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Connected coverings and an application to oriented matroids

David Forge[a], J.L. Ramírez Alfonsín[b,*,1]

[a] *Université Pierre et Marie Curie, Paris 6, case 189-Combinatoire, 4, place Jussieu, 75252 Paris Cedex 05, France*

[b] *Instituto de Matemáticas, Universidad Nacional Autónoma de México, Area de la Investigacion Científica, Circuito Exterior, C.U., Mexico D.F. 04510*

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Abstract

In this paper we are interested in the following question: what is the smallest number of circuits, $s(n,r)$, that is sufficient to determine every uniform oriented matroid of rank r on n elements? We shall give different upper bounds for $s(n,r)$ by using special coverings called *connected coverings*.

1. Introduction

Let n,r be positive integers with $n>r$. Let $\mathcal{M}_{n,r}$ be the *uniform oriented matroid* having as basis (as circuits) all r-subsets (all $(r+1)$-subsets) of $\{1,\ldots,n\}$. In this paper, we are interested in the following question. What is the smallest number of circuits that is sufficient to determine $\mathcal{M}_{n,r}$? We denote by $s(n,r)$ such a number. The best known upper bound for $s(n,r)$ is given by Hamidoune and Las Vergnas [4]. They proved that $s(n,r)\leqslant\binom{n-1}{r}$.

We will achieve different upper bounds for $s(n,r)$ by giving a relation between $s(n,r)$ and covering numbers. In particular, we will be interested in a special covering number called the *connected covering number*.

This paper is self-contained and is organised as follows. In Section 2, we give some basic definitions of oriented matroids. Also, we show that upper bounds for a connected covering number with special parameters lead to upper bounds for $s(n,r)$.

* Correspondence address: Université Pierre et Marie Curie, Paris 6, case 189-Combinatoire, 4, place Jussieu, 75252 Paris Cedex 05, France. Tel.: +33 1 44277187; fax: +33 1 44272724; e-mail: ramirez@ec.ibp.fr.

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In Section 3, we present two methods for finding upper bounds for connected covering numbers. We will be able to prove that one of those methods gives an upper bound $b(n,r)$ (for $s(n,r)$) such that $\lim_{n\to\infty} b(n,r)/\binom{n-1}{r} \to \frac{1}{2}$, for fixed integer r. In Section 4, we give other upper bounds based on upper bounds for *covering numbers*. In fact, we find an upper bound $d(n,r)$ (for $s(n,r)$) such that $\lim_{n\to\infty} d(n,r)/\binom{n}{r} \to \frac{1}{r}$, for fixed integer r. Finally, in Section 5, we compute the values of different connected coverings given in previous sections for $7 \leqslant n \leqslant 14$ and $2 \leqslant r \leqslant n-1$.

2. Definitions and notations

We recall some basic definitions of oriented matroids (for further details see [1]). A *signed set* is a set X together with a partition into two distinguished subsets X^+ and X^-. The opposite of X is the signed set $-X$ such that $(-X)^+ = X^-$ and $(-X)^- = X^+$. An *oriented matroid* $\mathcal{M}$ on a finite set E is defined by its collection $\mathcal{C}$ of *signed circuits*, i.e. signed subsets of E satisfying the following two properties:

(1) For all $C_1 \in \mathcal{C}$, $C_1 \neq \emptyset$ and $-C_1 \in \mathcal{C}$, and for all $C_1, C_2 \in \mathcal{C}$, $C_2 \subseteq C_1$ implies $C_2 = C_1$ or $-C_1$.

(2) *Elimination property.* For all $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq -C_2$ and all $x \in (C_1^+ \cap C_2^-)$, there exists $C_3 \in \mathcal{C}$ such that $C_3^+ \subseteq (C_1^+ \cup C_2^+) \backslash x$ and $C_3^- \subseteq (C_1^- \cup C_2^-) \backslash x$.

By ignoring signs, a (non-oriented) *underlying matroid* $\underline{\mathcal{M}}$ is clearly attached to each oriented matroid $\mathcal{M}$. The cocircuits of $\underline{\mathcal{M}}$ can be signed in a natural way in order to obtain an oriented matroid $\mathcal{M}^*$ having the dual $\underline{\mathcal{M}}^*$ of $\underline{\mathcal{M}}$ as underlying matroid. The *bases* of $\mathcal{M}$ are the maximal subsets of E which contain no circuit, that is, they are the bases of $\underline{\mathcal{M}}$. The *rank function* of $\mathcal{M}$ is the rank function of $\underline{\mathcal{M}}$ and is denoted by r.

