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Diagonal Bases in Orlik-Solomon Type Algebras

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Abstract. To encode an important property of the "no broken circuit bases" of the Orlik-Solomon-Terao algebras, András Szenes has introduced a particular type of bases, the so called "diagonal basis." We prove that this definition extends naturally to a large class of algebras, the so called χ-algebras. Our definitions make also use of an "iterative residue formula" based on the matroidal operation of contraction. This formula can be seen as the combinatorial analogue of an iterative residue formula introduced by Szenes. As an application we deduce nice formulas to express a pure element in a diagonal basis.

Keywords: arrangement of hyperplanes, broken circuit, cohomology algebra, matroid, Orlik-Solomon algebra

1. Introduction

We denote by $\mathcal{M} = \mathcal{M}([n])$ a matroid of rank r on the ground set $[n] := \{1, 2, \dots, n\}$. Let V be a vector space of dimension d over some field $\mathbb{K}$. A (central) arrangement (of hyperplanes) in V, $\mathcal{A}_{\mathbb{K}} = \{H_1, \dots, H_n\}$, is a finite listed set of codimension one vector subspaces. Given an arrangement $\mathcal{A}_{\mathbb{K}}$, we suppose always fixed a family of linear forms $\{\theta_{H_i} \in V^*\colon H_i \in \mathcal{A}_{\mathbb{K}}, \mathrm{Ker}(\theta_{H_i}) = H_i\}$, where V^* denotes the dual space of V. We denote by $L(\mathcal{A}_{\mathbb{K}})$ the *intersection lattice of* $\mathcal{A}_{\mathbb{K}}$: i.e., the set of intersections of hyperplanes in $\mathcal{A}_{\mathbb{K}}$, partially ordered by reverse inclusion. There is a matroid $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ on the ground set $[n]$ determined by $\mathcal{A}_{\mathbb{K}}$: a subset $D \subset [n]$ is a *dependent set* of $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ iff there are scalars $\zeta_i \in \mathbb{K}, i \in D$, not all nulls, such that $\sum_{i \in D} \zeta_i \theta_{H_i} = 0$. A *circuit* is a minimal dependent set with respect to inclusion.

If $\mathbb{K}$ is an ordered field, an additional structure is obtained: to every circuit C, $\sum_{i \in C} \zeta_i \theta_{H_i} = 0$, we associate a partition (determined up to a factor ± 1) $C^+ = \{i \in$

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$C\colon \zeta_i > 0\}, C^- = \{i \in C\colon \zeta_i < 0\}$. With this new structure $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ is said a (*realizable*) *oriented matroid* and denoted by $\boldsymbol{\mathcal{M}}(\mathcal{A}_{\mathbb{K}})$. Set $\underline{\boldsymbol{\mathcal{M}}}(\mathcal{A}_{\mathbb{K}}) = \mathcal{M}(\mathcal{A}_{\mathbb{K}})$. Oriented matroids on a ground set $[n]$, denoted $\boldsymbol{\mathcal{M}}([n])$, are a very natural mathematical concept and can be seen as the theory of generalized hyperplane arrangements, see [2].

Set $\mathfrak{M}(\mathcal{A}_{\mathbb{K}}) = V \setminus \bigcup_{H\in\mathcal{A}_{\mathbb{K}}} H$. The manifold $\mathfrak{M}(\mathcal{A}_{\mathbb{C}})$ plays an important role in the Aomoto-Gelfand theory of multidimensional hypergeometric functions (see [14] for a recent introduction from the point of view of arrangement theory). Let K be a commutative ring. In [10–12] the determination of the cohomology K-algebra $H^*\big(\mathfrak{M}(\mathcal{A}_{\mathbb{C}}); K\big)$ from the matroid $\mathcal{M}(\mathcal{A}_{\mathbb{C}})$ is accomplished by first defining the Orlik-Solomon K-algebra $\mathrm{OS}(\mathcal{A}_{\mathbb{C}})$ in terms of generators and relators which depends only on the matroid $\mathcal{M}(\mathcal{A}_{\mathbb{C}})$, and then by showing that this algebra is isomorphic to $H^*\big(\mathfrak{M}(\mathcal{A}_{\mathbb{C}}); K\big)$. The Orlik-Solomon algebras have been then intensively studied. Descriptions of developments from the early 1980's to the end of 1999, together with the contributions of many authors, can be found in [8, 19].

Aomoto suggested the study of the (graded) $\mathbb{K}$-vector space $\mathrm{AO}(\mathcal{A}_{\mathbb{K}})$, generated by the basis $\{Q(\mathcal{B}_{\mathrm{I}})^{-1}\}$, where I is an independent set of $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$, $\mathcal{B}_{\mathrm{I}} := \{H_i \in \mathcal{A}_{\mathbb{K}}\colon i \in I\}$, and $Q(\mathcal{B}_{\mathrm{I}}) = \prod_{i\in I} \theta_{H_i}$ denotes the corresponding defining polynomial. To answer to a conjecture of Aomoto, Orlik and Terao have introduced in [13] a commutative $\mathbb{K}$-algebra, $\mathrm{OT}(\mathcal{A}_{\mathbb{K}})$, isomorphic to $\mathrm{AO}(\mathcal{A}_{\mathbb{K}})$ as a graded $\mathbb{K}$-vector space in terms of the equations $\{\theta_H\colon H \in \mathcal{A}_{\mathbb{K}}\}$.

