The Garden of Eden Theorem for Sofic Shifts

Francesca Fiorenzi *

Dipartimento di Matematica, Università di Roma "La Sapienza"
Piazzale Aldo Moro 2 - 00185 Roma
e–mail fiorenzi@mat.uniroma1.it

Abstract

We consider the Garden of Eden theorem of Moore and Myhill in the case of the shifts of $A^{\mathbf{Z}}$. The theorem still holds in the case of irreducible shifts of finite type; this follows easily from a theorem of Lind and Marcus. We give examples showing that neither the finite type condition nor the irreducibility condition can be removed.

1. The *Garden of Eden (GOE)* theorem concerns *cellular automata (CA)* on the plane grid. More precisely, in this setting the "universe" is the lattice of integers $\mathbf{Z}^2$ of Euclidean plane $\mathbf{R}^2$. The *set of states* is a finite set A (also called the *alphabet*) and a *configuration* is a function $c : \mathbf{Z}^2 \to A$. Time t goes on in discrete steps and represents a *transition function* $\tau : A^{\mathbf{Z}^2} \to A^{\mathbf{Z}^2}$ (if c is the configuration at time t, then $\tau(c)$ is the configuration at time $t+1$), which is deterministic and *local*. *Locality* means that the new state at a point $\gamma \in \mathbf{Z}^2$ at time $t+1$ only depends on the states of certain fixed points in the neighborhood of γ at time t. More precisely, if c is the configuration reached from the automaton at time t then $\tau(c)_{|\gamma} = \delta(c_{|N_\gamma})$, where $\delta : A^N \to A$ is a function defined on the configurations with support the finite set N (the neighborhood of the point $(0,0) \in \mathbf{Z}^2$), and $N_\gamma = \gamma + N$ is the neighborhood of γ obtained from N by translation. For these structures, Moore [Moo] has given a sufficient condition for the existence of the so–called *GOE patterns*, that is those configurations with finite support that cannot be reached at time t from another configuration starting at time $t-1$ and hence can only appear at time $t=0$. Moore's condition (i.e. the existence of *mutually erasable patterns* – a sort of non–injectivity of the transition function on the "finite" configurations) was also proved to be necessary by Myhill [My]. This equivalence between "local

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injectivity" and "local surjectivity" of the transition function is the classical well–known GOE theorem.

This theorem holds much more generally for finitely generated amenable groups ([CeMaSca], see also [MaMi]). In this paper we consider CA having $\mathbf{Z}$ as universe; in this case the set of all configurations is the set $A^{\mathbf{Z}}$ of all bi–infinite words on the finite alphabet A. As it is well known, this is a compact metric space and the local functions are the functions that are both continuous and commute with the natural action of $\mathbf{Z}$ on $A^{\mathbf{Z}}$ [LinMar, Theorem 6.2.9]. We investigate the extent to which the GOE theorem holds for the closed and $\mathbf{Z}$–invariant subsets, the so–called *shifts* or *subshifts*.

Instead of the non–existence of mutually erasable patterns, we deal with the notion of *pre–injectivity* (a function $\tau : X \subseteq A^{\mathbf{Z}} \to A^{\mathbf{Z}}$ is *pre–injective* if whenever two configurations $c, \bar{c} \in X$ differ only on a finite non–empty subset of $\mathbf{Z}$, then $\tau(c) \neq \tau(\bar{c})$); this notion has been introduced by Gromov in [G]. In fact, these two properties are equivalent for local functions defined on the full shift $A^{\mathbf{Z}}$, but in the case of proper subshifts the former may be meaningless. On the other hand, the non–existence of GOE patterns is equivalent to the non–existence of GOE configurations, that is to the surjectivity of the transition function. Hence, in this language, the GOE theorem states that τ is surjective if and only if it is pre–injective. We call *Moore's property* the implication surjective $\Rightarrow$ pre–injective and *Myhill's property* the inverse one. We call *Moore–Myhill property (MM–property)* the union of these two properties; it can be easily seen that this last is an invariant of the shift.

2. Let X be a subset of $A^{\mathbf{Z}}$.

Definition 2.1 The *language of* X is the set $L(X)$ of all the finite words appearing in some bi–infinite word of X.

We denote with $X_n = L(X) \cap A^n$ the set of the words in $L(X)$ with length n.

