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Note

How is a chordal graph like a supersolvable binary matroid?

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To the memory of Claude Berge

Abstract

Let G be a finite simple graph. From the pioneering work of R.P. Stanley it is known that the cycle matroid of G is supersolvable iff G is chordal (rigid): this is another way to read Dirac's theorem on chordal graphs. Chordal binary matroids are in general not supersolvable. Nevertheless we prove that, for every supersolvable binary matroid M, a maximal chain of modular flats of M canonically determines a chordal graph.

MSC: primary 05B35; secondary 05CXX

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1. Introduction and notations

Throughout this note M denotes a matroid of rank r on the ground set $[n] := \{1, 2, \ldots, n\}$. We refer to [7,9] as standard sources for matroid theory. We recall and fix some notation of matroid theory. The restriction of M to a subset $X \subseteq [n]$ is denoted $M|X$. A matroid M is said to be *simple* if all circuits have at least three elements. A matroid M is *binary* if the symmetric difference of any two different circuits of M is a union of disjoint circuits. Graphic and cographic matroids are extremely important examples of binary matroids. The dual of M is denoted M^*. Let $\mathscr{C} = \mathscr{C}(M)$ [resp. $\mathscr{C}^* = \mathscr{C}^*(M) = \mathscr{C}(M^*)$] be the set of circuits [resp. cocircuits] of M. Let $\mathscr{C}_\ell := \{C \in \mathscr{C} : |C| \leqslant \ell\}$. In the following the singleton $\{x\}$ is denoted by x. We will denote by

$$\mathrm{cl}(X) := X \cup \{x \in [n] : \exists C \in \mathscr{C}, C \setminus X = x\},$$

the *closure* in M of a subset $X \subseteq [n]$. We say that $X \subseteq [n]$ is a *flat* of M if $X = \mathrm{cl}(X)$. The set $\mathscr{F}(M)$ of flats of M, ordered by inclusion, is a geometric lattice. The *rank* of a flat $F \in \mathscr{F}$, denoted $r(F)$, is equal to m if there are $m+1$ flats in a maximal chain of flats from $\emptyset$ to F. The flats of rank 1, 2, 3 and $r-1$ are called *points*, *lines*, *planes*, and *hyperplanes*, respectively. A line L with two elements is called *trivial* and a line with at least three elements is called nontrivial (a binary matroid has no line

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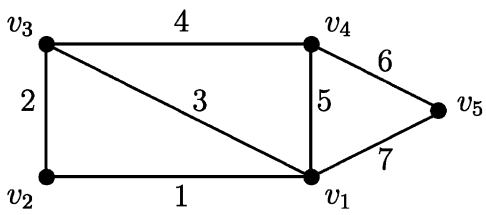

Fig. 1. Graph G_0.

with more than three points). Given a set $X \subseteq [n]$, let $r(X) := r(\mathrm{cl}(X))$. A pair F, F' of flats is called *modular* if

$$r(F) + r(F') = r(F \vee F') + r(F \wedge F').$$

A flat $F \in \mathscr{F}$ is *modular* if it forms a modular pair with every other flat $F' \in \mathscr{F}$. The notion of supersolvable lattices was introduced and studied by Stanley [8]. In the particular case of geometric lattices the definition can be read as follows.

Definition 1.1 (*Stanley [8]*). A matroid M on $[n]$ of rank r is *supersolvable* if there is a maximal chain of modular flats $\mathscr{M}$

$$\mathscr{M} := F_0(=\emptyset) \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F_r(=[n]).$$

We call $\mathscr{M}$ an *M-chain* of M. To the M-chain $\mathscr{M}$ we associate the partition $\mathscr{P}$ of $[n]$

$$\mathscr{P} := F_1 \uplus \cdots \uplus (F_i \backslash F_{i-1}) \uplus \cdots \uplus (F_r \backslash F_{r-1}).$$

We call $\mathscr{P}$ an *M-partition* of M.

We recall that a graph G is said to be *chordal* (or *rigid* or *triangulated*) if every cycle of length at least four has a chord. Chordal graphs are treated extensively in Chapter 4 of [6]. The notion of a "chordal matroid" has also been recently explored in the literature, see [2].

