Semi–strongly irreducible shifts

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Abstract

If $\mathbb{Z}$ is the group of integers, A a finite alphabet and $A^{\mathbb{Z}}$ the set of all functions $c : \mathbb{Z} \to A$, the equivalence between pre–injectivity and surjectivity of a local function holds for irreducible shifts of finite type of $A^{\mathbb{Z}}$ (see [Fio1]). In [Fio2] we give a definition of *strong irreducibility* that, together with the finite type condition, allows us to prove the above equivalence for strongly irreducible shifts of finite type in A^{Γ}, if Γ is an amenable group. In this paper, we define *semi–strong irreducibility* for a shift. This property allows us to prove the implication "pre–injective $\Rightarrow$ surjective" for a local function on a semi–strongly irreducible shift of finite type of A^{Γ}, if Γ has non–exponential growth. As a by–product, we prove that *the entropy of a proper subshift of a semi–strongly irreducible shift X is strictly smaller than the entropy of X.*

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1 Introduction

The notion of a *cellular automaton* has been introduced by Ulam [U] and von Neumann [vN]. In this classical setting, the "universe" is the lattice of integers $\mathbb{Z}^n$ of Euclidean space $\mathbb{R}^n$. The *set of states* is a finite set A (also called the *alphabet*) and a *configuration* is a function $c : \mathbb{Z}^n \to A$. Time t goes on in discrete steps and represents a *transition function* $\tau : A^{\mathbb{Z}^n} \to A^{\mathbb{Z}^n}$ (if c is the configuration at time t, then $\tau(c)$ is the configuration at time $t+1$), which is deterministic and *local*. *Locality* means that the new state at a point $\gamma \in \mathbb{Z}^n$ at time $t+1$ only depends on the states of certain fixed points in the neighborhood of γ at time t. For these structures, Moore [Moo] has given a sufficient condition for the existence of the so–called *Garden of Eden (GOE)* patterns, that is those configurations with finite support that cannot be reached at time t from another configuration starting at time $t-1$ and hence can only appear at time $t = 0$. Moore's condition (the existence of *mutually erasable patterns*, that

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is, of two different patterns with the same support and such that each pair of configurations extending them and coinciding out of the support, have the same image under the transition function), was also proved to be necessary by Myhill [My]. This equivalence between "local injectivity" and "local surjectivity" of the transition function is the classical well–known *GOE theorem.*

The purpose of this work is to consider this kind of problems in the more general framework of symbolic dynamics theory, with particular regard to GOE–like theorems restricted to the *subshifts* of the space A^Γ (where Γ is a finitely generated group and A is a finite alphabet).

More precisely, if Γ is a finitely generated group, we consider the space A^Γ (and we call it *full A–shift*) of functions defined on Γ with values in a finite alphabet A. This space is naturally endowed with a metric and hence with an induced topology, this topology being equivalent to the usual product topology, where the topology in A is the discrete one. We call *subshift*, *shift space* or simply *shift*, a subset X of A^Γ which is Γ–invariant and topologically closed. A function $\tau : X \to A^\Gamma$ is *local* if the value of $\tau(c)$, where $c \in X$ is a configuration, at a point $\gamma \in \Gamma$ only depends on the values of c at the points of a fixed finite neighborhood of γ.

In Section 2 we formally define all these notions, recalling from [Fio2] many basic results on the subshifts of A^Γ. We give the notion *shift of finite type.* As in the one–dimensional case (i.e. the case $\Gamma = \mathbb{Z}$), such a shift has an useful "overlapping" property that will be necessary in Section 4. Then we give the fundamental notion of *irreducibility* for a shift. This notion is well-known in the one–dimensional case. It means that given any pair of words u, v in the *language* of the shift (i.e. the set of all finite words appearing in some bi–infinite configuration), there is a word w such that the concatenation uwv still belongs to the language.

Finally, the notion of *entropy* as defined by Gromov in [G] is given. We prove that if the group has non–exponential growth, the entropy of a subshift of A^Γ can be calculated relative to a suitable sequence of disks in Γ with increasing radius.

The GOE–theorem has been proved by Machì and Mignosi [MaMi] more generally for local functions in which the space of configurations is the whole A–shift A^Γ and the group Γ has non–exponential growth. More recently it has been proved by Ceccherini–Silberstein, Machì and Scarabotti [CeMaSca] for the wider class of the amenable groups. Instead of the non–existence of mutually erasable patterns we deal here with the notion of *pre–injectivity* (a function $\tau : X \subseteq A^\Gamma \to A^\Gamma$ is *pre–injective* if whenever two configurations of X differ only on a finite non–empty subset of Γ, then their images under τ are different). This notion has been introduced by Gromov in [G]. In fact, it is easy to prove that these two properties are equivalent for local functions defined on the full shift, but in the case of proper subshifts the former may be meaningless. On the other hand, the non–existence of GOE patterns is equivalent to the non–existence of GOE configurations (see [MaMi, Theorem 5]), that is the surjectivity of the local function. Hence, in this language, the

GOE–theorem states that τ is surjective if and only if it is pre–injective. In the one–dimensional case we have that this equivalence holds for irreducible shifts of finite type of $A^{\mathbb{Z}}$(see [Fio1]). Moreover, in [Fio1] we have proved that Myhill's implication holds for irreducible sofic shifts of $A^{\mathbb{Z}}$ (recall that a one–dimensional shift is *sofic* if it is the set of labels of all bi–infinite walks on a labeled graph). On the other hand, we give there a counterexample of an irreducible sofic shift $X \subseteq A^{\mathbb{Z}}$ but not of finite type for which the inverse implication does not hold.

Concerning general shifts over amenable groups, from a result of Gromov [G] under much more general hypotheses it follows (see [Fio2, Section 3]) that the GOE theorem holds for local function on shifts of *bounded propagation* contained in A^{Γ}, if Γ is amenable. In [Fio2, Section 4] we give the notion of *strong irreducibility* and we generalize the result of Gromov, proving that it holds for strongly irreducible shifts of finite type of A^{Γ}.

The main difference between irreducibility and strong irreducibility is easily seen in the one–dimensional case. Here the former property states that given any word u, v in the language of the shift, there exists a third word w such that the word uwv is still in the language. Strong irreducibility says that we can arbitrarily fix the length of this word (but it must be greater than a fixed constant depending on the shift). The same properties for a generic shift refers to the way in which two different patterns in the language of the shift may appear simultaneously in a global configuration. For irreducibility we have that two patterns always appear simultaneously in some configuration if we translate their supports. Strong irreducibility states that if the supports of the patterns are far enough, then it is not necessary to translate them in order to find a configuration in which the patterns both appear.

