Coverings of the vertices of a graph by small cycles

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Abstract. Given a graph G with n vertices, we call $c_k(G)$ the minimum number of elementary cycles of length at most k necessary to cover the vertices of G. We bound $c_k(G)$ from the minimum degree and the order of the graph.

1. Introduction and definitions

Let $G = (V, E)$ be a simple non oriented graph and $N \subset V$ a subset of V. The order of the graph is the number of vertices of the graph. A path $P[a, b]$ of G is a path with extremities a and b; such a path is N-**alternated** if a is a vertex of N and P does not contain 2 consecutive vertices not in N. Similarly, a cycle C is N-**alternated** if it does not contain 2 consecutive vertices not in N. If there is no ambiguity we will just say alternated.
The triangle graph of G, denoted $T(G) = (V, E')$, is the graph on the set of vertices V whose set of edges is the set of edges of the triangles of G. We recall that for p and q two non zero integers $K_{p,q}$ is the complete bipartite graph with partite sets of cardinalities p and q. Similarly for p, q and r three non zero integers, $K_{p,q,r}$ is the complete tripartite graph with partite sets of cardinalities p, q and r. For any graph G let $\alpha(G)$ be the cardinality of a maximum stable set of G.For unexplained terminology, see [1].
In this work we consider coverings of the vertices of a graph by elementary cycles. A covering of G is a family of cycles of G such that each vertex of G is at least in one cycle of the family. If the minimum degree is at least half of the order of the graph then by Dirac's lemma [2] we know that the graph is hamiltonian. More generally many authors bounded the minimum number of cycles necessary to cover the

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vertices, in function of the minimum degree and the order of the graph. Let us recall somme results. Kouider and Lonc [4] proved:
"Let G be a graph of order n and $s \geq 2$ be an integer. If for every stable set $S \subset V$ of cardinality s we have $\sum_{x \in S} deg_G x \geq n$, then the vertex set V can be covered by at most $s-1$ cycles, edges or vertices. If G is supposed 2-connected we can take only cycles."
Here we fix an integer k and we consider only cycles of length at most k. We denote by $c_k(G)$ the minimum number of such cycles necessary to cover the vertices. We bound $c_k(G)$ in function of the minimum degree and the order of the graph.
Among other, we get the three following results. For $\delta \geq \frac{n}{2} + \frac{k-3}{3}$, we have $c_k(G) \leq \lceil \frac{n}{k-1} \rceil$ and this bound is almost tight. For $\delta \geq \frac{16n}{k} + \frac{5k}{6}$, $c_k(G)$ has a bound of order $\frac{16n}{5k}$. And for $\delta \geq \frac{8n}{k} + \frac{2k}{3}$, $c_k(G)$ has a bound of order $\frac{3n}{k}$. These results and other will be precised further.

2. Results

2.1. Case $k = 3$ and $k = 4$.

Proposition 2.1. *Let G be a graph with t_1 vertex disjoint triangles. Then*

$$\frac{n-t_1}{2} \leq c_3(G) \leq n - 2t_1.$$

Proof. The t_1 disjoint triangles cover $3t_1$ vertices and the $n - 3t_1$ remaining vertices are covered at worst 1 by 1 and at most 2 by 2. This gives $\frac{n-3t_1}{2} + t_1 \leq c_3(G) \leq (n - 3t_1) + t_1$. □

Proposition 2.2. *Let G be a graph of order n and $T(G)$ its triangle graph. Then*

$$c_3(G) \leq \frac{n + \alpha(T(G))}{2}.$$

Proof. Let S be an independent set of maximum size in $T(G)$. Let M be a maximum matching in the complement of S. So vertices saturated by M can be covered by triangles 2 by 2. The remaining vertices of $T(G)$ form a stable set $X' = V \setminus (S \cup V(M))$. By maximality of S we have for every subset X'' of X', $|N(X'') \cap S| \geq |X''|$. By König-Hall theorem there is a matching in $S \cup X'$ which saturates X'. So we have a covering by at most $\dfrac{n-(\alpha(T(G))-|X'|)}{2} + \alpha(T(G)) - |X'| \leq \dfrac{n+\alpha(T(G))}{2}$. $\square$

Let us take the graph G with n vertices contructed by adding an edge in the 2 vertices part in the $K_{2,n-2}$. We have $T(G) = G$, $\alpha(T(G)) = n-2$ and $c_3(G) = n-2$ while the bound given by the preceeding proposition is $n-1$.
For $k = 4$, there are graphs with n vertices such that $c_4(G) \geq 2\lceil \dfrac{n-3}{3} \rceil$ for example, consider a graph H which is a cycle ($x_1, x_2, x_3, .., x_6$) with a diametral chord $[x_2, x_5$]; the graph G is composed by s copies of H such that the common vertices to any pair of copies are exactly x_1, x_2, x_3.