Definition 1.2 (*Barahona and Grötschel [1, p. 53]*). Let M be an arbitrary matroid (not necessarily simple or binary). A circuit C of M has a *chord* i_0 if there are two circuits C_1 and C_2 such that $C_1 \cap C_2 = i_0$ and $C = C_1 \Delta C_2$. In this case, we say that the chord i_0 *splits* the circuit C into the circuits C_1 and C_2. We say that a matroid is *ℓ-chordal* if every circuit with at least ℓ elements has a chord. A simple matroid M is *chordal* if it is 4-chordal.

In this paper, we always suppose that the edges of a graph G are labelled with the integers of $[n]$. If nothing else is indicated we suppose that G is a connected graph. Let $M(G)$ be the *cycle matroid* of the graph G: i.e., the elementary cycles of G, as subsets of $[n]$, are the circuits of $M(G)$. In the same way, the minimal cutsets of a connected graph G (i.e, a set of edges that disconnect the graph) are the circuits of a matroid on $[n]$, called the *cocycle matroid* of G. A matroid is *graphic* (resp. *cographic*) if it is the cycle (resp. cocycle) matroid of a graph. The cocycle matroid of G is dual to the cycle matroid of G and both are binary. The cocycle matroids of the complete graph K_5 and of the complete bipartite graph $K_{3,3}$ are examples of binary but not graphic matroids; see Section 13.3 in [7] for details. The Fano matroid is an example of a supersolvable binary matroid that is neither graphic nor cographic. Finally, note that an elementary cycle C of G has a chord iff C seen as a circuit of the matroid $M(G)$ has a chord.

Example 1.3. Consider the chordal graph $G_0 = G_0(V, [7])$ in Fig. 1 and the corresponding cycle matroid $M(G_0)$. It is clear that

$$\mathscr{M} := \emptyset \subsetneq \{1\} \subsetneq \{1, 2, 3\} \subsetneq \{1, 2, 3, 4, 5\} \subsetneq [7]$$

is an M-chain. The associated M-partition is

$$\mathscr{P} := \{1\} \uplus \{2, 3\} \uplus \{4, 5\} \uplus \{6, 7\}.$$

The linear order of the vertices is such that for every i in $\{2, 3, 4, 5\}$ the neighbors of the vertex v_i contained in the set $\{v_1, \ldots, v_{i-1}\}$ form a clique; this is Dirac's well-known characterization of chordal graphs (see [5,6]). This is also a characterization of graphic supersolvable matroids (see Proposition 2.8 in [8]). That is, a graphic matroid $M(G)$ is supersolvable iff the vertices of G can be labeled as $v_1, v_2, \ldots, v_m$ such that, for every $i = 2, \ldots, m$, the neighbors of v_i contained in the set $\{v_1, \ldots, v_{i-1}\}$ form a clique. We say that a linear order of the vertices of G with the above properties is an *S-label* of the vertices of G.

Ziegler proved that every supersolvable binary matroid without a Fano submatroid is graphic (Theorem 2.7 in [10]). In the next section we answer the following natural question:

- *For a generic binary matroid, what are the relations between the notions of* "*chordal*" *and* "*supersolvable*"?

2. Chordal and supersolvable matroids

Lemma 2.1. *Let M be a simple binary matroid. The following two conditions are equivalent for every circuit C of M*:

(2.1.1) $C \subsetneq \mathrm{cl}(C)$,
(2.1.2) *C has a chord*.
For nonbinary matroids only implication (2.1.2) $\Rightarrow$ (2.1.1) *holds*.

Proof. If $i \in \mathrm{cl}(C) \backslash C$, then there is a circuit D such that $i \in D$ and $D \backslash i \subsetneq C$. As M is binary $D' = D \Delta C$ is also a circuit of M. So i is a chord of C. If i is a chord of C, then clearly $i \in \mathrm{cl}(C)$. Finally, in the uniform rank-two nonbinary matroid $U_{2,4}$, the set $C = \{1, 2, 3\}$ is a circuit without a chord but $C \subsetneq \mathrm{cl}(C) = [4]$. □

Theorem 2.2. *A binary supersolvable matroid M is chordal but the converse does not hold in general*.

