Orlik-Solomon Type Algebras

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Abstract. We introduce χ-algebras, and show that a χ-algebra has the NBC basis property. We also show that a certain ideal used in the construction has the so-called BC basis property. The Orlik-Solomon algebra of a matroid, the Orlik-Terao algebra of a set of vectors, and the Cordovil algebra of an oriented matroid are χ-algebras. We define a new χ-algebra from a set of vectors, close to the Orlik-Terao, Orlik-Terao, and Cordovil algebras, but nevertheless different. Our proof provides a unified short and elementary proof of the NBC basis property for these algebras.

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1. INTRODUCTION

In a vector space, a *(central) hyperplane arrangement* is a finite collection of codimension 1 subspaces. The *matroid of an hyperplane arrangement* can be defined by saying that a subset of the arrangement is independent if and only if the codimension of its intersection is equal to its cardinality. Manifolds defined as complements of complex hyperplane arrangements are important in the Aomoto-Gelfand theory of A-hypergeometric functions. In [5] [6] the cohomology algebra of a manifold of this form is shown to be isomorphic to the Orlik-Solomon algebra of the matroid of the arrangement. This result has motivated further research on Orlik-Solomon algebras. It is known that Orlik-Solomon algebras of matroids have the *NBC basis property*: the NBC subsets of a matroid constitute a basis of its Orlik-Solomon algebra. We refer the reader to [1] [7] for more details.

In Section 2, we construct χ-algebras as a generalization of Orlik-Solomon algebras. A slight modification of the usual construction permits to show, by a short and elementary proof, that a χ-algebra has the NBC basis property. Furthermore, we also obtain that an certain ideal, used in the construction, has the BC basis property. In Section 3, we verify that the Orlik-Solomon algebra of a matroid, and also two other algebras of the literature, namely the Orlik-Terao algebra of a set of vectors [8], introduced to answer a question of K. Aomoto and themselves, and the Cordovil algebra of an oriented matroid [3], defined to disprove a question of [8], are χ-algebras. Finally, in Section 4, we define a new χ-algebra of a set of vectors by mixing the definitions of Orlik-Solomon and Orlik-Terao algebras.

2. χ-ALGEBRAS

Let M be a matroid on a (finite) set E. We say that a subset $U \subseteq E$ of elements is *unidependent* if it contains exactly one circuit, denoted by C_U. For any $e \in C_U$ the subset $U \setminus e$ is independent. This property characterizes unidependents among dependents: *a dependent D is unidependent if and only if there is $e \in D$ such that $D \setminus e$ is independent.* This equivalence, which follows easily from circuit elimination in matroids, will be crucial in Lemma 2.1 and 2.3.

In the sequel E is linearly ordered. Let X be an independent of M. We say that an element $e \in E \setminus X$ is *(externally) active* with respect to X if $X \cup e$ contains a circuit with smallest element e. An independent set with at least one active element is said to be *active*, and *inactive* otherwise. We denote by $\alpha(X)$ the smallest active element with respect to an active independent X. Inactive independents are often called *NBC subsets* in the literature, since a subset of E is an inactive independent if and only if it contains no broken bircuit, where a *broken circuit* is any set obtained by removing the smallest element from a circuit. For a unidependent U, we denote by e_U the smallest element of C_U.

Let $\mathcal{E}$ be the associative algebra on a field K generated by the elements of

E, and satisfying the relations $e * e = 0$ for all $e \in E$ and $e' * e = a_{e,e'} e * e'$ with $0 \neq a_{e,e'} \in K$ for all $e < e'$ in E. Both the free exterior algebra and the free commutative algebra with squares zero generated by the elements of E are such algebras (take $a_{e,e'} = -1$ resp. $a_{e,e'} = 1$ for all $e, e' \in E$, $e < e'$).

The *standard form* of a subset $X \subseteq E$ is $X = \{x_1, x_2, \ldots, x_k\}$ with $x_1 < x_2 < \ldots < x_k$. In the sequel we will denote by the same letter - if no confusion results - a subset $\{x_1, x_2, \ldots, x_k\}$ of E in standard form and the corresponding (pure) element $x_1 * x_2 \ldots * x_k$ of $\mathcal{E}$. Most often we will write $x_1 x_2 \ldots x_k$ instead of $x_1 * x_2 \ldots * x_k$.

