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Covering the vertices of a graph with cycles of bounded length

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ABSTRACT

Let $c_k(G)$ be the minimum number of elementary cycles of length at most k necessary to cover the vertices of a given graph G. In this work, we bound $c_k(G)$ by a function of the order of G and its independence number.

1. Introduction

Throughout this paper, we consider a finite simple graph $G = (V, E)$ and we denote by n its order. The distance between two vertices u and v in G is denoted by $d_G(u, v)$, and is defined to be the length of a shortest path joining them in G. The size of a largest independent set of G is called the independence number of G and is denoted by α.

A *covering* of a graph G is a family of elementary cycles of G such that each vertex of G lies in at least one cycle of this family. For terms not defined here, we refer the reader to [1].

In the literature, many results dealing with the covering of a graph with cycles have appeared. Corrádi and Hajnal (in [3]) have proved a result conjectured a few years before by Erdös, which is that if G is a graph of order $n \geq 3k$ with minimum degree $\delta \geq 2k$, then G contains k vertex disjoint cycles. Later on, several authors have been, in some sense, inspired by this theorem and have sharpened it in many ways. In [9], Lesniak has discussed a variety of results dealing with the existence of disjoint cycles in a given graph.

In [5,10], Enomoto and Wang have relaxed the degree condition given by Erdös. They have independently established that a graph of order at least $3k$ in which $d(u) + d(v) \geq 4k - 1$ for every pair of non-adjacent vertices u and v contains k vertex disjoint cycles. In [4], Egawa et al. have proved that by taking three integers d, k, and n such that $k \geq 3$, $d \geq 4k - 1$ and $n \geq 3k$ and a graph G of order n, in which each pair of non-adjacent vertices x and y verifies $d(x) + d(y) \geq d$, then at least $\min(d, n)$ vertices of G can be covered with k vertex disjoint cycles.

However, in what precedes, the interest was in the independence of the cycles rather than the fact that they cover all the vertices of the graph. In [7], Kouider and Lonc have proved that the vertices of a 2-connected graph in which $\sum_{x \in S} d_G(x) \geq n$ for every independent set S of cardinality s can be covered with at most $s - 1$ cycles. In another paper[8], Kouider shows that the vertices of any κ connected graph are covered with at most $\lceil \alpha/\kappa \rceil$ cycles.

But in all these results, no bound for the length of the cycles taken in the covering is imposed. Recently, in [6], Forge and Kouider have laid down that the cycles taken in the covering are of length not exceeding k (where k is an integer fixed as a preliminary). They have denoted by $c_k(G)$ the cardinality of a minimum covering in which each cycle satisfies the previous

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condition. They have bounded $c_k(G)$ by a function of the minimum degree and the order of the graph G. They have shown that:

If p and k are two integers such that $2 \le p \le \frac{k}{8}$ and if G is a graph of order $n \ge \frac{2k}{3}(p-1)^2 + (p-1)$ and minimum degree δ at least $\frac{n}{p} + \frac{2k}{3}$, then

$$c_k(G) \le \frac{3n}{k} + \frac{\log \frac{k}{3}}{-\log(1 - \frac{1}{2(p-1)^2})} + \left(1 - \frac{3}{k}\right)(p-2) + 1.$$

In this work, we intend to bound $c_k(G)$ by a function of the independence number of the graph and its order and we show, among others, the Corollaries 2.8 and 2.9:

- Let G be a 2-connected graph of order n with independence number $\alpha > 1$ and k be an integer such that $k \ge 2\alpha + 1$. If $n > \alpha(\frac{k+1}{2})$ then $c_k(G) \le \frac{2n}{k+1} + \alpha(1 + \log \frac{k+1}{6})$.
- Let G be a 2-connected graph of order n with independence number α and k an integer such that $\frac{(k+1)}{2(\alpha+1)} \ge 2$. Then $c_k(G) \le \frac{n}{k - \frac{4}{3}(\alpha+1)} + \alpha \log \frac{k}{3}$ if $n > \alpha(k - \frac{4}{3}(\alpha+1))$.