2.2. **General case.** We will use intensively the following easy lemma.

Lemma 2.3. *Let $p \geq 2$ be an integer and $c \geq 1$ be a number. Let $G = (V,E)$ be a graph with minimum degree δ strictly more than $\dfrac{n}{p} + c - 1$. Let $x_1, x_2, \ldots, x_p$ be p vertices of G and $N_1, N_2, \ldots, N_p$ be subsets of V of cardinality at most c. Then there exist two vertices x_i and x_j $(1 \leq i < j \leq n)$ such that*
i) either x_i and x_j are adjacent;and, $x_i \notin N_j$ or $x_j \notin N_i$,
ii) or x_i and x_j have a common neighbor v outside $N_i \cup N_j$.

Proof. As $\delta > \dfrac{n}{p} + c - 1$, we have

$$\sum_i d_G(x_i) > n \; + \; p(c-1).$$

By hypothesis on the sets (N_i),

$$\sum_i (d_G(x_i) + 1 - n_i \;) > n.$$

This implies that the sets $(x_i \cup N(x_i)) \setminus N_i)_i$ are not disjoint. $\square$

Theorem 2.4. *Let $k \geq 5$ be an integer. Let $G = (V,E)$ be a 2-connected graph with minimum degree δ at least $3\dfrac{n}{k-1}$ then $c_k(G) \leq \dfrac{n}{2} + \dfrac{k}{6} -$ $min(\; \delta, k + 1/4 \;) + 1$.*

Proof. By Dirac's Lemma there exists a cycle C_0 of length at least 2δ. If the length of C_0 is at most k, we take C_0 as a cycle of the covering and $N = V \setminus V(C_0)$. If not we make the following construction.

Let C be a cycle of length $\ell(C)$ at least $k+1$. Let $p = \lfloor \frac{k+1}{3} \rfloor$ and $\{c_1, \dots, c_p\}$ be p vertices of C mutually at distance at least 3 in the cycle C. Since $\delta > 3\frac{n}{k-1} - 1$, then by lemma 2.3, there are 2 vertices c_i and c_j who are neighbors or with a common neighbor. If c_i and c_j have a common neighbor outside the cycle C then we get two smaller cycles than C with the sum of their size equal to $|C| + 4$. In the case they are neighbors or have a common neighbor on the cycle C we get two smaller cycles than C with the sum of their size equal to $|C| + 2$. In all these cases, we get by taking the bigger of the two cycles a cycle of length at least $\frac{\ell(C)}{2}$ and strictly less then $\ell(C)$. Starting with the cycle C and iterating, if necessary, this construction we get a cycle of length ℓ_1 satisfying $\frac{k+1}{2} \leq \ell_1 \leq k$. In any case the first cycle covers $min(\ 2\delta, \frac{k+1}{2}\ )$ vertices.

Let N the set of uncovered vertices. If $|N| \geq \frac{n+1}{\delta+1}$ then N contains two vertices a and b at distance at most 2.

a)If $[a, b]$ is an edge, let us consider a cycle C which contains the edge $[a, b]$. If the length of C is greater than k, we use the preceding construction. In any case, we get a cycle of length at most k containing $[a, b]$.

b)If a and b have a common neighbor c, as G is 2-connected there exists a cycle C which contains the path $[a, c, b]$. If the length of C is greater than k, we use the preceding construction by considering the neighborhoods in $G \setminus C$. As $\delta - 1 \geq 3\frac{n}{k-1} - 1$, in any case, by lemma 2.3, we get a cycle of length at most k containing $[a, c, b]$.