Let χ be a mapping of the set of subsets of E into K. We define the χ-*boundary* of a subset $X = \{x_1, x_2 \ldots x_k\}$ in standard form of E by

$$\partial X = \partial_\chi(X) = \sum_{i=1}^{i=k} (-1)^i \chi(X \setminus x_i)(X \setminus x_i)$$

We extend ∂ to $\mathcal{E}$ by linearity.

Let $\mathcal{I} = \mathcal{I}_\chi(M)$ be the (right) ideal of $\mathcal{E}$ generated by the (χ-)boundaries of circuits of M. We say that

$$\mathcal{A} = \mathcal{A}_\chi(M) = \mathcal{E}/\mathcal{I}_\chi(M)$$

is a χ-*algebra* if χ satisfies the following properties (UC1) and (UC2):

(UC1) *for any independent X of M we have $\chi(X) \neq 0$*

(UC2) *for any unidependent U of M there is $a \in K$, $a \neq 0$, such that*

$$\partial U = a(\partial C_U)(U \setminus C_U)$$

It can be observed that (UC2) implies that $\chi(U) = 0$ for a unidependent U containing no basis of M. Values of a χ on other dependents are irrelevant. In the examples of Sections 3 and 4, we have $\chi(D) = 0$ for any dependent D.

Lemma 2.1

If $\mathcal{A}_\chi(M)$ is a χ-algebra, then the inactive independents of M linearly generate $\mathcal{A}_\chi(M)$.

Proof

Let X be an active independent, and $e \in E \setminus X$ be active with respect to X. Then the set $X \cup e$ is unidependent. By (UC2) we have $\partial(X \cup e) = 0$ in $\mathcal{A}$. Using this relation, since $\chi(X) \neq 0$ by (UC1), we can express X as a linear combination of the independents $\{X \cup e \setminus f \mid f \in C_{X \cup e} \setminus e\}$, which are all smaller than X in the lexicographical ordering. Lemma 2.1 follows by induction. □

This proof, by lexicographic compression, is an immediate extension of proofs of the literature [1] [7].

We now slightly transform the definition of $\mathcal{A}$ as follows. Let $\mathcal{J} = \mathcal{J}(M)$ be the ideal of $\mathcal{E}$ generated by the circuits of M. Set $\mathcal{E}' = \mathcal{E}'(M) = \mathcal{E}/\mathcal{J}$. Clearly the set of independents of M constitutes a basis of $\mathcal{E}'$, the *algebra of independents* of M. Let $\mathcal{I}' = \mathcal{I}'_\chi(M)$ be the ideal of $\mathcal{E}'$ generated by the boundaries of the circuits of M.

Lemma 2.2

If χ satisfies (UC1) then $\mathcal{A}_\chi = \mathcal{E}'/\mathcal{I}'_\chi$.

Proof

Let C be an ordered circuit and and $e \in C$. Since $ee = 0$, we have $(\partial C)e = \pm\chi(C \setminus e)(C \setminus e)e$. Hence there is $a \in K$, $a \neq 0$, such that $(\partial C)e = \pm a\chi(C \setminus e)C$. If χ satisfies (UC1) we have $\chi(C \setminus e) \neq 0$, hence C is in $\mathcal{I}$. It follows that $\mathcal{J} \subseteq \mathcal{I}$. Therefore $\mathcal{A} = \mathcal{E}/\mathcal{I} = (\mathcal{E}/\mathcal{J})/(\mathcal{I}/\mathcal{J}) = \mathcal{E}'/\mathcal{I}'$. □

Lemma 2.3

If χ satisfies (UC2) then the boundaries of the unidependents of M linearly generate the ideal $\mathcal{I}'(M)$.

Proof

By definition $\mathcal{I}'$ is linearly generated in $\mathcal{E}'$ by products $(\partial C)X$ with C circuit and X independent. A product $(\partial C)X$ is a linear combination of products $(C \setminus e)X$ with $e \in C$. If $C \cup X$ is not unidependent, then all $(C \setminus e) \cup X$ contain a circuit implying $(C \setminus e)X = 0$ in $\mathcal{E}'$, hence $(\partial C)X = 0$. Suppose $C \cup X$ is unidependent. If $C \cap X \neq \emptyset$, then for $e \in C \cap X$ we have $C \subseteq (C \setminus e) \cup X$, hence $(C \setminus e)X = 0$ in $\mathcal{E}'$, and for $e \in C \setminus X$ there is $f \in (C \setminus e) \cap X$, hence $(C \setminus e)X = 0$ since $ff = 0$. Again $(\partial C)X = 0$. Therefore we may suppose $C \cap X = \emptyset$. In this case, by (UC2), we have $(\partial C)X = a\partial(C \cup X)$. Lemma 2.3 follows. □

We say that a unidependent $U \subseteq E$ is *inactive* if there is a (necessarily unique) active independent X such that $U = X \cup \alpha(X)$.