2. Covering the vertices with cycles of length at most k

Let k be an integer and G a graph of order n. We want to cover G with the minimum number of cycles of length at most k.

Each time we have a cycle in G, we check its length. If it is less than or equal to k then this cycle is taken in the covering; otherwise, a chord may reduce its length. Therefore, we should assume that $k \ge 2\alpha + 1$ so that any cycle of length larger than k has at least one chord.

In what follows, we show that according to the prescribed value of k we can guarantee the existence in G of a cycle of length not only at most k but at least a fraction of k as well.

Proposition 2.1. *Let G be a graph of order n and independence number α and let k be an integer such that $k \ge 2\alpha + 1$. If G has a cycle of length more than k, then it has a cycle of length at least $\frac{k+1}{2}$ and at most k.*

Proof. Indeed, if C is a cycle of G of length $l(C)$ at least $k+1 \ge 2\alpha+2$, then there are at least $\alpha+1$ independent vertices on C and thus at least two of these vertices (say x and y) are adjacent. Furthermore, $2 \le d_C(x, y) \le \frac{l(C)}{2}$. The chord (x, y) divides the cycle C into two smaller cycles; the bigger, C_1, is of length $l(C_1)$ between $\frac{l(C)}{2}$ and $l(C) - 1$. We repeat the same construction until we get a cycle C_i such that $\frac{k+1}{2} \le l(C_i) \le k$. □

If we increase the lower bound for k in the previous theorem then the lower bound of the length for the cycle is increased.

Proposition 2.2. *Let G be a graph of order n with independence number α and let k be an integer such that $k \ge 4\alpha + 3$. If G possesses a cycle of length at least $\frac{2k}{3}$, then it has a cycle of length at least $\frac{2k}{3}$ and at most k.*

Proof. Let C be a cycle of G of length $l \ge \frac{2k}{3}$.

If $l \le k$ then C is a cycle of length between $\frac{2k}{3}$ and k.

In the case where $l > k$, we are going to construct a cycle of length at least $\frac{2k}{3}$ and strictly smaller than l. Clearly by iterating the construction we will finally get a cycle of length between $\frac{2k}{3}$ and k.

Consider an orientation O on the cycle. We will use $d_O(x, y)$ as the distance on the cycle using the orientation O. Consider, among all possible sets $\{v_1, \ldots, v_{\alpha+1}\}$ of $(\alpha+1)$ distinct vertices such that $d_O(v_i, v_{i+1}) = 2$ for $1 \le i \le \alpha$, the one that contains two adjacent vertices v_1 and v_i (adjacent in G) at minimum distance on C.

- If $d_O(v_1, v_i) \le \frac{l}{3}$ then we have the desired cycle.
- If not, then consider the following set: $S = \{v_2, \ldots v_{\alpha+1}, v_{\alpha+2}\}$ where $d_O(v_{\alpha+1}, v_{\alpha+2})$ is also 2 on C. Let v_j and v_r be two adjacent vertices of S (as $|S| = \alpha + 1$). We cannot have $j \ge i$; otherwise, since $d_O(v_j, v_r) \ge d_O(v_1, v_i) > \frac{l}{3}$ then $d_O(v_1, v_{\alpha+2}) \ge d_O(v_1, v_i) + d_O(v_j, v_r) \ge \frac{2l}{3}$ but $d_O(v_1, v_{\alpha+2}) \le \frac{l}{2}$ (because $l \ge 4(\alpha+1)$). We get $\frac{l}{2} \ge \frac{2l}{3}$ which is a contradiction. Thus the segments $[v_1, v_i]$ and $[v_j, v_r]$ of C do intersect in at least two vertices. Let $l_1 = d_O(v_1, v_j)$, $l_2 = d_O(v_j, v_i)$ and $l_3 = d_O(v_i, v_r)$. We have $l_1 + l_2 + l_3 \le \frac{l}{2}$ and $l_1 + 2l_2 + l_3 \ge \frac{2l}{3}$. It follows that $l_2 \ge \frac{l}{6}$ and consequently the cycle $C' = (v_1, v_i) \bigcup [v_i, v_j] \bigcup (v_j, v_r) \bigcup [v_r, v_1]$ is of length $l' \ge \frac{2l}{3}$. Let us note that the vertex set of C' is strictly contained in the vertex set of C as it does not contain the neighbor v_1^+ of v_1. So $l' < l$. This completes the proof. □

More generally, for an integer $c \ge 2$ and for $k \ge 2c(\alpha+1) - 1$, we have the following result.

