# Joint Operator Pricing and Network Selection Game in Cognitive Radio Networks: Equilibrium, System Dynamics and Price of Anarchy

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***Abstract*—This paper addresses the joint pricing and network selection problem in cognitive radio networks. The problem is formulated as a Stackelberg game where first the Primary and Secondary operators set the network subscription price to maximize their revenue. Then, users perform the network selection process, deciding whether to pay more for a guaranteed service, or use a cheaper, best-effort secondary network, where congestion and low throughput may be experienced.**

**We derive optimal stable price and network selection settings. More specifically, we use the *Nash equilibrium* concept to characterize the equilibria for the price setting game. On the other hand, a *Wardrop equilibrium* is reached by users in the network selection game, since in our model a large number of users must determine individually the network they should connect to. Furthermore, we study network users' dynamics using a population game model, and we determine its convergence properties under replicator dynamics, a simple yet effective selection strategy.**

**Numerical results demonstrate that our game model captures the main factors behind cognitive network pricing and network selection, thus representing a promising framework for the design and understanding of cognitive radio systems.**

***Index Terms*: - Cognitive Radio Networks, Pricing, Network Selection, Stackelberg Game, Population Game Model, Replicator Dynamics.**

## I. INTRODUCTION

Cognitive radio networks (CRNs), also referred to as *xG* networks, are envisioned to deliver high bandwidth to mobile users via heterogeneous wireless architectures and dynamic spectrum access techniques [1], [2]. In CRNs, a *Primary* (or licensed) User (PU) has a license to operate in a certain spectrum band; his access is generally controlled by the Primary Operator (PO) and should not be affected by the operations of any other unlicensed user. On the other hand, the Secondary Operator (SO) has no spectrum license; therefore, *Secondary* Users (SUs) must implement additional functionalities to share the licensed spectrum band without interfering with primary users.

In this work, we consider a cognitive radio scenario which consists of primary and secondary networks, as well as a large set of cognitive users, and we focus on a fundamental issue concerning such systems, i.e. whether it is better for a CR user to act as a primary user, paying the Primary operator for costlier, dedicated network resources with Quality of Service guarantees, or act as a secondary user (paying the Secondary operator), sharing the spectrum holes left available by licensed users and facing lower costs with degraded performance guarantees. At the same time, we consider the *pricing* problem of both Primary and Secondary operators, who compete with each other, setting access prices to maximize their revenues.



The joint pricing and cognitive radio network selection problem is modeled as a *Stackelberg* (leader-follower) game, where first the Primary and Secondary operators set their access prices in order to maximize their revenues. In this regard, we study both practical cases where (1) the Primary and Secondary operators fix access prices at the same time, and (2) the Primary operator exploits his dominant position by playing first, anticipating the choices of the Secondary operator. Then, network users react to the prices set by the operators, choosing which network they should connect to, therefore acting either like primary or secondary users.

The solution provides an insight on how rational users will distribute among existing access solutions (higher-price primary networks vs. lower-price secondary networks), i.e., the proportion of players who choose different strategies.

We adopt a fluid queue approximation approach (as in [3], [4], [5], [6], [7]) to study the steady-state performance of these users, focusing on *delay* as QoS metric. Besides considering *static* traffic equilibrium settings, we further formulate the network selection process of cognitive radio users as a *population game* [8], which provides a powerful framework for characterizing the strategic interactions among large numbers of agents, whose behavior is modeled as a *dynamic* adjustment process. More specifically, we study the cognitive users' behavior according to *replicator dynamics* [8], [9], since such users adapt their choices and strategies based on the observed network state.

We provide equilibrium and convergence properties of the proposed game, and derive optimal stable price and network selection settings.

More specifically, we use the *Nash equilibrium* concept to characterize the equilibria of the pricing game between a finite number of decision makers (viz., the Primary and Secondary operators). In addition to that, we further determine the *Wardrop equilibrium* for the network selection game, in which a large number of users must choose individually the network they should connect to. Such equilibrium is characterized by two properties, namely *traffic equilibrium* (the total costs perceived by users on all used networks are equal) and *system optimum principle* (the average delay/cost is minimum) [10].

Numerical results obtained in different network scenarios illustrate that our game captures the main factors behind cognitive network pricing and selection, thus representing a promising framework for the design and performance evaluation of cognitive radio systems.

In summary, in an effort to understand the pricing and networking selection issues that characterize CRNs, our work makes the following contributions:

- the proposition of a novel game theoretical model where Primary and Secondary operators set access prices, and users select which network to connect to, based both on the total delay and the experienced cost.
- The computation of equilibrium points for our game, as well as relevant performance metrics, including the Price of Anarchy and the Price of Stability.
- The analysis of a dynamic model, based on population games, which further illustrates how players converge to the equilibrium in a dynamic context under an easily implementable, distributed strategy (viz., replicator dynamics), along with formal, detailed proofs of its convergence.

The rest of this paper is organized as follows: related work is reviewed in Section II. The network model for the proposed joint pricing and network selection game is described in Section III; the equilibrium points of such game, as well as its Price of Anarchy and Price of Stability, are derived in Sections IV and V, respectively. The dynamic network selection model, based on population games and replicator dynamics, is presented in Section VI, and its convergence properties are demonstrated in Section VII. Numerical results are discussed in Section VIII, while Section IX concludes this work.

## II. Related Work

In this section, we first review the most notable works on spectrum pricing and access in cognitive radio networks [3], [4], [11], [12], [13], [14], [15], [16], [17], [18]. Then, we discuss relevant works that use evolutionary games to study the users behavior in CR as well as in heterogeneous wireless networks [19], [20], [21], [22], [23].

In [11], the authors provide a systematic overview on CR networking and communications, by looking at the key functions of the physical, MAC and network layers involved in a CR design, and by studying how these layers are crossly related. In [3], the authors consider the decision-making process of SUs who have the choice of either acquiring a dedicated spectrum (paying a price) or using the primary user band for free, and they characterize the resulting Nash equilibrium for the single-band case. This work differs from ours in two main aspects: 1) the CR users already arrive at the system as secondary or primary ones; SUs have the choice between dedicated or PU band, and 2) the users' behavior is studied based on queueing theory. The work in [4] considers a CRN where multiple secondary users (SUs) contend for spectrum usage, using random access, over available primary user channels, focusing on SUs' queueing delay performance. A fluid queue approximation approach is adopted to study the steady-state delay performance of SUs. In [12], the authors analyze the price competition between PUs who can lease out their unused bandwidth to secondaries in exchange for a fee, considering bandwidth uncertainty and spatial reuse. The problem of dynamic spectrum leasing in a secondary market of CRNs is considered in [14], where secondary service providers lease spectrum from spectrum brokers to provide service to SUs.

Recent works have considered *evolutionary games* to study the users' behavior in cognitive radio and heterogeneous wireless networks [19], [20], [21], [22], [23].

In [19], the authors use evolutionary game theory to investigate the dynamics of user behavior in heterogeneous wireless access networks (i.e., WMANs, cellular networks, and WLANs). The evolutionary game solution is compared to the Nash equilibrium, and a set of algorithms (i.e., population evolution and reinforcement learning algorithms) are proposed to implement the evolutionary network selection game model. In [20], the dynamics of a multiple-seller, multiple-buyer spectrum trading market is modeled as an evolutionary game, in which PUs want to sell and SUs want to buy spectrum opportunities. Secondary users evolve over time, buying the spectrum opportunities that optimize their performance in terms of transmission rate and price. In [21], the authors propose a distributed framework for spectrum access, with and without complete network information (i.e., channel statistics and user selections). In the first case, an evolutionary game approach is proposed, in which each SU compares its payoff with the system average payoff to evolve its spectrum access decision over time. For the incomplete information case, a learning mechanism is proposed, in which each SU estimates its expected throughput locally and learns to adjust its channel selection strategy adaptively. The problem of opportunistic spectrum access in CSMA/CA-based cognitive radio networks is also addressed in [22] from an evolutionary game theoretic angle.

In our preliminary works [24], [25], we addressed the pricing and network selection problems in cognitive radio networks. However, in [24], we assumed that the PO and SO use *separate* frequency bands, which greatly simplifies the problem, and we did not study the impact of the order in which operators set prices on the quality of the reached equilibria. The work in [25] differs from the one presented here in that it considered uniquely Primary operators, and a finite set of SUs, which are characterized by elastic traffic demands that can be transmitted over one or multiple frequency spectra.

Unlike previous works, which study the interaction between two well-defined sets of users (primary and secondary ones) who *already* performed the choice of using the primary or the secondary network, our paper tackles a fundamental issue in CRNs. In fact, we model the users' decision process that occurs *before* such users enter the CRN, thus assessing the economic interest of deploying secondary (xG) networks. Such choice depends on the trade-off between *cost* and *performance guarantees* in such networks. At the same time, we derive the optimal price setting for both Primary and Secondary operators that play before network users, in order to maximize their revenue. We use enhanced game theoretical tools, derived from population game theory, to model the network selection dynamics, providing convergence conditions and equilibrium settings.