Proof. Let $\mathscr{M} := \emptyset \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F_r = [n]$ be an M-chain of M. Suppose by induction that the restriction of M to F_{r-1} is chordal. The result is clear in the case that $C^* := [n] \backslash F_{r-1}$ is a singleton. Suppose that $|C^*| > 1$ and consider a circuit C of M not contained in the modular hyperplane F_{r-1}. Then there are two elements $i, j \in C \cap C^*$ and the line $\mathrm{cl}(\{i, j\})$ meets F_{r-1}. So $C \subsetneq \mathrm{cl}(C)$ and we know from Lemma 2.1 that C has a chord.

A counterexample of the converse is $M^*(K_{3,3})$, the cocycle matroid of the complete bipartite graph $K_{3,3}$. It is easy to see from its geometric representation that it is chordal but not supersolvable (see [10] and page 514 in [7] for its geometric representation). □

Definition 2.3 (*Crapo [4]*). Let M be an arbitrary matroid and consider an integer $\ell \geqslant 2$. The matroid M is *ℓ-closed* if the following two conditions are equivalent for every subset $X \subseteq [n]$:

(2.3.1) X is closed,
(2.3.2) for every subset Y of X with at most ℓ elements we have $\mathrm{cl}(Y) \subseteq X$.
We note that condition (2.3.2) is equivalent to

(2.3.2′) for every circuit C of M with at most $\ell + 1$ elements

$$|C \cap X| \geqslant |C| - 1 \implies C \subseteq X.$$

Definition 2.4. Let $\mathscr{C}'$ be a subset of $\mathscr{C}$, the set of circuits of M. Let $\mathrm{cl}_\Delta(\mathscr{C}')$ denote the smallest subset of $\mathscr{C}$ such that:

(2.4.1) $\mathscr{C}' \subseteq \mathrm{cl}_\Delta(\mathscr{C}')$,
(2.4.2) whenever a circuit C splits into two circuits C_1 and C_2 that are in $\mathrm{cl}_\Delta(\mathscr{C}')$ then C is also in $\mathrm{cl}_\Delta(\mathscr{C}')$.

Theorem 2.5. *For every simple binary matroid M the following three conditions are equivalent*:

(2.5.1) *M is ℓ-closed*,
(2.5.2) *M is $(\ell + 2)$-chordal*,
(2.5.3) $\mathscr{C}(M) = \mathrm{cl}_\Delta(\mathscr{C}_{\ell+1})$.

Proof. (2.5.2) $\Longleftrightarrow$ (2.5.3): This equivalence is a direct consequence of the definitions.

(2.5.1) $\Longrightarrow$ (2.5.2): Consider a circuit C with at least $\ell+2$ elements and suppose for a contradiction that C is not chordal. From Lemma 2.1 we know that $\mathrm{cl}(C)=C$. Pick an element $i\in C$. Then the set $X=C\backslash i$ is not closed but every subset Y of X with at most ℓ elements is closed which is a contradiction.

(2.5.3) $\Longrightarrow$ (2.5.1): Let X be a subset of $[n]$ and suppose that for every circuit C with at most $\ell+1$ elements such that $|C\cap X|\geqslant|C|-1$, we have $C\subseteq X$; see (2.3.2′). To prove that X is closed it is enough to prove that for every circuit C such that $|C\cap X|\geqslant|C|-1$, we have $C\subseteq X$. Suppose that the result is true for every circuit with at most m elements and let D be a circuit with $m+1$ elements such that $D\backslash d\subset X$ with $d\in D$. By hypothesis there are circuits $C_1,C_2\in \mathrm{cl}_{\Delta}(\mathscr{C}_{\ell+1})$ such that $C_1\cap C_2=i$ and $D=C_1\Delta C_2$. Suppose w.l.o.g that $d\in C_1$. We have $C_2\backslash i\subset X$ and since $|C_2|\leqslant m$ which implies that $i\in C_2\subset X$. We have that $C_1\backslash d\subset X$ and $|C_1|\leqslant m$ which implies that $C_1\subset X$. This gives that $D\subseteq X$ and concludes the proof. □

We make use of the following elementary but useful proposition which is a particular case of Proposition 3.2 in [8]. The reader can easily check it from Brylawski's characterisation of modular hyperplanes [3].

