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Semi–Strong Irreducibility for Shifts and Moore–Myhill Property

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Abstract

If $\mathbf{Z}$ is the group of integers, A a finite alphabet and $A^{\mathbf{Z}}$ the set of all functions $c : \mathbf{Z} \to A$, the equivalence between pre–injectivity and surjectivity of a local function holds for irreducible shifts of finite type of $A^{\mathbf{Z}}$ (see [Fio1]). In [Fio2] we give a definition of *strong irreducibility* that, together with the finite type condition, allows us to prove the above equivalence for the strongly irreducible shifts of finite type in A^Γ, if Γ is an amenable group. In this paper, we define the *semi–strong irreducibility* for a shift; this property allows us to prove the implication “pre–injective $\Rightarrow$ surjective” for a local function on a semi–strongly irreducible shift of finite type of A^Γ, if Γ has non–exponential growth. We also see that in the one–dimensional case irreducibility and semi–strong irreducibility are equivalent. As a by–product, we generalize a well–known result about the entropy of a shift: *the entropy of a proper subshift of a semi–strongly irreducible shift X is strictly smaller than the entropy of X.*

Key Words: irreducible shifts, amenable groups, entropy. AMS Classification: 37B10 – 43A07.

1. If Γ is a finitely generated group, one can consider the space A^Γ (the so–called *full A–shift*) of functions defined on Γ with values in a finite *alphabet* A. An element $c \in A^\Gamma$ is also called a *configuration.* This space is naturally endowed with a compact topology; a subset X of A^Γ which is Γ–invariant and topologically closed is called *subshift*, *shift space* or simply *shift.* In this setting a *local function* $\tau : X \to A^\Gamma$ is a continuous function commuting with the natural action of Γ on X.

In Section 2 we state a few basic results about the subshifts of A^Γ. We give the fundamental notion of a *shift of finite type* and that of *irreducibility* for a shift, generalizing the corresponding concepts that are well–known in the one–dimensional case (i.e. the case $\Gamma = \mathbf{Z}$). In particular, we will see that a shift of finite type has a useful “overlapping” property; this will be used in Section 3.

*The results of this paper are taken from my PhD thesis written under the supervision of Prof. Antonio Machì at the University of Rome “La Sapienza”.

The notion of *entropy* as defined by Gromov in [G] is given. We prove that if the group has non–exponential growth, the entropy of a subshift of A^Γ can be calculated relative to a suitable sequence of disks in Γ with increasing radius.

Finally, the notion of *pre–injective* function is given; it is a sort of injectivity of the function on the set of the "finite" configurations. We consider under which hypotheses pre–injectivity of a local function defined on a shift implies surjectivity.

From a result of Gromov [G] under much more general hypotheses it follows (see [Fio2]) that the equivalence between pre–injectivity and surjectivity holds for local function on shifts of *bounded propagation* contained in A^Γ, if Γ is amenable. In [Fio2] we generalize this result, proving that it holds for *strongly irreducible* shifts of finite type of A^Γ.

In the one–dimensional case the above equivalence holds for irreducible shifts of finite type of $A^{\mathbf{Z}}$. Moreover, using this result it is proved in [Fio1] that the implication "pre–injective $\Rightarrow$ surjective" holds when the local function is defined on an irreducible *sofic shift* of $A^{\mathbf{Z}}$ (that is, the set of the bi–infinite words which are labels of all bi–infinite paths in a finite graph). On the other hand, in [Fio1] we give a counterexample of an irreducible sofic shift $X \subseteq A^{\mathbf{Z}}$ but not of finite type for which the inverse implication does not hold.