These two irreducibility conditions are not equivalent, not even in the one–dimensional case. Hence our general result about strongly irreducible shifts of finite type is strictly weaker than the one–dimensional one[1]. In the attempt of using weaker hypotheses to prove our result, in Section 4 a new notion of irreducibility, the *semi–strong irreducibility*, is introduced. This property states that if the supports of the patterns are far enough (provided that one of this is a ball), then translating them "a little" we find a configuration in which the patterns both appear. The reason of this choice lies in the fact that using the Pumping Lemma, we prove that a sofic subshift of $A^{\mathbb{Z}}$ which is irreducible has a property quite similar to semi–strong irreducibility. Indeed such a shift has the property that between two words of the language (and not, in general, between a pattern and a word), we can write a third word "almost" of the length we want (provided that it is long enough): we must allow it to be "a little" longer or "a little" shorter. The length of this difference is bounded and only depends on the shift.

In Section 4 we prove that the implication "pre–injective $\Rightarrow$ surjective" holds for semi–strongly irreducible subshifts of finite type of A^{Γ}, if Γ has non–

[1] Indeed it can be easily seen by trivial counterexamples that none of the two implications "pre–injective $\Rightarrow$ surjective" and "surjective $\Rightarrow$ pre–injective" holds in general for local functions defined on irreducible shifts of finite type of $A^{\mathbb{Z}^2}$.

exponential growth. This result is a consequence of Corollary 4.7 which is an interesting result about shifts entropy: under the above conditions, *the entropy of a proper subshift of a semi–strongly irreducible shift $X \subseteq A^\Gamma$ is strictly weaker than the entropy of X.*

2 Shifts on Cayley graphs

In this section, we give the basic notion of *Cayley graph* of a finitely generated group. We define a *shift space* as a suitable subset of the set of all functions defined from this graph with values in a finite set A. Moreover, we recall the definition of *growth* of a finitely generated group Γ.

Let Γ be a finitely generated group and $\mathcal{X}$ a fixed finite set of generators for Γ. Then each $\gamma \in \Gamma$ can be written as

$$\gamma = x_{i_1}^{\delta_1} x_{i_2}^{\delta_2} \dots x_{i_n}^{\delta^n} \tag{1}$$

where the x_{i_j}'s are generators and $\delta_j \in \mathbb{Z}$. We define the *length of γ (with respect to $\mathcal{X}$)* as the natural number

$$\|\gamma\|_{\mathcal{X}} := \min\{|\delta_1| + |\delta_2| + \cdots + |\delta_n| \mid \gamma \text{ is written as in (1)}\}.$$

A decomposition for γ as in (1) such that $\|\gamma\|_{\mathcal{X}} = |\delta_1| + |\delta_2| + \cdots + |\delta_n|$ is called *minimal representation of* γ. The group Γ is naturally endowed with a metric space structure, with the distance given by

$$\mathrm{dist}_{\mathcal{X}}(\alpha, \beta) := \|\alpha^{-1}\beta\| \tag{2}$$

and we denote by $D_n^{\mathcal{X}}$ the ball of Γ centered at 1 and with radius n. Notice that $D_1^{\mathcal{X}}$ is the set $\mathcal{X} \cup \mathcal{X}^{-1}$. The asymptotic properties of the group being independent on the choice of the set of generators $\mathcal{X}$, from now on we fix a set $\mathcal{X}$ which is also symmetric (i.e. $\mathcal{X}^{-1} = \mathcal{X}$) and we omit the index $\mathcal{X}$ in all the above definitions.

For each $\gamma \in \Gamma$, this set D_n provides, by left translation, a *neighborhood of* γ, that is the set $\gamma D_n = D(\gamma, n)$, where $D(\gamma, n)$ is the disk of radius n centered at γ.

Given a subset $E \subseteq \Gamma$ and for each $n \in \mathbb{N}$ we denote by

$$E^{+n} := \bigcup_{\alpha \in E} D(\alpha, n), \;\; E^{-n} := \{\alpha \in E \mid D(\alpha, n) \subseteq E\} \text{ and } \partial_n E := E^{+n} \setminus E^{-n}$$

the *n–closure of E*, the *n–interior of E* and the *n–boundary of E*, respectively. By

$$\partial_n^+ E := E^{+n} \setminus E \text{ and } \partial_n^- E := E \setminus E^{-n}$$

the *n–external boundary of E* and the *n–internal boundary of E*, respectively. For all these sets, we will omit the index n if $n = 1$.

The *Cayley graph* of Γ, is the graph in which Γ is the set of vertices and there is an edge from γ to $\bar{\gamma}$ if there exists a generator $x \in \mathcal{X}$ such that $\bar{\gamma} = \gamma x$. Obviously this graph depends on the presentation of Γ. For example, we may look at the classical cellular decomposition of Euclidean space $\mathbb{R}^n$ as the Cayley graph of the group $\mathbb{Z}^n$ with the presentation $\langle a_1, \dots, a_n \mid a_i a_j = a_j a_i \rangle$.

If $\mathbf{G} = (\mathcal{V}, \mathcal{E})$ is a graph with set of vertices $\mathcal{V}$ and set of edges $\mathcal{E}$, the *graph distance* (or *geodetic distance*) between two vertices $v_1, v_2 \in \mathcal{V}$ is the minimal length of a path from v_1 to v_2. Hence the distance defined in (2) coincides with the graph distance on the Cayley graph of Γ. Indeed a minimal representation of an element $\gamma \in \Gamma$ represents a path of minimal length from 1 to γ.

We recall (see for example [Mil] or [CeMaSca]) that the function $g : \mathbb{N} \to \mathbb{N}$ defined by

$$g(n) := |D_n|$$

which counts the elements of the disk D_n, is called *growth function* of Γ (with respect to $\mathcal{X}$). One can prove that the limit

$$\lambda := \lim_{n\to\infty} g(n)^{\frac{1}{n}}$$

always exists. If $\lambda > 1$ then, for all sufficiently large n,

$$g(n) \geq \lambda^n,$$

and the group Γ has *exponential growth*. If $\lambda = 1$, we distinguish two cases. Either there exists a polynomial $p(n)$ such that for all sufficiently large n

$$g(n) \leq p(n),$$

in which case Γ has *polynomial growth*, or Γ has *intermediate growth* (i.e. $g(n)$ grows faster than any polynomial in n and slower then any exponential function x^n with $x > 1$). Moreover, it is possible to prove that the type of growth is a property of the group Γ (i.e. it does not depend on the choice of a set of generators). For this reason we deal with the *growth of a group*. This notion has been indipendently introduced by Milnor [Mil], Efremovič [E] and Švarc [Š].