We can cover the vertices 2 by 2 until we get a set N of uncovered vertices of cardinality $|N| < \frac{n+1}{\delta+1} < \frac{k-1}{3}$ that we will cover 1 by 1.

So we used at most $\frac{n}{2} + \frac{k}{6} -\ min(\ \delta, k + 1/4\ ) + 1$ cycles for a covering of V. $\square$

Theorem 2.5. *Let $k \geq 4$ be an integer. Let $G = (V, E)$ be a 2-connected graph of order n with minimum degree δ strictly more than $\frac{n}{2} + \frac{k-3}{2}$ then $c_k(G) \leq \lceil \frac{n}{k-1} \rceil$.*

Proof. It is known that the fact that $\delta \geq \frac{n}{2}$ implies that the graph is hamiltonian. Let $v_1, v_2, \ldots, v_n, v_1$ be an hamiltonian cycle in G. Since $\delta > \frac{n}{2} + \frac{k-3}{2}$, by lemma 2.3, any two vertices v_i and v_{i+k-2} are adjacent or have a common neighbour outside the interval $[v_i, v_{i+k-2}]$. So we have a cycle C_i composed of the interval $[v_i, \ldots, v_{i+k-2}]$ and eventually one more vertex. This implies that the cycles $C_{1+j(k-1)}$ for j from 0 to $\lceil\frac{n}{k-1}\rceil - 1$, cover V. □

For n even and $4 \leq k \leq \frac{2n}{3}$, the bound of the last theorem is almost tight for the tripartite graph $K_{r,r,\alpha}$ with $\alpha = k-3$ and $r = \frac{n-\alpha}{2}$. For k odd we have $c_k(G) = \lceil\frac{n-\alpha}{k-1}\rceil \geq \frac{n}{k-1} - 1$.
By using more difficult constructions, we will give smaller coverings when the minimum degree is bigger then in the previous theorem. The construction is based on the existence of an alternated cycle of length between k and some fraction of k.
The first step is to show the existence of an N-alternated cycle of size at least this fraction of k (but maybe bigger than k) in Corollary 2.7. The second step will be to break this cycle until we get a cycle of size at most k in Theorem 2.8.

Lemma 2.6. *Let p be an integer, $c \geq 3$ be a positive number. If $\delta > \frac{n+1}{p} + c - 3$ then at least one of the two following assertions is true.*
(1)There exists an alternated cycle of length at least c.
(2)There exists a covering of N by at most $p-1$ alternated paths.

Proof. Let P_1 be an alternated path starting with a vertex a_1 of N of maximum cardinality. If the $(P_j)_{1\leq j<i}$ are already defined and if they dont cover N, then let $P_i[a_i, b_i]$ be an $(N \setminus \cup_{j<i} P_j)$-alternated path of maximum cardinality. By this process we construct a covering of N by say r paths. The vertices a_i are necessarily distinct by construction.
We may suppose that assertion (2) is false and we shall show that (1) is true. So we have $r \geq p$. Let N_i be the set of $c-2$ vertices following a_i in P_i if $\ell(P_i) \geq c-1$ and N_i be $P_i \setminus a_i$ otherwise.
By the hypothesis on δ and r and Lemma 2.3, there are two vertices a_i and a_j which either are adjacent or have a common neighbor outside $N_i \cup N_j$. If the vertices a_i and a_j are adjacent or if their common neighbor is outside $P_i \cup P_j$ it contradicts the maximality of P_i or P_j.

And if their common neighbor u is in $P_i \cup P_j \setminus (N_i \cup N_j)$. Let us say it is in P_i. We have a cycle $P_i[a_i, u] \cup [u, a_i]$ which is of length at least c. We get assertion (1) and this completes the proof. $\square$

Corollary 2.7. *Let $p \geq 2$ be an integer and c a positive number. If $|N| \geq (p-1)^2 c + p - 1$ and $\delta > \dfrac{n+1}{p} + c - 3$ then there exists an alternated cycle of length at least c.*

Proof. From the previous lemma, we may suppose we have an N-alternated path P of order at least $\dfrac{|N|}{p-1} \geq (p-1)c+1$. In the path P there exists p vertices,$v_1, v_2, \ldots, v_p$, of N mutually at distance at least $c-1$. We choose an orientation on P. For $1 \leq i \leq p-1$, let N_i be the set of $c-2$ vertices following the vertex v_i. By the hypothesis on the minimum degree, there are two of them v_i and v_j which are either adjacent or have a common neighbor outside the two intervals N_i and N_j. This gives an N-alternated cycle of length at least c. $\square$

Now we show that once we have a circuit of size at least k we can obtain a circuit of size at most k but bigger than a fraction of k.