Proposition 2.2 [LinMar, Theorem 6.1.21] *The set X is a shift if and only if there exists a set $\mathcal{F}$ of finite words on A such that X is the set of all bi–infinite words avoiding every word of $\mathcal{F}$.*

If X is a shift, a set $\mathcal{F}$ as in the Proposition 2.2 is called *a set of forbidden blocks for X.*

Definition 2.3 A shift is *of finite type* if it admits a finite set of forbidden blocks and is *irreducible* if for each pair of words $u, v \in L(X)$, there exists a word $w \in L(X)$ such that $uwv \in L(X)$.

Definition 2.4 Let X be a subshift of $A^{\mathbf{Z}}$; a function $\tau : X \to A^{\mathbf{Z}}$ is *M–local* if there exists $\delta : X_{2M+1} \to A$ such that for every $c \in X$ and $z \in \mathbf{Z}$

$$(\tau(c))_{|z} = \delta(c_{|z-M}, \dots, c_{|z+M}).$$

A relevant class of one–dimensional subshifts of finite type, is that of *edge shifts*. This class is strictly tied up the one of finite graphs. This relation allows us to study the properties of an edge shift (possible quite complex) studying the properties of its graph.

Definition 2.5 Let $\mathbf{G}$ be a finite directed graph with edge set $\mathcal{E}$. The *edge shift* $X_{\mathbf{G}}$ is the subshift of $\mathcal{E}^{\mathbf{Z}}$ defined by

$$X_{\mathbf{G}} := \{(e_z)_{z\in\mathbf{Z}} \mid \mathbf{t}(e_z) = \mathbf{i}(e_{z+1}) \text{ for all } z \in \mathbf{Z}\}$$

where, for $e \in \mathcal{E}$, $\mathbf{i}(e)$ denotes the initial vertex of e and $\mathbf{t}(e)$ the terminal one.

It is easy to see that every edge shift is a shift of finite type with set of forbidden blocks $\{ef \mid \mathbf{t}(e) \neq \mathbf{i}(f)\}$.

The class of *sofic shifts* has been introduced by Weiss in [Wei] as the smallest class of shifts containing the shifts of finite type and closed under *factorization* (i.e. the image under a local map of a sofic shift is sofic). Equivalently, one can see that a shift is sofic if it is the set of all *labels* of the bi–infinite paths in a finite *labeled graph* (or *finite–state automaton*). In automata theory, this corresponds to the notion of *regular language*. Indeed a *language* (i.e. a set of finite words over a finite alphabet) is *regular* if it is the set of all labels of finite paths in a labeled graph.

Definition 2.6 Let A be a finite alphabet; a *labeled graph* $\mathcal{G}$ is a pair $(\mathbf{G}, \mathcal{L})$, where $\mathbf{G}$ is a finite graph with edge set $\mathcal{E}$, and the *labeling* $\mathcal{L} : \mathcal{E} \to A$ assigns to each edge e of $\mathbf{G}$ a label $\mathcal{L}(e)$ from A.

If $\xi = \dots e_{-1}e_0e_1 \dots$ is a configuration of the edge shift $X_{\mathbf{G}}$, define the *label of the path* ξ to be

$$\mathcal{L}(\xi) = \dots \mathcal{L}(e_{-1})\mathcal{L}(e_0)\mathcal{L}(e_1) \dots \in A^{\mathbf{Z}}.$$

The set of labels of all configurations in $X_{\mathbf{G}}$ is denoted by

$$X_{\mathcal{G}} := \{c \in A^{\mathbf{Z}} \mid c = \mathcal{L}(\xi) \text{ for some } \xi \in X_{\mathbf{G}}\} = \mathcal{L}(X_{\mathbf{G}}).$$

Definition 2.7 A shift $X \subseteq A^{\mathbf{Z}}$ is *sofic* if $X = X_{\mathcal{G}}$ for some labeled graph $\mathcal{G}$. A *presentation* of a sofic shift X is a labeled graph $\mathcal{G}$ for which $X_{\mathcal{G}} = X$.

Obviously each edge shift is sofic. Indeed if $\mathbf{G}$ is a graph with edge set $\mathcal{E}$, we can consider the identity function on $\mathcal{E}$ as a labeling $\mathcal{L} : \mathcal{E} \to \mathcal{E}$ so that $X_{\mathbf{G}} = X_{(\mathbf{G},\mathrm{id}_{\mathcal{E}})}$. Moreover, each shift of finite type is sofic (see, for example, [LinMar, Theorem 3.1.5]), but the converse does not necessarily hold.