Let A be a finite set (with at least two elements) and let Γ be (the Cayley graph of) a finitely generated group. A *configuration* is an element of A^Γ, that is a function $c : \Gamma \to A$ assigning to each point of the graph a letter of A. We denote by $c_{|\alpha}$ the value of $c \in A^\Gamma$ at $\alpha \in \Gamma$. On A^Γ, we have a natural metric and hence a topology which is equivalent to the usual product topology, where the topology in A is the discrete one. By Tychonoff's theorem, A^Γ is also compact.

If $c_1, c_2 \in A^\Gamma$ are two configurations, we define the distance

$$\mathrm{dist}(c_1, c_2) := \frac{1}{n+1}$$

where n is the least natural number such that $c_1 \neq c_2$ in D_n (i.e. the least natural number such that $c_{1|\alpha} \neq c_{2|\alpha}$ for some $\alpha \in D_n$). If such an n does not exist, that is if $c_1 = c_2$, we set their distance equal to zero. Notice that $c_1 = c_2$ on D_n if and only if $\mathrm{dist}(c_1, c_2) \leq \frac{1}{n+2}$.

The group Γ acts on A^Γ on the right as follows:

$$(c^\gamma)_{|\alpha} := c_{|\gamma\alpha}$$

for $c \in A^\Gamma$ and $\gamma, \alpha \in \Gamma$.

With this, a *shift* is a subset X of A^Γ which is topologically closed and Γ–invariant (i.e. $X^\Gamma = X$). As we shall see later, this topological definition is equivalent to the classical (well-known in the Euclidean case, that is the case $\Gamma = \mathbb{Z}^n$) combinatorial one.

For $X \subseteq A^\Gamma$ and $E \subseteq \Gamma$, we set

$$X_E := \{c_{|E} \mid c \in X\}.$$

A *pattern of* X is an element of X_E where E is a non–empty finite subset of Γ. The set E is the *support of* the pattern. A *block of* X is a pattern of X with support a disk. The *language of* X is the set $L(X)$ of all the blocks of X. If X is a subshift of $A^\mathbb{Z}$, a configuration is a bi–infinite word and a block of X is a finite word appearing in some configuration of X.

Hence a pattern with support E is a function $p : E \to A$. If $\gamma \in \Gamma$, we have that the function $\bar{p} : \gamma E \to A$ defined as $\bar{p}_{|\gamma\alpha} = p_{|\alpha}$ (for each $\alpha \in E$), is the pattern obtained copying p on the translated support γE. Moreover, if X is a shift, we have that $\bar{p} \in X_{\gamma E}$ if and only if $p \in X_E$. For this reason, in the sequel we do not make distinction between p and $\bar{p}$ (when the context makes it possible). In the one–dimensional case, for example, a word $a_1 \ldots a_n$ is simply a finite sequence of symbols for which we do not specify (if it is not necessary), if the support is the interval $[1, n]$ or the interval $[-n, -1]$.

Let X be a shift. A function $\tau : X \to A^\Gamma$ is *M–local* if there exists $\delta : X_{D_M} \to A$ such that for every $c \in X$ and $\gamma \in \Gamma$

$$(\tau(c))_{|\gamma} = \delta((c^\gamma)_{|D_M}) = \delta(c_{|\gamma\alpha_1}, c_{|\gamma\alpha_2}, \ldots, c_{|\gamma\alpha_m}),$$

where $D_M = \{\alpha_1, \ldots, \alpha_m\}$. A function is *local* if it is M–local for some M. Hence τ is local if the value of $\tau(c)$ at a point $\gamma \in \Gamma$ only depends on the values of c at the points of the neighborhood γD_M of γ and this value is given by the "local rule" δ.

In this definition, we have assumed that the alphabet of the shift X is the same as the alphabet of its image $\tau(X)$. In this assumption there is no loss of generality because if $\tau : X \subseteq A^\Gamma \to B^\Gamma$, one can always consider X as a shift over the alphabet $A \cup B$.

Let $\tau : X \to A^\Gamma$ be a local function. If c is a configuration of X and E is a subset of Γ, $\tau(c)_{|E}$ only depends on $c_{|E^{+M}}$. Thus we have a family of functions $(\tau_{E^{+M}} : X_{E^{+M}} \to \tau(X)_E)_{E \subseteq \Gamma}$. This notation will be useful in the sequel.

There is a characterization of local functions which in the one–dimensional case is known as the Curtis–Lyndon–Hedlund theorem (see [LinMar, Theorem 6.2.9]). A shift being compact, it is easy to see that it holds for a general local function. It states that *a function* $\tau : X \to A^\Gamma$ *is local if and only if it is continuous and commutes with the* Γ*–action* (i.e. for each $c \in X$ and each $\gamma \in \Gamma$, one has $\tau(c^\gamma) = \tau(c)^\gamma$). From this result, it is clear that *the composition of two local functions is still local*, as it can also be easily seen directly from the definition.

Now, fix $\gamma \in \Gamma$ and consider the function $X \to A^\Gamma$ that associates with each $c \in X$ its translated configuration c^γ. In general, this function does not commute with the Γ–action (and therefore it is not local). Indeed, if Γ is not abelian and $\gamma\alpha \neq \alpha\gamma$, then $(c^\gamma)^\alpha \neq (c^\alpha)^\gamma$. However, as proved in [Fio2, Section 2], this function is continuous.

Observe that if X is a subshift of A^Γ and $\tau : X \to A^\Gamma$ is a local function, then, by the (generalized) Curtis–Lyndon–Hedlund theorem, the image $Y := \tau(X)$ is still a subshift of A^Γ. Indeed Y is closed (or, equivalently, compact) and Γ–invariant:

$$Y^\Gamma = (\tau(X))^\Gamma = \tau(X^\Gamma) = \tau(X) = Y.$$

Moreover, if τ is injective then $\tau : X \to Y$ is a homeomorphism. If $c \in Y$ then $c = \tau(\bar{c})$ for a unique $\bar{c} \in X$ and we have

$$\tau^{-1}(c^\gamma) = \tau^{-1}(\tau(\bar{c})^\gamma) = \tau^{-1}(\tau(\bar{c}^\gamma)) = \bar{c}^\gamma = (\tau^{-1}(c))^\gamma$$

that is, τ^{-1} commutes with the Γ–action. Hence τ^{-1} is local and the well–known theorem (see [R]), stating that the inverse of an invertible local function defined on $\mathbb{Z}^n$ is still local, holds also in this more general setting. In the one–dimensional case, Lind and Marcus [LinMar, Theorem 1.5.14] give a direct proof of this fact.