Proposition 2.3. *Let G be a graph of order n with independence number α. Let c and k be two integers such that $c \ge 2$ and $k \ge 2c(\alpha+1) - 1$. If G possesses a cycle of length at least $(1 - \frac{2}{3c})k$, then it has a cycle of length at least $(1 - \frac{2}{3c})k$ and at most k.*

Proof. We use the definitions and techniques of the preceding proof. Let C be a cycle of G of length $l \geq (1-\frac{2}{3c})k$.

If $l \leq k$ then C is as desired.

Otherwise, consider, among all possible sets $\{v_1, \ldots, v_{\alpha+1}\}$ of $(\alpha+1)$ vertices such that $d_O(v_i, v_{i+1}) = 2$ for $1 \leq i \leq \alpha$, the one that contains two adjacent vertices v_1 and v_i at minimum distance on C.

- If $d_O(v_1, v_i) \leq \frac{2l}{3c}$ then we have the desired cycle.
- If $d_O(v_1, v_i) > \frac{2l}{3c}$ then consider the following set: $S = \{v_2, \ldots v_{\alpha+1}, v_{\alpha+2}\}$, where $d_O(v_{\alpha+1}, v_{\alpha+2})$ is also 2 on C. Let v_j and v_r be two adjacent vertices of S. We have $j < i$; otherwise, on one hand $d_O(v_j, v_r) \geq d_O(v_1, v_i) > \frac{2l}{3c}$ and then $d_O(v_1, v_{\alpha+2}) \geq d_O(v_1, v_i) + d_O(v_j, v_r) \geq \frac{4l}{3c}$, and on the other hand $d_O(v_1, v_{\alpha+2}) \leq \frac{l}{c}$ (since $l \geq 2c(\alpha+1)$). We get $\frac{4l}{3c} \leq \frac{l}{c}$ which is a contradiction. Thus the segments $[v_1, v_i]$ and $[v_j, v_r]$ of the cycle C do intersect in at least two vertices. Let $l_1 = d_O(v_1, v_j)$, $l_2 = d(v_j, v_i)$ and $l_3 = d_O(v_i, v_r)$. We have: $l_1 + l_2 + l_3 \leq \frac{l}{c}$ and $l_1 + 2l_2 + l_3 \geq \frac{4l}{3c}$. So $l_2 \geq \frac{l}{3c}$ and as a result the cycle $C' = (v_1, v_i) \bigcup [v_i, v_j] \bigcup (v_j, v_r) \bigcup [v_r, v_1]$ is of length l', such that $l - 1 \geq l' \geq (1-\frac{2}{3c})l$, as desired.

In the previous propositions, we supposed that a cycle exists to begin the construction. The next proposition of [2] ensures the existence (maybe by adding conditions) of at least a cycle in G of sufficient length.

Proposition 2.4. *Let G be a graph of independence number α; then G possesses a cycle, an edge or a vertex whose removal reduces its independence number by at least 1. Therefore, G can be covered with at most α disjoint cycles, edges or vertices.*

Proof. The proposition is obviously true for edgeless graphs; so we assume that the graph G has edges. Let P be a longest path in G and let x be one of its endpoints. All the neighbors of x are on P; otherwise we get a contradiction. Two cases may occur:

(1) x is not of degree 1 in G. Then we consider u the furthermost neighbor of x on P. The cycle C made of the segment $[x, u]$ on P and the edge (x, u) contains x and all of its neighbors. Thus if we remove it, we get a graph with smaller independence number: $\alpha(G - C) \leq \alpha(G) - 1$.

(2) x is of degree 1 in G. Then by suppressing the vertex x and its neighbor x' we get $\alpha(G - \{x, x'\}) \leq \alpha(G) - 1$.