A "combinatorial analogue" of the algebra of Orlik-Terao was introduced in [7]: to every oriented matroid $\boldsymbol{\mathcal{M}}$ was associated a commutative $\mathbb{Z}$-algebra, denoted by $\mathbb{A}(\boldsymbol{\mathcal{M}})$.

Here we consider a large class of algebras, the so called χ-algebras, that contain the three just mentioned algebras: Orlik-Solomon, Orlik-Terao and the algebras $\mathcal{A}_\chi(\mathcal{M})$, see [9] or Definition 2.1 below. Following Szenes [15], we define a particular type of bases of $\mathcal{A}_\chi$, the so called "diagonal basis", see Definition 2.7. There is a natural example of these bases, the "no circuit basis". We construct the dual bases of these bases, see Theorem 2.8. Our definitions make also use of an "iterative residue formula" based on the matroidal operation of contraction, see Equation (2.6). This formula can be seen as the combinatorial analogue of an iterative residue formula introduced by Szenes, [15]. As applications we deduce nice formulas to express a pure element in a diagonal basis. We prove also that the χ-algebras verify a splitting short exact sequence, see Theorem 2.5. This theorem generalizes for the χ-algebras previous similar theorems of [7, 12].

We use [17, 18] as a general reference in matroid theory. We refer to [2] and [12] for good sources of the theory of oriented matroids and arrangements of hyperplanes, respectively.

2. Diagonal bases

Let $\mathrm{IND}_\ell(\mathcal{M}) \subset \binom{[n]}{\ell}$ be the family of independent sets of cardinal ℓ of the matroid $\mathcal{M}$ and set $\mathrm{IND}(\mathcal{M}) = \bigcup_{\ell\in\mathbb{N}} \mathrm{IND}_\ell(\mathcal{M})$. We denote by $\mathfrak{C} = \mathfrak{C}(\mathcal{M})$ the set of circuits of $\mathcal{M}$. For shortening of the notation the singleton set $\{x\}$ is denoted by x. When the smallest element α of a circuit C, $|C| > 1$, is deleted, the remaining set, $C \setminus \alpha$, is said to be a *broken circuit*. (Note that our definition is slightly different of the standard one. In the standard definition $C \setminus \alpha$ can be empty.) A *no broken circuit* set of a matroid $\mathcal{M}$ is an independent subset of $[n]$ which does not contain any broken circuit. Let $\mathrm{NBC}_\ell(\mathcal{M}) \subset \binom{[n]}{\ell}$ be the set of the no broken circuit sets of cardinal ℓ of $\mathcal{M}$. Set $\mathrm{NBC}(\mathcal{M}) = \bigcup_{\ell \in \mathbb{N}} \mathrm{NBC}_\ell(\mathcal{M})$. We denote by $L(\mathcal{M})$ the lattice of flats of $\mathcal{M}$. $\big($We remark that the lattice map $\phi\colon L(\mathcal{A}_{\mathbb{K}}) \to L(\mathcal{M}(\mathcal{A}_{\mathbb{K}}))$, determined by the one-to-one correspondence $\phi'\colon H_i \longleftrightarrow \{i\}$, $i = 1, \ldots, n$, is a lattice isomorphism.$\big)$ For an independent set I, let $c\ell(I)$ be the closure of I in $\mathcal{M}$.

Fix a set $E = \{e_1, \ldots, e_n\}$. Let $\mathcal{E} = \mathbb{K} \oplus \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ be the graded algebra over the field $\mathbb{K}$ generated by the elements $1, e_1, \ldots, e_n$ and satisfying the relations $e_i^2 = 0$ for all $e_i \in E$ and $e_j \cdot e_i = \beta_{i,j} e_i \cdot e_j$ with $\beta_{i,j} \in \mathbb{K} \setminus 0$ for all $i < j$. Both the exterior algebra (take $\beta_{i,j} = -1$) and the commutative algebra with squares zero (take $\beta_{i,j} = 1$) are such algebras and will be the only ones to be used in the examples. Let $X^\sigma = (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(m)})$, $\sigma \in \mathfrak{S}_m$, denote the ordered set $i_{\sigma(1)} \prec \cdots \prec i_{\sigma(m)}$. When necessary we see the set $X = \{i_1, \ldots, i_m\}$, as the ordered set X^{id}. Set $X^\sigma \setminus x :=$ $(i_{\sigma(1)}, \ldots, \widehat{x}, \ldots, i_{\sigma(m)})$. If $Y^\beta = (j_{\beta(1)}, \ldots, j_{\beta(m')})$ and $X \cap Y = \emptyset$, set $X^\sigma * Y^\beta$ the concatenation $(i_{\sigma(1)}, \ldots, i_{\sigma(m)}, j_{\beta(1)}, \ldots, j_{\beta(m')})$. In the sequel we will denote by e_X the (pure) element $e_{i_1} e_{i_2} \cdots e_{i_m}$ of $\mathcal{E}$. Fix a mapping $\chi\colon 2^{[n]} \to \mathbb{K}$. Let us also define χ for ordered sets by $\chi(X^\sigma) = \mathrm{sgn}(\sigma)\chi(X)$, where $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ.