Theorem 2.8. *Let p and k be two integers such that $2 \leq p \leq \dfrac{k}{8}$. Let $G = (V, E)$ be a graph with minimum degree at least $\dfrac{n}{p} + \dfrac{2}{3}k$, and $N \subset V$. If G has an N-alternated cycle of length at least $\dfrac{2}{3}k$ then it has an N-alternated cycle of length between $\dfrac{2}{3}k$ and k.*

Proof. We may suppose that we have an alternated cycle $C = \{a_1, \ldots, a_\ell\}$ of length ℓ at least $k+1$ and so $\ell \geq 8p+1$ (1). We will construct from the cycle C a cycle of length between $\dfrac{2}{3}k$ and $\ell - 1$. For any vertex a_i of C we define N_i the interval of the vertices of C at distance on C at least $\dfrac{\ell}{2} - \dfrac{k}{3} + 1$ from a_i. The cardinality of N_i is between $\dfrac{2}{3}k - 3$ and $\dfrac{2}{3}k - 1$. We may suppose that there is no vertex a_i of $C \cap N$ with a neighbor in $C \setminus (N_i \cup \{a_{i-1}, a_{i+1}\})$ otherwise we have the desired cycle of length at least $\dfrac{2}{3}k$.

Let $v_1, \ldots, v_p$ be p vertices of $C \cap N$ and mutually at distance at least 3 ($l \geq 4p$). By lemma 2.3, there exist v_i and v_j which are either adjacent and $v_i \notin N_j$, or, have a common neighbor. By the previous remark they can not be adjacent. So there exist two vertices a_i and a_{i+x} (with $x \geq 3$) of $C \cap N$ with a common neighbor u in $V \setminus C$. Let us choose a couple such that x is minimum. We may suppose that $\frac{\ell}{3} < x \leq \frac{\ell}{2}$ otherwise we have the desired cycle. Now, the interval $[a_{i+1}, a_{i+4p}]$ contains at least p vertices of $C \cap N$ and mutually at distance at least 3. By hypothesis on the minimum degree, this interval contains two vertices a_j and a_{j+y} with a common neighbor v in $V \setminus C$. By the minimality of x and the hypothesis on p, we have $\frac{\ell}{3} \leq y$ and $y \leq 4p+1 \leq \frac{k}{2}+1 < 2x$; the two segments $[a_i, a_{i+x}]$ and $[a_j, a_{j+y}]$ intersect in at least two vertices otherwise, by (1), we have $\frac{2\ell}{3} \leq x+y \leq \frac{\ell}{2}$ which is a contradiction. Let $\ell_1 = j-i$, $\ell_2 = i+x-j$ and $\ell_3 = j+y-i-x$. We get $x = \ell_1 + \ell_2 > \frac{\ell}{3}$, *and* $y = \ell_2 + \ell_3 > \frac{\ell}{3}$ and $\ell_1 + \ell_2 + l_3 \leq 4p \leq \frac{k}{2}$. It follows that $\ell_2 \geq \frac{k}{6}$. The cycle $C_2 = (a_{j+y}, a_{j+y+1}, \ldots, a_i, u, a_{i+x}, a_{i+x-1}, \ldots, a_j, v, a_{j+y})$ is as desired: $l(C_2) \geq l_2+4+(\ell-4p-1) \geq \frac{2}{3}k$. This completes the proof. $\square$

Using similar proofs we can get the following more general results.

Theorem 2.9. *Let t and k be two positive numbers such that $t \geq 4$. Let $G = (V, E)$ be a graph with minimum degree at least $\frac{tn}{k} + \frac{k}{2}$ and $N \subset V$. If G has an N-alternated cycle of length at least $\frac{k}{2}$ then it has an N-alternated cycle of length between $\frac{k}{2}$ and k.*

We remark that in our construction the condition $t \geq 4$ is necessary. Now we shall give for $t \geq \frac{16}{3}$ a generalization of Theorem 2.8.