The difference between irreducibility and strong irreducibility lies in the way in which two different "finite" configurations of the shift may appear simultaneously in a global configuration. These two irreducibility conditions are not equivalent, not even in the one–dimensional case. Hence our general result about strongly irreducible shifts of finite type is strictly weaker than the one–dimensional one [1]. In the attempt of using weaker hypotheses to prove our result, in Section 3 a new notion of irreducibility, the *semi–strong irreducibility*, is given. This notion is motivated by the fact that a sofic subshift of $A^{\mathbf{Z}}$ is irreducible if and only if is semi–strongly irreducible. Furthermore, in Section 3 we prove that the implication "pre–injective $\Rightarrow$ surjective" holds for semi–strongly irreducible subshifts of finite type of A^Γ, if Γ has non–exponential growth. This result is a consequence of Corollary 3.9 which generalizes a well–known result about shifts entropy; indeed we prove that, under the above conditions, *the entropy of a proper subshift of a semi–strongly irreducible shift $X \subseteq A^\Gamma$ is strictly weaker than the entropy of X.*

2. Let Γ be a finitely generated group; we refer to [CeMaSca] and [MaMi] for the notation and for the notions and properties of Cayley graphs and growth functions. We denote by D_n the ball of Γ centered at 1 and with radius n.

The group Γ acts on A^Γ on the right as follows:

$$(c^\gamma)_{|\alpha} := c_{|\gamma\alpha}$$

[1] Indeed it can be easily seen by trivial counterexamples that none of the two implications "pre–injective $\leftrightarrows$ surjective" holds in general for local functions defined on irreducible shifts of finite type of $A^{\mathbf{Z}^2}$.

for $c \in A^\Gamma$ and $\gamma, \alpha \in \Gamma$ (where $c_{|\alpha}$ is the value of c at α).

With this, a *shift* is a subset X of A^Γ which is topologically closed and Γ–invariant (i.e. $X^\Gamma = X$); as we shall see later, this topological definition is equivalent (in the Euclidean case) to the classical combinatorial one.

For $X \subseteq A^\Gamma$ and $E \subseteq \Gamma$, we set

$$X_E := \{c_{|E} \mid c \in X\};$$

a *pattern of* X is an element of X_E where E is a non–empty finite subset of Γ. The set E is the *support of* the pattern; a *block of* X is a pattern of X with support a disk. The *language of* X is the set $L(X)$ of all the blocks of X. If X is a subshift of $A^{\mathbf{Z}}$, a configuration is a bi–infinite word and a block of X is a finite word appearing in some configuration of X.

Let X be a shift; a function $\tau : X \to A^\Gamma$ is *M–local* if there exists $\delta : X_{D_M} \to A$ such that for every $c \in X$ and $\gamma \in \Gamma$

$$(\tau(c))_{|\gamma} = \delta((c^\gamma)_{|D_M}) = \delta(c_{|\gamma\alpha_1}, c_{|\gamma\alpha_2}, \ldots, c_{|\gamma\alpha_m}),$$

where $D_M = \{\alpha_1, \ldots, \alpha_m\}$; a function is *local* if it is M–local for some M. Let $\tau : X \to A^\Gamma$ be local; if c is a configuration of X and E is a subset of Γ, $\tau(c)_{|E}$ only depends on $c_{|E^{+M}}$. Thus we have a family of functions $(\tau_{E^{+M}} : X_{E^{+M}} \to \tau(X)_E)_{E\subseteq\Gamma}$, this notation will be useful in the sequel.

There is a characterization of local functions which in the one–dimensional case is known as the Curtis–Lyndon–Hedlund theorem; a shift being compact, it holds for a general local function. It states that *a function* $\tau : X \to A^\Gamma$ *is local if and only if it is continuous and commutes with the* Γ*–action* (i.e. for each $c \in X$ and each $\gamma \in \Gamma$, one has $\tau(c^\gamma) = \tau(c)^\gamma$). From this result, it is clear that *the composition of two local functions is still local*, as it can also be easily seen directly from the definition.

Observe that if X is a subshift of A^Γ and $\tau : X \to A^\Gamma$ is a local function, then, by the (generalized) Curtis–Lyndon–Hedlund theorem, the image $Y := \tau(X)$ is still a subshift of A^Γ. Indeed Y is closed (or, equivalently, compact) and Γ–invariant. Moreover, if τ is injective then it is a homeomorphism (and commutes with the shift). In this case X and Y are *conjugate*.