The χ-*boundary* of an element $e_X \in \mathcal{E}$ is given by the equation

$$\partial e_X = \sum_{p=1}^{m} (-1)^p \chi(X \setminus i_p) e_{X \setminus i_p}.$$

We extend ∂ to $\mathcal{E}$ by linearity. It is easy to see that for $\sigma \in \mathfrak{S}_{|X|}$ we have

$$\partial e_X = \mathrm{sgn}(\sigma) \sum_{p=1}^{m} (-1)^p \chi(X^\sigma \setminus i_{\sigma(p)}) e_{X \setminus i_{\sigma(p)}},$$

and also for any $x \notin X$,

$$\pm \partial e_{X \cup x} = (-1)^{m+1} \chi(X) e_X + \sum_{p=1}^{m} (-1)^p \chi(X \setminus i_p * x) e_{X \setminus i_p \cup x}.$$

Given an independent set I, an element $a \in c\ell(I) \setminus I$ is said *active* in I if a is the minimal element of the unique circuit contained in $I \cup a$. We say that a subset $U \subset [n]$ is a *unidependent* of $\mathcal{M}$, if it contains a unique circuit, denoted $C(U)$. Note that U is unidependent iff $\mathrm{rk}(U) = |U| - 1$. We say that a unidependent set U is an *inactive unidependent* if $\min C(U)$ is the minimal active element of $U \setminus \min C(U)$. Let us remark that U is a unidependent of $\mathcal{M}$ iff for some (or every) $x \in U$, $\mathrm{rk}(x) \neq 0$, $U \setminus x$ is a unidependent of $\mathcal{M}/x$.

Definition 2.1. [9] *Let $\mathfrak{I}_\chi(\mathcal{M})$ be the (right) ideal of $\mathcal{E}$ generated by the χ-boundaries $\{\partial e_C\colon C \in \mathfrak{C}(\mathcal{M}), |C| > 1\}$ and the set $\{e_i\colon \{i\} \in \mathfrak{C}(\mathcal{M})\}$. We say that $\mathbb{A}_\chi(\mathcal{M}) :=$ $\mathcal{E}/\mathfrak{I}_\chi(\mathcal{M})$ is a χ-algebra if χ satisfies the following two properties*:

(UC1) $\chi(I) \neq 0$ *if and only if I is independent.*

(UC2) *For any two unidependents U and U' of $\mathcal{M}$ with $U' \subset U$ there is a scalar $\varepsilon_{_{U,U'}} \in \mathbb{K} \setminus 0$, such that $\partial e_U = \varepsilon_{_{U,U'}} (\partial e_{U'}) e_{U \setminus U'}$.*

Remark 2.1. From (UC2) we conclude that $\mathfrak{I}_\chi(\mathcal{M})$ has the basis

$$\{e_D \colon D \text{ dependent of } \mathcal{M}\} \cup \{\partial e_U \colon U \text{ inactive unidependent of } \mathcal{M}\},$$

and that $\boldsymbol{nbc} := \{[I]_{\mathbb{A}} \colon I \in \mathrm{NBC}(\mathcal{M})\}$ is a basis of the vector space $\mathbb{A} = \mathbb{A}_\chi(\mathcal{M})$. This fundamental property was first discovered for the Orlik-Solomon algebras [12], and then also for other classes of χ-algebras, see [7,13] and the following example for more details. Note also that this implies that $[X]_{\mathbb{A}} \neq 0$ iff X is an independent set of $\mathcal{M}$.

Example 2.1. [9] Recall the three usual χ-algebras. Let $\mathcal{E}$ be the graded algebra over the field $\mathbb{K}$ generated by the elements $1, e_1, \dots, e_n$ and satisfying the relations $e_i^2 = 0$ for all $e_i \in E$ and $e_j \cdot e_i = \beta_{i,j} e_i \cdot e_j$ where $\beta_{i,j}$ denotes a non null scalar fixed for every pair $i < j$.

- Let $\mathcal{E}$ be the exterior algebra (taking $\beta_{i,j} = -1$). Setting $\chi(I^\sigma) = \mathrm{sgn}(\sigma)$ for every independent set I of a matroid $\mathcal{M}$ and every permutation $\sigma \in \mathfrak{S}_{|I|}$, we obtain the Orlik-Solomon algebra, $\mathrm{OS}(\mathcal{M})$.
- Let $\mathcal{A}_{\mathbb{K}} = \{H_i \colon H_i = \mathrm{Ker}(\theta_i), i = 1, 2, \dots, n\}$ be a hyperplane arrangement and $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ its associated matroid. For every flat $F := \{f_1, \dots, f_k\} \subset [n]$ of $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ we choose a basis B_F of the vector subspace of $(\mathbb{K}^d)^*$ generated by $\{\theta_{f_1}, \dots, \theta_{f_k}\}$. By taking for $\mathcal{E}$ the free commutative algebra with squares null (taking $\beta_{i,j} = 1$) and taking for any $\{i_1, \dots, i_\ell\} = I \in \mathrm{IND}_\ell$, $\chi(I) = \det(\theta_{i_1}, \dots, \theta_{i_\ell})$, where the vectors are expressed in the basis $B_{c\ell(I)}$, we obtain the algebra $\mathrm{OT}(\mathcal{A}_{\mathbb{K}})$, defined in [13].
- Let $\boldsymbol{\mathcal{M}}([n])$ be an oriented matroid. For every flat F of $\boldsymbol{\mathcal{M}}([n])$, we choose (determined up to a factor ± 1) a basis signature in the restriction of $\boldsymbol{\mathcal{M}}([n])$ to F. We define a *signature of the independents of an oriented matroid* $\boldsymbol{\mathcal{M}}([n])$ as a mapping, $\mathrm{sgn} \colon \mathrm{IND}(\boldsymbol{\mathcal{M}}) \to \{\pm 1\}$, where $\mathrm{sgn}(I)$ is equal to the basis signature of I in the restriction of $\boldsymbol{\mathcal{M}}([n])$ to $c\ell(I)$. By taking for $\mathcal{E}$ the free commutative algebra over the rational field $\mathbb{Q}$ with squares zero (take $\beta_{i,j} = 1$) and taking $\chi(I) = \mathrm{sgn}(I)$ (resp. $\chi(X) = 0$) for every independent (resp. dependent) set of the matroid, we obtain the algebra $\mathbb{A}(\boldsymbol{\mathcal{M}}) \oplus_{\mathbb{Z}} \mathbb{Q}$, where $\mathbb{A}(\boldsymbol{\mathcal{M}})$ denotes the $\mathbb{Z}$-algebra defined in [7].