In automata theory, a finite–state automaton is *deterministic* if, given a state Q and a letter a, there is at most one successive state $\bar{Q}$ (determined by Q and a). This corresponds, in the finite graph representing the automata, to the fact that from a vertex i (the state) there is at most one edge carrying the label a. Although this restrictive condition, one can prove that for each regular language there is a deterministic automaton accepting it. This property holds true for sofic shifts: each sofic shift admits a deterministic presentation. A *minimal deterministic presentation* of a sofic shift is a deterministic presentation with the least possible number of vertices. We will see that the irreducibility of the sofic shift implies the existence of only one (up to labeled graphs isomorphism) minimal deterministic presentation of it.

Definition 2.8 A labeled graph $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ is *deterministic* if, for each vertex i of $\mathbf{G}$, the edges starting at i carry different labels.

Proposition 2.9 [LinMar, Theorem 3.3.2] *Every sofic shift has a deterministic presentation.*

Definition 2.10 A *minimal deterministic presentation* of a sofic shift X is a deterministic presentation of X having the least number of vertices among all deterministic presentations of X.

One can prove that any two minimal deterministic presentations of an irreducible sofic shift are isomorphic as labeled graphs (see [LinMar, Theorem 3.3.18]), so that one can speak of *the* minimal deterministic presentation of an irreducible sofic shift.

In the following propositions, we clarify the relation between the irreducibility of a sofic (or edge) shift and the strong connectedness of its presentations.

Proposition 2.11 [LinMar, Lemma 3.3.10] *Let X be an irreducible sofic shift and $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ the minimal deterministic presentation of X. Then $\mathbf{G}$ is a strongly connected graph.*

Proposition 2.12 [LinMar, Proposition 2.2.14] *If $\mathbf{G}$ is a strongly connected graph, then the edge shift $X_{\mathbf{G}}$ is irreducible.*

As a consequence of this two facts, we have the following corollary that will be useful in the sequel.

Corollary 2.13 *Let X be an irreducible sofic shift and $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ the minimal deterministic presentation of X. Then the edge shift $X_{\mathbf{G}}$ is irreducible.*

Now we give the definition of entropy for a shift. Lind and Marcus [LinMar, Chapter 4] give its basic properties also stating the principal result of the Perron–Frobenius theory to compute it.

Definition 2.14 Let X be a shift. The *entropy of* X is defined as

$$\mathrm{ent}(X) = \lim_{n\to\infty} \frac{\log |X_n|}{n}.$$

In order to see that the above limit is well–defined, observe that $|X_{n+m}| \leq |X_n||X_m|$; hence setting $a_n := \log |X_n|$, we have that the sequence $(a_n)_{n\in\mathbf{N}}$ is sub–additive, namely $a_{n+m} \leq a_n + a_m$. We want to prove that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}.$$

Fix $m \geq 1$; then there exist $q, r \in \mathbf{N}$ such that $n = mq + r$ and we have $a_n \leq qa_m + a_r$ so that $\frac{a_n}{n} \leq \frac{qa_m}{n} + \frac{a_r}{n}$. Now

$$\lim_{n\to\infty} \left(\frac{qa_m}{n} + \frac{a_r}{n}\right) = \lim_{n\to\infty} \left(\frac{q}{n}a_m + \frac{a_r}{n}\right) = \frac{a_m}{m}$$

and then $\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m}$. Hence

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \inf_{m\geq 1} \frac{a_m}{m} \leq \liminf_{n\to\infty} \frac{a_n}{n}.$$

Using combinatorial methods, it is easy to see that the entropy of a sofic shift X coincides with the entropy of the edge shift $X_{\mathbf{G}}$ of a deterministic presentation of X, as stated in the following theorem.

Proposition 2.15 [LinMar, Proposition 4.1.13] *Let X be a sofic shift and let $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ be a deterministic presentation of X. Then* $\mathrm{ent}(X) = \mathrm{ent}(X_{\mathbf{G}})$.

Hence we have the following consequence.

Corollary 2.16 *Let X be an irreducible sofic shift and let $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ be the minimal deterministic presentation of X. Then* $\mathrm{ent}(X) = \mathrm{ent}(X_{\mathbf{G}})$.

Another fundamental result about entropy, is the following.