Theorem 2.10. *Let t and k be two positive numbers such that $t \geq \frac{16}{3}$. Let $G = (V, E)$ be a graph with minimum degree at least $\frac{tn}{k} + k(1 - \frac{8}{3t})$ and $N \subset V$. If G has an N-alternated cycle of length at least $k(1 - \frac{8}{3t})$ then it has an N-alternated cycle of length between $k(1 - \frac{8}{3t})$ and k.*

Theorem 2.11. *Let p and k be two integers such that $2 \leq p \leq \frac{k}{8}$. Let $G = (V, E)$ be a graph of order $n \geq (p-1)^2 \frac{2k}{3} + (p-1)$ with minimum degree δ at least $\frac{n}{p} + \frac{2}{3}k$. Then the number $c_k(G)$ verifies:*

$$c_k(G) \leq \frac{3n}{k} + \frac{log \frac{k}{3}}{-log(1 - \frac{1}{2(p-1)^2})} + (1 - \frac{3}{k})(p-2) + 1.$$

Proof. **Step 1** We call N the set of uncovered vertices. At the beginning, from Corollary 2.7 and Theorem 2.8, we know that there exists a cycle of G which covers at least $\frac{2k}{3}$ vertices of V. From Corollary 2.7 and Theorem 2.8, while $|N| \geq (p-1)^2 \frac{2k}{3} + p - 1$ there exists a cycle in G which covers at least $\frac{k}{3}$ vertices of N. Then at most $\frac{3n}{k} - 2(p-1)^2 - \frac{3}{k}(p-2)$ cycles are necessary to cover the first $n - (p-1)^2 \frac{2k}{3} - (p-1) + 1$ vertices.

Step 2 We have that $2(p-1)^2 \leq |N| - (p-1) < (p-1)^2 \frac{2k}{3}$. By Corollary 2.7 and Theorem 2.8, we can find an N-alternated cycle of length at least $\frac{|N| - (p-1)}{(p-1)^2}$. So it covers at least $\frac{|N| - (p-1)}{2(p-1)^2}$ vertices of N. Let denote by $n'(t)$ the number of uncovered vertices after using t cycles in step 2. Let $\alpha = 1 - \frac{1}{2(p-1)^2}$. From the preceeding remarks, we have that

$$n'(t+1) \leq \alpha n'(t) + \frac{1}{2(p-1)}.$$

This inequality is equivalent to

$$n'(t+1) - (p-1) \leq \alpha(n'(t) - (p-1)).$$

It follows that

$$n'(t) - (p-1) \leq \alpha^t(n'(o) - (p-1)).$$

We continue to apply Corollary 2.7 and Theorem 2.8 until we get $n'(t_0) < 2(p-1)^2 + (p-1)$. So t_0 is bounded by the smallest solution of the inequation $\alpha^{t_0}(n'(0) - (p-1)) < 2(p-1)^2$. We get that

$$t_0 \leq \lceil \frac{log \frac{k}{3}}{|log(\alpha)|} \rceil \leq \frac{log \frac{k}{3}}{|log(\alpha)|} + 1.$$

Step 3 After step 2 is over, we have $|N| \leq 2(p-1)^2+(p-2)$ uncovered vertices. As the minimum degree is bigger then $\frac{2k}{3} \geq p$ we have that any vertex x is contained in a C_3 or a C_4 (by applying Lemma 2.3 to $N(x)$). So we can recover the remaining $2(p-1)^2+(p-2)$ vertices 1 by 1.
So after the 3 steps we need at most $\frac{3n}{k} - 2(p-1)^2 - \frac{3}{k}(p-2) + \frac{log\frac{k}{3}}{-log(1-\frac{1}{2(p-1)^2})} + 2(p-1)^2 + (p-2) = \frac{3n}{k} + \frac{log\frac{k}{3}}{-log(1-\frac{1}{2(p-1)^2})} + (1-\frac{3}{k})(p-2) + 1$ cycles. □