A *basis orientation* of an oriented matroid $\mathcal{M}$ is a mapping Φ of the set of ordered bases of $\mathcal{M}$ to $\{-1, 1\}$ satisfying the following two properties:

(1) Φ is alternating and

(2) for any two ordered bases of $\mathcal{M}$ of the form $(e, x_2, \ldots, x_r)$ and $(f, x_2, \ldots, x_r)$, $e \neq f$, we have $\Phi(f, x_2, \ldots, x_r) = -C(e)C(f)\Phi(e, x_2, \ldots, x_r)$, where C is one of the two opposite signed circuits of $\mathcal{M}$ in the set $(e, f, x_2, \ldots, x_r)$ and $C(f)$ (and $C(e)$) denote the sign corresponding to element f (and e) in C.

Las Vergnas [5,6] proved that every oriented matroid $\mathcal{M}$ has exactly two basis orientations and these two basis orientations are opposite, Φ and $-\Phi$. Lawrence [7] gave a complete characterization of oriented matroids in terms of an alternating function Φ called *chirotope* (see also [2] for another description of oriented matroids).

Remark. Let C be a circuit and B a basis of $\mathcal{M}_{n,r}$ with $B \subseteq C$. Given the sign of B the signature of C allows us to sign the other r basis contained in C.

We now relate the number $s(n,r)$ with some special covering designs. In order to do that, we need the following definitions.

A (n,m,p) *covering* is a family of m-subsets, called *blocks*, of $\{1,\ldots,n\}$ such that each p-subset is contained in at least one of the blocks, $(n\geqslant m\geqslant p)$ (a detailed survey of results on the covering numbers can be found in [3]).

A (n,m,p) covering is a *connected covering* if the blocks cannot be partitioned into two sets A and B such that $W_A\cap W_B=\emptyset$ where

$$W_A=\{D\mid |D|=p \text{ and } D\subseteq S \text{ for some block } S\in A\},$$

$$W_B=\{D\mid |D|=p \text{ and } D\subseteq S \text{ for some block } S\in B\}.$$

We will say that the set of blocks $B=\{b_1,\ldots,b_s\}$ of a (n,m,p) covering forms a *connected component* if B cannot be partitioned into two sets as above. The number of blocks is the (connected) covering's *size*, and the minimum size of such a covering (connected covering) is called the *covering number* (the *connected covering number*), denoted by $C(n,m,p)$ $(CC(n,m,p))$.

A consequence of the above remark is the following theorem.

Theorem 2.1. *Let n,r be nonnegative integers with $n\geqslant r+1$. Then $s(n,r)\leqslant CC(n,r+1,r)$.*

Proof. Let $\hat{C}=\hat{C}(n,r+1,r)$ be a $(n,r+1,r)$ connected covering. We shall give a procedure to sign all the basis of $\mathcal{M}_{n,r}$ using at most $|\hat{C}|$ circuits. Let $B=\{b_1,\ldots,b_{|\hat{C}|}\}$ be the blocks of $\hat{C}$. We may refer to b_i as either a block of $\hat{C}$ or as a circuit of $\mathcal{M}_{n,r}$ (since $|b_i|=r+1$). Without loss of generality, suppose that b_1 is a block that contains $\{1,\ldots,r\}$.

Procedure

[1] Put $\Phi(1,\ldots,r)=1$ (or -1). Sign the other r basis (r-subsets) contained in the circuit b_1 and put $B=B\backslash b_1$ and $\bar{B}=b_1$.

Repeat

[2] Find a block b_i in B such that $|b_i\cap b_j|=r$ for some $b_j\in\bar{B}$ (always possible since $\hat{C}$ is connected).

[3] Sign the $r+1$ basis (r-subsets) contained in b_i and put $B=B\backslash b_i$ and $\bar{B}=\bar{B}\cup b_i$ (there are maybe some basis signed already).

Until $|B|=0$.

Note that in the end of the procedure all basis are signed since $\hat{C}$ is a covering. Hence, the above procedure outputs a signature of all basis of $\mathcal{M}_{n,r}$ by using no more than $|\hat{C}|$ blocks (circuits). □

The upper bound given by Hamidoune and Las Vergnas [4] is easily derived from Theorem 2.1 by showing that the sets, $HV_1(n,r)=$ all $(r+1)$-subsets that contains a fixed integer j, $1 \leqslant j \leqslant n$ and $HV_2(n,r)$ = all $(r+1)$-subsets whose largest two elements are consecutive, are $(n,r+1,r)$ connected coverings and $|HV_i(n,r)| = \binom{n-1}{r}$ for $i = 1,2$.

3. Upper bounds for $CC(n, r+1, r)$

In this section we will study the number $CC(n,r+1,r)$. In particular, we will be interested in the case when $n > r+1$. Of course, $CC(n,r+1,r) \geqslant C(n,r+1,r)$ as it is shown in the following proposition.