For every $X \subset [n]$, we denote by $[X]_{\mathbb{A}}$ or shortly by e_X when no confusion will result, the residue class in $\mathbb{A}_\chi(\mathcal{M})$ determined by the element e_X. Since $\mathfrak{I}_\chi(\mathcal{M})$ is a homogeneous ideal, $\mathbb{A}_\chi(\mathcal{M})$ inherits a grading from $\mathcal{E}$. More precisely we have $\mathbb{A}_\chi(\mathcal{M}) = \mathbb{K} \oplus \mathbb{A}_1 \oplus \cdots \oplus \mathbb{A}_r$, where $\mathbb{A}_\ell = \mathcal{E}_\ell / \mathcal{E}_\ell \cap \mathfrak{I}_\chi(\mathcal{M})$ denotes the subspace of $\mathbb{A}_\chi(\mathcal{M})$ generated by the elements $\{[I]_{\mathbb{A}} \colon I \in \mathrm{IND}_\ell(\mathcal{M})\}$. Set $\boldsymbol{nbc}_\ell := \{[I]_{\mathbb{A}} \colon I \in \mathrm{NBC}_\ell(\mathcal{M})\}$ and $\boldsymbol{nbc} := \bigcup_{\ell=0} \boldsymbol{nbc}_\ell$. From Remark 2.1 we conclude that $\boldsymbol{nbc}_\ell$ is a basis of the vector space $\mathbb{A}_\ell$.

Proposition 2.2. *Let $\mathbb{A}_\chi(\mathcal{M})$ be a χ-algebra. For any non loop element x of $\mathcal{M}([n])$, we define the two maps*:

$$\chi_{\mathcal{M} \setminus x} \colon 2^{[n] \setminus x} \to \mathbb{K} \quad by \quad \chi_{\mathcal{M} \setminus x}(I) = \chi(I), \quad and \tag{2.1}$$

$$\chi_{\mathcal{M} / x} \colon 2^{[n] \setminus x} \to \mathbb{K} \quad by \quad \chi_{\mathcal{M} / x}(I) = \chi(I * x). \tag{2.2}$$

Then $\mathbb{A}_{\chi_{\mathcal{M}/x}}(\mathcal{M}/x)$ *and* $\mathbb{A}_{\chi_{\mathcal{M}\setminus x}}(\mathcal{M}\setminus x)$ *are* χ*-algebras.*

Proof. The deletion case being trivial, we will just prove the contraction case. We have to show that $\chi_{\mathcal{M}/x}$ verifies properties (UC1) and (UC2). The first property is verified since a set I is independent in $\mathcal{M}/x$ iff $I\cup x$ is independent in $\mathcal{M}$. To see that the second property is also verified, let U and U' be two unidependents of $\mathcal{M}/x$ (iff $U\cup x$ and $U'\cup x$ are two unidependents of $\mathcal{M}$). We know that $\partial e_{U\cup x} = \varepsilon_{U\cup x, U'\cup x}(\partial e_{U'\cup x})e_{U\setminus U'}$. We denote ∂' the boundary defined by $\chi_{\mathcal{M}/x}$ and so we will show that there is a scalar $\varepsilon_{U,U'}$ such that $\partial' e_U = \varepsilon_{U,U'}(\partial' e_{U'})e_{U\setminus U'}$. Let $X, X' \subset [n]$ be two disjoint subsets then $e_X e_{X'} = \beta_{X,X'} e_{X\cup X'}$, where $\beta_{X,X'} = \prod_{e_i\in X, e_j \in X', i>j} \beta_{i,j}$. We have with $U = (i_1, \dots, i_m)$ and $U' = (j_1, \dots, j_k)$:

$$\pm\,\partial e_{U\cup x} = \sum_{p=1}^{m} (-1)^p \chi(U\setminus i_p * x) e_{U\cup x\setminus i_p} + (-1)^{m+1}\chi(U) e_U,$$

$$\partial' e_U = \sum_{p=1}^{m} (-1)^p \chi(U\setminus i_p * x) e_{U\setminus i_p},$$

$$\pm\,(\partial e_{U'\cup x}) e_{U\setminus U'} = \sum_{p=1}^{k} (-1)^p \chi(U'\setminus j_p * x)\beta_{U'\cup x\setminus j_p, U\setminus U'} e_{U\cup x\setminus j_p} + (-1)^{k+1}\chi(U')\beta_{U', U\setminus U'} e_U,$$

$$(\partial' e_{U'}) e_{U\setminus U'} = \sum_{p=1}^{k} (-1)^p \chi(U'\setminus j_p * x)\beta_{U'\setminus j_p, U\setminus U'} e_{U\setminus j_p}.$$