## III. Network Model

We now detail the network model, which is illustrated in Figure 1: a cognitive radio wireless system which consists of a secondary (xG) network that coexists with a primary network at the same location and on the same spectrum band.

We consider an *overlay model* (focusing on the "interference avoidance" approach [26], [27] to cognitive radio) as in [3], [20], [28], where Secondary Users periodically sense the radio spectrum, intelligently detect occupancy in the different frequency bands and then opportunistically communicate over the spectrum holes left available by Primary Users, thus avoiding interference with active primary users. In other words, our model is an overlay CR where secondary users opportunistically access primary users' spectrum only when it is not occupied. As in [3], we further consider perfect primary user detection at the secondary users and zero interference tolerance at each of the primary and secondary users.

We assume that users arrive at this system following a Poisson process with rate $\lambda$, and the maximum achievable transmission rate of the wireless channel (licensed to the PO and opportunistically used by the SO) is denoted by $C$. The total traffic $\lambda$ admitted in the network must not exceed its capacity $C$; this can be obtained, for example, using admission control techniques, which are out of the scope of this paper. All these assumptions are commonly adopted in several recent works like [4], [5], [6], [7].

Each arriving user must choose whether to join the primary network (paying a higher subscription cost) or the xG one (which has a lower subscription cost), based on criteria to be specified below, i.e., a combination of cost and QoS (service time/latency).

Finally, let us denote by $\lambda_P$ the overall transmission rate of primary users (i.e., those who choose the primary network) and by $\lambda_S$ the rate of secondary users, so that $\lambda = \lambda_P + \lambda_S$. Table I summarizes the basic notation used in our game model.

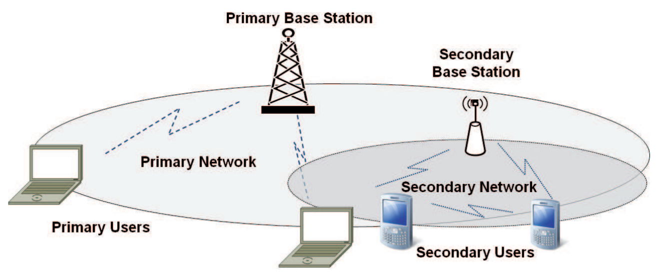



Fig. 1. CRN scenario with a primary network and a secondary (xG) network. Arriving users must decide whether to join the primary network, paying a subscription fee ($p_1$) for guaranteed QoS, or the xG network (which has a lower subscription cost, $p_2 < p_1$, and less performance guarantees), based on the expected cost and congestion levels.

TABLE I
Basic Notation

| | |
|---|---|
| $\lambda$ | Total traffic accepted in the network |
| $C$ | Wireless channel capacity |
| $\alpha$ | Weighting parameter of delay wrt access cost |
| $\lambda_P$ | Total traffic transmitted by Primary Users |
| $\lambda_S$ | Total traffic transmitted by Secondary Users |
| $X_P$ | Fraction of Primary Users |
| $X_S$ | Fraction of Secondary Users |
| $p_1, p_2$ | Price charged by the PO/SO |
| $K$ | Constant, velocity of convergence |

We now define users' *cost functions* as well as the *utility functions* of Primary and Secondary operators. We assume that the total cost incurred by a network user is a combination of the service time (delay, or latency) experienced in the network, and the cost for the player to access such network.

We underline that a similar model is used in [3], where the average cost incurred by a Secondary User (SU) consists of two components: (1) the price ($\tilde{C}$) of the dedicated spectrum band, and (2) an average delay cost ($\frac{1}{\mu}$), where $\mu$ is the service time. The average delay cost is weighted by a parameter $\alpha$, which represents the delay vs. monetary cost tradeoff of the SUs. To further support our choice, another similar model is considered by Anshelevich et al. in [29] for a different networking context. The authors set the player's cost for using an edge $e$ in the network as a combination of a cost function $c_e(x)$ and a latency function $d_e(x)$; the goal of each user in such game is to minimize the sum of his cost and latency. The same model is also used in [30]. Finally, note that in [19] the authors consider two components, namely throughput (the allocated capacity to a player, which is obviously related to the delay experienced by such user) and the corresponding price (see equations (2) and (3) in [19]).

In this work, we consider a fluid queue approximation approach, which permits to study the steady-state delay performance of both PUs and SUs. To this aim, and without loss of generality, we assume that the wireless channel is modeled as a M|M|1 queue, with service rate $C$ and arrival rate $\lambda$. Recall that both the primary and secondary networks operate on the same channel; the Primary and Secondary operators fix the prices $p_1$ and $p_2$, respectively, for accessing their services. Therefore, the total cost perceived by primary users is given by:

$$Cost_{PU} = \frac{\alpha}{C - \lambda_P} + p_1, \tag{1}$$

where parameter $\alpha$ weights the relative importance of the experienced delay with respect to the access cost. Note that primary users are affected *exclusively* by the traffic transmitted by primary users ($\lambda_P$), and not by the traffic of secondary users ($\lambda_S$), since usually, in a cognitive radio network, primary users have strict priority over secondary users; these latter must therefore implement spectrum sensing and spectrum handover strategies to avoid any interference towards primary users, and can transmit only in the spectrum holes left unoccupied by these ones.

As mentioned previously, we consider perfect primary user detection at the secondary users and zero interference tolerance at each of the primary and secondary users.

For this reason, secondary users' performance is affected by the *whole traffic*, transmitted by both primary and secondary users; such users are characterized by the following cost function:

$$Cost_{SU} = \frac{\alpha}{C - (\lambda_P + \lambda_S)} + p_2 = \frac{\alpha}{C - \lambda} + p_2. \quad (2)$$

As for operators' utilities, they correspond to the total revenue obtained by pricing users. As a consequence, the Primary operator's utility function is expressed as follows:

$$U_P = p_1 \lambda_P. \quad (3)$$

Correspondingly, the Secondary operator's utility function is:

$$U_S = p_2 \lambda_S = p_2(\lambda - \lambda_P). \quad (4)$$

To summarize, network users *minimize* the perceived cost, which is expressed as $Cost_{PU} = \frac{\alpha}{C-\lambda_P} + p_1$ (see equation (1)) if they choose the primary network, and $Cost_{SU} = \frac{\alpha}{C-\lambda} + p_2$ (see equation (2)) if they act as secondary users. As for Primary/Secondary operators, they try to *maximize* the total revenue obtained by pricing primary ($U_P = p_1 \lambda_P$) or secondary users ($U_S = p_2 \lambda_S$), respectively. Users' cost functions as well as operators' utilities are also reported in Tables II and III, respectively.

TABLE II
PRIMARY AND SECONDARY USER'S COST FUNCTIONS

| | |
|---|---|
| Primary User (PU) | $Cost_{PU} = \frac{\alpha}{C-\lambda_P} + p_1$ |
| Secondary User (SU) | $Cost_{SU} = \frac{\alpha}{C-\lambda} + p_2$ |

TABLE III
PRIMARY AND SECONDARY OPERATOR'S UTILITY FUNCTIONS

| | |
|---|---|
| Primary Operator (PO) | $U_P = p_1 \lambda_P$ |
| Secondary Operator (SO) | $U_S = p_2 \lambda_S$ |

## IV. EQUILIBRIUM COMPUTATION

In this section, we derive the equilibrium points of our game, namely: (i) the equilibrium traffic sent by primary and secondary users, (ii) steady-state Primary/Secondary operator's utilities, as well as (iii) equilibrium prices set by the PO/SO.

We consider two practical cases: (1) both operators fix their access price *at the same time*, trying to maximize their own revenue (Section IV-A), and (2) the PO plays *before* the SO, anticipating the strategy of this latter, thus exploiting his dominant position (Section IV-B). We will refer to the first case as the *TOGETHER* scenario, while the latter will be referred to as the *BEFORE* scenario. Note that when the Primary and Secondary operators play at the same time, we have a Cournot duopoly competition between such operators. However, in the original Cournot duopoly, production quantities (outputs) and prices are linear, while in this work we consider a nonlinear system which requires non-standard studies that cannot rely on existing results. On the other hand, when the Primary operator plays before the Secondary, anticipating his choices, we have a Stackelberg game model between the operators.

The *Nash* equilibrium concept will be used for the price setting game, since we have a finite number of decision makers, i.e., the two network operators. More precisely, a Nash Equilibrium is a set of players' (here, operators') strategies, each of which maximizes the player's revenue, and such that none of the actors has an incentive to deviate unilaterally. For this reason the corresponding network configurations are said to be stable.

On the other hand, a *Wardrop* equilibrium [31] is reached by CR users in the network selection game, since in our model a large number of users must determine individually the network they should connect to. Such equilibrium satisfies the two Wardrop's principles, namely traffic equilibrium (the total costs perceived by users on all used networks are equal) and system optimum principle (the average delay/cost is minimum).