Proposition 3.1. *Let n,r be nonnegative integers with $n \geqslant r+1$. Then*

$$C(n,r+1,r) \geqslant \frac{\binom{n}{r}}{r+1} =: C^*(n,r)$$

and

$$CC(n,r+1,r) \geqslant \frac{\binom{n}{r}-1}{r} =: CC^*(n,r).$$

Proof. (a) Since each block covers exactly $r+1$ of the $\binom{n}{r}$ r-subsets.

(b) Let $b_1,\ldots,b_s$ be the blocks of a $(n,r+1,r)$ connected covering such that $b_1,\ldots,b_i$ is connected for $i=2,\ldots,s$. So, b_1 covers exactly $r+1$ r-subsets and b_i covers at most r r-subsets not covered by $b_1,\ldots,b_{i-1}$ for $i=2,\ldots,s$. Hence, $r+1+(s-1)r \geqslant \binom{n}{r}$ or equivalently $s \geqslant \binom{n}{r} - 1/r$. □

Notice that $\binom{n}{r} - 1/r \leqslant CC(n,r+1,r) \leqslant \binom{n-1}{r}$. So, the upper bound is approximately r times the lower bound.

Also, note that the lower bounds $C^*(n,r)$ and $CC^*(n,r)$ may not always be attainable. For instance, by using exhaustive enumeration of possibilities, it can be shown that for the case $n=5$ and $r=3$ (and $n=6$ and $r=4$) the minimal connected covering $\bar{C}_1$ (and $\bar{C}_2$) are such that $|\bar{C}_1(5,4,3)| = CC^*(5,3)+1$ (and $|\bar{C}_2(6,5,4)| = CC^*(6,4)+1$), see Table 1; in Section 5.

Theorem 3.2. *Let n,r be nonnegative integers with $n > r+1$. Then*

$$CC(n,r+1,r) \leqslant \sum_{\substack{i=2 \\ i\text{-even}}}^{2\lfloor (n-r+1)/2 \rfloor} \binom{n-i}{r-1} + \left\lfloor \frac{n-r}{2} \right\rfloor.$$

Proof. Let $\bar{S}(n,r+1,r)$ be the set of all $(r+1)$-subsets in $\{1,\ldots,n\}$ such that the last two elements are consecutive and the last element has the same parity as n. We claim that $\bar{S}(n,r+1,r) \cup \{1,\ldots,r\}$ is a $(n,r+1,r)$ covering. Let $U_{n,r}$ be the set of all r-tuples in $\{1,\ldots,n\}$. Let $B = \{b_1,\ldots,b_r\} \in U_{n,r} \backslash \{1,\ldots,r\}$ and let b' be the greatest integer in $\{1,\ldots,n\} \backslash B$ with $b' < b_r$. Hence, if b_r and n have the same parity then B is contained in the block $B \cup \{b'\}$; otherwise, B is contained in the block $B \cup \{b_r + 1\}$. However, $\bar{S}(n, r+1, r) \cup \{1, \ldots, r\}$ is not connected. In fact, $\bar{S}(n, r+1, r) \cup \{1, \ldots, r\}$ has $\lfloor (n-r+1)/2 \rfloor$ connected components, the blocks in which the last two elements are $n-2i$ and $n-2i-1$ for $i = 0, \ldots, \lfloor (n-r-1)/2 \rfloor$. So, a connected covering is given by $S(n, r+1, r) = \bar{S}(n, r+1, r) \cup \{1, 2, \ldots, r-1, n-2i-2, n-2i\}_{0 \leqslant i \leqslant \lfloor (n-r-1)/2 \rfloor}$. So, we have,

$$|S(n, r+1, r)| = \binom{n-2}{r-1} + \binom{n-4}{r-1} + \cdots + \binom{n - 2\lfloor \frac{n-r+1}{2} \rfloor}{r-1} + \left\lfloor \frac{n-r}{2} \right\rfloor,$$

hence,

$$|S(n, r+1, r)| = \sum_{\substack{i=2 \\ i\text{-even}}}^{2\lfloor (n-r+1)/2 \rfloor} \binom{n-i}{r-1} + \left\lfloor \frac{n-r}{2} \right\rfloor. \quad \square$$

Corollary 3.3. *Let $r > 1$ be a fixed integer and let $S(n, r+1, r)$ be the $(n, r+1, r)$ connected covering given in Theorem 3.2. Then*

$$\lim_{n \to \infty} \frac{|S(n, r+1, r)|}{\binom{n-1}{r}} \to \frac{1}{2}.$$

Proof. It is known that

$$\binom{n}{r} = \sum_{j=r-1}^{n-1} \binom{j}{r-1}.$$

So, by Theorem 3.2,

$$\frac{\binom{n}{r}}{2} + \left\lfloor \frac{n-r}{2} \right\rfloor = \frac{1}{2} \sum_{j=1}^{n-r+1} \binom{n-i}{r-1} + \left\lfloor \frac{n-r}{2} \right\rfloor \geqslant \sum_{\substack{j=2 \\ j\text{-even}}}^{2\lfloor \frac{n-r+1}{2} \rfloor} \binom{n-i}{r-1} + \left\lfloor \frac{n-r}{2} \right\rfloor$$