Lemma 2.4

If χ satisfies (UC1), then any boundary of a unidependent of M is a linear combination of boundaries of inactive unidependents in $\mathcal{E}'$.

Proof

The proof uses lexicographic compression. Let $U \subseteq E$ be a unidependent, then $X = U \setminus e_U$ is independent and active. If U is not inactive, we have $\alpha(X) < e_U$. Let $D = U \cup \alpha(X) = \{x_1, x_2, \ldots, x_k\}$ in standard form. By an obvious extension of the classical relation $\partial\partial = 0$ of topological algebra, we have $\sum_{i=1}^{i=k}(-1)^i \partial(D \setminus x_i) = 0$. Let m be the integer such that $x_m = \alpha(X)$. For $i < m$ the set $D \setminus x_i$ is dependent, but not unidependent, hence $\partial(D \setminus x_i) = 0$ in $\mathcal{E}'$. We have $U = D \setminus x_m$. For $i > m$ the set $D \setminus x_i$ is unidependent, and lexicographically smaller than U. Hence the boundary of any non inactive unidependent is a linear combination of boundaries of lexicographically smaller unidependents. Applying inductively this reduction, we obtain Lemma 2.4. □

With above notation we have

Theorem 2.5

Let M be a matroid on a linearly ordered set, and $\mathcal{A}_\chi(M)$ be a χ-algebra. Then the inactive independents of M constitute a basis of $\mathcal{A}_\chi(M)$, and the boundaries of the inactive unidependents of M constitute a basis of $\mathcal{I}'_\chi(M)$.

Proof

By Lemma 2.1, $\dim(\mathcal{A})$ is at most the number of inactive independents. By Lemmas 2.3 and 2.4, $\dim(\mathcal{I}')$ is at most the number of active independents. By Lemma 2.2, $\dim(\mathcal{A}) + \dim(\mathcal{I}') = \dim(\mathcal{E}')$, which is the number of independents, hence both $\dim(\mathcal{A})$ and $\dim(\mathcal{I}')$ achieve their bounds. □

We have referred to the first property proved in Theorem 2.5 as the NBC basis property. Consistently, we will say that the second one is the *BC basis property.* We point out that the BC basis property is not intrinsic to the χ-algebra itself, but concerns an ideal used in its construction.

3. THREE KNOWN χ-ALGEBRAS

We apply Section 2 to three algebras of the literature : the Orlik-Solomon algebra of a matroid [5] [6], the Orlik-Terao algebra of a set of vectors [8], and the Cordovil algebra of an oriented matroid [3].

Proposition 3.1

The Orlik-Solomon algebra of a matroid, the Orlik-Terao algebra of a set of vectors, and the Cordovil algebra of an oriented matroid, are χ-algebras.

Proof

Let K be a field. In each case we specify the relevant matroid M, algebra $\mathcal{E}$ over K, and mapping χ, and check the equality of the considered algebra with $\mathcal{A}_\chi(M)$, and properties (UC1), (UC2).

1. *The Orlik-Solomon algebra*

Let M be a matroid on an ordered set E. The Orlik-Solomon algebra $OS(M)$ is the quotient of the free exterior algebra $\mathcal{E}$ generated by E over K, by the ideal generated by boundaries (in the classical sense) of circuits of M [5] [6] [7].

We claim that $OS(M)$ is the χ-algebra obtained for M and $\mathcal{E}$ as in the above definition, and $\chi_1 = \chi$ defined for $X \subseteq E$ by $\chi(X) = 1$ if X is independent and by $\chi(X) = 0$ otherwise. We have clearly $OS(M) = \mathcal{A}_\chi(M)$. Property (UC1) holds by definition, and checking (UC2) is a one-line proof (in this case $a = \pm 1$).

2. *The Orlik-Terao algebra*

Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in a vector space over K. The Orlik-Terao algebra $OT(\mathcal{V})$ is the quotient of the free commutative algebra $\mathcal{E}$, with squares zero, generated by $\mathcal{V}$ over K, by the ideal generated by the elements of $\mathcal{E}$ of the form $\sum_{j=1}^{j=k} \lambda_{i_j} v_{i_1} v_{i_2} \ldots v_{i_{j-1}} v_{i_{j+1}} \ldots v_{i_k}$ for any minimal non-trivial linear dependency $\sum_{j=1}^{j=k} \lambda_{i_j} v_{i_j} = 0$ among the vectors of $\mathcal{V}$ [8].