As mentioned above, it is easy to prove that the topological definition of a shift space is equivalent to the following combinatorial one involving the avoidance of certain *forbidden blocks*; this fact is well–known in the Euclidean case (see [LinMar, Theorem 6.1.21]): *a subset* $X \subseteq A^\Gamma$ *is a shift if and only if there exists a subset* $\mathcal{F} \subseteq \bigcup_{n\in\mathbf{N}} A^{D_n}$ *such that* $X = X_{\mathcal{F}}$*, where*

$$X_{\mathcal{F}} := \{c \in A^\Gamma \mid {c^\alpha}_{|D_n} \notin \mathcal{F} \text{ for every } \alpha \in \Gamma, n \in \mathbf{N}\}.$$

We now give the fundamental notion of a shift of finite type. The basic definition is in terms of forbidden blocks: a shift is *of finite type* if it admits a finite set of forbidden blocks. In a sense we may say that a shift is of finite type if we can decide whether or not a configuration belongs to the shift only by checking its blocks of a fixed size only depending on the shift.

Since a finite set $\mathcal{F}$ of forbidden blocks of X has a maximal support, we can always assume that in a shift of finite type each block of $\mathcal{F}$ has the disk D_M as support (indeed a block that contains a forbidden block is also forbidden). In this case the shift X is called *M–step* and the number M is called the *memory of X*. If X is a subshift of $A^{\mathbf{Z}}$, we define the memory of X as the number M, where $M+1$ is the maximal length of a forbidden word.

For the shifts of finite type in $A^{\mathbf{Z}}$ we have (see [LinMar, Theorem 2.1.8]), the following useful property: *a shift $X \subseteq A^{\mathbf{Z}}$ is an M–step shift of finite type if and only if whenever $uv, vw \in L(X)$ and $|v| \geq M$, then $uvw \in L(X)$.*

It is easy to prove that this "overlapping" property holds more generally for subshifts of finite type of A^Γ; this has the following useful consequence.

Proposition 2.1 *Let X be an M–step shift of finite type and let E be a finite subset of Γ; if $p_1, p_2 \in X_{E^{+2M}}$ are two patterns that agree on $\partial^+_{2M} E$, then there exist two extensions $c_1, c_2 \in X$ of p_1 and p_2, respectively, that agree on $\complement E = \Gamma \setminus E$.*

Where, given a subset $E \subseteq \Gamma$ and for each $n \in \mathbf{N}$ we denote by

$$E^{+n} := \bigcup_{\alpha \in E} D(\alpha, n), \;\; E^{-n} := \{\alpha \in E \mid D(\alpha, n) \subseteq E\} \text{ and } \partial_n E := E^{+n} \setminus E^{-n}$$

the *n–closure of E*, the *n–interior of E* and the *n–boundary of E*, respectively; by

$$\partial^+_n E := E^{+n} \setminus E \text{ and } \partial^-_n E := E \setminus E^{-n}$$

the *n–external boundary of E* and the *n–internal boundary of E*, respectively. For all these sets, we will omit the index n if $n = 1$.

The natural generalization for a generic shift of the notion of irreducibility which is well–known in the one–dimensional case is the following: a shift $X \subseteq A^\Gamma$ is *irreducible* if for each pair of patterns $p_1 \in X_E$ and $p_2 \in X_F$, there exists an element $\gamma \in \Gamma$ such that $E \cap \gamma F = \emptyset$ and a configuration $c \in X$ such that $c_{|E} = p_1$ and $c_{|\gamma F} = p_2$. In other words, a shift is irreducible if whenever we have $p_1, p_2 \in L(X)$, there exists a configuration $c \in X$ in which these two blocks appear simultaneously on disjoint supports.

Next we give the definition of entropy for a shift. This definition involves the existence of a suitable sequence of sets that for groups of non–exponential growth (that is in the case in which the cardinality of the ball D_n grows slower then any exponential function), can be taken as balls centered at 1 and with increasing radius.

Definition 2.2 Let $(E_n)_{n\geq 1}$ be a sequence of subsets of Γ such that $\bigcup_{n\in\mathbf{N}} E_n = \Gamma$ and such that the *Følner condition* holds:

$$\lim_{n\to\infty} \frac{|\partial E_n|}{|E_n|} = 0; \tag{1}$$

if $X \subseteq A^\Gamma$ is a shift, the *entropy of X respect to* (E_n) is given by

$$\mathrm{ent}(X) := \limsup_{n\to\infty} \frac{\log |X_{E_n}|}{|E_n|}.$$

Condition (1) is necessary in order to prove that the *entropy is invariant under conjugacy*; other aspects of its importance will be seen in Section 3. A sequence as in (1) is called *amenable* or *Følner's sequence*.