Proposition 2.6. *Let M be a supersolvable matroid and*

$$\mathscr{M}:=F_0\subsetneq\cdots\subsetneq F_{r-1}\subsetneq F_r$$

an M-chain. Let F be a flat of M. Then $M|F$, the restriction of M to the flat F, is a supersolvable matroid and $\{F_i\cap F: F_i\in\mathscr{M}\}$ is the set of (*modular*) *flats of an $M|F$-chain.*

Definition 2.7. Let $\mathscr{P}=P_1\uplus\cdots\uplus P_r$ be an M-partition of a supersolvable matroid M. We associate to $(M,\mathscr{P})$ a graph $G_{\mathscr{P}}$ such that:

- $V(G_{\mathscr{P}})=\{P_i: i=1,2,\ldots,r\}$ is the vertex set of $G_{\mathscr{P}}$,
- $\{P_i,P_j\}$ is an edge of $G_{\mathscr{P}}$ iff there is a nontrivial line L of M meeting P_i and P_j.
  We call $G_{\mathscr{P}}$ the *S-graph* of the pair $(M,\mathscr{P})$.

Note that every nontrivial line L of the binary supersolvable matroid M meets exactly two P_i's and if L meets P_i and P_j, with $i<j$, necessarily $|P_i\cap L|=1$ and $|P_j\cap L|=2$. Indeed $F_{j-1}=\bigcup_{\ell=1}^{j-1}P_\ell$ is a modular flat disjoint from P_j, so $|F_{j-1}\cap L|=1$. This simple property will be used extensively in the proof of Theorem 2.10. Given a chordal graph G with a fixed S-labeling, we get an associated supersolvable matroid $M(G)$ and an associated M-partition $\mathscr{P}$. We say that $G_{\mathscr{P}}$, the S-graph determined by $(M(G),\mathscr{P})$, is the *derived S-graph* of G for this S-labeling.

Remark 2.8. Note that the derived S-graph $G_{\mathscr{P}}$ of a chordal graph G is a subgraph of G. Indeed set $V(G_{\mathscr{P}})=\{P_1,\ldots,P_m\}$ and consider the map $P_\ell\to v_{\ell+1}$, $\ell=1,\ldots,m$. Let $\{P_i,P_j\}$, $1\leqslant i<j\leqslant m$, be an edge of $G_{\mathscr{P}}$. From the definitions we see that $\{v_{i+1},v_{j+1}\}$ is necessarily an edge of G.

Example 2.9. Consider the S-labeling of the graph G_0 given in Fig. 1 and the associated M-partition $\mathscr{P}$ (see Example 1.3). The derived S-graph $G_{\mathscr{P}}$ is a path from P_1 to P_4. Consider now the M-partition of $M(G_0)$:

$$\mathscr{P}':=\{4\}\uplus\{3,5\}\uplus\{1,2\}\uplus\{6,7\}.$$

In this case the corresponding S-graph $G'_{\mathscr{P}'}$ is $K_{1,3}$ with P_2 being the degree-3 vertex. It is easy to prove that for any M-partition $\mathscr{P}$ of the cycle matroid of the complete graph K_ℓ, the S-graph $G_{\mathscr{P}}$ is the complete graph $K_{\ell-1}$.

Our main result is:

Theorem 2.10. *Let M be a simple binary supersolvable matroid with an M-partition $\mathscr{P}$. Then the S-graph $G_{\mathscr{P}}$ is chordal.*

Proof. Let $\mathscr{P}=P_1\uplus\cdots\uplus P_r$. We claim that P_r is a simplicial vertex of $G_{\mathscr{P}}$. Suppose that $\{P_r,P_i\}$ and $\{P_r,P_j\}$, $i<j$, are two different edges of $G_{\mathscr{P}}$ and that there are two nontrivial lines $L_1:=\{x,y,z\}$ and $L_2=\{x',y',z'\}$ where $x,y,x',y'\in P_r$ and $z\in P_i$, $z'\in P_j$. We will consider two possible cases:

- Suppose first, that two of the elements x,y,x',y' are equal; w.l.o.g., we can suppose $x=x'$. As M is binary the elements x,y,y' cannot be colinear, so $\mathrm{cl}(\{x,y,y'\})$ is a plane. From modularity of F_{r-1}, we know that $\mathrm{cl}(\{x,y,y'\})\cap F_{r-1}$ is a line.