After remarking that $\beta_{U'\cup x\setminus j_p, U\setminus U'}\beta^{-1}_{U'\setminus j_p, U\setminus U'} = \beta_{x, U\setminus U'}$ does not depend on j_p, we can deduce that $\partial' e_U = \varepsilon_{U,U'}(\partial' e_{U'}) e_{U\setminus U'}$ with $\varepsilon_{U\cup x, U'\cup x} = \pm \varepsilon_{U,U'}\beta_{x, U\setminus U'}$. ∎

Proposition 2.3. *For every non loop element x of $\mathcal{M}([n])$, there is a unique monomorphism of vector spaces,* $\mathfrak{i}_x\colon \mathbb{A}(\mathcal{M}\setminus x) \to \mathbb{A}(\mathcal{M})$, *such that, for every* $I \in \mathrm{IND}(\mathcal{M}\setminus x)$, *we have* $\mathfrak{i}_x(e_I) = e_I$.

Proof. By a reordering of the elements of the matroid $\mathcal{M}$ we can suppose that $x = n$. It is clear that

$$\mathrm{NBC}(\mathcal{M}\setminus x) = \big\{X : X \subset [n-1] \text{ and } X \in \mathrm{NBC}(\mathcal{M})\big\}.$$

So the proposition is a consequence of Equation (2.1). ∎

Proposition 2.4. *For every non loop element x of $\mathcal{M}([n])$, there is a unique epimorphism of vector spaces,* $\mathfrak{p}_x\colon \mathbb{A}(\mathcal{M}) \to \mathbb{A}(\mathcal{M}/x)$, *such that, for every* e_I, $I \in \mathrm{IND}(\mathcal{M})$, *we have*

$$\mathfrak{p}_x(e_I) := \begin{cases} e_{I\setminus x}, & \text{if } x \in I,\\ \frac{\chi(I\setminus y, x)}{\chi(I\setminus y, y)} e_{I\setminus y}, & \text{if there is } y \in I \text{ parallel to } x,\\ 0, & \text{otherwise}. \end{cases} \tag{2.3}$$

Proof. From Remark 2.1, it is enough to prove that $\mathfrak{p}_x(\partial e_U) = 0$, for all unidependent $U = (i_1, \dots, i_m)$. We recall that if $x \in U$ then $U \setminus x$ is a unidependent set of $\mathcal{M}/x$. There are only the following four cases:

- If U contains x but no y parallel to x then:

$$\pm\mathfrak{p}_x(\partial e_U) = \mathfrak{p}_x\Big((-1)^m\chi(U\setminus x)e_{U\setminus x} + \sum_{i_p\in U\setminus x}(-1)^p\chi(U\setminus\{i_p,x\}*x)e_{U\setminus i_p})\Big)$$
$$= \sum_{i_p\in U\setminus x}(-1)^p\chi(U\setminus\{i_p,x\}*x)e_{U\setminus\{i_p,x\}} = 0$$

  from Proposition 2.2.

- If U does not contain x but a y parallel to x then:

$$\pm\mathfrak{p}_x(\partial e_U) = \mathfrak{p}_x\Big((-1)^m\chi(U\setminus y)e_{U\setminus y} + \sum_{i_p\in U\setminus y}(-1)^p\chi(U\setminus\{i_p,y\}*y)e_{U\setminus i_p}\Big)$$
$$= \sum_{i_p\in U\setminus y}(-1)^p\chi(U\setminus\{i_p,y\}*y)\frac{\chi(U\setminus\{i_p,x\}*x)}{\chi(U\setminus\{i_p,y\}*y)}e_{U\setminus\{i_p,y\}} = 0$$

  like previously since $U\setminus y$ is again a unidependent of $\mathcal{M}/x$.

- If U contains x and a y parallel to x then:

$$\pm\,\mathfrak{p}_x(\partial e_U) = \mathfrak{p}_x\Big(\chi(U\setminus\{x,y\}*y)e_{U\setminus x} - \chi(U\setminus\{x,y\}*x)e_{U\setminus y}\Big)$$
$$= \chi(U\setminus\{x,y\}*y)\frac{\chi(U\setminus\{x,y\}*x)}{\chi(U\setminus\{x,y\}*y)}e_{U\setminus\{x,y\}} - \chi(U\setminus\{x,y\}*x)e_{U\setminus\{x,y\}}$$
$$= 0.$$

- If U does not contain x nor a y parallel to x then:

$$\mathfrak{p}_x(\partial e_U) = \mathfrak{p}_x\Big(\sum_{i_p\in U}(-1)^p\chi(U\setminus i_p)e_{U\setminus i_p}\Big) = 0.$$