Therefore, at Wardrop equilibrium, primary and secondary users will both experience the same cost, that is, $Cost_{PU} = Cost_{SU}$, or:

$$\frac{\alpha}{C - \lambda_P} + p_1 = \frac{\alpha}{C - (\lambda_P + \lambda_S)} + p_2 = \frac{\alpha}{C - \lambda} + p_2. \quad (5)$$

This permits to compute the equilibrium traffic[1] for the primary network as a function of the prices set by both the PO and SO:

$$\lambda_P = \frac{\alpha\lambda - C(C - \lambda)(p_1 - p_2)}{\alpha - (C - \lambda)(p_1 - p_2)}, \quad (6)$$

with $0 \leq \lambda_P \leq \lambda$. The traffic sent by secondary users, $\lambda_S$, will therefore be equal to $\lambda - \lambda_P$. Note that, in order for the equilibrium condition (5) to hold and for equilibrium traffic $\lambda_P$ to be comprised in the $[0, \lambda]$ range, $p_1 - p_2$ must satisfy the condition $p_1 - p_2 < \frac{\alpha\lambda}{C(C-\lambda)}$. Furthermore, since there is a unique $\lambda_P$ value which satisfies condition (5), such value represents the unique Wardrop equilibrium point of the network selection game.

The corresponding equilibrium utility for the PO is given by the following expression:

$$U_P = p_1 \lambda_P = p_1 \cdot \frac{\alpha\lambda - C(C - \lambda)(p_1 - p_2)}{\alpha - (C - \lambda)(p_1 - p_2)}, \quad (7)$$

while the utility of the SO will be:

$$\begin{aligned} U_S &= p_2 \lambda_S = p_2(\lambda - \lambda_P) \\ &= p_2\lambda + p_2 \Big[ \frac{\alpha(C - \lambda)}{\alpha - (C - \lambda)(p_1 - p_2)} - C \Big]. \end{aligned} \quad (8)$$

[1]With a slight abuse of notation, we will denote equilibrium flows still by $\lambda_P$ and $\lambda_S$, since in the following we will almost exclusively refer to equilibrium game conditions.

Hereafter we compute equilibrium prices for both our considered scenarios.

### A. The Primary and Secondary operators fix their prices simultaneously (TOGETHER scenario)

In this scenario, both the Primary and Secondary operators fix their prices simultaneously, trying to maximize their own revenue. As a consequence, to maximize the utility function of the PO, it suffices to take the derivative of $U_P$ with respect to $p_1$, imposing its equality to zero:

$$\frac{\partial U_P}{\partial p_1} = C - \frac{\alpha(C-\lambda)[\alpha-(C-\lambda)(p_1-p_2)] + \alpha(C-\lambda)^2 p_1}{[\alpha-(C-\lambda)(p_1-p_2)]^2} = 0 \tag{9}$$

Hence, we can express the price $p_1$ as a function of $p_2$:

$$p_1 = p_2 + \frac{\alpha}{C-\lambda}\Big\{1 - \sqrt{\frac{(C-\lambda)}{\alpha C}[\alpha + (C-\lambda)p_2]}\Big\}. \tag{10}$$

Similarly, the Secondary operator aims at maximizing his revenue $U_S$; by deriving $U_S$ with respect to $p_2$ and imposing its equality to zero, we obtain:

$$\frac{\partial U_S}{\partial p_2} = (\lambda - C) + \frac{\alpha^2(C-\lambda) - \alpha(C-\lambda)^2 p_1}{[\alpha-(C-\lambda)(p_1-p_2)]^2} = 0, \tag{11}$$

and the expression of $p_2$ as a function of $p_1$ is given by:

$$p_2 = p_1 - \frac{1}{(C-\lambda)}\{\alpha - \sqrt{\alpha^2 - \alpha(C-\lambda)p_1}\}. \tag{12}$$

Finally, combining expressions (10) and (12) we obtain the equilibrium price values $p_1$ and $p_2$, which are function of $\alpha$, $C$ and $\lambda$:

$$p_1 = \alpha\frac{(3C^2-\lambda^2) - (C-\lambda)^2\sqrt{\frac{9C-5\lambda}{C-\lambda}}}{2(2C-\lambda)^2(C-\lambda)} \tag{13}$$

$$p_2 = \alpha\frac{C\sqrt{9C-5\lambda} - (3C-2\lambda)\sqrt{C-\lambda}}{2(2C-\lambda)^2\sqrt{C-\lambda}}, \tag{14}$$

with $p_1 \geq 0$ and $p_2 \geq 0$.

### B. The Primary operator plays before the Secondary (BEFORE scenario)

In this case, we have a Stackelberg game between operators, in which the Primary operator is the leader while the Secondary operator is the follower.

The PO will therefore anticipate the choice of the SO (who will set the price $p_2$ in order to maximize his utility), and will play his best strategy, setting the optimal value for $p_1$ taking into account the choice on $p_2$ operated by the SO.

To derive the equilibrium prices in such scenario, it suffices to take the derivative of $U_S$ with respect to the price $p_2$, obtaining $p_2$ in function of $p_1$ (see equation (12)). We next insert the expression of $p_2$ in (7), obtaining $U_P$ as a function of $p_1$:

$$U_P = p_1\Big\{C + \frac{\alpha(\lambda - C)}{\sqrt{\alpha^2 - \alpha(C-\lambda)p_1}}\Big\}.$$

Deriving $U_P$ with respect to the price $p_1$, we obtain $C + \frac{\sqrt{\alpha}(\lambda-c)[2\alpha-(C-\lambda)p_1]}{2[\alpha-(C-\lambda)p_1]^{3/2}}$; then, imposing that such derivative is null, we obtain the equilibrium value for $p_1$, which has the following expression:

$$p_1 = \frac{\alpha}{C-\lambda}\Big\{1 - (Z + \frac{h}{3})^2\Big\}, \tag{15}$$

where $Z = (\frac{h}{4})^{1/3}\Big[\Big(\sqrt{1+\frac{4}{27}h^2}+1\Big)^{2/3} + \Big(\sqrt{1+\frac{4}{27}h^2}-1\Big)^{2/3}\Big]$, and $h = \frac{C-\lambda}{2C}$.

If we combine such expression of $p_1$ with (12), we obtain the equilibrium price set by the Secondary operator:

$$p_2 = \frac{\alpha}{C-\lambda}(Z + \frac{h}{3})[1 - (Z + \frac{h}{3})]. \tag{16}$$

### C. Comments

Note that, in both the *TOGETHER* and *BEFORE* scenarios, equilibrium prices are unique. In fact, if we compute the second derivatives in both network scenarios ($\frac{\partial^2 U_P}{\partial {p_1}^2}$ and $\frac{\partial^2 U_S}{\partial {p_2}^2}$), they are both negative for all price values in the feasible region $p_1 - p_2 < \frac{\alpha\lambda}{C(C-\lambda)}$. Hence, the maximums, as well as the Nash equilibrium points, are unique.

Furthermore, equilibrium prices ($p_1$ and $p_2$) are directly proportional to $\alpha$, while equilibrium flows ($\lambda_P$ and $\lambda_S$) are independent of $\alpha$; this can be seen by substituting, in expression (6), $p_1 - p_2$, which is proportional to $\alpha$. As a consequence, operators' utilities grow proportionally to $\alpha$. All these trends will be illustrated in more detail in the Numerical Results section.

Finally, primary users' equilibrium traffic, $\lambda_P$, decreases with increasing $C$ values, while secondary users' traffic follows an opposite trend. As for operators' prices and utilities, they both decrease with $C$, as we will quantify in Section VIII.

## V. Price of Anarchy and Price of Stability

We now investigate the efficiency of the equilibria reached by operators and users in our joint pricing and network selection game, through the determination of the Price of Anarchy (PoA) and the Price of Stability (PoS). They both quantify the loss of efficiency as the ratio between the cost of a specific stable outcome/equilibrium and the cost of the optimal outcome, which could be designed by a central authority. In particular the PoA, first introduced in [32], considers the worst stable outcome (that with the highest cost), while the PoS [29] considers the best stable equilibrium (that with the lowest cost). However, we observe that in our game these two performance metrics coincide due to the uniqueness of the equilibrium reached by network users. For this reason, in the following we will refer exclusively to the first performance figure, the PoA, which has a particular importance in characterizing the efficiency of distributed game formulations.

To determine the optimal system-wide solution, we define the social welfare $S$ as the weighted average of the delays experienced by primary and secondary users; $S$ is therefore a function of the amount $x$ of traffic sent by primary users:

$$S(x) = \frac{\alpha x}{C-x} + \frac{\alpha(\lambda - x)}{C-\lambda}.$$

Note that $p_1$ and $p_2$ do not appear in the social welfare's expression, since all the prices paid by primary/secondary users (which represent for them a *disutility* or *cost*) correspond to a symmetric *utility* or *gain* for the Primary/Secondary operators, who collect this income in exchange for the network services they offer.

To minimize this quantity, it suffices to derive with respect to $x$ and impose its equality to zero, thus obtaining:

$$\frac{dS(x)}{dx} = \frac{\alpha C}{(C-x)^2} - \frac{\alpha}{C-\lambda} = 0,$$

which leads to $x_{min} = C - \sqrt{C(C-\lambda)}$.