$$= |S(n, r+1, r)|,$$

since for all $i = 2, \ldots, 2\lfloor (n-r+1)/2 \rfloor$, i-even we have $\binom{n-i}{r-1} < \binom{n-i+1}{r-1}$. On the other hand,

$$\frac{\binom{n-1}{r}}{2} = \frac{1}{2} \sum_{j=r-1}^{n-2} \binom{j}{r-1} \leqslant \sum_{\substack{j=2 \\ j\text{-even}}}^{2\lfloor (n-r+1)/2 \rfloor} \binom{n-j}{r-1} \leqslant |S(n, r+1, r)|.$$

The result follows since $\lim_{n \to \infty} \binom{n}{r} / \binom{n-1}{r} \to 1$ for fixed integer $r > 1$. $\square$

Corollary 3.4. *Let n be an integer with $n \geqslant 3$. Then there exists a $(n, 3, 2)$ connected covering $\bar{C}(n, 3, 2)$ such that $|\bar{C}(n, 3, 2)| = CC^*(n, 2) + \lceil \frac{n-5}{4} \rceil$.*

Proof. By Theorem 3.2 there exists a $(n,3,2)$ connected covering $\bar{C}(n,3,2)$ such that, if n is even then

$$|\bar{C}(n,3,2)| = n-2+\cdots+2+\frac{n-2}{2} = \frac{n-2}{2}\left(\frac{n-2}{2}+1\right)+\frac{n-2}{2}$$

$$= \frac{n(n-2)}{4}+\frac{n-2}{2} = \frac{n^2-4}{4}.$$

And, if n is odd then

$$|\bar{C}(n,3,2)| = n-2+\cdots+1+\frac{n-3}{2}$$

$$= \frac{n-1}{2}\left(\frac{n-1}{2}+1\right)+\frac{n-3}{2}-\frac{n-1}{2} = \frac{n^2-5}{4}. \qquad \square$$

We illustrate Theorem 3.2 with the following example. Let $n=6$ and $r=3$. Hence, $S(6,4,3)=\{3456, 2456, 2356, 1356, 1456, 1256, 1234, 1246\}$ with $|S(6,4,3)| = \binom{4}{2}+\binom{2}{2}+\lfloor\frac{4}{2}\rfloor = 8$ which is better than

$$HV_1(6,4)=\{1234, 1235, 1236, 1245, 1246, 1256, 1345, 1346, 1356, 1456\}$$

or

$$HV_2(6,4)=\{1234, 1245, 1256, 1345, 1356, 1456, 2345, 2356, 2456, 3456\}$$

with $|HV_i(6,4)| = \binom{5}{3} = 10$ for $i=1,2$. However, the minimum value $CC(6,4,3) = CC^*(6,3) = \lceil\binom{6}{3}-1/3\rceil = 7$ is attained, for instance, by the set $\{1234, 1235, 1236, 4561, 4562, 4563, 2345\}$ (note that 2345 is necessary for connectivity). From the former family, we may deduce another method to construct $(n,r+1,r)$ connected coverings.

Theorem 3.5. *Let n,r and k be positive integers with $n>r,k$. Let*

$$f_a = \begin{cases} \binom{n-k}{r+1} & \text{if } a=0, \\ \binom{k}{r+k-n+1} & \text{if } a=r+k-n, \end{cases} \qquad f_b = \begin{cases} \binom{k}{r+1} & \text{if } b=r, \\ \binom{n-k}{r-k+1} & \text{if } b=k, \end{cases}$$

where $a=\sup\{0,r+k-n\}$ and $b=\inf\{r,k\}$. Then

$$CC(n,r+1,r) \leqslant \begin{cases} \sum\limits_{\substack{i=1 \\ i-\text{odd}}}^{b-a} \binom{k}{a+i}\binom{n-k}{r+1-a-i} + \dfrac{b-a-1}{2} & \text{if } b-a \text{ is odd}, \\ \sum\limits_{\substack{i=1 \\ i-\text{odd}}}^{b-a-1} \binom{k}{a+i}\binom{n-k}{r+1-a-i} + f_b + \dfrac{b-a-2}{2} & \text{otherwise} \end{cases}$$

and

$$CC(n,r+1,r)\leqslant\begin{cases}\displaystyle\sum_{\substack{i=2\\ i\text{-even}}}^{b-a-1}\binom{k}{a+i}\binom{n-k}{r+1-a-i}+f_b\\ \displaystyle\quad+f_a+\frac{b-a-1}{2} & \textit{if } b-a \textit{ is odd},\\ \displaystyle\sum_{\substack{i=2\\ i\text{-even}}}^{b-a}\binom{k}{a+i}\binom{n-k}{r+1-a-i}\\ \displaystyle\quad+f_a+\frac{b-a}{2} & \textit{otherwise.}\end{cases}$$