If X is a subshift of $A^{\mathbf{Z}}$, we choose as E_n the interval $[1,n]$ (or equivalently, in order to have $\bigcup_{n\in\mathbf{N}} E_n = \Gamma$, the interval $[-n,n]$), so that X_{E_n} is the set of words of X of length n (one can prove that in this one–dimensional case, the above maximum limit is a limit and coincides with $\inf_{m\geq 1} \frac{\log |X_{E_m}|}{m}$; see [Fio1]).

In general, if Γ is a group of non–exponential growth, we choose as E_n a suitable disk centered at $1 \in \Gamma$; indeed, setting $a_h = |D_h|$, we have (by definition of non–exponential growth) $\lim_{h\to\infty} \sqrt[h]{a_h} = 1$, and hence $\liminf_{h\to\infty} \frac{a_{h+1}}{a_h} = 1$; it follows that for a suitable sequence $(a_{h_k})_k$ we have $\lim_{k\to\infty} \frac{a_{h_k+1}}{a_{h_k}} = 1$. Hence $\liminf_{k\to\infty} \frac{a_{h_k+1}}{a_{h_k-1}} = 1$ and for a suitable sequence $(a_{h_{k_n}})_n$ we have $\lim_{n\to\infty} \frac{a_{h_{k_n}+1}}{a_{h_{k_n}-1}} = 1$, i.e. we find a sequence of disks $E_n := D_{h_{k_n}}$ such that $\lim_{n\to\infty} \frac{|E_n^+|}{|E_n^-|} = 1$. Hence $\frac{|\partial E_n|}{|E_n|} = \frac{|E_n^+ \setminus E_n^-|}{|E_n|} \leq \frac{|E_n^+ \setminus E_n^-|}{|E_n^-|} = \frac{|E_n^+|}{|E_n^-|} - 1 \longrightarrow_{n\to\infty} 0$.

A function $\tau : X \subseteq A^\Gamma \to A^\Gamma$ is *pre–injective* if whenever $c_1, c_2 \in X$ and $c_1 \neq c_2$ only on a finite non–empty subset of Γ, then $\tau(c_1) \neq \tau(c_2)$. Recall that a group Γ is *amenable* if it admits a Γ–invariant probability measure (with this hypothesis on the group, it is proved in [CeMaSca] that *a transition function on the full shift A^Γ is pre–injective if and only of is surjective*).

3. As proved in [Fio1], the equivalence "pre–injective $\Leftrightarrow$ surjective" holds for transition functions defined on an irreducible subshifts of finite type of $A^{\mathbf{Z}}$. If Γ is amenable, it holds for *strongly irreducible* subshifts of finite type of A^Γ (see [Fio2]). In this section we define another form of irreducibility: the *semi–strong irreducibility*. For sofic shifts in the one–dimensional case this notion is equivalent to irreducibility; if Γ has non–exponential growth it allows us to prove the implication "pre–injective $\Rightarrow$ surjective" for local functions defined on subshifts of finite type.

Definition 3.3 A shift X is called (M,k)*–irreducible* (where M, k are natural numbers such that $M \geq k$) if for each pair of finite sets $E, \alpha D \subseteq \Gamma$ (the second one is a ball centered at α) such that $\mathrm{dist}(E, \alpha D) > M$ and for each pair of patterns $p_1 \in X_E$ and $p_2 \in X_{\alpha D}$, there exists a configuration $c \in X$ such that $c = p_1$ in E and $c = p_2$ in $\alpha\varepsilon D$ (that is the disk centered at $\alpha\varepsilon$), where $\varepsilon \in \Gamma$ is such that $\|\varepsilon\| \leq k$. The shift X is called *semi–strongly irreducible* if it is (M,k)–irreducible for some $M, k \in \mathbf{N}$.