The optimal social welfare is therefore equal to:

$$\begin{aligned} S(x_{min}) &= \alpha\Big[\frac{C - \sqrt{C(C-\lambda)}}{\sqrt{C(C-\lambda)}} + \frac{\lambda - C + \sqrt{C(C-\lambda)}}{C-\lambda}\Big] \\ &= 2\alpha\Big[\sqrt{\frac{C}{C-\lambda}} - 1\Big]. \qquad (17)\end{aligned}$$

Recall that the total traffic transmitted by primary users at the Wardrop equilibrium is given by expression (6), and the equilibrium traffic for secondary users is $\lambda_s = \lambda - \lambda_p$.

The (average) total delay experienced by primary/secondary users at equilibrium is therefore equal to:

$$TD_E = \alpha\frac{\lambda_p}{C-\lambda_p} + \alpha\frac{\lambda_s}{C-\lambda}, \qquad (18)$$

while the Price of Anarchy (PoA) is defined as the ratio between the cost of the worst (here, the unique) equilibrium and the social optimum, $PoA = \frac{TD_E}{S(x_{min})}$.

Hereafter, we derive the closed-form expressions for the $PoA$ in both the considered scenarios (i.e., the *TOGETHER* and *BEFORE* scenarios). To this aim, it is sufficient to use equilibrium expressions for $\lambda_P$ and $\lambda_S$ in both scenarios.

### A. PoA for the TOGETHER scenario (the PO and SO play together)

The total delay of cognitive users at equilibrium ($TD_E^T$) can be expressed as follows:

$$\begin{aligned} TD_E^T &= \alpha\frac{\lambda_p}{C-\lambda_p} + \alpha\frac{\lambda_s}{C-\lambda} = \frac{\alpha\lambda}{C-\lambda} - (p_1 - p_2)\lambda_p \\ &= \frac{\alpha C(9C-5\lambda) - \alpha(3C-2\lambda)\sqrt{(C-\lambda)(9C-5\lambda)}}{(2C-\lambda)[(C-\lambda)+\sqrt{(C-\lambda)(9C-5\lambda)}]}. \qquad (19)\end{aligned}$$

Therefore, the Price of Anarchy can be calculated as:

$$\begin{aligned} PoA_T &= \frac{TD_E^T}{S(x_{min})} \\ &= \frac{C(9C-5\lambda)\sqrt{C-\lambda} - (3C-2\lambda)(C-\lambda)\sqrt{9C-5\lambda}}{2(2C-\lambda)[(C-\lambda)+\sqrt{(C-\lambda)(9C-5\lambda)}][\sqrt{C}-\sqrt{C-\lambda}]}. \qquad (20)\end{aligned}$$

### B. PoA for the BEFORE scenario (the PO plays before the SO)

In this case, the total delay of cognitive users at equilibrium ($TD_E^B$) can be expressed as:

$$\begin{aligned} TD_E^B &= \alpha\frac{\lambda_p}{C-\lambda_p} + \alpha\frac{\lambda_s}{C-\lambda} = \frac{\alpha\lambda}{C-\lambda} - (p_1 - p_2)\lambda_p \\ &= \alpha\Big[-2 + \frac{C}{C-\lambda}(Z + \frac{h}{3}) + \frac{1}{Z + \frac{h}{3}}\Big], \qquad (21)\end{aligned}$$

where

$$Z = (\frac{h}{4})^{1/3}\Big[\Big(\sqrt{1 + \frac{4}{27}h^2} + 1\Big)^{2/3} + \Big(\sqrt{1 + \frac{4}{27}h^2} - 1\Big)^{2/3}\Big],$$

and $h = \frac{C-\lambda}{2C}$.

The Price of Anarchy is therefore equal to:

$$\begin{aligned} PoA_B &= \frac{TD_E^B}{S(x_{min})} \\ &= \frac{\sqrt{C-\lambda}}{2(\sqrt{C-\lambda} - \sqrt{C})}\Big[-2 + \frac{C}{C-\lambda}(Z + \frac{h}{3}) + \frac{1}{Z + \frac{h}{3}}\Big]. \qquad (22)\end{aligned}$$

Note that both expressions (20) and (22) are independent of $\alpha$.

## VI. Cognitive Users' Behavior: Replicator dynamics

After having characterized the *static*, steady-state equilibria reached by network operators and users in the joint pricing and spectrum selection game, in this section we further focus on modeling the *dynamic* behavior of network users.

To this aim, we use population dynamics (and, in particular, *replicator dynamics*) to model the behavior of users who decide which network they should connect to, since such dynamics models network users who adapt their choices and strategies based on the observed state of the system (in terms of costs and congestion, in our case).

Before introducing replicator dynamics for our network selection game, we must first define some relevant game theoretic concepts.

### A. Introduction to Population Games and Replicator Dynamics

Hereafter we briefly introduce population games and replicator dynamics; for more details, the reader is referred to the book by W. H. Sandholm [8].

*1) Population Games:* A population game $G$, with $Q$ non-atomic classes of players (i.e., network users) is defined by a mass and a strategy set for each class, and a payoff function for each strategy. By a non-atomic population, we mean that the contribution of each member of the population is very small; this is the case in our game, where a large set of users compete for CRN's bandwidth resources. We denote the set of classes by $\mathcal{Q} = \{1, \dots, Q\}$, where $Q \geq 1$. The class $q$ has mass $m^q$. Let $S^q$ be the set of strategies available for players of class $q$, where $S^q = \{1, \dots, s^q\}$. These strategies can be thought of as the actions that members of $q$ could possibly take (i.e., connecting to the primary or the secondary network).

During the game play, each player of class $q$ selects a strategy from $S^q$. The mass of players of class $q$ that choose the strategy $n \in S^q$ is denoted by $x_n^q$, where $\sum_{n\in S^q} x_n^q = m^q$. We denote the vector of strategy distributions being used by the entire population by $x = \{x^1, \ldots, x^Q\}$, where $x^i = \{x_1^i, \ldots, x_{s^i}^i\}$. The vector $x$ can be thought of as the state of the system.

The marginal payoff function (per mass unit) of players of class $q$ who play strategy $n$ when the state of the system is $x$ is denoted by $F_n^q(x)$, usually referred to as *fitness* in evolutionary game theory, which is assumed to be continuous and differentiable. The total payoff of the players of class $q$ is therefore $\sum_{n\in S^q} F_n^q(x)x_n^q$.

*2) Replicator Dynamics:* The replicator dynamics describes the behavior of a large population of agents who are randomly matched to play normal form games. It was first introduced in biology by Taylor and Jonker [33] to model the evolution of species, and it is also used in the economics field. Recently, such dynamics has been applied to many networking problems, like routing and resource allocation [34], [35].

Given $x_n^q$, which represents the proportion of players of class $q$ that choose strategy $n$, as illustrated before, the replicator dynamics can be expressed as follows:

$$\dot{x}_n^q = x_n^q \Big[F_n^q(x) - \frac{1}{m^q}\sum_{n\in S^q} F_n^q(x)x_n^q\Big], \qquad (23)$$

where $\dot{x}_n^q$ represents the derivative of $x_n^q$ with respect to time.

In fact, the ratio $\dot{x}_n^q/x_n^q$ measures the evolutionary success (the rate of increase) of a strategy $n$. This ratio can be also expressed as the difference in fitness $F_n^q(x)$ of the strategy $n$ and the average fitness $\frac{1}{m^q}\sum_{n\in S^q} F_n^q(x)x_n^q$ of the class $q$.

An important concept in population games and replicator dynamics is *Wardrop* equilibrium [31], which we introduced in Section IV. In this context, a state $\hat{x}$ is a Wardrop equilibrium if for any class $q \in \mathcal{Q}$, all strategies being used by the members of $q$ yield the same marginal payoff to each member of $q$, whereas the marginal payoff that would be obtained by members of $q$ is lower for all strategies not used by class $q$.

### *B. Cognitive Users' Behavior in the Network Selection Game: Replicator Dynamics*

Having reviewed the mathematical tools we will rely on, we now focus on the cognitive radio scenario illustrated in Section III, introducing replicator dynamics for the network selection game. In particular, we consider a population game $G$ with a non-atomic set of players ($q = 1$), which is defined by a *strategy set* denoted by $\mathcal{S} = \{s_p, s_s\}$, identical for all players, and a *payoff function* for each strategy; $s_p$ means that the player chooses the *primary* network, and $s_s$ that the player chooses the *secondary* network, using the spectrum holes left free by primary users.

Our goal is to determine the dynamic network selection settings ($X_P$ and $X_S = 1 - X_P$), i.e., the fraction of players that choose the primary and secondary network, respectively, based on the equilibrium prices set by Primary and Secondary operators. Hence, the total traffic accepted in the primary network is equal to $\lambda_P = \lambda X_P$, and the one accepted in the secondary network is $\lambda_S = \lambda X_S$.

The proposed replicator dynamics provides a means to analyze how players can "learn" about their environment, and converge towards an equilibrium choice. Replicator dynamics is also useful to investigate the speed of convergence of strategy adaptation to reach a stable solution in the game. A mathematical analysis to bound such speed is provided in Section VII. In this case, CR users need to know some information, viz. the total cost (the service delay plus the price charged by the PO/SO, respectively) and the size of the populations ($X_P$, $X_S$) that already performed such selection, before undertaking the best choice based on the system state.