Theorem 2.12. *Let p and k be two integers such that $2 \leq p \leq \frac{k}{8}$. Let $G=(V,E)$ be a graph of order n such that $2(p-1)^2 \leq n-(p-1) \leq (p-1)^2\frac{2k}{3}$ with minimum degree δ at least $\frac{n}{p}+\frac{2}{3}k$. Then the number $c_k(G)$ verifies:*

$$c_k(G) \leq \frac{log\frac{n-(p-1)}{2(p-1)^2}}{-log(1-\frac{1}{2(p-1)^2})} + 2(p-1)^2 + p - 2.$$

Proof. The proof is the same as the previous one, starting at step 2 and replacing $n'(0)$ by n. □

If we take $k = 8p$, we deduce by using the majoration $|log(1-x)| \geq x$ for a number $0 \leq x < 1$ from the preceeding theorem the following result.

Corollary 2.13. *Let k be an integer and let $G=(V,E)$ be a graph with minimum degree δ at least $\frac{8n}{k}+\frac{2}{3}k-2$. Then*

$$c_k(G) \leq \frac{3n}{k} + \frac{(k-8)^2}{32}log(\frac{k}{3}) + \frac{k}{8} \quad \textit{if } n > \frac{k(k-8)^2}{96} + \frac{k}{8} - 1 \quad \textit{and}$$

$$c_k(G) \leq \frac{(k-8)^2}{32}(1+log(\frac{k}{3}))+\frac{k}{8} \quad \textit{if } \frac{(k-8)^2}{32} \leq n-(\frac{k}{8}-1) \leq \frac{k(k-8)^2}{96}.$$

By similar proofs, by using Theorem 2.9 and Theorem 2.10, we get the following results:

Theorem 2.14. *Let k be an integer and let $G = (V, E)$ be a graph with minimum degree at least $\frac{4n}{k} + \frac{k}{2}$. Then the number $c_k(G)$ verifies:*

$$c_k(G) \leq \frac{4n}{k} + \frac{(k-4)^2}{8} log\frac{k}{2} + \frac{k}{4} - 1 \text{ } if \text{ } n > \frac{k(k-4)^2}{32} + \frac{k}{4} - 1$$

and

$$c_k(G) \leq \frac{(k-4)^2}{8}(1+log(\frac{k}{2}))+\frac{k}{4} \text{ } if \text{ } \frac{(k-4)^2}{8} \leq n-(\frac{k}{4}-1) \leq \frac{k(k-4)^2}{32}.$$

Theorem 2.15. *Let t and k be two numbers such that $t \geq \frac{16}{3}$ and let $G = (V, E)$ be a graph of order $n \geq \frac{(3t-8)k(k-t)^2}{3t^3} + \frac{k-t}{t}$ and with minimum degree at least $\frac{tn}{k} + k(1 - \frac{8}{6t})$. Then the number $c_k(G)$ verifies:*

$$c_k(G) \leq \frac{2n}{k}\frac{3t}{3t-8} + \frac{log\frac{(3t-8)k}{6t}}{-log(1-\frac{1}{2(\frac{k}{t}-1)^2})} + (\frac{k}{t} - 2). \quad \square$$

By replacing the *log* by a bound, we get the simpler bound:

$$c_k(G) \leq \frac{2n}{k}\frac{3t}{3t-8} + 2\frac{(k-t)^2}{t^2}log\frac{(3t-8)k}{6t} + (\frac{k}{t} - 2).$$

For example if we take $k = 16p$, we deduce directly from the preceeding theorem the following result.

Corollary 2.16. *Let k be an integer and let $G = (V, E)$ be a graph of order $n \geq (\frac{k}{16})^2$ and with minimum degree δ at least $\frac{16n}{k} + \frac{5k}{6} - 2$. Then*

$$c_k(G) \leq \frac{12n}{5k} + \frac{k^2}{128}log(\frac{5k}{12}) + \frac{k}{16}. \quad \square$$

For the complete bipartite graph $K_{\delta,n-\delta}$ and k even,we have $c_k = 2\lceil\frac{n-\delta}{k}\rceil$. By taking $\delta = \frac{16n}{k} + \frac{5k}{6} - 2$ we obtain that c_k is of order $\frac{15n}{8k}$ which is not too far from the $\frac{12n}{5k}$ of the last Corollary.

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