Proof. Let n,r and k be integers with $n>r,k$. We will construct two $(n,r+1,r)$ connected covering, $M_k^1(n,r+1,r)$ and $M_k^2(n,r+1,r)$ as follows. Partition $\{1,\dots,n\}$ into sets A and B such that $|A|=k$ and $|B|=n-k$. Let T_i be the set of the r-subsets, t_i of $\{1,\dots,n\}$ with $|t_i\cap A|=i$, $0\leqslant i\leqslant k$ and $|t_i\cap B|=r-i$, $0\leqslant r-i\leqslant n-k$. Hence, the set of all r-subsets of $\{1,\dots,n\}$, $U_{n,r}$, is given by $U_{n,r}=\bigcup_{i=a}^{b}T_i$ where $a=\sup\{0,r+k-n\}$ and $b=\inf\{r,k\}$. Let $L_i=\{i\text{-subsets of } A\}\times\{(r+1-i)\text{-subsets of } B\}$ for $a\leqslant i\leqslant b$. It is clear that for each $t_i\in T_i$ there exists a $(r+1)$-subset $l_i\in L_i$ with $t_i\subset l_i$ and/or a $(r+1)$-subset $l_{i+1}\in L_{i+2}$ with $t_i\subset l_{i+1}$. Note that the $(r+1)$-subsets l_i and l_{i+1} can always be constructed except when $|B|=r-i$ and $|A|=i$ respectively. Hence,

$$\bar{M}_k^1=\bar{M}_k^1(n,r+1,r)=\begin{cases}\displaystyle\bigcup_{\substack{i=1\\ i\text{-odd}}}^{b-a}L_{a+i} & \text{if } b-a \text{ is odd},\\ \displaystyle\bigcup_{\substack{i=1\\ i\text{-odd}}}^{b-a-1}L_{a+i}\cup L_b & \text{otherwise}\end{cases}$$

and

$$\bar{M}_k^2=\bar{M}_k^2(n,r+1,r)=\begin{cases}\displaystyle\bigcup_{\substack{i=0\\ i\text{-even}}}^{b-a-1}L_{a+i}\cup L_b & \text{if } b-a \text{ is odd},\\ \displaystyle\bigcup_{\substack{i=0\\ i\text{-even}}}^{b-a}L_{a+i} & \text{otherwise,}\end{cases}$$

are $(n,r+1,r)$ covering. Note that each L_i is a connected component of $\bar{M}_k^i$, $i=1,2$. However, $\bar{M}_k^i$, $i=1,2$ is not connected since $|l_i\cap l_j|\leqslant r-1$ for either all $1\leqslant i\neq j\leqslant k$, i,j-odd or all $1\leqslant i\neq j\leqslant k$, i,j-even. Let

$$M_k^1(n,r+1,r)=\begin{cases}\displaystyle\bar{M}_k^1\cup\bigcup_{\substack{i=2\\ i\text{-even}}}^{b-a-1}\{e_1,\dots,e_i,e_1',\dots,e_{r+1-i}'\} & \text{if } b-a \text{ is odd},\\ \displaystyle\bar{M}_k^1\cup\bigcup_{\substack{i=2\\ i\text{-even}}}^{b-a-2}\{e_1,\dots,e_i,e_1',\dots,e_{r+1-i}'\} & \text{otherwise}\end{cases}$$

and

$$M_k^2(n,r+1,r)=\begin{cases}\bar{M}_k^2 \cup \bigcup_{\substack{i=1\\ i-\text{odd}}}^{b-a-2} \{e_1,\dots,e_i,e'_1,\dots,e'_{r+1-i}\} & \text{if } b-a \text{ is odd},\\ \bar{M}_k^2 \cup \bigcup_{\substack{i=1\\ i-\text{odd}}}^{b-a-1} \{e_1,\dots,e_i,e'_1,\dots,e'_{r+1-i}\} & \text{otherwise},\end{cases}$$

where $e_1,\dots,e_i$ and $e'_1,\dots,e'_{r+1-i}$ are the first i and $r+1-i$ elements in A and B, respectively. Clearly, $M_k^i(n,r+1,r)$, $i=1,2$ is a $(n,r+1,r)$ connected covering. Moreover,

$$|M^1(n,r+1,r)|=\begin{cases}\displaystyle\sum_{\substack{i=1\\ i-\text{odd}}}^{b-a}\binom{k}{a+i}\binom{n-k}{r+1-a-i} + \frac{b-a-1}{2} & \text{if } b-a \text{ is odd},\\ \displaystyle\sum_{\substack{i=1\\ i-\text{odd}}}^{b-a-1}\binom{k}{a+i}\binom{n-k}{r+1-a-i} + f_b + \frac{b-a-2}{2} & \text{otherwise}\end{cases}$$