This result leads us to say that two subshifts $X, Y \subseteq A^\Gamma$ are *conjugate* if there exists a local bijective function between them (namely a *conjugacy*). The *invariants* are the properties of a shift invariant under conjugacy.

As mentioned above, it is easy to prove that the topological definition of a shift space is equivalent to the following combinatorial one involving the avoidance of certain blocks (therefore called *forbidden blocks*). This fact is well–known in the Euclidean case (see [LinMar, Theorem 6.1.21]). Let $\mathcal{F}$ be a set of blocks, we denote by $\mathsf{X}_\mathcal{F}$ the set of all configurations of A^Γ avoiding each block of $\mathcal{F}$. With these notations we have that *a subset* $X \subseteq A^\Gamma$ *is a shift if and only if there exists a set of blocks* $\mathcal{F}$ *such that* $X = \mathsf{X}_\mathcal{F}$. In this case, $\mathcal{F}$ is a *set of forbidden blocks* of X.

We now give the fundamental notion of a shift of finite type. The basic definition is in terms of forbidden blocks: a shift is *of finite type* if it admits a finite set of forbidden blocks. In a sense we may say that a shift is of finite type if we can decide whether or not a configuration belongs to the shift only by checking its blocks of a fixed size (where this size only depends on the shift).

Since a finite set $\mathcal{F}$ of forbidden blocks of X has a maximal support, we can always assume that in a shift of finite type each block of $\mathcal{F}$ has the disk D_M as support (indeed a block that contains a forbidden block is also forbidden). In this case the shift X is called *M–step* and the number M is called the *memory of X*. If X is a subshift of $A^{\mathbb{Z}}$, we define the memory of X as the number M, where $M+1$ is the maximal length of a forbidden word.

For the shifts of finite type in $A^{\mathbb{Z}}$ we have (see [LinMar, Theorem 2.1.8]), the following useful property: *a shift $X \subseteq A^{\mathbb{Z}}$ is an M–step shift of finite type if and only if whenever $uv, vw \in L(X)$ and $|v| \geq M$, then $uvw \in L(X)$.*

It is easy to prove that this "overlapping" property holds more generally for M–step subshifts of finite type of A^{Γ}: *if E is a subset of Γ and $c_1, c_2 \in X$ are two configurations that agree on $\partial^{+}_{2M}E$, then the configuration $c \in A^{\Gamma}$ that agrees with c_1 on E and with c_2 on $\complement E$ is still in X.*

It is also easy to see that this property has the following useful consequence.

Proposition 2.1 *Let X be an M–step shift of finite type and let E be a finite subset of Γ. If $p_1, p_2 \in X_{E^{+2M}}$ are two patterns that agree on $\partial^{+}_{2M}E$, then there exist two extensions $c_1, c_2 \in X$ of p_1 and p_2, respectively, that agree on $\complement E$.*

The natural generalization for a generic shift of the notion of irreducibility which is well–known in the one–dimensional case is the following: a shift $X \subseteq A^{\Gamma}$ is *irreducible* if for each pair of blocks p_1 and p_2 of X, there exists a configuration $c \in X$ such that $c_{|E} = p_1$ and $c_{|F} = p_2$, where $E, F \subseteq \Gamma$ are disjoint translations of the supports of p_1 and p_2 respectively.

In other words, a shift is irreducible if whenever we have $p_1, p_2 \in L(X)$, there exists a configuration $c \in X$ in which these two blocks appear simultaneously (on disjoint supports). This definition could seem weaker than the one–dimensional one, in fact in this latter we establish that each word $u \in L(X)$ must always appear in a configuration on the left of each other word of the language. But, as proved in [Fio2, Section 2], the two definitions agree.

The *entropy* of a shift is the first invariant we deal with in the present work. It is a concept introduced by Shannon [Sha] in information theory that involves probabilistic concepts. Later Adler, Konheim and McAndrew [AdlKoM] introduced the *topological entropy* for *dynamical systems*. The entropy we deal with is a special case of topological entropy and is independent on probabilities.

In this section we give the general definition of entropy for a generic shift. We will see that this definition involves the existence of a suitable sequence of sets that, in the case of non–exponential growth of the group can be taken as balls centered at 1 and with increasing radius.

Definition 2.2 Let $(E_n)_{n\geq 1}$ be a sequence of subsets of Γ such that $\bigcup_{n\in\mathbb{N}} E_n = \Gamma$ and such that the *Følner condition* holds:

$$\lim_{n\to\infty} \frac{|\partial E_n|}{|E_n|} = 0. \tag{3}$$

If $X \subseteq A^\Gamma$ is a shift, the *entropy of X respect to* (E_n) is given by

$$\mathrm{ent}(X) := \limsup_{n\to\infty} \frac{\log |X_{E_n}|}{|E_n|}.$$

Condition (3) is necessary in order to prove that the *entropy is invariant under conjugacy* (see [Fio2, Theorem 2.12]). Other aspects of its importance will be seen in Section 4.

If X is a subshift of $A^{\mathbb{Z}}$, we choose as E_n the interval $[1,n]$ (or equivalently, in order to have $\bigcup_{n\in\mathbb{N}} E_n = \Gamma$, the interval $[-n,n]$), so that X_{E_n} is the set of words of X of length n (in [Fio1] we prove that in this one–dimensional case, the above maximum limit is a limit and coincides with $\inf_{m\geq 1} \frac{\log|X_{E_m}|}{m}$).

In general, if Γ is a group of non–exponential growth, we choose as E_n a suitable disk centered at $1 \in \Gamma$. Indeed, setting $a_h = |D_h|$, we have (by definition of non–exponential growth) $\lim_{h\to\infty} \sqrt[h]{a_h} = 1$, and hence $\liminf_{h\to\infty} \frac{a_{h+1}}{a_h} = 1$. It follows that for a suitable sequence $(a_{h_k})_k$ we have $\lim_{k\to\infty} \frac{a_{h_k+1}}{a_{h_k}} = 1$. Hence $\liminf_{k\to\infty} \frac{a_{h_k+1}}{a_{h_k-1}} = 1$ and for a suitable sequence $(a_{h_{k_n}})_n$ we have $\lim_{n\to\infty} \frac{a_{h_{k_n}+1}}{a_{h_{k_n}-1}} = 1$, i.e. we find a sequence of disks $E_n := D_{h_{k_n}}$ such that $\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_{n-1}|} = 1$. Being $D_h^+ = D_{h+1}$ and $D_{h-1} \subseteq D_h^-$ we also have that $\lim_{n\to\infty} \frac{|E_n^+|}{|E_n^-|} = 1$. Hence $\frac{|\partial E_n|}{|E_n|} = \frac{|E_n^+ \setminus E_n^-|}{|E_n|} \leq \frac{|E_n^+ \setminus E_n^-|}{|E_n^-|} = \frac{|E_n^+|}{|E_n^-|} - 1 \longrightarrow_{n\to\infty} 0$.