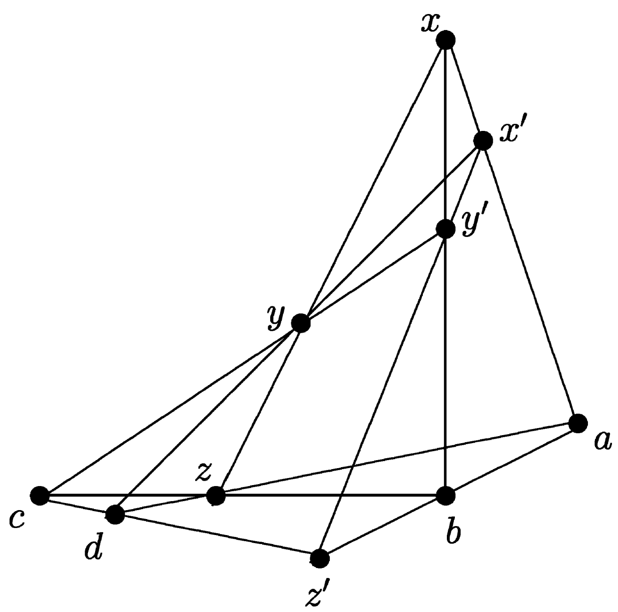

Fig. 2.

So the line $\mathrm{cl}(\{y, y'\})$ meets the modular hyperplane F_{r-1} in a point a. Now the line $\{z, z', a\}$ is a nontrivial line which meets P_i and P_j. Then by definition $\{P_i, P_j\}$ is an edge of $G_{\mathscr{P}}$.

- Suppose now that the elements x, y, x', y' are different. Then as M is binary we have $r(\{x, y, x', y'\}) = 4$. From modularity of F_{r-1}, we know that $r(\mathrm{cl}(\{x, y, x', y'\}) \cap F_{r-1}) = 3$. Then the six lines $\mathrm{cl}(\{\alpha, \beta\})$, for α and β in $\{x, y, x', y'\}$ meet F_{r-1} in six coplanar points; let these points be labelled as in Fig. 2. Let P_ℓ be the set that contains a. We will consider three subcases.
  - Suppose first that $i < j < \ell$. From the property given immediately after Definition 2.7, we have that c is also in P_ℓ. Consider the modular flat $F_{\ell-1} = \bigcup_{h=1}^{\ell-1} P_h$. We know that the plane $\mathrm{cl}(\{a, c, z, z'\})$ meets $F_{\ell-1}$ in a line, so $\mathrm{cl}(\{z, z'\})$ is a nontrivial line meeting P_i and P_j and so $\{P_i, P_j\}$ is an edge of $G_{\mathscr{P}}$.
  - Suppose now that $\ell < i < j$. Then the nontrivial line $\{a, d, z\}$ meets P_i and P_ℓ and we have $d \in P_i$. So the nontrivial line $\{c, d, z'\}$ meets P_i and P_j and $\{P_i, P_j\}$ is an edge of $G_{\mathscr{P}}$.
  - Suppose finally that $i \leqslant \ell \leqslant j$. The nontrivial line $\{a, d, z\}$ meets P_i and P_ℓ so $d \in P_\ell$. The nontrivial line $\{c, d, z'\}$ meets P_ℓ and P_j and necessarily we have $c \in P_j$. We conclude that the nontrivial line $\{b, c, z\}$ meets P_i and P_j and $\{P_i, P_j\}$ is an edge of $G_{\mathscr{P}}$.

By induction we conclude that $G_{\mathscr{P}}$ is chordal. □

We say that two M-chains

$$\mathscr{M} := \emptyset \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F_r = [n]$$

and

$$\mathscr{M}' := \emptyset \subsetneq \cdots \subsetneq F'_{r-1} \subsetneq F'_r = [n]$$

are related by an *elementary deformation* if they differ by at most one flat. We say that two M-chains are *equivalent* if they can be obtained from each other by elementary deformations.