and

$$|M^2(n,r+1,r)|=\begin{cases}\displaystyle\sum_{\substack{i=2\\ i-\text{even}}}^{b-a-1}\binom{k}{a+i}\binom{n-k}{r+1-a-i} + f_b + f_a + \left\lfloor\frac{b-a-1}{2}\right\rfloor & \text{if } b-a \text{ is odd},\\ \displaystyle\sum_{\substack{i=2\\ i-\text{even}}}^{b-a}\binom{k}{a+i}\binom{n-k}{r+1-a-i} + f_a + \frac{b-a}{2} & \text{otherwise},\end{cases}$$

with

$$f_a=\begin{cases}\binom{n-k}{r+1} & \text{if } a=0,\\ \binom{k}{r+k-n+1} & \text{if } a=r+k-n,\end{cases}\qquad f_b=\begin{cases}\binom{k}{r+1} & \text{if } b=r,\\ \binom{n-k}{r-k+1} & \text{if } b=k,\end{cases}$$

where $a=\sup\{0,r+k-n\}$ and $b=\inf\{r,k\}$. □

In fact, we may do it much better than Theorem 3.5 as follows.

Theorem 3.6. *Let n, r and k be positive integers with $n > r, k$. Then*

$$CC(n,r+1,r) \leqslant \begin{cases} \sum_{\substack{i=1\\ i-\text{odd}}}^{b-a} E(n,r,a+i) + f_b + \frac{b-a-1}{2} & \text{if } b-a \text{ is odd,} \\ \sum_{\substack{i=1\\ i-\text{odd}}}^{b-a-1} E(n,r,a+i) + \frac{b-a-2}{2} & \text{otherwise} \end{cases}$$

and

$$CC(n,r+1,r) \leqslant \begin{cases} \sum_{\substack{i=2\\ i\text{-even}}}^{b-a-1} E(n,r,a+i) + f_b + f_a + \frac{b-a-1}{2} \\ \quad \text{if } b-a \text{ is odd,} \\ \sum_{\substack{i=2\\ i-\text{even}}}^{b-a} E(n,r,a+i) + f_a + \frac{b-a}{2} & \text{otherwise,} \end{cases}$$

with

$$E(n,r,i) = CC(k,i,i-1)\binom{n-k}{r+1-i} + CC(n-k,r+1-i,r-i)\binom{k}{i} \\ -CC(k,i,i-1)CC(n-k,r+1-i,r-i),$$

$$f_a = \begin{cases} CC(n-k,r+1,r) & \text{if } a=0, \\ CC(k,r+k-n+1,r+k-n) & \text{if } a=r+k-n, \end{cases}$$

$$f_b = \begin{cases} CC(k,r+1,r) & \text{if } b=r, \\ CC(n-k,r-k+1,r-k) & \text{if } b=k, \end{cases}$$

where $a = \sup\{0, r+k-n\}$ and $b = \inf\{r,k\}$.

Proof. Let n, r and k be integers with $n > r, k$. We may construct two $(n, r+1, r)$ connected coverings, $N_k^1(n,r+1,r)$ and $N_k^2(n,r+1,r)$ (similarly as in Theorem 3.5) as follows. Partition $\{1,\ldots,n\}$ into sets A and B such that $|A| = k$ and $|B| = n-k$. Let $M_i = \bar{C}(n-k, r+1-i, r-i) \times \{i\text{-subsets of } A\}$ and $\bar{M}_i = \bar{C}(k,i,i-1) \times \{(r+1-i)\text{-subsets of } B)\}$ for $a \leqslant i \leqslant b$ $a = \sup\{0, r+k-n\}$ and $b = \inf\{r,k\}$ where $\bar{C}(l, p+1, p)$ is a $(l, p+1, p)$ connected covering with$|\bar{C}(l,p+1,p)| = CC(l,p+1,p)$. It is clear that for each r-subset t of $\{1,\ldots,n\}$ with $|t \cap A| = i$ (or $|t \cap A| = i-1$) there exists a $(r+1)$-subset $m_i \in M_i$ (or $\bar{m}_i \in \bar{M}_i$) such that $t \subset m_i$ (or $t \subset \bar{m}_i$). Note that $|M_i| = CC(k,i,i-1)\binom{n-k}{r+1-i} + CC(n-k,r+1-i,r-i)\binom{k}{i} - CC(k,i,i-1)CC(n-k,r+1-i,r-i)$. From here, the rest of the proof can be deduced by using similar arguments as in Theorem 3.5. □

4. Other upper bounds

In this section, we will give upper bounds of $CC(n,r+1,r)$ in terms of $C(n,r+1,r)$.