3 The Garden of Eden theorem and the Moore–Myhill property

Let τ be a local function. Recall from [Moo] and [My] that two different patterns with the same support are called τ*–mutually erasable* if each pair of configurations extending them and coinciding out of the support, have the same image under τ. This notion is used in the original works of Moore and Myhill. Indeed they prove that a local function τ on the full shift $A^{\mathbb{Z}^2}$ admits two mutually erasable patterns if and only if it admits a *GOE pattern*, that is a pattern without pre–image. In this section we restate the GOE theorem using the notion of *pre–injective* function. This notion has been introduced by Gromov in [G] and it is equivalent to that of non–existence of mutually erasable patterns

The *Garden of Eden (GOE)* theorem is the union of the following two theorems.

Theorem 3.1 (E. F. Moore - 1962) *If $\tau : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ is a local function and there exist two τ–mutually erasable patterns, then there exists a GOE pattern.*

Theorem 3.2 (J. Myhill - 1963) *If $\tau : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ is a local function and there exists a GOE pattern, then there exist two τ–mutually erasable patterns.*

Now we recall the definition of *amenability* for a group Γ. Using a characterization of it due to Følner (see [F], [Gr] and [N]), Ceccherini–Silberstein, Machì and Scarabotti have proved that the GOE theorem holds for local functions defined on the full shift A^Γ (see [CeMaSca]).

Definition 3.3 A group Γ is called *amenable* if it admits a Γ–invariant probability measure, that is a function $\mu : \mathcal{P}(\Gamma) \to [0,1]$ such that for $A, B \subseteq \Gamma$ and for every $\gamma \in \Gamma$

- $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$ (*finite additivity*)
- $\mu(\gamma A) = \mu(A)$ (Γ*–invariance*)
- $\mu(\Gamma) = 1$ (*normalization*).

Theorem 3.4 (Følner) *A group Γ is amenable if and only if for each finite subset $F \subseteq \Gamma$ and each $\varepsilon > 0$ there exists a finite subset $K \subseteq \Gamma$ such that*

$$\frac{|KF \backslash K|}{|K|} < \varepsilon.$$

This characterization is equivalent to the following one.

For each pair of finite subsets $F, H \subseteq \Gamma$ with $1 \in H$ and each $\varepsilon > 0$ there exists a finite subset $K \supseteq H$ such that

$$\frac{|KF \backslash K|}{|K|} < \varepsilon.$$

Indeed, suppose that there exists $\bar{K}$ such that

$$\frac{|\bar{K}HF \backslash \bar{K}|}{|\bar{K}|} < \varepsilon.$$

We have that $\bar{K} \subseteq \bar{K}H$ and hence

$$\frac{|\bar{K}HF \backslash \bar{K}H|}{|\bar{K}H|} \leq \frac{|\bar{K}HF \backslash \bar{K}|}{|\bar{K}|} < \varepsilon.$$

It suffices to set $K := \bar{K}H$.

Using this characterization we can prove the existence of a (nested) sequence $(E_n)_{n\geq 1}$ satisfying condition (3) in Definition 2.2.

Theorem 3.5 *Let Γ be an amenable group. Then there exists a sequence of finite sets $(E_n)_{n\geq 1}$ such that:*

- $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n \subseteq \ldots$
- $\bigcup_{n\geq 1} E_n = \Gamma$,

- $\lim_{n\to\infty} \frac{|\partial_M E_n|}{|E_n|} = 0$.

PROOF First, notice that in Følner condition there is no loss of generality if we suppose $1 \in K$. Now we construct, by induction, a nested sequence $1 \in K_1 \subseteq \ldots \subseteq K_n \subseteq \ldots$ such that, for each $n \geq 1$

$$\frac{|K_n(D_n^{+M})\backslash K_n|}{|K_n|} < \frac{1}{n}.$$

Let K_1 be a finite subset $1 \in K_1 \subseteq \Gamma$ such that

$$\frac{|K_1(D_1^{+M})\backslash K_1|}{|K_1|} < 1$$

whose existence is guaranteed by Theorem 3.4. Suppose to have found K_n, there exists $K_{n+1} \supseteq K_n$ such that

$$\frac{|K_{n+1}(D_{n+1}^{+M})\backslash K_{n+1}|}{|K_{n+1}|} < \frac{1}{n+1}.$$

Observe that

- $K_n(D_n^{+M}) = (K_n D_n)^{+M}$
- $K_n \subseteq K_n(D_n^{-M}) \subseteq (K_n D_n)^{-M}$
- $K_n \subseteq K_n D_n$

hence

$$\frac{|(K_nD_n)^{+M}\backslash (K_nD_n)^{-M}|}{|K_nD_n|} \leq \frac{|K_n(D_n^{+M})\backslash K_n|}{|K_n|} < \frac{1}{n}.$$

Setting $E_n := K_n D_n$ we have the stated properties because $D_n \subseteq K_n$. □

A sequence as in Theorem 3.5 is called *amenable* (or *Følner sequence*). From now on we fix an amenable sequence $(E_n)_{n\geq 1}$ (the one found at the end of Section 2, if Γ has non–exponential growth) and the entropy of a shift will be defined with respect to $(E_n)_{n\geq 1}$. Notice that condition (3) implies $\lim_{n\to\infty} \frac{|\partial_M^+ E_n|}{|E_n|} = 0$ and $\lim_{n\to\infty} \frac{|\partial_M^- E_n|}{|E_n|} = 0$.

Using the existence of an amenable sequence in the amenable group Γ, Ceccherini–Silberstein, Machì and Scarabotti [CeMaSca] have generalized the GOE theorem to local functions defined on the whole shift A^Γ.

In order to consider GOE–like theorems not in the whole of A^Γ but in a subshift $X \subseteq A^\Gamma$, notice first that two patterns of X are not necessarily extendible by the same configuration of X. Therefore it could happen that two patterns with support F for which there does not exist a common extension $c_{|\complement F}$, are τ–mutually erasable although the function τ is bijective. The notion that seems to be a good generalization of the non–existence of mutually erasable patterns, is that of *pre–injectivity*. It can be easily seen that if $X = A^\Gamma$ then the non–existence of τ–mutually erasable patterns is equivalent to the pre–injectivity of τ.