As illustrated in Section III, the goal of each cognitive radio user is to *minimize* a weighted sum of his delay (latency) and price paid to the network operator (either primary or secondary), $\alpha$ being the parameter which permits to give more weight to delay with respect to the paid price. Hence, we can formalize the network selection game as follows:

$$\begin{aligned}\dot{X_P} &= KX_P\Big[\frac{-\alpha}{C-\lambda X_P} - p_1 - \\ &- \Big(\frac{-\alpha X_P}{C-\lambda X_P} - X_P \cdot p_1 - (1-X_P)(\frac{\alpha}{C-\lambda} + p_2)\Big)\Big] = \\ &= KX_P(1-X_P)\Big[-p_1 + p_2 + \frac{\alpha}{C-\lambda} - \frac{\alpha}{C-\lambda X_P}\Big],\end{aligned} \qquad (24)$$

where $\dot{X_P}$ represents the derivative of $X_P$ with respect to time.

This equation has the same structure as the replicator dynamics (see equation (23)): the first term ($F_n^q(x) \equiv \frac{-\alpha}{C-\lambda X_P} - p_1$) corresponds to the total cost (the service delay plus the price charged by the PO) perceived by users that choose to connect to the primary network, using a M|M|1 approximation; the second term ($\frac{1}{m^q}\sum_{n\in S^q} F_n^q(x)x_n^q \equiv \frac{-\alpha X_P}{C-\lambda X_P} - X_P \cdot p_1 - (1-X_P)(\frac{\alpha}{C-\lambda} + p_2)$) represents the average cost/delay incurred by the fraction $X_P$ of primary users as well as by the fraction $X_S$ of secondary users (recall that $p_1$ and $p_2$ are the prices charged by the Primary and Secondary operator, respectively).

In particular, the speed of variation of $X_P$ is proportional to the population size $X_P$ (via the proportionality coefficient $K$), which models the willingness of the population to change strategy.

A similar equation can be written for Secondary Users, thus we can express the replicator dynamics for such SUs as follows:

$$\dot{X_S} = KX_S(1-X_S)\Big[p_1 - p_2 - \frac{\alpha}{C-\lambda} + \frac{\alpha}{(C-\lambda) + \lambda X_S}\Big]. \qquad (25)$$

Obviously, by comparing these two expressions it can be verified that condition $X_p + X_s = 1$ holds.

It can be demonstrated [8] that Wardrop equilibria are the stationary points of equations (24) and (25). As we will show in the next section, it can be easily proved that the unique non-trivial fixed point of such dynamics coincides with the Wardrop equilibrium point of the CR users' network selection game already determined in Section IV.

## VII. Convergence analysis of Replicator Dynamics

This section provides an in-depth analysis on the replicator dynamics given by (24)[2]. To this end, we rewrite it in a discretized version as follows:

$$X_P(t+1) = X_P(t)+kX_P(t)[1-X_P(t)]\left[A-\frac{1}{B-X_P(t)}\right], \tag{26}$$

where $k = K\alpha/\lambda$, $A = \lambda(-p_1/\alpha + p_2/\alpha + \frac{1}{C-\lambda})$ and $B = C/\lambda$.

The above dynamics has three fixed points, among which $0$ and $1$ are *trivial* fixed points corresponding to the case where all users either act as secondary or primary users, respectively. $X_P^* = B - 1/A$ is the only non-trivial fixed point, which is also the Wardrop equilibrium of the game; its expression is equal to $X_P^* = \frac{\lambda_P}{\lambda}$, where $\lambda_P$ is the equilibrium flow already derived for the static game in Section IV (see expression (6)).

In the subsequent analysis, we investigate the convergence of the replicator dynamics to $X_P^*$. We start by establishing the following auxiliary lemma.

**Lemma 1.** *Under the condition that* $K(A - \frac{1}{B-1}) \leq 1$, *it holds that*

- $X_P(t+1)$ *is non-decreasing w.r.t.* $X_P(t)$ *for* $X_P(t) \in [0, X_P^*)$ *and non-increasing w.r.t.* $X_P(t)$ *for* $X_P(t) \in (X_P^*, 1]$;
- $X_P(t+1) > X_P(t)$, $\forall X_P(t) < X_P^*$ *and* $X_P(t+1) < X_P(t)$, $\forall X_P(t) > X_P^*$.

*Proof.* The proof of the first part is straightforward by checking the derivative $\partial X_P(t+1)/\partial X_P(t)$. Specifically, it can be checked that under the condition that $K(A - \frac{1}{B-1}) \leq 1$, $\partial X_P(t+1)/\partial X_P(t) > 0$ when $X_P(t) \in [0, X_P^*)$ and $\partial X_P(t+1)/\partial X_P(t) < 0$ when $X_P(t) \in (X_P^*, 1]$. The second part follows readily from (26). □

The following theorem establishes the convergence of the replicator dynamics to the non-trivial fixed point $X_P^*$.

**Theorem 1.** *Under the condition that* $K(A - \frac{1}{B-1}) \leq 1$, *the replicator dynamics depicted in* (26) *converges to the non-trivial fixed point* $X_P^*$ *for any initial state* $0 < X_P(0) < 1$.

*Proof.* Consider an arbitrary sequence of update steps commencing from an initial vector $X_P(0)$. We distinguish the following two cases:

- *Case 1:* $0 < X_P(0) \leq X_P^*$. In this case (recall that $X_P^*$ is a fixed point of (26)), it follows from Lemma 1 that: (1) $X_P(t) \leq X_P^*, \forall t$ and (2) $X_P(0) \leq X_P(1) \leq \cdots \leq X_P(t-1) \leq X_P(t) \leq \cdots$, i.e., $X_P(t)$ is a non-decreasing sequence. Since $X_P(t)$ is also bounded by $X_P^*$, it follows that it must converge to a limit. Since there is no fixed point other than $X_P^*$ in the range $(0, X_P^*]$, this limit must be $X_P^*$.
- *Case 2:* $X_P^* < X_P(0) < 1$. This case can be proved in a similar manner. In fact (recall that $X_P^*$ is a fixed point of (26)), it follows from Lemma 1 that: (1) $X_P(t) > X_P^*, \forall t$ and (2) $X_P(0) \geq X_P(1) \geq \cdots \geq X_P(t-1) \geq X_P(t) \geq \cdots$, i.e., $X_P(t)$ is a non-increasing sequence. Since $X_P(t)$ is also bounded by $X_P^*$, it follows that it must converge to a limit. Since there is no fixed point other than $X_P^*$ in the range $[X_P^*, 1)$, this limit must be $X_P^*$.

Combining the above analysis, the replicator dynamics is ensured to converge to the non-trivial fixed point $X_P^*$ for any initial state $0 < X_P(0) < 1$. □

The above theorem essentially illustrates that with a conservative strategy (i.e., small $K$), the replicator dynamics is ensured to converge to the Wardrop equilibrium.

**Remark.** The above theorem establishes the sufficient condition for the convergence of the replicator dynamics to the unique non-trivial fixed point, which is also the Wardrop equilibrium. It follows straightforwardly that under the same condition, the equilibrium is also stable in that any deviated point from it will be dragged back under the replicator dynamics. In fact, $X_P^*$ is an evolutionary stable equilibrium. Meantime, it follows from the theorem that the two trivial fixed points $0$ and $1$ are not stable, in the sense that any deviation from them will drag the system to $X_P^*$.

It is also worth pointing out that Theorem 1 provides only a *sufficient* condition for the convergence and may be too stringent in some cases.

We further investigate the stability and the convergence speed of the replicator dynamics in the following theorem, following the guidelines of [36].

**Theorem 2.** *Under the condition that* $K(A - \frac{1}{B-1}) < 1$, *the non-trivial fixed point* $X_P^*$ *is exponentially stable under the replicator dynamics depicted in* (26), *i.e., there exists* $0 \leq k' < 1$ *such that* $|X(t) - X_P^*| \leq (k')^t |X(0) - X_P^*|$.

*Proof.* We show that the replicator dynamics $X_P(t) \rightarrow X_P(t+1)$ in (26) is a contraction.

The contraction is defined as follows: let $(X, d)$ be a metric space, $f$: $X \rightarrow X$ is a contraction if there exists a constant $k' \in [0, 1)$ such that $\forall x, y \in X$, $d(f(x), f(y)) \leq k' d(x, y)$, where $d(x, y) = ||x - y|| = \max_i ||x_i - y_i||$.

To that end, note that:

$$d(f(x), f(y)) = ||f(x)-f(y)|| \leq \left|\left|\frac{\partial f}{\partial x}\right|\right| \cdot ||x-y|| = \left|\left|\frac{\partial f}{\partial x}\right|\right| d(x, y).$$

If the Jacobian $\left|\left|\frac{\partial f}{\partial x}\right|\right| \leq k'$, then $f$ is a contraction.