A shift X is *M–irreducible* if for every pair of finite sets $E, F \subseteq \Gamma$ such that $\mathrm{dist}(E, F) > M$ and every pair of patterns $p_1 \in X_E$ and $p_2 \in X_F$, there exists a configuration $c \in X$ such that $c = p_1$ in E and $c = p_2$ in F. X is *strongly irreducible* if it is M–irreducible for some $M \in \mathbf{N}$. Hence the difference between semi–strong irreducibility and strong irreducibility lies in the fact that in the former the support of the second pattern must be a ball and the configuration c merging the two patterns moves this support "slightly". Notice that this motion is a translation and hence it makes sense to say that the configuration c restricted to $\alpha\varepsilon D$ coincides with $p_2 \in X_{\alpha D}$. Moreover, under the previous hypotheses, the translated disk $\alpha\varepsilon D$ is still contained in $(\alpha D)^{+M}$; indeed if $D = D_r$ and $\gamma \in \alpha\varepsilon D_r$, then $\mathrm{dist}(\gamma, \alpha) \leq \mathrm{dist}(\gamma, \alpha\varepsilon) + \mathrm{dist}(\alpha\varepsilon, \alpha) \leq r + \|\varepsilon^{-1}\| \leq r + k$. In particular $E \cap \alpha\varepsilon D = \emptyset$.

In Definition 3.3 is in fact essential that, given a finite set $F \subseteq \Gamma$, there exists $\alpha \in \Gamma\backslash\{1\}$ such that the translated set αF is still contained in F^{+M}. If the group is not abelian, the set αF may be quite far from F. On the other hand the set $F\alpha$ is α–close to F, but it is not, in general, obtained from F by translation [1]. This is why we require that the second set in Definition 3.3 be a ball centered at α; then we consider the new center $\alpha\varepsilon$ (which is ε–near α). The ball $\alpha\varepsilon D$ having the same radius as αD, is obtained by translating αD. As we have seen, if Γ has non–exponential growth we can fix a suitable sequence $(E_n)_n$ of balls centered at 1 with property (1) of Section 2. Hence if M is large enough we have $\varepsilon E_n \subseteq E_n^{+M}$.

In the special case $\Gamma = \mathbf{Z}$ a shift $X \subseteq A^{\mathbf{Z}}$ is (M, k)–irreducible if for each $n \geq M$ and each pair of words $u, v \in L(X)$, there exists a word $w \in L(X)$ with $n - k \leq |w| \leq n + k$, such that $uwv \in L(X)$.

In the one–dimensional case irreducibility and semi–strong irreducibility are equivalent, as follows from the well–known Pumping Lemma.

Lemma 3.4 (Pumping Lemma) *Let L be a regular language. There exists $M \geq 1$ such that if $uwv \in L$ and $|w| \geq M$, there exists a decomposition*

$$w = xyz$$

with $0 < |y| \leq M$ so that for each $n \in \mathbf{N}$ we have $uxy^nzv \in L$.

Moreover, one can take as M the number of vertices of a graph accepting L.

Corollary 3.5 *If $X \subseteq A^{\mathbf{Z}}$ is a sofic shift, then*

$$X \text{ irreducible} \iff X \text{ semi–strongly irreducible.}$$

Proof If X is irreducible, we claim that X is (M, M)–strongly irreducible, where M is given by the Pumping Lemma. If $n \geq M$ and $u, v \in L(X)$, there exists $w \in L(X)$ such that $uwv \in L(X)$. We distinguish two cases.

[1] Consider, for example, the free group $\mathbf{F}_2$ generated by a and b. If $F = \{a^n, b^n\}$ with $n > M$, it is easily seen that for no $\alpha \neq 1$ we have $\alpha F = \{\alpha a^n, \alpha b^n\} \subseteq F^{+M}$.

If $|w| > n+M$, then $w = x_1 y_1 z_1$ with $0 < |y_1| \leq M$ and if $w_1 := x_1 z_1$, then $uw_1 v \in L(X)$ and $|w| - M \leq |w_1| \leq |w| - 1$. If $|w_1| \leq n+M$, we have $|w_1| \geq |w| - M > n > n - M$. If $|w_1| > n+M$, we repeat the above construction to obtain, for some $i \geq 1$, an element w_i such that $uw_i v \in L(X)$ and $n - M < |w_i| \leq n+M$.

The second case, when $|w| < n - M$, is analogous. $\square$

We now prove our main results. The following proposition holds in the case of strongly irreducible shifts of finite type in A^Γ, where the group Γ is amenable (see [Fio2]).