Proposition 2.11. *Every two M-chains of the same matroid M are equivalent.*

Proof. We prove it by induction on the rank. The result is clear for $r = 2$. Suppose it is true for all matroids of rank at most $r - 1$. Consider two different M-chains

$$\mathscr{M} := \emptyset \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F_r = [n],$$

$$\mathscr{M}' := \emptyset \subsetneq \cdots \subsetneq F'_{r-1} \subsetneq F'_r = [n].$$

Let F_ℓ be the flat of highest rank of the M-chain $\mathscr{M}$ contained in F'_{r-1}. We know that $F_j \cap F'_{r-1}$, $j = 0, 1, \ldots, r$, is a modular flat of the matroid M and that

$$r(F_j \cap F'_{r-1}) = j - 1, \quad \text{for } j = \ell + 2, \ldots, r - 1.$$

Let $\mathscr{M}_0 := \mathscr{M}$ and for $i = 1, \ldots, r - 1 - \ell$, let $\mathscr{M}_i$ be the M-chain

$$\emptyset \subsetneq \cdots \subsetneq F_l \subsetneq F_{\ell+2} \cap F'_{r-1} \subsetneq \cdots F_{\ell+i+1} \cap F'_{r-1} \subsetneq F_{\ell+i+1} \cdots \subsetneq [n].$$

We have clearly by, definition, that for $i = 0, \ldots, r - 2 - \ell$, the M-chains $\mathscr{M}_i$ and $\mathscr{M}_{i+1}$ are equivalent. This sequence of equivalences shows that $\mathscr{M}$ is equivalent to $\mathscr{M}_{r-1-\ell}$. Finally, note that the two M-chains $\mathscr{M}'$ and $\mathscr{M}_{r-1-\ell}$ have the same component of rank $r - 1$, which by the induction hypothesis implies that $\mathscr{M}'$ is equivalent to $\mathscr{M}_{r-1-\ell}$. We have obtained the equivalence of $\mathscr{M}$ and $\mathscr{M}'$ which concludes the proof. $\square$

Remark 2.12. Proposition 2.11 can be used to obtain all the S-labels of a given chordal graph G from a fixed one. If G is doubly connected the number of M-chains of $M(G)$ is equal to the half the number of such labelings, see [8, Proposition 2.8].

It is natural to ask if, given a chordal graph G, there is a supersolvable matroid M together with an M-partition $\mathscr{P}$ such that $G = G_{\mathscr{P}}$. Can the matroid M be supposed graphic? The next proposition gives a positive answer to these questions.

Proposition 2.13. *Let $G = (V, E)$ be a chordal graph with an S-labeling $v_1, \ldots, v_m$ of its vertices, and $\widetilde{G}$ the extension of G by a vertex v_0 adjacent to all the vertices, i.e.*

$$V_{\widetilde{G}} = V_G \cup v_0 \quad \textit{and} \quad E_{\widetilde{G}} = E_G \cup \{\{v_i, v_0\},\ i = 1, \ldots, m\}.$$

Then $G_{\widetilde{\mathscr{P}}}$, the derived S-graph of $\widetilde{G}$ for the S-labeling $v_0, v_1, \ldots, v_m$ is isomorphic to G.

Proof. As v_0 is adjacent to every vertex v_i, $i = 1, \ldots, m$, it is clear that $v_0, v_1, \ldots, v_m$ is an S-labeling of $\widetilde{G}$. Let $\mathscr{P}$ and $\widetilde{\mathscr{P}}$ denote the corresponding M-partitions of the graphic matroids $M(G)$ and $M(\widetilde{G})$. We have $\mathscr{P} = P_1 \uplus \cdots \uplus P_{m-1}$ and $\widetilde{\mathscr{P}} = \widetilde{P}_1 (= \{v_0, v_1\}), \uplus \widetilde{P}_2 \uplus \cdots \uplus \widetilde{P}_m$ with $\widetilde{P}_i = P_{i-1} \cup \{v_o, v_i\}$, for $i = 2, \ldots, m$. Now we can see that if $\{v_i, v_j\}$, $0 \leqslant i < j \leqslant m-1$, is an edge of G then $\{\widetilde{P}_i, \widetilde{P}_j\}$ is an edge of $G_{\widetilde{\mathscr{P}}}$. From Remark 2.8 we get that reciprocally $G_{\widetilde{\mathscr{P}}}$ is a subgraph of G. $\square$

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