Theorem 2.17 [LinMar, Corollary 4.4.9] *If X is an irreducible sofic shift and Y is a proper subshift of X, then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

As far as irreducible shifts of finite type are concerned, we have the following result that stems from the work of Hedlund [H] and Coven and Paul [CovP].

Theorem 2.18 [LinMar, Theorem 8.1.16] *Let X be an irreducible shift of finite type, Y a shift and $\tau : X \to Y$ a local function. Then τ is pre–injective if and only if* $\mathrm{ent}(X) = \mathrm{ent}(\tau(X))$.

Corollary 2.19 (MM–property for irreducible subshifts of finite type of $A^{\mathbf{Z}}$) *An irreducible subshift of finite type of $A^{\mathbf{Z}}$ has the MM–property.*

PROOF If τ is pre–injective, then by Theorem 2.18 we have $\text{ent}(X) = \text{ent}(\tau(X))$. By Theorem 2.17, there does not exist a proper subshift of X whose entropy equals that of X. Thus $\tau(X) = X$ and τ is surjective. Conversely, if τ is surjective we have $\text{ent}(X) = \text{ent}(\tau(X))$ and Theorem 2.18 applies. □

We now prove that a result similar to Theorem 2.18 holds for irreducible sofic shifts.

Theorem 2.20 *Let X be an irreducible sofic shift, Y a shift and $\tau : X \to Y$ a local function. Let $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ be the minimal deterministic presentation of X. Then $\tau \circ \mathcal{L}$ is pre–injective if and only if $\text{ent}(X) = \text{ent}(\tau(X))$.*

PROOF The labeled graph $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ being a presentation of X, we have $X = X_{\mathcal{G}} = \mathcal{L}(X_{\mathbf{G}})$. By Corollary 2.13, $X_{\mathbf{G}}$ is an irreducible shift of finite type. Moreover $\tau \circ \mathcal{L} : X_{\mathbf{G}} \to Y$ is a local function; thus, by Theorem 2.18, $\tau \circ \mathcal{L}$ is pre–injective if and only if $\text{ent}(X_{\mathbf{G}}) = \text{ent}(\tau(\mathcal{L}(X_{\mathbf{G}}))) = \text{ent}(\tau(X))$. By Corollary 2.16, $\text{ent}(X_{\mathbf{G}}) = \text{ent}(X)$ and the claim is proved. □

Corollary 2.21 (Myhill–property for irreducible sofic shifts) *Let X be an irreducible sofic shift and $\tau : X \to X$ a transition function. Then τ pre–injective implies τ surjective.*

PROOF Let $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ be the minimal deterministic presentation of X; we prove that if $\tau \circ \mathcal{L}$ is not pre–injective, then τ is not pre–injective either. Suppose that there exist two bi–infinite paths $c_1, c_2 \in X_{\mathbf{G}}$ which are different only on a finite path and such that $\tau(\mathcal{L}(c_1))) = \tau(\mathcal{L}(c_2))$. Then one can write c_1 and c_2, respectively, as:

$$c_1 : \qquad \cdots \xrightarrow{e_{-2}} i_{-1} \xrightarrow{e_{-1}} i_0 \xrightarrow{e_0} i_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{n-1}} i_n \xrightarrow{e_n} i_{n+1} \xrightarrow{e_{n+1}} \cdots$$

and

$$c_2 : \qquad \cdots \xrightarrow{e_{-2}} i_{-1} \xrightarrow{e_{-1}} i_0 \xrightarrow{f_0} j_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} j_n \xrightarrow{f_n} i_{n+1} \xrightarrow{e_{n+1}} \cdots,$$

with $e_0 \neq f_0$. Setting $a_i := \mathcal{L}(e_i)$ and $b_i := \mathcal{L}(f_i)$, the graph $\mathcal{G}$ being deterministic we have $a_0 \neq b_0$ and hence

$$\mathcal{L}(c_1) = a_{-2}a_{-1}\ a_0a_1 \ldots a_{n-1}a_n\ a_{n+1} \cdots$$

and

$$\mathcal{L}(c_2) = a_{-2}a_{-1}\ b_0b_1 \ldots b_{n-1}b_n\ a_{n+1} \ldots$$

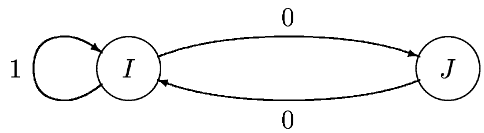
are two configurations in X which differ only on a finite (non empty) set and whose images under τ are equal. Therefore τ is not pre–injective.