Definition 3.6 A function $\tau : X \subseteq A^\Gamma \to A^\Gamma$ is called *pre–injective* if whenever $c_1, c_2 \in X$ and $c_1 \neq c_2$ only on a finite non–empty subset of Γ, then $\tau(c_1) \neq \tau(c_2)$.

One can prove (see [MaMi, Theorem 5]) that a local function on A^Γ is surjective if and only if there are no GOE patterns. A shift being compact, it is easy to prove that this property holds also for the local functions between shifts. Hence we can state the GOE theorem as follows: *if Γ is an amenable group and $\tau : A^\Gamma \to A^\Gamma$ is a local function, then τ is pre–injective if and only if it is surjective.*

In the following definition we introduce an interesting property concerning shifts.

Definition 3.7 A shift $X \subseteq A^\Gamma$ has the *Moore–Myhill property* (briefly *MM–property*), if every local function $\tau : X \to A^\Gamma$ is pre–injective if and only if it is surjective. The *Moore–property* is surjective $\Rightarrow$ pre–injective and the *Myhill–property* is pre–injective $\Rightarrow$ surjective.

In the sequel we will distinguish between these properties and the GOE–theorems for a local function. Indeed the former are properties of a single shift. For example it is easy to see that the composition of two local pre–injective functions is still a (local) pre–injective function and hence one can prove that *the MM–property is invariant under conjugacy.* On the other hand, we will speak of GOE–theorem whenever we have a GOE–like theorem for a local function between two possibly different shifts.

4 Semi–strongly irreducible shifts

As proved in [Fio1], the MM–property holds for irreducible subshifts of finite type of $A^{\mathbb{Z}}$. If Γ is amenable, it holds for *strongly irreducible* subshifts of finite type of A^Γ (see [Fio2]). In this section we define another form of irreducibility: the *semi–strong irreducibility.* If Γ has non–exponential growth it allows us to prove the implication "pre–injective $\Rightarrow$ surjective" for local functions defined on a subshift of finite type.

Definition 4.1 A shift X is called *(M,k)–irreducible* (where M, k are natural numbers such that $M \geq k$) if for each pair of finite sets $E, \alpha D \subseteq \Gamma$ (the second one is a ball centered at α) such that $\mathrm{dist}(E, \alpha D) > M$ and for each pair of patterns $p_1 \in X_E$ and $p_2 \in X_{\alpha D}$, there exists a configuration $c \in X$ such that $c = p_1$ in E and $c = p_2$ in $\alpha\varepsilon D$ (that is the disk centered at $\alpha\varepsilon$), where $\varepsilon \in \Gamma$ is such that $\|\varepsilon\| \leq k$. The shift X is called *semi–strongly irreducible* if it is (M,k)–irreducible for some $M, k \in \mathbb{N}$.

Recall from [Fio2] that a shift X is *M–irreducible* if for every pair of finite sets $E, F \subseteq \Gamma$ such that $\mathrm{dist}(E, F) > M$ and every pair of patterns $p_1 \in X_E$ and $p_2 \in X_F$, there exists a configuration $c \in X$ such that $c = p_1$ in E and $c = p_2$ in F. X is *strongly irreducible* if it is M–irreducible for some $M \in \mathbb{N}$. Hence the difference between semi–strong irreducibility and strong irreducibility lies in the fact that in the former the support of the second pattern must be a ball and the configuration c merging the two patterns moves this support "slightly". Notice that this motion is a translation and hence it makes sense to say that the configuration c restricted to $\alpha\varepsilon D$ coincides with $p_2 \in X_{\alpha D}$. Moreover, under the previous hypotheses, the translated disk $\alpha\varepsilon D$ is still contained in $(\alpha D)^{+M}$. Indeed if $D = D_r$ and $\gamma \in \alpha\varepsilon D_r$, then $\mathrm{dist}(\gamma, \alpha) \leq \mathrm{dist}(\gamma, \alpha\varepsilon) + \mathrm{dist}(\alpha\varepsilon, \alpha) \leq r + \|\varepsilon^{-1}\| \leq r + k$. In particular $E \cap \alpha\varepsilon D = \emptyset$.

In Definition 4.1 is in fact essential that, given a finite set $F \subseteq \Gamma$, there exists $\alpha \in \Gamma\backslash\{1\}$ such that the translated set αF is still contained in F^{+M}. If the group is not abelian, the set αF may be quite far from F. On the other hand the set $F\alpha$ is α–close to F, but it is not, in general, obtained from F by translation[2]. This is why we require that the second set in Definition 4.1 be a ball centered at α. Then we consider the new center $\alpha\varepsilon$ (which is ε–near α). The ball $\alpha\varepsilon D$ having the same radius as αD, is obtained by translating αD. As we have seen, if Γ has non–exponential growth we can fix a suitable sequence $(E_n)_n$ of balls centered at 1 with property (3) of Definition 2.2. Hence if M is large enough we have $\varepsilon E_n \subseteq E_n^{+M}$.

In the one–dimensional case irreducibility is a property quite similar to that of semi–strong irreducibility, as clarified in Corollary 4.3. To this aim, we restate as follows the well–known Pumping Lemma.

Lemma 4.2 (Pumping Lemma) *Let L be an infinite regular language. There exists $M \geq 1$ such that if $uwv \in L$ and $|w| \geq M$, there exists a decomposition*

$$w = xyz$$

with $0 < |y| \leq M$ so that for each $n \in \mathbb{N}$ we have $uxy^nzv \in L$.

Moreover, one can take as M the number of vertices of a graph accepting L.

Corollary 4.3 *If $X \subseteq A^{\mathbb{Z}}$ is a sofic shift, then X is irreducible if and only if there exist $M, k \in \mathbb{N}$ such that for each $n \geq M$ and each pair of words $u, v \in L(X)$, there exists a word $w \in L(X)$ with $n - k \leq |w| \leq n + k$ and such that $uwv \in L(X)$.*

PROOF If X is irreducible, let $M = k$, where M is given by the Pumping Lemma. If $n \geq M$ and $u, v \in L(X)$, there exists $w \in L(X)$ such that $uwv \in L(X)$. We distinguish two cases.

If $|w| > n + M$, then $w = x_1y_1z_1$ with $0 < |y_1| \leq M$ and if $w_1 := x_1z_1$, then $uw_1v \in L(X)$ and $|w| - M \leq |w_1| \leq |w| - 1$. If $|w_1| \leq n + M$, we

[2] Consider, for example, the free group $\mathbf{F}_2$ generated by a and b. If $F = \{a^n, b^n\}$ with $n > M$, it is easily seen that for no $\alpha \neq 1$ we have $\alpha F = \{\alpha a^n, \alpha b^n\} \subseteq F^{+M}$.

have $|w_1| \geq |w| - M > n > n - M$. If $|w_1| > n + M$, we repeat the above construction to obtain, for some $i \geq 1$, an element w_i such that $uw_iv \in L(X)$ and $n - M < |w_i| \leq n + M$.

