co., Reading, Mass, 1979.

[18] W. Gottschalk, Some General Dynamical Notions. Recent Advances in Topological Dynamics, 120-125. Lecture Notes in Math. **318**, Springer, Berlin, 1973.

[19] R.I. Grigorchuk and P. de la Harpe, On problems related to growth, entropy and spectrum in group theory, J. Dynam. Control Systems **3** (1997), 51-89.

[20] M. Gromov, Endomorphisms of symbolic algebraic varieties, J. Eur. Math. Soc. (JEMS) 1 (1999), no. 2, 109–197.

[21] W. Krieger, On sofic systems. I; II. Israel J. Math. 48 (1984), no. 4, 305–330; Israel J. Math. 60 (1987), no. 2, 167–176.

[22] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding. Cambridge University Press, Cambridge, 1995.

[23] A. Machì and F. Mignosi, Garden of Eden Configurations for Cellular Automata on Cayley Graphs on Groups, SIAM J. Disc. Math. **6** (1993), 44-56.

[24] E.F. Moore, Machine Models of Self-Reproduction, Proc. Symp. Appl. Math., AMS Providence R.I. **14** (1963), 17-34.

[25] J. Myhill, The converse of Moore's Garedn of Eden Theorem, Proc. Amer. Math. Soc. **14** (1963), 685-686.

[26] J. von Neumann, The Theory of Self-Reproducing Automata (A. Burks ed.), University of Illinois Press, Urbana and London, 1966.

[27] F. Scarabotti, On a lemma of Gromov and the entropy of a graph. European J. Combin. (to appear).

[28] S. Ulam, Processes and Transformations, Proc. Int. Cong. Mathem. **2** (1952), 264-275.

[29] B. Weiss, Subshifts of Finite Type and Sofic Systems, Monatsh. Math. **77** (1973), 462-474.

[30] B. Weiss, Sofic Groups and Dynamical Systems, Ergodic theory and harmonic analysis (Mumbai, 1999). Ser. A 62 (2000), no. 3, 350–359.

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