Thus, if τ is pre–injective, then $\tau \circ \mathcal{L}$ is also pre–injective; by Theorem 2.20, we have $\text{ent}(X) = \text{ent}(\tau(X))$. X being irreducible and by Theorem 2.17, $\tau(X)$ cannot be a proper subshift of X. Hence $\tau(X) = X$, i.e. τ is surjective. □

3. In this section we give an example of an irreducible sofic shift not of finite type for which the transition function is surjective but not pre–injective (that is, Moore–property no longer holds in general if the *finite type* condition is dropped). Our example will be the *even shift* X_e, that is the subshift of $\{0,1\}^{\mathbf{Z}}$ with forbidden blocks

$$\{10^{2n+1}1 \mid n \geq 0\}.$$

The shift X_e is sofic, indeed it is accepted by the following labeled graph.



We define a transition function τ as follows:

$$\tau(c)_{|z} = \delta(c_{|z-2}, c_{|z-1}, c_{|z}, c_{|z+1}, c_{|z+2})$$

where δ is the local rule:

$$\delta(a_1a_2a_3a_4a_5) = \begin{cases} 1 & \text{if } a_1a_2a_3 = 000 \text{ or } a_1a_2a_3 = 111 \text{ or } a_1a_2a_3a_4a_5 = 00100, \\ 0 & \text{otherwise.} \end{cases}$$

First we prove a Lemma.

Lemma 3.22 *If a block 0^n1 with $n \geq 3$, has a pre–image under τ of length $n+5$ in the language of X_e*

a_1	a_2	a_3	a_4	$\dots$	a_{n+1}	a_{n+2}	a_{n+3}	a_{n+4}	a_{n+5}
		0	0	$\dots$	0	0	1		

,

then this pre–image can be only of type

1. *(i)* $a_1a_2\ xx\ (1-x)(1-x)\dots 11\ 00\ 11\ 000a_{n+4}a_{n+5}$,
   (ii) $a_1a_2\ xx\ (1-x)(1-x)\dots 11\ 00\ 11\ 00100$,
   (iii) $a_1a_2\ (1-x)(1-x)\ xx\dots 00\ 11\ 00\ 111a_{n+4}a_{n+5}$,
   when n is even and for a suitable $x \in \{0,1\}$;
2. *(i)* $a_1a_2\ (1-x)\ xx\dots 11\ 00\ 11\ 000a_{n+4}a_{n+5}$,
   (ii) $a_1a_2\ (1-x)\ xx\dots 11\ 00\ 11\ 00100$

(iii) $a_1a_2\ x\ (1-x)(1-x)\ldots 00\ 11\ 00\ 111a_{n+4}a_{n+5}$
when n is odd and for a suitable $x \in \{0,1\}$.

PROOF We prove the statement by induction on $n \geq 3$. Assume that $\tau(a_1a_2a_3a_4a_5a_6a_7a_8) = 0001$; we distinguish three cases.

- $a_4a_5a_6 = 000$

a_1	a_2	a_3	0	0	0	a_7	a_8
		0	0	0	1		

.

Then $a_3 = 1$ otherwise $\delta(a_3a_4a_5a_6a_7) = \delta(0000a_7) = 1 \neq 0$.

- $a_4a_5a_6a_7a_8 = 00100$

a_1	a_2	a_3	0	0	1	0	0
		0	0	0	1		

.

Then, for the same reasons as above, $a_3 = 1$.

- $a_4a_5a_6 = 111$

a_1	a_2	a_3	1	1	1	a_7	a_8
		0	0	0	1		

.

Then $a_3 = 0$ otherwise $\delta(a_3a_4a_5a_6a_7) = \delta(1111a_7) = 1 \neq 0$.

Now let us suppose that the statement is true for n and that we have $\tau(a_1\ldots a_{n+6}) = 0^{n+1}1$:

a_1	a_2	a_3	a_4	$\ldots$	a_{n+2}	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}
		0	0	$\ldots$	0	0	1		

.

If n is even, by the inductive hypothesis one has either

$$a_4\ldots a_{n+4} = xx\ (1-x)(1-x)\ldots 11\ 000$$

or

$$a_4\ldots a_{n+6} = xx\ (1-x)(1-x)\ldots 11\ 00100$$

or

$$a_4\ldots a_{n+4} = (1-x)(1-x)\ xx\ldots 00\ 111$$

for a suitable $x \in \{0,1\}$.