By some algebraic operations, we can bound the Jacobian as

$$||J||_\infty = \max_{X_P(t)\in(0,1)} \left|\frac{\partial X_P(t+1)}{\partial X_P(t)}\right| \leq 1 - K(A - \frac{1}{B-1}).$$

Hence, since the condition $K(A - \frac{1}{B-1}) < 1$ holds, i.e., $||J||_\infty \leq k' \triangleq 1 - K(A - \frac{1}{B-1}) < 1$, $X_P^*$ is exponentially stable where $k'$ is the exponential convergence speed. □

[2]Note that the same analysis can be conducted for (25).

## VIII. Numerical Results

In this section, we analyze and discuss the numerical results obtained from solving our joint pricing and spectrum access game model in different cognitive radio scenarios. More in detail, we measure the sensitivity of the operators' utilities and prices, as well as users' equilibrium flows and costs, to different parameters like the total traffic $\lambda$ accepted in the network and the channel capacity $C$.

Before doing so, let us first consider an example of a primary operator utility function ($U_P$). Figure 2 shows this latter as a function of the price $p_1$ set by the Primary Operator (the price $p_2$ has been fixed to the Nash equilibrium value), with $\alpha = 1$, $C = 100$ and $\lambda = 10$. By simply deriving and using the second order derivative test, it can be proved that the PO's revenue has a global maximum, as illustrated in the figure, since for small $p_1$ values the incoming primary traffic is priced too low, resulting in a low PO revenue, while for high $p_1$ values few users choose the primary network, thus diminishing its profitability.

### A. Effect of the traffic accepted in the network ($\lambda$)

We first consider a CRN scenario with maximum channel capacity $C = 100$ and total accepted traffic $\lambda$ varying in the $[0, 100]$ range. The parameter $\alpha$, which expresses the relative importance of the experienced delay with respect to the access cost, is set to 1, unless otherwise stated.

Figures 3(a) and 3(b) show the prices set at the Nash equilibrium by the Primary ($p_1$) and the Secondary operator ($p_2$), respectively, in the two considered scenarios (the PO and SO play *TOGETHER*, the PO plays *BEFORE* the SO, anticipating the choices of this latter). The difference between the prices set by the operators in these two scenarios can be better appreciated in Figures 4(a) and 4(b) for the PO and SO, respectively. All numerical results illustrated in Figures 3 and 4 are summarized in Table IV.

It can be observed (Figure 4(a)) that in the *BEFORE* scenario, the PO sets a higher price than in the *TOGETHER* scenario, until the network is overloaded ($\lambda \leq 80$); above this threshold, the price set by the PO in the former scenario is lower than in the latter. As for the price set by the Secondary operator (Figure 4(b)), it is always higher in the *BEFORE* than in the *TOGETHER* scenario, and such difference increases consistently for increasing $\lambda$ values. This is the reason why the PO in the *BEFORE* scenario can lower his price while still attracting the large majority of network users, as we will show in the following.

The corresponding equilibrium traffic sent by primary ($\lambda_P$) and secondary users ($\lambda_S$) is illustrated in Figures 5(a) and 5(b) as a function of $\lambda$, for both the considered scenarios.

We can observe that:

- The traffic accepted (and consequently, the overall fraction of users) in the primary network, $\lambda_P$, always increases with the offered traffic, until finally, when $\lambda \to C$, all users choose the primary network. This is due to the superior attractiveness of such network (in terms of the delay experienced by users) with respect to the secondary one, since resources are licensed to primary users and SUs always observe a higher delay than PUs.
- Furthermore, concerning $\lambda_P$, in the *BEFORE* scenario the PO admits (slightly) less traffic than the SO, when $\lambda < 80\%$ of the total capacity $C$ (Figure 5(a)); this is due to the fact that the equilibrium price $p_1$ set by the PO in such scenario is higher than in the *TOGETHER* case (see Figure 4(a)), which in turn makes $\lambda_P$ decrease. In the high traffic regime, the PO increasingly attracts more

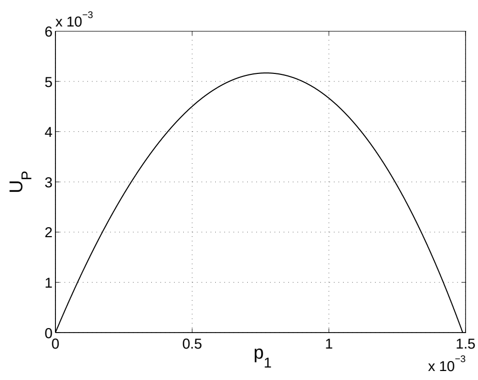
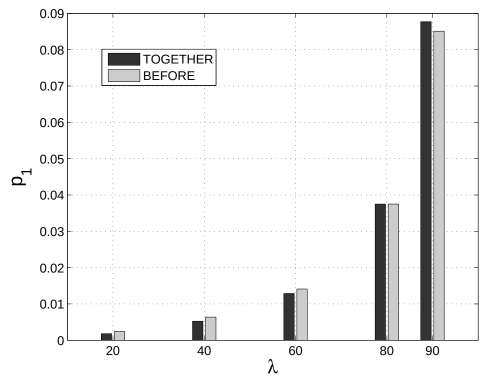
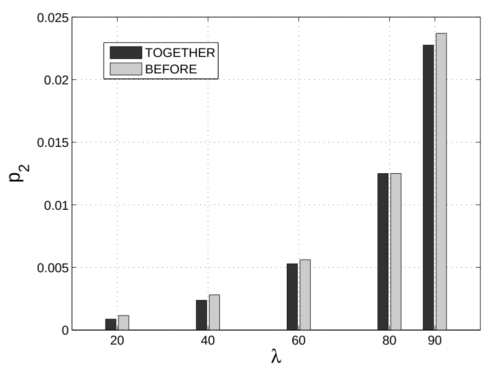



Fig. 2. Primary Operator's utility ($U_P$) as a function of the imposed price $p_1$ in the *TOGETHER* scenario. Price $p_2$ has been fixed to the Nash equilibrium value.




(a)




(b)

Fig. 3. (a) Equilibrium price $p_1$ set by the Primary operator and (b) Equilibrium price $p_2$ set by the Secondary operator, as a function of the total traffic $\lambda$ offered to the network for both the *BEFORE* and *TOGETHER* scenarios.

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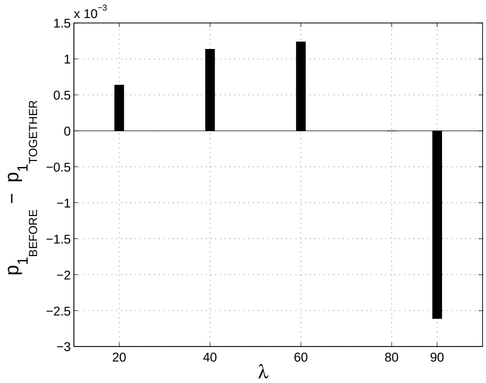
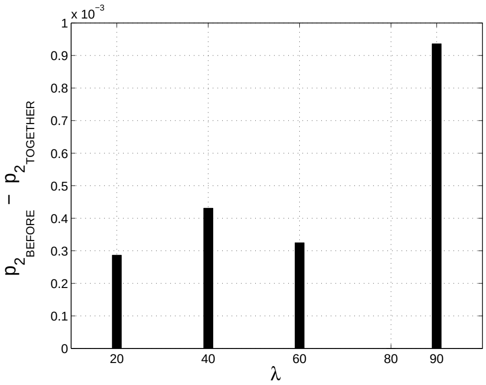
TABLE IV
EQUILIBRIUM PRICES $p_1$ AND $p_2$ SET BY THE PO/SO (AS WELL AS THEIR DIFFERENCE), FOR DIFFERENT VALUES OF THE TOTAL TRAFFIC $\lambda$ OFFERED TO THE NETWORK FOR BOTH THE *BEFORE* AND *TOGETHER* SCENARIOS.

| $\lambda$ | $p1_{TOGETHER} \times 10^{-3}$ | $p1_{BEFORE} \times 10^{-3}$ | $p2_{TOGETHER} \times 10^{-3}$ | $p2_{BEFORE} \times 10^{-3}$ | $\Delta p_1 \times 10^{-3}$ | $\Delta p_2 \times 10^{-3}$ |
|---|---|---|---|---|---|---|
| 20 | 1.806 | 2.441 | 0.868 | 1.154 | 0.635 | 0.286 |
| 40 | 5.242 | 6.375 | 2.374 | 2.805 | 1.133 | 0.431 |
| 60 | 12.885 | 14.122 | 5.288 | 5.613 | 1.237 | 0.325 |
| 80 | 37.5 | 37.5 | 12.5 | 12.5 | 0 | 0 |
| 90 | 87.724 | 85.112 | 22.761 | 23.697 | - 2.612 | 0.936 |

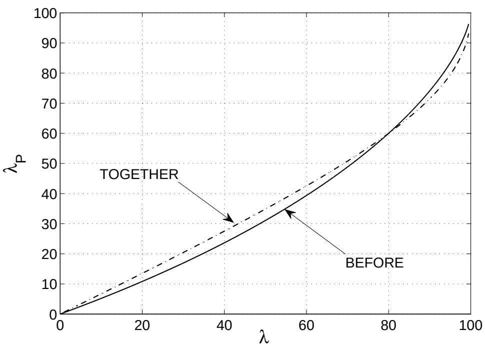



(a)

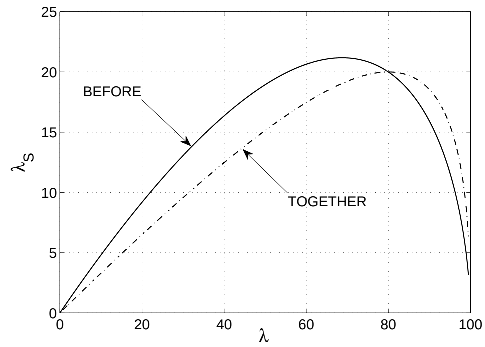



(b)

Fig. 4. (a) Difference in the equilibrium prices $p_1$ set by the Primary operator in the *TOGETHER* and *BEFORE* scenarios, and (b) difference in the equilibrium prices $p_2$ set by the Secondary operator in the same scenarios.