The second case, when $|w| < n - M$, is analogous. □

Hence, by Corollary 4.3, one has that the notion of semi–strong irreducibility given in Definition 4.1 is satisfied by irreducible sofic shifts of $\mathbb{Z}$ whenever the support E mentioned in this definition is a disk as well.

We now prove our main results. The following proposition holds in the case of strongly irreducible shifts of finite type in A^Γ, where the group Γ is amenable (see [Fio2]).

Proposition 4.4 *Let Γ be a group of non–exponential growth. Let X be a semi–strongly irreducible shift of finite type and let $\tau : X \to A^\Gamma$ be a local and pre–injective function. Then* $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$.

PROOF Suppose that the memory of X is M, that X is (M,k)–irreducible and that τ is M–local. Set $Y := \tau(X)$ and fix an amenable sequence of disks $(E_n)_n$. We have

$$|Y_{E_n^{+2M}}| \leq |Y_{E_n}||A|^{|\partial_{2M}^+ E_n|}.$$

Thus

$$\frac{\log |Y_{E_n^{+2M}}|}{|E_n|} \leq \frac{\log |Y_{E_n}|}{|E_n|} + \frac{|\partial_{2M}^+ E_n| \log |A|}{|E_n|}.$$

Being $\lim_{n\to\infty} \frac{|\partial_{2M}^+ E_n|}{|E_n|} = 0$, we have $\limsup_{n\to\infty} \frac{\log |Y_{E_n^{+2M}}|}{|E_n|} \leq \mathrm{ent}(Y)$. Let $l = l(k)$ be the number of ε's such that $\|\varepsilon\| \leq k$ and suppose that $\mathrm{ent}(Y) < \mathrm{ent}(X)$. Then

$$\limsup_{n\to\infty} \frac{\log |Y_{E_n^{+2M}}|}{|E_n|} < \limsup_{n\to\infty} \frac{\log |X_{E_n}|}{|E_n|} = \limsup_{n\to\infty} \frac{\log(\frac{|X_{E_n}|}{l})}{|E_n|}.$$

Then there exists $n \in \mathbb{N}$ such that $\frac{\log |Y_{E_n^{+2M}}|}{|E_n|} < \frac{\log(\frac{|X_{E_n}|}{l})}{|E_n|}$ that is $|Y_{E_n^{+2M}}| < \frac{|X_{E_n}|}{l}$. Fix $v \in X_{\partial_{2M}^+ E_n^{+M}}$. Since $\mathrm{dist}(\partial_{2M}^+ E_n^{+M}, E_n) = M + 1 > M$ for each $u \in X_{E_n}$ there exists $\varepsilon \in D_k$ and a pattern $p \in X_{E_n^{+3M}}$ that agrees with u on εE_n and with v on $\partial_{2M}^+ E_n^{+M}$. Thus

$$|\{p \in X_{E_n^{+3M}} \mid p_{|\partial_{2M}^+ E_n^{+M}} = v\}| \geq \frac{|X_{E_n}|}{l} > |Y_{E_n^{+2M}}|.$$

Since $\tau_{E_n^{+3M}} : X_{E_n^{+3M}} \to Y_{E_n^{+2M}}$ is surjective, there exist two patterns $p_1, p_2 \in X_{E_n^{+3M}}$ such that $p_1 \neq p_2$ but $p_1 = v = p_2$ on $\partial_{2M}^+ E_n^{+M}$ and $\tau_{E_n^{+3M}}(p_1) = \tau_{E_n^{+3M}}(p_2)$. By Proposition 2.1, there exist two configurations $c_1, c_2 \in X$ which extend p_1 and p_2 and which agree outside E_n^{+M}. It is easy to prove that $\tau(c_1) = \tau(c_2)$, and hence that τ is not pre–injective. □

Recall (see [Fio2, Lemma 4.3]), that if Γ is a finitely generated group, there exists a sequence of disks $(F_j)_{j\in\mathbb{N}}$ obtained by translating a disk D and at distance $> M$ such that $\bigcup_{j\in\mathbb{N}} F_j^{+R} = \Gamma$ for a suitable $R > 0$. We call the above sequence a (D, M, R)–*net*. The following lemma is an essential result used in the proof of the equivalence between pre–injectivity and surjectivity in the case of strongly irreducible shifts of finite type and of amenable groups (see [Fio2]).

Lemma 4.5 *Let Γ be an amenable group and let $(E_n)_n$ be a fixed amenable sequence of Γ. Let $(F_j)_{j\in\mathbb{N}}$ be a $(D_r, 2M, R)$–net, let X be an M–irreducible shift and let Y be a subset of X such that $Y_{F_j} \subset X_{F_j}$ for every $j \in \mathbb{N}$. Then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

For semi–strongly irreducible shifts, Lemma 4.5 does not necessarily hold. Consider, for example, the shift $X = \{\ldots 010101\ldots, \ldots 101010\ldots\} \subseteq A^{\mathbb{Z}}$. It is of finite type and $(2,2)$–irreducible, but $\mathrm{ent}(X) = 0$.

The following lemma is similar to Lemma 4.5 but as one can see the hypotheses are quite stronger.

Lemma 4.6 *Let Γ be a group of non–exponential growth and let $(E_n)_n$ be a fixed amenable sequence of disks . Let $(F_j)_{j\in\mathbb{N}} = (D(\beta_j, r))_{j\in\mathbb{N}}$ be a $(D_r, 2M, R)$–net, let X be an (M,k)–irreducible shift and let Y be a subset of X such that for each $j \in \mathbb{N}$, there exists a pattern $p_j \in X_{F_j}$ for which $p_j \notin Y_{D(\beta_j\varepsilon, r)}$ whenever $\varepsilon \in D_k$. Then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