We claim that $OT(\mathcal{V})$ is the χ-algebra obtained as follows. Let E be any linear ordering of $\mathcal{V}$, the algebra $\mathcal{E}$ be as above, and the matroid M be the matroid of linear dependencies of the vectors in $\mathcal{V}$. To define $\chi_2 = \chi$, we fix a basis B_F for any flat F of the matroid M. Then for X independent in M we define $\chi(X)$ as the determinant of the vectors in X in standard order with respect to the vectors of B_F in standard order, where F is the geometric closure of X in the matroid M. For $D \subseteq E$ dependent, we set $\chi(D) = 0$.

Let $C = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ in standard form be a circuit of M. By an elementary property of determinants, we have $\sum_{j=1}^{j=k} (-1)^j \chi(C \setminus v_{i_j}) v_{i_j} = 0$. By the minimality property of circuits, a linear combination of the vectors in C is determined up to a non zero scalar factor. This proportionality shows that the ideal considered by Orlik-Terao is equal to $\mathcal{I}_\chi(M)$, hence $OT(\mathcal{V}) = \mathcal{A}_\chi(M)$.

Property (UC1) is clear. To prove (UC2), let U be a unidependent of M. Let $C = C_U$ be the unique circuit contained in U. As above, determinants provide a linear relation between vectors in U and a linear relation between vectors in C. In the relation between vectors of U the coefficients of vectors in $U \setminus C$ are $= 0$ (since all vectors in C occurs in the corresponding determinants). Hence, again due to the minimality of C, these two relations are the same up to a scalar factor, and coefficients of corresponding vectors are proportional. By definition of χ, the coefficients of the relation between the vectors of U are exactly the coefficients occurring in ∂U. It follows that the non zero terms of ∂U are equal to constant times the corresponding terms of ∂C, where the correspondance is the commutative multiplication by $U \setminus C$. Hence $\partial U = a(\partial C)(U \setminus C)$, i.e. (UC2) holds.

3. *The Cordovil algebra*

Let OM be an oriented matroid on a set E. The Cordovil algebra $C(OM)$ is the quotient of the free commutative algebra $\mathcal{E}$, with squares zero, generated by E over K, by the ideal generated by the elements of $\mathcal{E}$ of the form $\sum_{e\in C} sg_C(e)(C\setminus e)$for any signed circuit C of OM with signature sg_C [3].

We claim that $C(OM)$ is the χ-algebra obtained as follows. We choose any linear ordering on E. Let the algebra $\mathcal{E}$ be as above, and the matroid M be the underlying matroid of OM. To define $\chi_3 = \chi$, we fix a basis signature independently in all restrictions of OM to a flat F of M (we recall that a basis signature of an oriented matroid is determined up to a factor ± 1). Then for X independent in M we define $\chi(X)$ as the sign of X in standard form for the chosen basis signature of the submatroid of OM on the geometric closure of X in M. For $D \subseteq E$ dependent, we set $\chi(D) = 0$.

Let $C = \{x_1, x_2, \ldots, x_k\}$ be a (signed) circuit of OM, in standard form. By the pivot property of the basis signature of an oriented matroid [2][4] for any $1 \le i < j \le k$ we have $\chi(C\setminus x_i)\chi(C\setminus x_j) = (-1)^{j-i} sg_C(x_i) sg_C(x_j)$. It follows that $\sum_{e\in C} sg_C(e)(C\setminus e) = \pm\partial_\chi C$. This proportionality shows that the ideal considered by Cordovil is equal to $\mathcal{I}_\chi(M)$, hence $C(OM) = \mathcal{A}_\chi(M)$.