(a)




(b)

Fig. 5. Equilibrium traffic sent by primary ($\lambda_P$) and secondary users ($\lambda_S$) as a function of the total traffic, $\lambda$, accepted in the network, for both the *TOGETHER* and *BEFORE* scenarios.

traffic due to the significantly lower delay experienced in the primary network, while the SO increases $p_2$ in an effort to increase his utility in spite of the customer rush towards the primary network (more specifically, fewer clients choose the SO, who reacts by raising his access price $p_2$ in order to increase his revenue, reaction which in turn accentuates this phenomenon).

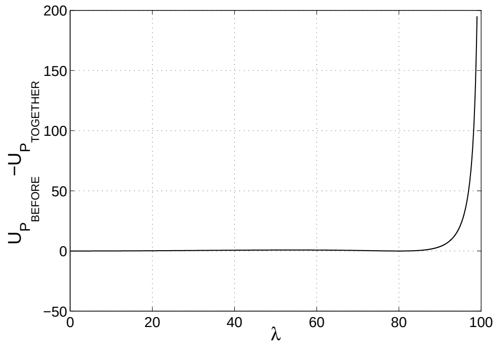
- Concerning $\lambda_S$, its derivative with respect to $\lambda$ is always decreasing: it is increasingly less attractive to be a secondary user than a primary one, since for increasing $\lambda$ values the delay tends to dominate in the total cost perceived by the user.

We now focus our analysis on operators' utility, which we recall is defined as the product of the price set by the operator and the total flow transmitted by users that choose such operator. Figures 6(a) and 6(b) show, respectively, the difference in utilities for the Primary ($\Delta U_P$) and Secondary operator ($\Delta U_S$) in the *TOGETHER* and *BEFORE* scenarios.

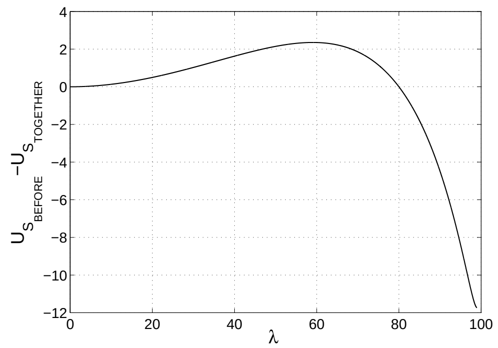
It can be observed that it is increasingly more convenient for the PO to be a leader, anticipating the SO, and this is reflected in the utility, which consistently grows for increasing $\lambda$ values. At the same time, for low and medium $\lambda$ values ($\lambda < 0.8C$), even the SO obtains a higher utility in the *BEFORE* scenario. This means that in such scenario, both operators achieve an economic advantage at the expense of the total price paid by cognitive radio users.

### B. Effect of the channel capacity ($C$)

We now consider a variation of this network scenario, doubling the channel capacity $C$ to 200; the total traffic admitted in the primary network is illustrated in Figure 7. The trend is the same as already shown in Figure 5(a), and




(a)




(b)

Fig. 6. (a) Difference in utilities $U_P$ of the Primary operator when he plays *BEFORE* and *TOGETHER* with the SO. (b) Difference in utilities $U_S$ of the secondary operator in the same scenarios.

a similar behavior can be observed for the secondary traffic, which is not reported for the sake of brevity.

On the other hand, Figure 8 shows the equilibrium traffic sent by primary users as a function of the wireless channel capacity $C$, with $\lambda$ fixed to 100. It can be observed that $\lambda_P$ tends to $\frac{\lambda}{2}$ ($= 50$ in this case) in the *BEFORE* scenario, and to $\frac{2\lambda}{3}$ ($\approx 66.6$) in the *TOGETHER* scenario[3]. This behavior is in line with what already observed in Figure 7, since when $\lambda$ is consistently lower than $C$, the Primary operator who plays before the SO (*BEFORE* scenario) tends to admit less traffic than this latter.

[3]It suffices to compute the limit for $C \to \infty$ of $\lambda_P$ in expression (6), substituting the equilibrium values $p_1, p_2$ for both the considered scenarios. Note that such limit is independent of $\alpha$.

We further illustrate in Figure 9 the chosen price as well as the utility perceived by the Primary operator, in both the considered scenarios, for increasing values of the channel capacity $C$ and a total accepted traffic $\lambda$ fixed to 100 (note that the prices $p_1$ set by the PO, illustrated in Figure 9(a), almost overlap in the two considered scenarios). A similar trend can be observed for both the price and utility of the Secondary operator (see Figure 10).

In summary, as the available capacity increases, operators fix increasingly lower prices, achieving a lower total revenue.

The impact of $C$ on the Price of Anarchy is further investigated in the following subsection VIII-C.

### *C. Efficiency of the reached equilibria: Price of Anarchy (PoA)*

We now measure the efficiency of the equilibria reached by the system. The Price of Anarchy (PoA), which in our game coincides with the Price of Stability due to the uniqueness of the equilibria reached by operators and users, is illustrated in Figure 11 for both the *TOGETHER* ($PoA_T$) and *BEFORE* scenarios ($PoA_B$).

When both operators play together, the PoA is equal to 1 for both extreme cases ($\lambda = 0$ and $\lambda = C$). Furthermore, it has a maximum equal to 1.0127 for $\frac{\lambda}{C} = \frac{2}{3}$, which means that, in such scenario, the equilibrium reached by the system is only $\approx 1.3\%$ worse (in terms of the overall experienced delay) with respect to the socially optimal solution. In the *BEFORE* scenario, the PoA is also low, but the trend exhibited by such performance figure differs from the previous scenario, since the PoA tends to infinity for $\lambda$ approaching the channel capacity $C$. This is due to the fact that the total cost for users at equilibrium increases significantly faster than the social welfare, especially for high $\lambda$ values.

As a consequence, such situation should be avoided by market controllers either 1) by controlling the admitted traffic $\lambda$, imposing that it does not exceed a predefined fraction of the available channel capacity, or 2) by preventing the *BEFORE*

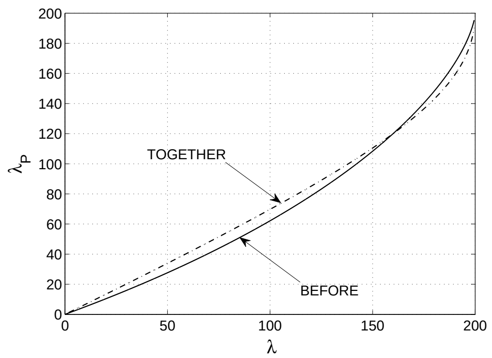



Fig. 7. Equilibrium traffic sent by primary users ($\lambda_P$) as a function of the total traffic, $\lambda$, accepted in the network, for both the *TOGETHER* and *BEFORE* scenarios. The total channel capacity is $C = 200$.

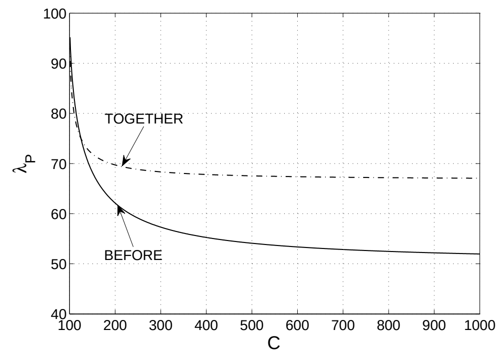



Fig. 8. Equilibrium traffic sent by primary users ($\lambda_P$) as a function of the channel capacity $C$ for both the *TOGETHER* and *BEFORE* scenarios. The total traffic offered to the network, $\lambda$, is fixed and equal to 100.