Proof Let $N(n)$ be the number of F_j's such that $F_j^{+M} \subseteq E_n$ and denote by $F_{j_1}, \ldots, F_{j_N}$ these disks. Set $\xi := |X_{D^{+M}}|$ and denote by $P_{j_m} \subseteq X_{E_n}$ the set of the blocks p of X_{E_n} such that $p_{|D(\beta_{j_m}\varepsilon, r)} = p_{j_m}$ for some $\varepsilon \in D_k$. We prove that

$$|X_{E_n} \setminus \bigcup_{i=1}^{N} P_{j_i}| \le (1 - \xi^{-1})^N |X_{E_n}| \tag{4}$$

by induction on $m \in \{1, \ldots, N\}$. We have

$$|X_{E_n}| \le |X_{F_{j_1}^{+M}}||X_{E_n \setminus F_{j_1}^{+M}}|$$

and hence

$$|X_{E_n}| \le \xi |X_{E_n \setminus F_{j_1}^{+M}}|.$$

Since X is (M,k)–irreducible and since $\mathrm{dist}(F_{j_1}, E_n \setminus F_{j_1}^{+M}) > M$, given a pattern $p \in X_{E_n \setminus F_{j_1}^{+M}}$ there exists a pattern $\bar{p}$ defined on all E_n that coincides with p on $E_n \setminus F_{j_1}^{+M}$ and with p_{j_1} on some $D(\beta_{j_1}\varepsilon, r)$. Then

$$|X_{E_n \setminus F_{j_1}^{+M}}| \le |P_{j_1}|.$$

Hence

$$\frac{1}{\xi}|X_{E_n}| \le |P_{j_1}|$$

so that

$$|X_{E_n} \backslash P_{j_1}| \leq |X_{E_n}| - \frac{1}{\xi}|X_{E_n}| = (1-\frac{1}{\xi})|X_{E_n}|.$$

Suppose that (4) holds for $m-1$. We have

$$|X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}| \leq \xi |\{p \in X_{E_n \backslash F_{j_m}^{+M}} \mid p_{|D(\beta_{j_i}\varepsilon, r)} \neq p_{j_i}$$

$$\text{for each } i = 1, \dots, m-1 \text{ and each } \varepsilon|\}.$$

Moreover, since X is (M,k)–irreducible,

$$|\{p \in X_{E_n \backslash F_{j_m}^{+M}} \mid p_{|D(\beta_{j_i}\varepsilon, r)} \neq p_{j_i} \text{ for each } i = 1, \dots, m-1 \text{ and each } \varepsilon\}| \leq$$

$$\leq |\{p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i} \mid p_{|D(\beta_{j_m}\varepsilon, r)} = p_{j_m} \text{ for some } \varepsilon\}|.$$

Hence

$$\frac{1}{\xi}|X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}| \leq |\{p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i} \mid p_{|D(\beta_{j_m}\varepsilon, r)} = p_{j_m} \text{ for some } \varepsilon\}|$$

and then

$$|X_{E_n} \backslash \bigcup_{i=1}^{m} P_{j_i}| = |\left(X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}\right) \backslash P_{j_m}| \leq$$

$$\leq |\left(X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}\right) \backslash \{p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i} \mid p_{|D(\beta_{j_m}\varepsilon, r)} = p_{j_m} \text{ for some } \varepsilon\}| \leq$$

$$\leq |X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}| - \frac{1}{\xi}|X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}| \leq (1-\frac{1}{\xi})(1-\xi^{-1})^{m-1}|X_{E_n}|.$$

Hence (4) holds, and since $|Y_{E_n}| \leq |X_{E_n} \backslash \bigcup_{m=1}^{N} P_{j_m}|$, we have

$$\frac{\log|Y_{E_n}|}{|E_n|} \leq \frac{N(n)\log(1-\xi^{-1})}{|E_n|} + \frac{\log|X_{E_n}|}{|E_n|}. \tag{5}$$

Observe that

$$E_n \subseteq \bigcup_{i=1}^{N} F_{j_i}^{+R} \cup (E_n \backslash E_n^{-(R+2r+M)})$$

and hence we have

$$|E_n| \leq N(n)|D^{+R}| + |E_n \backslash E_n^{-(R+2r+M)}|$$

so that

$$1 \leq \frac{N(n)}{|E_n|}|D^{+R}| + \frac{|\partial_{R+2r+M}^{-} E_n|}{|E_n|}.$$

Taking the minimum limit and being $\lim_{n\to\infty}\frac{|\partial^-_{R+2r+M}E_n|}{|E_n|}=0$,

$$\zeta := \liminf_{n\to\infty}\frac{N(n)}{|E_n|} > 0.$$

Taking the maximum limit in (5), it follows

$$\mathrm{ent}(Y) \le \zeta\log(1-\xi^{-1}) + \mathrm{ent}(X) < \mathrm{ent}(X). \quad \square$$

The following statement is an easy consequence of Lemma 4.6 and generalizes the result in [LinMar, Corollary 4.4.9].

Corollary 4.7 *Let Γ be a group of non–exponential growth and let X be a semi–strongly irreducible subshift of A^Γ. If Y is a proper subshift of X then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

PROOF Let X be (M,k)–irreducible. If $Y \subset X$, there exists a configuration $c \in X$ which does not belong to Y and then there exists a disk D_r such that $c_{|D_r} \in X_{D_r}\backslash Y_{D_r}$. Let $(F_j)_{j\in\mathbb{N}} = (D(\beta_j,r))_{j\in\mathbb{N}}$ be a $(D_r,2M,R)$–net. Then $c_{|D_r} \notin Y_{D(\beta_j\varepsilon,r)}$ whenever $\varepsilon \in D_k$. By Lemma 4.6, $\mathrm{ent}(Y) < \mathrm{ent}(X)$. $\square$

Proposition 4.8 *Let Γ be a group of non–exponential growth. Let X be a shift, let Y be a semi–strongly irreducible shift and let $\tau : X \to Y$ be a local function such that* $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y)$*. Then τ is surjective.*

PROOF Let X and Y be as in the above hypotheses and let $\tau : X \to Y$ be a local function. Clearly $\tau(X)$ is a subshift of Y. By Corollary 4.7, we have that if $\tau(X) \subset Y$, then $\mathrm{ent}(\tau(X)) < \mathrm{ent}(Y)$. $\square$

Theorem 4.9 *Let Γ be a group of non–exponential growth, let X be a semi–strongly irreducible shift of finite type and let Y be a semi–strongly irreducible shift. If $\tau : X \to Y$ is a local function and* $\mathrm{ent}(X) = \mathrm{ent}(Y)$*, then τ pre–injective implies τ surjective.*

PROOF If τ is pre–injective we have, by Proposition 4.4, that $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$. Then $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y)$ so that, by Proposition 4.8, τ is surjective. $\square$

Hence we may conclude with the following (partial) generalization of the result of [Fio2] about strongly irreducible shifts of finite type.

Corollary 4.10 *Let Γ be a group of non–exponential growth. A semi–strongly irreducible subshift of finite type of A^Γ has the Myhill–property.*

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