∎

Theorem 2.5. *For every element x of a simple $\mathcal{M}([n])$, there is a splitting short exact sequence of vector spaces*

$$0 \to \mathbb{A}(\mathcal{M}\setminus x) \xrightarrow{\mathfrak{i}_x} \mathbb{A}(\mathcal{M}) \xrightarrow{\mathfrak{p}_x} \mathbb{A}(\mathcal{M}/x) \to 0. \tag{2.4}$$

Proof. From the definitions we know that $\mathfrak{p}_x \circ \mathfrak{i}_x$ is the null map, so $\mathrm{Im}(\mathfrak{i}_x) \subset \mathrm{Ker}(\mathfrak{p}_x)$. We will prove the equality $\dim(\mathrm{Ker}(\mathfrak{p}_n)) = \dim(\mathrm{Im}(\mathfrak{i}_n))$. By a reordering of the elements of $[n]$ we can suppose that $x = n$. The minimal broken circuits of $\mathcal{M}/n$ are the minimal sets X such that either X or $X \cup \{n\}$ is a broken circuit of $\mathcal{M}$ (see [5, Proposition 3.2.e]). Then

$$\mathrm{NBC}(\mathcal{M}/n) = \big\{X : X \subset [n-1] \ \text{ and } \ X\cup\{n\} \in \mathrm{NBC}(\mathcal{M})\big\} \quad \text{and}$$

$$\mathrm{NBC}(\mathcal{M}) = \mathrm{NBC}(\mathcal{M} \setminus n) \biguplus \big\{ I \cup n \colon I \in \mathrm{NBC}(\mathcal{M}/n) \big\}. \tag{2.5}$$

So $\dim(\mathrm{Ker}(\mathfrak{p}_n)) = \dim(\mathrm{Im}(\mathfrak{i}_n))$. There is a morphism of modules

$$\mathfrak{p}_n^{-1} \colon \mathbb{A}(\mathcal{M}/n) \to \mathbb{A}, \quad \text{where} \quad \mathfrak{p}_n^{-1}([I]_{\mathbb{A}(\mathcal{M}/n)}) := [I \cup n]_{\mathbb{A}}, \forall I \in \mathrm{NBC}(\mathcal{M}/n).$$

It is clear that $\mathfrak{p}_n \circ \mathfrak{p}_n^{-1}$ is the identity map. From Equation (2.5) we conclude that the exact sequence (2.4) splits. ∎

Similarly to [15] (see also [4]), we now construct, making use of iterated contractions, the dual basis $\boldsymbol{nbc}_\ell^* = (b_i^*)$ of the basis $\boldsymbol{nbc}_\ell = (b_j)$. More precisely $\boldsymbol{nbc}_\ell^*$ is the basis of $\mathbb{A}_\ell^*$ the vector space of the linear forms such that $b_i^*(b_j) = \delta_{ij}$ (the Kronecker delta).

We associate to the ordered independent set $I^\sigma := (i_{\sigma(1)}, \ldots, i_{\sigma(p)})$ of $\mathcal{M}$ the linear form on $\mathbb{A}_\ell$, $\mathfrak{p}_{I^\sigma} : \mathbb{A}_\ell \to \mathbb{K}$,

$$\mathfrak{p}_{I^\sigma} := \mathfrak{p}_{e_{i_{\sigma(1)}}} \circ \mathfrak{p}_{e_{i_{\sigma(2)}}} \circ \cdots \circ \mathfrak{p}_{e_{i_{\sigma(p)}}}. \tag{2.6}$$

We call $\mathfrak{p}_{I^\sigma}$ the *iterated residue* with respect to the ordered independent set I^σ. (It is clear that the map $\mathfrak{p}_{I^\sigma}$ depends on the order chosen on I^σ and not only on the underlying set I.) We associate to I^σ the flag of flats of $\mathcal{M}$,

$$\mathbf{Flag}(I^\sigma) := c\ell\big(\{i_{\sigma(p)}\}\big) \subsetneq c\ell\big(\{i_{\sigma(p)}, i_{\sigma(p-1)}\}\big) \subsetneq \cdots \subsetneq c\ell(I).$$

Proposition 2.6. *Let $J \in \mathrm{IND}_\ell(\mathcal{M})$ then we have $\mathfrak{p}_{I^\sigma}(e_J) \neq 0$ iff there is a unique permutation $\tau \in \mathfrak{S}_\ell$ such that $\boldsymbol{Flag}(J^\tau) = \boldsymbol{Flag}(I^\sigma)$. And in this case we have $\mathfrak{p}_{I^\sigma}(e_J) = \chi(I^\sigma)/\chi(J^\tau)$. In particular we have $\mathfrak{p}_{I^\sigma}(e_I) = 1$ for any independent set I and any permutation σ.*

Proof. The first equivalence is very easy to prove in both direction. To obtain the expression of $\mathfrak{p}_{I^\sigma}(e_J)$ we just need to iterate ℓ times the residue. This gives:

$$\begin{aligned} \mathfrak{p}_{I^\sigma}(e_J) = & \frac{\chi(J \setminus j_{\tau(\ell)} * i_{\sigma(\ell)})}{\chi(J \setminus j_{\tau(\ell)} * j_{\tau(\ell)})} \times \frac{\chi(J \setminus \{j_{\tau(\ell)}, j_{\tau(\ell-1)}\} * i_{\sigma(\ell-1)} * i_{\sigma(\ell)})}{\chi(J \setminus \{j_{\tau(\ell)}, j_{\tau(\ell-1)}\} * j_{\tau(\ell-1)} * i_{\sigma(\ell)})} \\ & \times \cdots \times \frac{\chi(I^\sigma)}{\chi(j_{\tau(1)} * I^\sigma \setminus i_{\sigma(1)})}. \end{aligned}$$