The second part can be deduced by induction. □

We note that the preceding proposition implies that if $n \geq 3\alpha$, then there exists a cycle of length at least n/α. By combining all the foregoing, and by supposing moreover that G is 2-connected with a vertex set large enough and with $\frac{k}{\alpha}$ large enough, then we can cover G with at most a number of order $\frac{n}{(1-\frac{2}{3c})k}$ of cycles of length at most k, as stated in the following result:

Theorem 2.5. *Let G be a 2-connected graph of order n with independence number $\alpha > 1$. Let c and k be two integers such that $c \geq 2$ and $k \geq 2c(\alpha+1) - 1$. If $n \geq \alpha(1-\frac{2}{3c})k$, then*

$$c_k(G) \leq \frac{n}{(1-\frac{2}{3c})k} + \alpha \log \frac{(1-\frac{2}{3c})k}{3} + \alpha.$$

Proof. The proof is composed of three steps depending on the size of N, the set of uncovered vertices. In the first step, $|N| \geq \alpha(1-\frac{2}{3c})k$ and there exists a cycle of length at least $(1-\frac{2}{3c})k$ and at most k. When $|N|$ is no longer greater than $\alpha(1-\frac{2}{3c})k$ we go to the next step. In step 2, while $|N| \geq 3\alpha$, there exists a cycle of length at least $|N|/\alpha$ and at most k. In Step 3, while $|N| \geq \alpha$ we cover the remaining vertices two by two, and then only one by one.

Step 1. While $|N| \geq \alpha(1-\frac{2}{3c})k$, then by Proposition 2.4, we have a cycle of length at least $\frac{|N|}{\alpha} \geq (1-\frac{2}{3c})k$. If the length of the cycle is greater than k then, by Proposition 2.3, we know how to reduce it, obtaining in any case a cycle which covers at least $(1-\frac{2}{3c})k$ vertices of N. At the end of this step, at most $\frac{n}{(1-\frac{2}{3c})k} - \alpha$ cycles would be used.

Now $|N| < (1-\frac{2}{3c})k\alpha$.

Step 2. While $|N| \geq 3\alpha$, then by Proposition 2.4 we can find a cycle in the induced subgraph $G[N]$ of length at least $|N|/\alpha$ and by Proposition 2.3 we can reduce its length. We then obtain a cycle of length at least $|N|/\alpha$ and at most k. The number of cycles used in this step is given by the number i of iterations carried out until $|N|$ becomes $< 3\alpha$. After the first iteration, there remain at most $|N| - \frac{|N|}{\alpha} = |N|(1-\frac{1}{\alpha})$ uncovered vertices. After i iterations, there are at most $|N|(1-\frac{1}{\alpha})^i$ uncovered vertices. We stop when $|N|(1-\frac{1}{\alpha})^i$ becomes smaller than 3α. Since $|N| < (1-\frac{2}{3c})k\alpha$, it is sufficient to stop for i satisfying $(1-\frac{2}{3c})k\alpha(1-\frac{1}{\alpha})^i \leq 3\alpha$. It follows that $i \leq \frac{\log \frac{3}{(1-\frac{2}{3c})k}}{\log(1-\frac{1}{\alpha})} \leq \alpha \log \frac{(1-\frac{2}{3c})k}{3}$, using that $\log(1-\frac{1}{\alpha}) < -\frac{1}{\alpha}$.

When this step is over, we have $|N| < 3\alpha$.

Step 3. While $|N|$ is greater than α, we can cover its vertices two by two (by Proposition 2.4) and since the considered graph G is 2-connected, then every edge lies in a cycle. If the length of this cycle is greater than k then we know how to reduce it (Proposition 2.3). Thus we obtain at most α new cycles in the covering.

And finally, when $|N| \leq \alpha$ we can cover the vertices one by one and for the same aforementioned reasons, we get at most α additional cycles in the covering. In short, we have a covering of G by at most $\frac{n}{(1-\frac{2}{3c})k} + \alpha \log \frac{(1-\frac{2}{3c})k}{3} + \alpha$ cycles. □

Remark 2.6.