In any case we have $a_4 = a_5$. If $a_3 = a_4$, then $\delta(a_3a_4a_5a_6a_7) = \delta(a_4a_4a_4a_6a_7) = 1 \neq 0$. Thus $a_3 \neq a_4$.

It follows, in the three cases, that either

$$a_1\ldots a_{n+6} = a_1a_2\ (1-x)\ xx\ (1-x)(1-x)\ldots 11\ 000a_{n+5}a_{n+6}$$

or

$$a_1 \dots a_{n+6} = a_1 a_2 \ (1-x) \ xx \ (1-x)(1-x) \dots 11 \ 00100$$

or

$$a_1 \dots a_{n+6} = a_1 a_2 \ x \ (1-x)(1-x) \ xx \dots 00 \ 111 a_{n+5} a_{n+6}.$$

If n is odd, by the inductive hypothesis we have either

$$a_4 \dots a_{n+4} = (1-x) \ xx \dots 11 \ 000$$

or

$$a_4 \dots a_{n+6} = (1-x) \ xx \dots 11 \ 00100$$

or

$$a_4 \dots a_{n+4} = x \ (1-x)(1-x) \dots 00 \ 111$$

for a suitable $x \in \{0,1\}$.

In any case $a_4 \neq a_5 = a_6$. If $a_3 \neq a_4$, then $a_3 a_4 a_5 = a_5 a_4 a_5$ so that $a_4 = 1$ (otherwise we had a forbidden block). For the same reason, $a_2 = a_3 = 0$. This implies $\delta(a_2 a_3 a_4 a_5 a_6) = \delta(00100) = 1 \neq 0$. Thus $a_3 = a_4$.

It follows, in the three cases, that either

$$a_1 \dots a_{n+6} = a_1 a_2 \ (1-x)(1-x) \ xx \dots 11 \ 000 a_{n+5} a_{n+6}$$

or

$$a_1 \dots a_{n+6} = a_1 a_2 \ (1-x)(1-x) \ xx \dots 11 \ 00100$$

or

$$a_1 \dots a_{n+6} = a_1 a_2 \ xx \ (1-x)(1-x) \dots 00 \ 111 a_{n+5} a_{n+6}.$$

Then the statement is still true for $0^{n+1}1$. □

Proposition 3.23 *The local function τ is a transition function, that is $\tau(X_e) \subseteq X_e$.*

PROOF $\tau(X_e)$ being a subshift of $\{0,1\}^{\mathbf{Z}}$, it suffices to prove that no forbidden block 10^n1 with n odd, has a pre–image of length $n+6$ in the language of X_e. First we prove that there is no block $a_1 a_2 a_3 a_4 a_5 a_6 a_7$ of length 7 such that $\tau(a_1 a_2 a_3 a_4 a_5 a_6 a_7) = 101$:

a_1	a_2	a_3	a_4	a_5	a_6	a_7
		1	0	1		

.

We distinguish two cases.

- $a_3 a_4 = 00$

a_1	a_2	0	0	a_5	a_6	a_7
		1	0	1		

Then $a_2 = 1$ otherwise $\delta(a_2a_3a_4a_5a_6) = \delta(000a_5a_6) = 1 \neq 0$. Then $\delta(a_1a_2a_3a_4a_5) = \delta(a_1100a_5a_6) = 0 \neq 1$.

- $a_3a_4a_5 = 111$

a_1	a_2	1	1	1	a_6	a_7
		1	0	1		

Then $a_2 = 0$ otherwise $\delta(a_2a_3a_4a_5a_6) = \delta(1111a_6) = 1 \neq 0$. Thus $\delta(a_1a_2a_3a_4a_5) = \delta(a_10111a_6) = 0 \neq 1$. We have proved that no block of length 7 goes to 101 under τ.