After simplification we obtain the announced formula. And finally the last result comes from the fact that if $I = J$ then clearly $\tau = \sigma$. ∎

Remark 2.2. The fact that $\mathfrak{p}_{I^\sigma}(e_J)$ is null depends on the permutation σ. For example, for any simple matroid of rank 2 we have $\mathfrak{p}_{13}(e_{12}) = 0$ and $\mathfrak{p}_{31}(e_{12}) \neq 0$. But if $\mathfrak{p}_{I^\sigma}(e_J) \neq 0$ then its value does not depend on σ. We mean by this that if there are two permutations σ and σ' such that $\mathfrak{p}_{I^\sigma}(e_J) \neq 0$ and $\mathfrak{p}_{I^{\sigma'}}(e_J) \neq 0$ then $\mathfrak{p}_{I^\sigma}(e_J) = \mathfrak{p}_{I^{\sigma'}}(e_J)$.

Definition 2.7. [15] *We say that the subset $\mathbb{I}_\ell \subset \big\{[I]_{\mathbb{A}} \colon I \in IND_\ell(\mathcal{M})\big\}$ is a* diagonal basis *of $\mathbb{A}_\ell$ if and only if the following three conditions hold*:

(2.7.1) *For every $[I]_{\mathbb{A}} \in \mathbb{I}_\ell$ there is a fixed permutation of the set I denoted $\sigma_I \in \mathfrak{S}_\ell$;*

(2.7.2) $|\mathbb{I}_\ell| \geq dim(\mathbb{A}_\ell)$;
(2.7.3) *For every* $[I]_{\mathbb{A}}, [J]_{\mathbb{A}} \in \mathbb{I}_\ell$ *and every permutation* $\tau \in \mathfrak{S}_\ell$, *the equality* $\boldsymbol{Flag}(J^\tau) = \boldsymbol{Flag}(I^{\sigma_I})$ *implies* $J = I$.

Theorem 2.8. *Suppose that* $\mathbb{I}_\ell$ *is a diagonal basis of* $\mathbb{A}_\ell$. *Then* $\mathbb{I}_\ell$ *is a basis of* $\mathbb{A}_\ell$ *and* $\mathbb{I}^*_\ell := \{\mathfrak{p}_{I^{\sigma_I}} : [I]_{\mathbb{A}} \in \mathbb{I}_\ell\}$ *is the dual basis of* $\mathbb{I}_\ell$.

Proof. Pick two elements $[I]_{\mathbb{A}}, [J]_{\mathbb{A}} \in \mathbb{I}_\ell$. Note that $\mathfrak{p}_{I^{\sigma_I}}(e_J) = \delta_{IJ}$ (the Kronecker delta), from condition (2.7.2) and Proposition 2.6. The elements of $\mathbb{I}_\ell$ are linearly independent: suppose that $[J] = \sum \zeta_j [I_j]$, $\zeta_j \in \mathbb{K} \setminus 0$; then $1 = \mathfrak{p}_{J^{\sigma_J}}([J]) = \mathfrak{p}_{J^{\sigma_J}}\left(\sum \zeta_j [I_j]\right) = 0$, a contradiction. It is also clear that $\mathbb{I}^*_\ell$ is the dual basis of $\mathbb{I}_\ell$. ∎

The following result gives an interesting explanation of results of [6] and [7].

Corollary 2.9. $\boldsymbol{nbc}_\ell(\mathcal{M})$ *is a diagonal basis of* $\mathbb{A}_\ell$ *where* σ_I *is the identity for every* $[I]_{\mathbb{A}} \in \boldsymbol{nbc}_\ell(\mathcal{M})$. *For a given* $[J]_{\mathbb{A}} \in \mathbb{A}_\ell$, *suppose that*

(2.9.2) $[J]_{\mathbb{A}} = \sum \xi(I,J)[I]_{\mathbb{A}}$, *where* $[I]_{\mathbb{A}} \in \boldsymbol{nbc}_\ell(\mathcal{M})$ *and* $\xi(I,J) \in \mathbb{K}$.

Then the following two statements are equivalent:

- $\xi(I,J) \neq 0$,
- $\boldsymbol{Flag}(I) = \boldsymbol{Flag}(J^\tau)$ *for some permutation* τ.

If $\xi(I,J) \neq 0$ *we have* $\xi(I,J) = \frac{\chi(I)}{\chi(J^\tau)}$. *In particular, if* $\mathbb{A}$ *is the Orlik-Solomon algebra then* $\xi(I,J) = \mathrm{sgn}(\tau)$.