Proposition 3.6 *Let Γ be a group of non–exponential growth. Let X be a semi–strongly irreducible shift of finite type and let $\tau : X \to A^\Gamma$ be a local and pre–injective function. Then* $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$.

PROOF Suppose that the memory of X is M, that X is (M,k)–irreducible and that τ is M–local. Set $Y := \tau(X)$ and fix an amenable sequence of disks $(E_n)_n$; we have

$$|Y_{E_n^{+2M}}| \leq |Y_{E_n}||A|^{|\partial^+_{2M} E_n|}.$$

Thus

$$\frac{\log |Y_{E_n^{+2M}}|}{|E_n|} \leq \frac{\log |Y_{E_n}|}{|E_n|} + \frac{|\partial^+_{2M} E_n| \log |A|}{|E_n|}.$$

Being $\lim_{n\to\infty} \frac{|\partial^+_{2M} E_n|}{|E_n|} = 0$, we have $\limsup_{n\to\infty} \frac{\log |Y_{E_n^{+2M}}|}{|E_n|} \leq \mathrm{ent}(Y)$. Let $l = l(k)$ be the number of ε's such that $\|\varepsilon\| \leq k$ and suppose that $\mathrm{ent}(Y) < \mathrm{ent}(X)$; then

$$\limsup_{n\to\infty} \frac{\log |Y_{E_n^{+2M}}|}{|E_n|} < \limsup_{n\to\infty} \frac{\log |X_{E_n}|}{|E_n|} = \limsup_{n\to\infty} \frac{\log(\frac{|X_{E_n}|}{l})}{|E_n|}.$$

Then there exists $n \in \mathbf{N}$ such that $\frac{\log |Y_{E_n^{+2M}}|}{|E_n|} < \frac{\log(\frac{|X_{E_n}|}{l})}{|E_n|}$ that is $|Y_{E_n^{+2M}}| < \frac{|X_{E_n}|}{l}$. Fix $v \in X_{\partial^+_{2M} E_n^{+M}}$; since $\mathrm{dist}(\partial^+_{2M} E_n^{+M}, E_n) = M + 1 > M$ for each $u \in X_{E_n}$ there exists $\varepsilon \in D_k$ and a pattern $p \in X_{E_n^{+3M}}$ that agrees with u on εE_n and with v on $\partial^+_{2M} E_n^{+M}$. Thus

$$|\{p \in X_{E_n^{+3M}} \mid p_{|\partial^+_{2M} E_n^{+M}} = v\}| \geq \frac{|X_{E_n}|}{l} > |Y_{E_n^{+2M}}|.$$

Since $\tau_{E_n^{+3M}} : X_{E_n^{+3M}} \to Y_{E_n^{+2M}}$ is surjective, there exist two patterns $p_1, p_2 \in X_{E_n^{+3M}}$ such that $p_1 \neq p_2$ but $p_1 = v = p_2$ on $\partial^+_{2M} E_n^{+M}$ and $\tau_{E_n^{+3M}}(p_1) = \tau_{E_n^{+3M}}(p_2)$. By Proposition 2.1, there exist two configurations $c_1, c_2 \in X$ which extend p_1 and p_2 and which agree outside E_n^{+M}. It is easy to prove that $\tau(c_1) = \tau(c_2)$, and hence that τ is not pre–injective. $\square$

Recall (see [Fio2, Lemma 4.3]), that if Γ is a finitely generated group, there exists a sequence of disks $(F_j)_{j\in\mathbf{N}}$ obtained by translating a disk D and at distance $> M$ such that $\bigcup_{j\in\mathbf{N}} F_j^{+R} = \Gamma$ for a suitable $R > 0$. We call the above sequence a (D, M, R)–*net*. The following lemma is an essential result used in the proof of the equivalence between pre–injectivity and surjectivity in the case of strongly irreducible shifts of finite type and of amenable groups (see [Fio2, Lemma 4.4]).