Theorem 4.1. *Let n and r be positive integers with $n>r+1$. Then $CC(n,r+1,r)\leqslant 2C(n,r+1,r)$.*

Proof. Let $\bar{V}=\bar{V}(n,r+1)$ be a $(n,r+1,r)$ covering with $|\bar{V}|=C(n,r+1,r)$. Suppose that $\bar{V}$ has $C_1,\dots C_s$, $s\geqslant 1$ connected components. We claim that it is always possible to form a $(n,r+1,r)$ connected covering $V=V(n,r+1,r)$ with $|V|=|\bar{V}|+s$. Indeed, given a connected component C_k there always exists an r-subset b such that $b\subset b_1$ for some $b_1\in C_k$ and an element $e\in\{1,\dots,n\}$ such that $b\cup\{e\}\notin C_k$ (otherwise, C_k is a connected covering since any r-subset is in C_k). Then $b\cup\{e\}$ contains at least one r-subset b' such that $b'\in C_i$ for some $1\leqslant i\neq k\leqslant s$. So, by adding block $b\cup\{e\}$ to V we reduce its number of components to at least $s-1$, and so on. The result follows since $|V|=C(n,r+1,r)+s\leqslant 2C(n,r+1,r)$. □

Lemma 4.2. *Let n and r be positive integers with $n>r\geqslant 2$. Then $CC(n,r+1,r)\leqslant \sum_{i=r+1}^{n-1}C(i,r,r-1)$.*

Proof. Let $S_{n,r}$ be the matrix with columns (and rows) all $(r+1)$-subsets (r-subsets) of $\{1,\dots,n\}$ in lexicographic order from left to right (from top to bottom) respectively with $s_{i,j}=1$ if the ith (r subset) row is contained in the jth ($(r+1)$-subset) column, $s_{i,j}=0$ otherwise.

We construct recursively a connected covering $W=W(n,r+1,r)$ as follows: consider a $(n-1,r,r-1)$ covering $W_1(n-1,r,r-1)$ and a $(n-1,r+1,r)$ connected covering $W_2(n-1,r+1,r)$. Since

$$S_{n,r}=\begin{pmatrix} S_{n-1,r-1} & 0 \\ I & S_{n-1,r}\end{pmatrix}$$

then $W=W_1+n\cup W_2$ is a $(n,r+1,r)$ connected covering where $W_1+n=\{w\cup\{n\}\mid w\in W_1\}$, and so on. □

The *Turán number* $T(n,l,r)$ is the minimum number of r-subsets of an $\{1,\dots,n\}$ such that every l-subset contains at least one of the r-subsets. It is easy to see that $C(v,k,t)=T(v,v-t,v-k)$, so covering numbers are just Turán numbers reordered. In 1941, Turán [10] determined $T(n,k,2)$ for any k. Rödl [8] proved that $C(n,m,p)=L(n,m,p)(1+o(1))$ where $L(n,m,p)=\lceil \frac{n}{m}\lceil\frac{n-1}{m-1}\lceil\cdots\lceil\frac{n-p+1}{m-p+1}\rceil\cdots\rceil\rceil\rceil$ and m,p are fixed and $n\to\infty$, (see [9] for further results on Turán numbers). We have the following corollary of Lemma 4.2.

Corollary 4.3. $CC(n,r+1,r)=(\binom{n}{r}/r)(1+o(1))$ *where r is fixed and $n\to\infty$.*

Proof. Let $W(n,r+1,r)$ be the $(n,r+1,r)$ connected covering as in Lemma 4.2. We claim that

$$|W(n,r+1,r)|=\left(\frac{\binom{n}{r}}{r}\right)(1+o(1))$$

where r is fixed and $n \to \infty$. Indeed, from Rödl's equality we have that

$$C(n,r+1,r) = \frac{\binom{n}{r}}{(r+1)}(1+o(1))$$

where r is fixed and $n \to \infty$. Hence,

$$|W(n,r+1,r)| = \sum_{i=r+1}^{n-1} C(i,r,r-1) = \sum_{i=r+1}^{n-1} \frac{\binom{i}{r-1}}{r}(1+o(1)) = \frac{\binom{n}{r}}{r}(1+o(1)),$$

where r is fixed and $n \to \infty$.

The result follows by considering

$$\left(\binom{n}{r} - 1\right)/r \leqslant CC(n,r+1,r) \leqslant |W(n,r+1,r)|. \quad \square$$

5. Numerical results

Here, we will compute some of the upper bounds for $CC(n,r+1,r)$ given in the previous sections to compare them together with $CC^*(n,r)$. First, we give Table 1 with minimal connected coverings for small values of n.

Before giving the table with upper bounds for $CC(n,r+1,r)$, for reader's convenience, we give a brief summary of notation, value and reference of some $(n,r+1,r)$ connected coverings given above.

Let n, r and k be integers with $n > r \geqslant k \geqslant 0$.