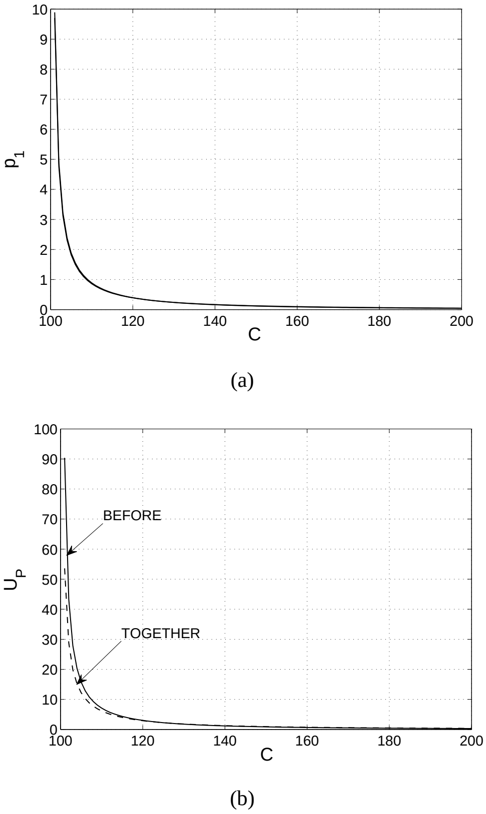



Fig. 9. (a) Primary operator's price $p_1$ and (b) utility $U_P$ as a function of the channel capacity $C$ for both the *TOGETHER* and *BEFORE* scenarios. The total traffic offered to the network, $\lambda$, is fixed and equal to 100. Note that prices $p_1$ practically overlap in the two considered scenarios.

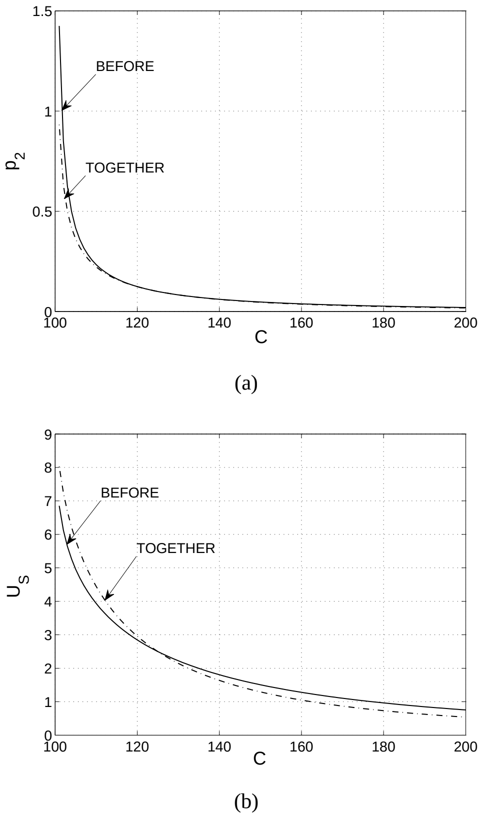



Fig. 10. (a) Secondary operator's price $p_2$ and (b) utility $U_S$ as a function of the channel capacity $C$ for both the *TOGETHER* and *BEFORE* scenarios. The total traffic offered to the network, $\lambda$, is fixed and equal to 100.

scenario to occur, imposing antitrust policies to limit dominant position abuse.

Figure 12 further reports the PoA as a function of the channel capacity $C$ for both the considered scenarios; $\lambda$ is fixed and equal to 100. It is not surprising that both curves decrease rapidly with $C$, since, as already observed in Figure 11, when $\lambda$ is consistently lower than $C$, the $PoA \to 1$ in both scenarios.

In summary, we can conclude that, apart from the limiting case illustrated before for very high traffic loads, the quality of the reached equilibria is indeed excellent: when the system is loaded at less than 95%, which is a reasonable operating region, the PoA is always less than 1.1, which means a loss of efficiency of 10% with respect to the social optimum. The system hence converges to a stable state which is globally very efficient.

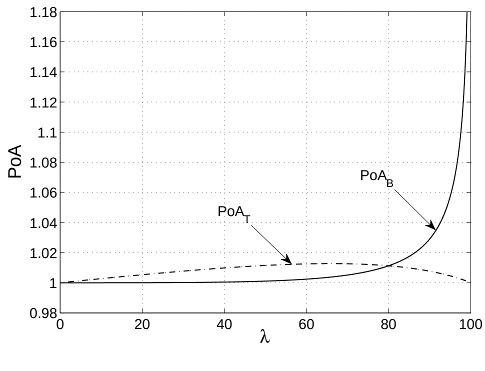



Fig. 11. The Price of Anarchy as a function of the total traffic offered to the network, $\lambda$, in both the *TOGETHER* ($PoA_T$) and *BEFORE* ($PoA_B$) scenarios.

### *D. Replicator Dynamics for the Network Selection Game*

We now analyze the convergence of the proposed replicator dynamics, fixing $\lambda = 30$ and $C = 100$. Figure 13 illustrates such convergence (expressed in steps needed in the replicator dynamics) of network users to a stationary solution, for different values of the parameter $K$ in equation (24), namely 1, 5 and 10. More specifically, the figure reports the fraction $X_P$ of users that choose the primary network. We consider both cases where the initial fraction of such users is close to zero (Figure 13(a)) and one (Figure 13(b)).

Note that the speed of convergence to the unique stable equilibrium point of the dynamics ($X_P^* \approx 0.68$, in such scenario) increases for increasing $K$ values. Furthermore, when $p_1$ and $p_2$ are equilibrium price values, we observe that the convergence conditions demonstrated in Theorems 1 and 2 for our proposed replicator dynamics (see the previous section) are always satisfied.

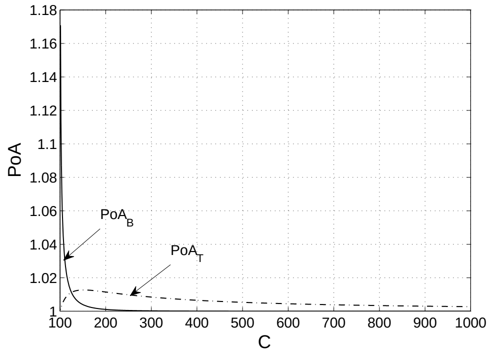
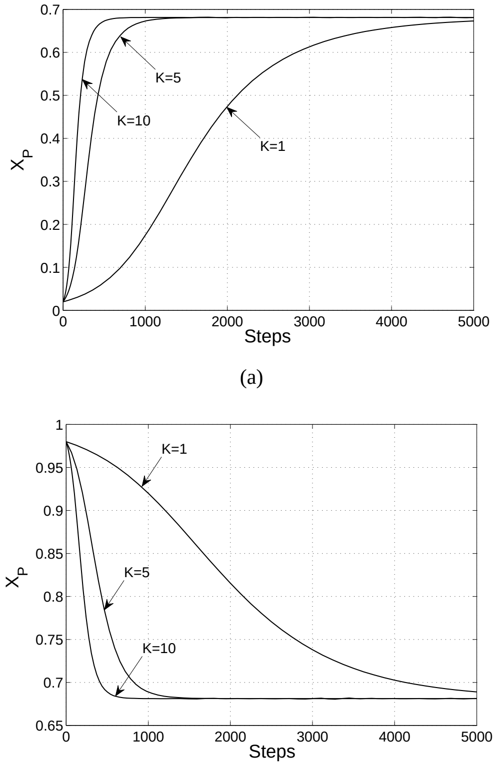



Fig. 12. The Price of Anarchy as a function of the channel capacity $C$ for both the *TOGETHER* ($PoA_T$) and *BEFORE* ($PoA_B$) scenarios. The total traffic offered to the network, $\lambda$, is fixed and equal to 100.




(b)

Fig. 13. Convergence of Primary Users to the stationary point ($X_P^* \approx 0.68$). The initial point is (a) *lower* or (b) *higher* than the equilibrium.

## IX. Conclusion

In this paper, we tackled a fundamental problem related to Cognitive Radio Networks, i.e., the joint pricing and Primary/Secondary network selection process. More specifically, we considered a CRN scenario which is composed of primary/secondary networks and a set of Cognitive Radio users who must decide whether to subscribe to the primary network for guaranteed bandwidth or to access the secondary network, paying a lower price at the expense of possible service degradation (in terms of experienced delay and congestion). At the same time, we studied the pricing game between the Primary and Secondary operators, considering two practical cases where such operators fix their access price simultaneously, and where the PO anticipates the SO strategy, exploiting his dominant position.

We computed optimal, stable pricing values and network selection settings; furthermore, we studied network users' dynamics using a population game model, and we determined its convergence properties under replicator dynamics. Numerical results demonstrate that our game model captures the main factors behind cognitive network pricing and access network selection, thus representing a promising framework for the design and understanding of cognitive radio systems.

A key finding of the present study is that the advantage for the PO to play before the SO can be significant, especially in a high traffic regime; this has an adverse impact on customers' choices, since in such situation the equilibria reached by cognitive radio users drift away from the social optimum, and the Price of Anarchy tends to infinity. It is therefore important (e.g., for government, regulation authorities), to implement actions that prevent or limit such dominant position abuse, if possible.

Apart from this limiting case, which occurs exclusively for very high traffic regimes, we observe that the quality of the reached equilibria is excellent: when the system is loaded at less than 95%, which seems a reasonable operating region, the PoA is always less than 1.1 (regardless of the order in which operators fix their price), which means a loss of efficiency of 10% with respect to the social optimum. Hence, the system is guaranteed to converge to a stable state which is very efficient from a social point of view.

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