Lemma 3.7 *Let Γ be an amenable group and let $(E_n)_n$ be a fixed amenable sequence of Γ. Let $(F_j)_{j\in\mathbf{N}}$ be a $(D_r, 2M, R)$–net, let X be an M–irreducible shift and let Y be a subset of X such that $Y_{F_j} \subset X_{F_j}$ for every $j \in \mathbf{N}$. Then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

For semi–strongly irreducible shifts, Lemma 3.7 does not necessarily hold. Consider, for example, the shift $X = \{\dots 010101\dots, \dots 101010\dots\} \subseteq A^{\mathbf{Z}}$. It is of finite type and $(2,2)$–irreducible, but $\mathrm{ent}(X) = 0$.

The following lemma is similar to Lemma 3.7 but as one can see the hypotheses are quite stronger.

Lemma 3.8 *Let Γ be a group of non–exponential growth and let $(E_n)_n$ be a fixed amenable sequence of disks . Let $(F_j)_{j\in\mathbf{N}} = (D(\beta_j, r))_{j\in\mathbf{N}}$ be a $(D_r, 2M, R)$–net, let X be an (M, k)–irreducible shift and let Y be a subset of X such that for each $j \in \mathbf{N}$, there exists a pattern $p_j \in X_{F_j}$ for which $p_j \notin Y_{D(\beta_j\varepsilon, r)}$ whenever $\varepsilon \in D_k$. Then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

Proof Let $N(n)$ be the number of F_j's such that $F_j^{+M} \subseteq E_n$ and denote by $F_{j_1}, \dots, F_{j_N}$ these disks. Set $\xi := |X_{D^{+M}}|$ and denote by $P_{j_m} \subseteq X_{E_n}$ the set of the blocks p of X_{E_n} such that $p_{|D(\beta_{j_m}\varepsilon, r)} = p_{j_m}$ for some $\varepsilon \in D_k$; we prove that

$$|X_{E_n} \setminus \bigcup_{i=1}^{N} P_{j_i}| \leq (1 - \xi^{-1})^N |X_{E_n}| \tag{1}$$

by induction on $m \in \{1, \dots, N\}$. We have

$$|X_{E_n}| \leq |X_{F_{j_1}^{+M}}||X_{E_n \setminus F_{j_1}^{+M}}|$$

and hence

$$|X_{E_n}| \leq \xi |X_{E_n \setminus F_{j_1}^{+M}}|.$$

Since X is (M, k)–irreducible and since $\mathrm{dist}(F_{j_1}, E_n \setminus F_{j_1}^{+M}) > M$, given a pattern $p \in X_{E_n \setminus F_{j_1}^{+M}}$ there exists a pattern $\bar{p}$ defined on all E_n that coincides with p on $E_n \setminus F_{j_1}^{+M}$ and with p_{j_1} on some $D(\beta_{j_1}\varepsilon, r)$; then

$$|X_{E_n \setminus F_{j_1}^{+M}}| \leq |P_{j_1}|.$$

Hence

$$\frac{1}{\xi}|X_{E_n}| \leq |P_{j_1}|$$

so that

$$|X_{E_n} \backslash P_{j_1}| \leq |X_{E_n}| - \frac{1}{\xi}|X_{E_n}| = (1 - \frac{1}{\xi})|X_{E_n}|.$$

Suppose that (1) holds for $m-1$; we have

$$|X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}| \leq \xi |\{p \in X_{E_n \backslash F_{j_m}^{+M}} \mid p_{|D(\beta_{j_i}\varepsilon, r)} \neq p_{j_i}$$

$$\text{for each } i = 1, \ldots, m-1 \text{ and each } \varepsilon|\}.$$

Moreover, since X is (M,k)–irreducible,

$$|\{p \in X_{E_n \backslash F_{j_m}^{+M}} \mid p_{|D(\beta_{j_i}\varepsilon, r)} \neq p_{j_i} \text{ for each } i = 1, \ldots, m-1 \text{ and each } \varepsilon\}| \leq$$

$$\leq |\{p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i} \mid p_{|D(\beta_{j_m}\varepsilon, r)} = p_{j_m} \text{ for some } \varepsilon\}|.$$