(1) (Proposition 3.1) $CC^*(n,r) = (\binom{n}{r} - 1)/r$.

(2) (see [4]) $|HV| = |HV_1(n,r+1,r)| = |HV_2(n,r+1,r)| = \binom{n-1}{r}$.

(3) (Theorem 3.2)

$$|S| = |S(n,r+1,r)| = \sum_{\substack{i=2 \\ i\text{-even}}}^{2\lfloor (n-r+1)/2 \rfloor} \binom{n-i}{r-1} + + \lfloor (n-r)/r2 \rfloor.$$

(4) (Theorem 3.6, (see also Theorem 3.5))

$$|N_k^1| = |N_k^1(n,r+1,r)| \quad \text{and} \quad |N_k^2| = |N_k^2(n,r+1,r)|.$$

Table 1

n,r	$CC^*(n,r)$	Minimal connected covering
4,2	3	123, 124, 234
5,2	5	123, 124, 145, 235, 345
5,3	3	1234, 1235, 1245, 2345
6,2	7	126, 134, 156, 234, 235, 356, 456
6,3	7	1234, 1235, 1236, 1456, 2345, 2456, 3456
6,4	4	12345, 12346, 12356, 13456, 23456
7,2	10	123, 124, 145, 167, 246, 257, 347, 356, 467, 567

Table 2

n,r	$CC^*(n,r)$	HV	S	N_k^*	V
7,3	12	20	15	15(2,3)	24
7,4	9	15	12	10(3)	18
7,5	4	6	6	6(1,2,3)	12
8,2	14	21	15	14(2)	—
8,3	19	35	24	23(2)	28
8,4	18	35	26	24(3)	40
8,5	11	21	17	13(4)	24
8,6	5	7	7	7(1,2,3,4)	14
9,2	18	28	19	19(2)	—
9,3	28	56	37	37(2)	50
9,4	32	70	48	45(3)	60
9,5	25	56	42	37(3)	60
9,6	14	28	23	17(4)	32
9,7	5	8	8	8(1,2,3,4)	16
10,2	22	36	24	23(2)	—
10,3	40	84	53	52(2)	60
10,4	53	126	83	81(2)	102
10,5	51	126	88	77(3)	100
10,6	35	84	64	55(3,4)	90
10,7	17	36	30	21(5)	40
10,8	6	9	9	9(1,2,3,4,5)	—
11,2	27	45	29	29(2)	—
11,3	55	120	74	74(2)	94
11,4	83	210	133	130(2)	132
11,5	93	252	169	161(3)	200
11,6	77	210	150	125(3)	168
11,7	47	120	93	76(4)	126
11,8	21	45	38	26(5)	50
11,9	6	10	10	10(1,2,3,4,5)	—
12,2	33	55	35	34(2)	—
12,3	73	165	99	98(2)	114
12,4	124	330	204	202(2)	226
12,5	159	462	299	288(2)	264
12,6	154	462	317	285(3,6)	354
12,7	113	330	241	193(6)	252
12,8	62	165	130	103(4)	168
12,9	25	55	47	31(6)	—
12,10	7	11	11	11(1,2,3,4,5,6)	—
13,2	39	66	41	41(2)	—
13,3	95	220	130	130(2)	156
13,4	179	495	299	296(2)	314
13,5	258	792	500	492(2)	490
13,6	286	924	613	549(6)	528
13,7	245	792	556	477(6)	594
13,8	161	495	369	279(6)	370
13,9	80	220	176	136(5)	—
13,10	29	66	57	37(6)	—
13,11	7	12	12	12(1,2,3,4,5,6)	—
14,2	45	78	48	47(2)	—
14,3	121	286	166	165(2)	182
14,4	250	715	425	423(2)	470

Table 2. Continued.

14, 5	401	1287	795	784(2)	770
14, 6	501	1716	1110	1032(6)	1018
14, 7	491	1716	1166	959(7)	948
14, 8	376	1287	923	753(6)	964
14, 9	223	715	543	387(6)	—
14, 10	100	286	232	174(5)	—
14, 11	33	78	68	43(7)	—
14, 12	8	13	13	13(1, 2, 3, 4, 5, 6, 7)	—

(5) (Theorem 4.1) $|V| = |V(n, r+1, r)| = 2C(n, r+1, r)$.

In our calculations for the coverings $N_k = \min\{N_k^1, N_k^2\}$, we will treat $CC(n, r+1, r)$ as the minimal value that we could find rather than the absolute minimum value (it may be). We actually give $N_k^* = \min_{0 \leqslant k \leqslant \lfloor n/2 \rfloor} \{N_k\}$ and the integers l for which $N_k^* = N_l$ for each n and r.

For coverings V, we only write their values in the case when $C(n, r+1, r)$ is found in the tables given by Gordon et al. [3] (see Table 2).

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