(1) In order for the function $\log(1-\frac{1}{\alpha})$ to be defined, the case $\alpha = 1$ has been put aside. If this case occurs, then the 2-connected graph G is a clique and hence it can be covered with at most $\lceil \frac{n}{k} \rceil$ cycles.

(2) More generally, by taking just a non-zero integer c, the same bound holds on replacing $(1-\frac{2}{3c})k$ by $\gamma = \max((1-\frac{2}{3c})k, \frac{k+1}{2})$. Note that the greater c is, the closer γ and k are.

The previous bound for $c_k(G)$ remains even if n is not as large as assumed in the previous theorem. However, it can be improved.

Theorem 2.7. *Let G be a 2-connected graph of order n with independence number $\alpha > 1$. Let c and k be two integers such that $c \geq 1$, $k \geq 2c(\alpha+1)-1$ and $\gamma = \max((1-\frac{2}{3c})k, \frac{k+1}{2})$.*

If $n > \alpha\gamma$ then $c_k(G) \leq \frac{n}{\gamma} + \alpha(1+\log\frac{\gamma}{3})$, if $3\alpha < n \leq \alpha\gamma$ then $c_k(G) \leq \alpha(2+\log\frac{\gamma}{3})$, and if $n \leq 3\alpha$ then $c_k(G) \leq 2\alpha$.

Proof. The proof of the first case is analogous to the proof of Theorem 2.5.

The proofs of the other two cases are quite similar starting from Step 2 and Step 3 respectively in the proof of Theorem 2.5. □

For the complete graph K_n (n very large), we have $c_k(K_n) = \lceil \frac{n}{k} \rceil$ cycles, which is not so far from $\frac{n}{(1-\frac{2}{3c})k} + \alpha\log(\frac{(1-\frac{2}{3c})k}{3})$ given by Theorem 2.5 for $k \geq 2c(\alpha+1)-1$ and c very large.

From the hypothesis $k \geq 2c(\alpha+1)-1$ of Theorem 2.5, the first term $\frac{n}{(1-\frac{2}{3c})k}$ is not better than $\frac{n}{k-8/3}$.

We deduce naturally the following corollaries from Theorem 2.7. We obtain Corollary 2.8 by taking $c = 1$ and Corollary 2.9 by taking $c = \lceil \frac{(k+1)}{2(\alpha+1)} \rceil$.

Corollary 2.8. *Let G be a 2-connected graph of order n with independence number $\alpha > 1$. Let k be an integer such that $k \geq 2\alpha$.*

If $n > \alpha(\frac{k+1}{2})$ then $c_k(G) \leq \frac{2n}{k+1} + \alpha(1+\log\frac{k+1}{6})$, if $3\alpha < n \leq \alpha(\frac{k+1}{2})$ then $c_k(G) \leq \alpha(2+\log\frac{k+1}{6})$, and if $n \leq 3\alpha$ then $c_k(G) \leq 2\alpha$.

Corollary 2.9. *Let G be a 2-connected graph of order n with independence number α and k an integer such that $\frac{(k+1)}{2(\alpha+1)} \geq 2$. Then $c_k(G) \leq \frac{n}{k-\frac{4}{3}(\alpha+1)} + \alpha(\log\frac{k}{3}+1)$ if $n > \alpha(k-\frac{4}{3}(\alpha+1))$; $c_k(G) \leq \alpha(2+\log\frac{k}{3})$ if $3\alpha \leq n \leq \alpha(k-\frac{4}{3}(\alpha+1))$ and $c_k(G) \leq 2\alpha$ if $n \leq 3\alpha$.*

Proof. In the case $n > \alpha(k-\frac{4}{3}(\alpha+1))$, as $c \geq 2$, then $\gamma = (1-\frac{2}{3c})k$. Furthermore $c \geq \frac{(k+1)}{2(\alpha+1)}$, so we get $\gamma \geq (1-\frac{4(\alpha+1)}{3(k+1)})k \geq (k-\frac{4(\alpha+1)}{3})$.

Then the first inequality of the corollary follows. ■

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