Property (UC1) is clear. To prove (UC2), let $U = \{y_1, y_2, \ldots, y_\ell\}$ be a unidependent of M in standard form. Let $C = C_U = \{x_1, x_2, \ldots, x_k\}$ be the unique circuit contained in U, in standard form. By the pivot property of basis signatures, for any $1 \le i < j \le k$ we have $\chi(C\setminus x_i)\chi(C\setminus x_j) = (-1)^{j-i} sg_C(x_i) sg_C(x_j)$, and for any $1 \le i < j \le \ell$ we have $\chi(U\setminus y_i)\chi(U\setminus y_j) = (-1)^{j-i} sg_C(y_i) sg_C(y_j)$ where $sg_C(y_i) = 0$ if $y_i \notin C$. As above we have $\sum_{e\in C} sg_C(e)(C\setminus e) = \pm\partial_\chi C$. Using the commutativity of the product in $\mathcal{E}$, we see that $\sum_{e\in C} sg_C(e)(C\setminus e)(U\setminus C) = \pm\partial_\chi U$. Hence $\partial_\chi U = \pm(\partial_\chi C)(U\setminus C)$. □

Corollary 3.2 ([3] [5] [6] [8] for the NBC basis property)

The Orlik-Solomon algebra of a matroid, the Orlik-Terao algebra of a set of vectors, and the Cordovil algebra of an oriented matroid, have the NBC basis property. The ideal supplementing these algebras in the algebra of independents, as defined in Lemma 2.2, has the BC basis property.

4. A NEW χ-ALGEBRA

In the case of a vectorial matroid over $\mathbb{R}$, the Orlik-Solomon algebra has an appealing geometric interpretation as a signed volume calculus in affine spaces. Let $x_1, x_2, \ldots, x_k$ be points in $\mathbb{R}^n$. The wedge product $x_1 \wedge x_2 \wedge \ldots \wedge x_k$ can be interpreted as the signed volume of the simplex defined by $x_1, x_2, \ldots, x_k$ if these points are affinely independent, and 0 otherwise.

Boundary relations have the following interpretation. Let $x_1, x_2, \ldots, x_k$ be points of a minimal affine dependency. Suppose that these k points constitute the vertices of a convex polytope of dimension $k-2$. There are two ways to triangulate this convex polytope (to go from one to the other is the *switching transformation*, a fundamental tool in triangulation theory). The relation $\partial(x_1 \wedge x_2 \wedge \ldots \wedge x_k) = 0$ expresses that the sum of the volumes of the simplices in the two triangulations are equal (both, of course, equal to the volume of the polytope). When the points does not form a convex polytope, a similar interpretation can be given (there are altogether $[d/2]+1$ cases in dimension d).

The Orlik-Terao algebra and the Cordovil algebra can be interpreted in essentially the same way.

From the point of view of this geometric interpretation, we observe that in the Orlik-Solomon algebra the signed volume of a simplex is implicit in the wedge product, in the Orlik-Terao algebra the signed volume is explicit in χ, in the Cordovil algebra the sign is explicit in χ and the scalar volume is implicit in the commutative product. This observation strongly suggests to complete the possible choices by a fourth one, where the sign is implicit in a wedge product and the scalar volume is explicit in χ.

More precisely, let K be an ordered field, and $E = \{v_1, v_2, \ldots, v_n\}$ be a linearly ordered set of vectors in a vector space K^d. Let $\mathcal{E}$ be the free exterior algebra generated by E over K, and M be the matroid of linear dependencies of $\{v_1, v_2, \ldots, v_n\}$. For every flat F of M we fix a basis B_F. For $X \subseteq E$ independent in M, we set $\chi_4(X) = |Det(B_F, X)|$. For $X \subseteq E$ dependent, we set $\chi_4(X) = 0$. Let $\mathcal{A}_4 = \mathcal{A}_{\chi_4}(M)$.

Proposition 4.1

The algebra $\mathcal{A}_4$ is a χ-algebra.

Proof

Property (UC1) is clear. Let U be a unidependent of M. We have $\chi_4(X) = |\chi_2(X)|$. By the proof that the Orlik-Terao algebra is a χ-algebra, we know that there is $a \in K$ such that $\chi_2(U \setminus e) = \pm a\chi_2(C_U \setminus e)$ for all $e \in C_U$. Hence $\chi_4(U \setminus e) = |a|\chi_4(C_U \setminus e)$ for all $e \in C_U$. On the other hand since the Orlik-Solomon algebra is a χ-algebra, we know that $\partial_{\chi_1} U = \pm(\partial_{\chi_1} C_U)(U \setminus C_U)$. It follows that $\partial_{\chi_4} U = \pm a(\partial_{\chi_4} C_U)(U \setminus C_U)$. Hence (UC2) holds. □

Corollary 4.2

The algebra $\mathcal{A}_4$ has the NBC basis property. The ideal supplementing this algebra in the algebra of independents, as defined in Lemma 2.2, has the BC basis property. □

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