Hence

$$\frac{1}{\xi}|X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}| \leq |\{p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i} \mid p_{|D(\beta_{j_m}\varepsilon, r)} = p_{j_m} \text{ for some } \varepsilon\}|$$

and then

$$|X_{E_n} \backslash \bigcup_{i=1}^{m} P_{j_i}| = |\Big(X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}\Big) \setminus P_{j_m}| \leq$$

$$\leq |\Big(X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}\Big) \setminus \{p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i} \mid p_{|D(\beta_{j_m}\varepsilon, r)} = p_{j_m} \text{ for some } \varepsilon\}| \leq$$

$$\leq |X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}| - \frac{1}{\xi}|X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}| \leq (1 - \frac{1}{\xi})(1 - \xi^{-1})^{m-1}|X_{E_n}|.$$

Hence (1) holds, and since $|Y_{E_n}| \leq |X_{E_n} \backslash \bigcup_{m=1}^{N} P_{j_m}|$, we have

$$\frac{\log |Y_{E_n}|}{|E_n|} \leq \frac{N(n)\log(1 - \xi^{-1})}{|E_n|} + \frac{\log |X_{E_n}|}{|E_n|}. \tag{2}$$

Observe that

$$E_n \subseteq \bigcup_{i=1}^{N} F_{j_i}^{+R} \cup (E_n \backslash E_n^{-(R+2r+M)})$$

and hence we have

$$|E_n| \leq N(n)|D^{+R}| + |E_n \backslash E_n^{-(R+2r+M)}|$$

so that

$$1 \leq \frac{N(n)}{|E_n|}|D^{+R}| + \frac{|\partial^{-}_{R+2r+M} E_n|}{|E_n|};$$

taking the minimum limit and being $\lim_{n\to\infty}\frac{|\partial^-_{R+2r+M}E_n|}{|E_n|}=0$,

$$\zeta := \liminf_{n\to\infty}\frac{N(n)}{|E_n|} > 0.$$

Taking the maximum limit in (2), it follows

$$\mathrm{ent}(Y) \le \zeta\log(1-\xi^{-1}) + \mathrm{ent}(X) < \mathrm{ent}(X). \quad \square$$

The following statement is an easy consequence of Lemma 3.8 and generalizes the result in [LinMar, Corollary 4.4.9].

Corollary 3.9 *Let Γ be a group of non–exponential growth and let X be a semi–strongly irreducible subshift of A^Γ. If Y is a proper subshift of X then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

PROOF Let X be (M,k)–irreducible. If $Y \subset X$, there exists a configuration $c \in X$ which does not belong to Y and then there exists a disk D_r such that $c_{|D_r} \in X_{D_r}\setminus Y_{D_r}$. Let $(F_j)_{j\in\mathbf{N}} = (D(\beta_j, r))_{j\in\mathbf{N}}$ be a $(D_r, 2M, R)$–net; then $c_{|D_r} \notin Y_{D(\beta_j\varepsilon, r)}$ whenever $\varepsilon \in D_k$; by Lemma 3.8, $\mathrm{ent}(Y) < \mathrm{ent}(X)$. $\square$

Proposition 3.10 *Let Γ be a group of non–exponential growth. Let X be a shift, let Y be a semi–strongly irreducible shift and let $\tau : X \to Y$ be a local function such that* $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y)$. *Then τ is surjective.*

PROOF Let X and Y be as in the above hypotheses and let $\tau : X \to Y$ be a local function. Clearly $\tau(X)$ is a subshift of Y. By Corollary 3.9, we have that if $\tau(X) \subset Y$, then $\mathrm{ent}(\tau(X)) < \mathrm{ent}(Y)$. $\square$

Theorem 3.11 *Let Γ be a group of non–exponential growth, let X be a semi–strongly irreducible shift of finite type and let Y be a semi–strongly irreducible shift. If $\tau : X \to Y$ is a local function and* $\mathrm{ent}(X) = \mathrm{ent}(Y)$, *then τ pre–injective implies τ surjective.*

PROOF If τ is pre–injective we have, by Proposition 3.6, that $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$. Then $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y)$ so that, by Proposition 3.10, τ is surjective. $\square$

Hence we may conclude with the following (partial) generalization of the result of [Fio2] about strongly irreducible shifts of finite type.

Corollary 3.12 *Let Γ be a group of non–exponential growth. Let X be a semi–strongly irreducible subshift of finite type of A^Γ and let $\tau : X \to X$ be a transition function. Then τ pre–injective implies τ surjective.*

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