Let us now prove that no block $a_1 \ldots a_{n+6}$ of length $n+6$ has 10^n1 as image under τ, where $n \in \mathbf{N}$ is odd and strictly greater than 1. If

a_1	a_2	a_3	a_4	a_5	$\ldots$	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}
		1	0	0	$\ldots$	0	1		

,

by Lemma 3.22 we have $a_4a_5a_6 \ldots = x(1-x)(1-x)\ldots$, and being $\delta(a_1a_2a_3a_4a_5) = 1$, we distinguish two cases:

- $x = 0$

a_1	a_2	a_3	0	1	1	$\ldots$	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}
		1	0	0	0	$\ldots$	0	1		

Then $a_3 = 0$ (otherwise we had a forbidden block) and $a_2 = 1$ because $\delta(a_2a_3a_4a_5a_6) = \delta(a_20011) = 0$ and $\delta(00011) = 1$. Then $\delta(a_1a_2a_3a_4a_5) = \delta(a_11001) = 0 \neq 1$.

- $x = 1$

a_1	a_2	a_3	1	0	0	$\ldots$	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}
		1	0	0	0	$\ldots$	0	1		

If $a_3 = 0$ then $a_2 = 0$ and $\delta(a_2a_3a_4a_5a_6) = \delta(00100) = 1 \neq 0$. Thus $a_3 = 1$. Then $\delta(a_2a_3100) = \delta(a_21100)$ and $\delta(a_21100) = 0$ implies $a_2 = 0$. Thus $\delta(a_1a_2a_310) = \delta(a_10110) = 0 \neq 1$. Hence 10^n1 has no pre–image under τ. □

Proposition 3.24 *The transition function $\tau : X_e \to X_e$ is surjective.*

PROOF To prove the surjectivity of τ, it suffices to prove the non–existence of GOE words. To this aim, as it can be easily seen, it is enough to prove that each block of kind $10^{n_1}10^{n_2}\ldots10^{n_k}1$ (where $n_1, \ldots n_k$ are even integers), has a pre–image block. Indeed each word in $L(X_e)$ is contained in such a special word.

First we prove that every block of the type 10^n1 where n is even, has a pre–image $a_1 \ldots a_{n+6}$ in the language of X_e of length $n+6$

a_1	a_2	a_3	a_4	a_5	$\dots$	a_{n+2}	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}
		1	0	0	$\dots$	0	0	1		

,

in each of the three cases in which $a_{n+4} \mapsto 1$.
If $n = 0$

0	0	0	0	a_5	a_6
		1	1		

  ,

0	0	0	1	0	0
		1	1		

  ,

and

1	1	1	1	a_5	a_6
		1	1		

  .

If $n = 2$

a_1	a_1	1	0	0	0	a_5	a_6
		1	0	0	1		

  ,

a_1	a_1	1	0	0	1	0	0
		1	0	0	1		

  ,

and

0	0	0	1	1	1	a_5	a_6
		1	0	0	1		

  .

If $n \geq 4$, for a suitable $x \in \{0, 1\}$,

$1-x$	$1-x$	$1-x$	x	x	$\dots$	0	0	0	a_{n+5}	a_{n+6}
		1	0	0	$\dots$	0	0	1		

  .

Similarly

$1-x$	$1-x$	$1-x$	x	x	$\dots$	0	0	1	0	0
		1	0	0	$\dots$	0	0	1		

and, finally,

x	x	x	$1-x$	$1-x$	$\dots$	1	1	1	a_{n+5}	a_{n+6}
		1	0	0	$\dots$	0	0	1		

  .

Now, fix a word of kind $10^{n_1}10^{n_2}\dots10^{n_k}1$; we can construct a pre–image of this word starting from the first on the right block $10^{n_k}1$ (over the first on the right 1 we can write, arbitrarily, 000**, 111** or 00100). In this way we get a word $a_1 \dots a_5$ over the second on the left 1 and we can start from this word over 1 to construct a pre–image for the second on the right block $10^{n_{k-1}}1$, and so on:

							$\longleftarrow$						$\longleftarrow$				
$\dots$	b_1	b_2	b_3	b_4	b_5	$\dots$	a_1	a_2	a_3	a_4	a_5	$\dots$	*	*	*	*	*
$\dots$	0	0	1	0	0	$\dots$	0	0	1	0	0	$\dots$	0	0	1		

$\underbrace{b_4 \dots a_2}_{n_{k-1}}$ $\qquad$ $\underbrace{a_4 \dots *}_{n_k}$

In each of the possible choices we can find a block whose image under τ is our fixed word.

For what we have stated before, τ is surjective. □

Proposition 3.25 *The transition function $\tau : X_e \to X_e$ is not pre–injective.*

Proof Let us consider the configuration c_1:

$\dots$	0	0	0	0	0	**1**	**0**	**0**	1	0	0	0	0	0	$\dots$

and the configuration c_2:

$\dots$	0	0	0	0	0	**0**	**1**	**1**	1	0	0	0	0	0	$\dots$

.

These configurations are different only on a finite subset of $\mathbf{Z}$, but they have the same image under τ, that is the configuration

$\dots$	1	1	1	1	1	1	0	0	1	0	0	1	1	1	$\dots$

.

Thus τ is not pre–injective. □

4.

In this final section we show that the irreducibility condition in Corollary 2.19 cannot be dropped.

Counterexample 4.26 Myhill–property no longer holds for a subshift of finite type of $A^{\mathbf{Z}}$ but not irreducible.

Let X be the full shift over the alphabet $A = \{0, 1\}$; clearly X is irreducible and of finite type. Consider the set $\bar{X} \subseteq \{0, 1, 2\}^{\mathbf{Z}}$ given by the union $X \cup \{\bar{2}\}$, where $\bar{2}$ is the bi–infinite word constant in 2. The set $\bar{X}$ is a shift of finite type over the alphabet $\bar{A} = \{0, 1, 2\}$ with set of forbidden blocks:

$$\{02, 20, 12, 21\}.$$

Moreover $\bar{X}$ is not irreducible; indeed we have $1, 2 \in L(\bar{X})$ but for no word $w \in L(\bar{X})$ the word $1w2$ belongs to $L(\bar{X})$.

Consider the transition function $\tau : \bar{X} \to \bar{X}$ defined by:

$$\tau(c) = \begin{cases} c & \text{if } c \in X \\ \bar{0} & \text{if } c = \bar{2}. \end{cases}$$

Clearly τ is 1–local where the local rule is given by $\delta(a) = a$ if $a \neq 2$ and $\delta(2) = 0$. This function is not surjective because the word $\bar{2}$ has no pre–images, but it is pre–injective. Actually, if c_1 and c_2 are different configurations which only differ on a finite subset of $\mathbf{Z}$, then they must belong to X and so their images under τ are different. □

Counterexample 4.27 Moore–property no longer holds for a shift of finite type but not irreducible.

Let X be the shift over the alphabet $A = \{0, 1, 2\}$ with set of forbidden blocks $\{01, 02\}$. The shift X is not irreducible; indeed for no word $u \in L(X)$ the word $0u1$ belongs to $L(X)$.

Consider the transition function $\tau : X \to X$ defined by the local rule:

$$\delta(a_1 a_2 a_3) = \begin{cases} a_2 & \text{if } a_3 \neq 0 \\ 0 & \text{if } a_3 = 0. \end{cases}$$

The function τ is surjective because a generic word of X, for example,

$$\ldots 1211122121212212121\ 0\ 0000000000000 \ldots$$

has two pre–images:

$$\ldots 1211122121212212121\ \mathbf{1}\ 0000000000000 \ldots$$

and

$$\ldots 1211122121212212121\ \mathbf{2}\ 0000000000000 \ldots$$

This also shows that τ is not pre–injective. □

References

[CeMaSca] T. G. Ceccherini–Silberstein, A. Machì and F. Scarabotti, *Amenable Groups and Cellular Automata*, Ann. Inst. Fourier, Grenoble **49** 2 (1999), 673-685.

[CovP] E. Coven and M. Paul, *Endomorphisms of Irreducible Shifts of Finite Type*, Math. Systems Theory **8** (1974), 167-175.

[G] M. Gromov, *Endomorphisms of Symbolic Algebraic Varieties*, J. Eur. Math. Soc.**1** (1999), 109-197.

[H] G. A. Hedlund, *Endomorphisms and Automorphisms of the Shift Dynamical System*, Math. Systems Theory **3** (1969), 320-375.

[LinMar] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press 1995.

[Moo] E. F. Moore, *Machine Models of Self-Reproduction*, Proc. Symp. Appl. Math., AMS Providence R. I. **14** (1963), 17-34.

[MaMi] A. Machì and F. Mignosi, *Garden of Eden Configurations for Cellular Automata on Cayley Graphs on Groups*, SIAM J. Disc. Math. **6** (1993), 44-56.

[My] J. Myhill, *The converse of Moore's Garden of Eden Theorem*, Proc. Amer. Math. Soc. **14** (1963), 685-686.

[Wei] B. Weiss, *Subshifts of Finite Type and Sofic Systems*, Monats. Math. **77** (1973), 462-474.