Proof. By hypothesis (2.7.1) and (2.7.2) are true. We claim that $\boldsymbol{nbc}_\ell(\mathcal{M})$ verifies (2.7.3). Suppose for a contradiction that $J \neq I$, $[J]_{\mathbb{A}}, [I]_{\mathbb{A}} \in \boldsymbol{nbc}_\ell(\mathcal{M})$ and there is $\tau \in \mathfrak{S}_\ell$, such that $\mathbf{Flag}(J^\tau) = \mathbf{Flag}(I)$. Set $I = (i_1, \ldots, i_\ell)$ and $J = (j_{\tau(1)}, \ldots, j_{\tau(\ell)})$, and suppose that $j_{\tau(m+1)} = i_{m+1}, \ldots, j_{\tau(\ell)} = i_\ell$ and $i_m \neq j_{\tau(m)}$. Then there is a circuit C of $\mathcal{M}$ such that

$$i_m,\, j_{\tau(m)} \in C \subset \{i_m, j_{\tau(m)}, i_{m+1}, i_{m+2}, \ldots, i_\ell\}.$$

If $j_{\tau(m)} < i_m$ [resp. $i_m < j_{\tau(m)}$] we conclude that $I \notin \mathrm{NBC}_\ell(\mathcal{M})$ [resp. $J \notin \mathrm{NBC}_\ell(\mathcal{M})$] a contradiction. So $\boldsymbol{nbc}_\ell(\mathcal{M})$ is a diagonal basis of $\mathbb{A}_\ell$.

From Theorem 2.8 we conclude that $\boldsymbol{nbc}^*_\ell := \left\{\mathfrak{p}_I \colon [I]_{\mathbb{A}} \in \boldsymbol{nbc}\right\}$ is the dual basis of $\boldsymbol{nbc}$. Suppose now that $[J]_{\mathbb{A}} = \sum \xi_I [I]_{\mathbb{A}}$, where $[I]_{\mathbb{A}} \in \boldsymbol{nbc}_\ell(\mathcal{M})$ and $\xi_I \in k$. Then $\xi_I = \mathfrak{p}_I(e_J)$ and the remaining follows from Proposition 2.6. ∎

Making full use of the matroidal notion of iterated residue, see Equation (2.6), we are able to prove the following result very close to [16, Proposition 2.1].

Proposition 2.10. *Consider the set of vectors* $\mathcal{V} := \{v_1, \ldots, v_k\}$ *in the plane* $x_d = 1$ *of* $\mathbb{K}^d$. *Set* $\mathcal{A}_{\mathbb{K}} := \{H_i \colon H_i = \mathrm{Ker}(v_i) \subset (\mathbb{K}^d)^*, i = 1, \ldots, k\}$ *and let* $\mathrm{OT}(\mathcal{A}_{\mathbb{K}})$ *be its Orlik-Terao corresponding algebra. Fix a diagonal basis* $\mathbb{I}_\ell \subset \{[I]_{\mathbb{A}} \colon I \in \mathrm{IND}_\ell(\mathcal{M})\}$ *of* $\mathbb{A}_\ell$ *and let* $\mathbb{I}^*_\ell = \{\mathfrak{p}_{I^{\sigma_I}} : [I]_{\mathbb{A}} \in \mathbb{I}_\ell\}$ *be the corresponding dual basis. Then, for any* $e_J \in \mathbb{A}_\ell \setminus 0$, *we have*

$$\sum_{I \in \mathbb{I}_\ell} \mathfrak{p}_{I^{\sigma_I}}(e_J) = \sum_{I \in \mathbb{I}_\ell} \left\langle \mathfrak{p}_{I^{\sigma_I}}, e_J \right\rangle = 1.$$

Proof. We have for any $\ell+1$-subset of $\mathcal{V}$, $\sum_{p=1}^{\ell+1}(-1)^p\chi(U\setminus i_p)=0$. (This is the development of a determinant with two lines of 1.) For any rank ℓ unidependent $U=\{i_1,\dots,i_{\ell+1}\}$ of the matroid $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$, we have

$$\partial e_U=\sum_{p=1}^{\ell+1}(-1)^p\chi(U\setminus i_p)e_{U\setminus i_p}.$$

Since the sum of the coefficients in these relations is 0 and that these relations are generating, see Remark 2.1, we can deduce that the sum of the coefficients in any relation in $\mathrm{OT}(\mathcal{A}_{\mathbb{K}})$ is also equal to 0 which concludes the proof. ∎

Example 2.2. Consider the 6 points $p_1,\dots,p_6$ in the affine plane $z=1$ of $\mathbb{R}^3$, whose coordinates are indicated in Figure 1. Set $v_i:=\overrightarrow{(0,p_i)}$, $i=1,\dots,6$. And let $\mathcal{A}$ be the corresponding arrangement of $(\mathbb{R}^3)^*$, $\mathcal{A}:=\{H_i=\mathrm{Ker}(v_i), i=1,\dots,6\}$. Let $\mathcal{M}(\mathcal{A})$ [resp. $\boldsymbol{\mathcal{M}}(\mathcal{A})$] be the corresponding rank three [resp. oriented] matroid.

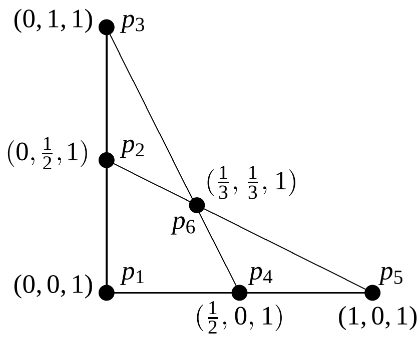

Figure 1.

Let $\mathbb{A}_\chi$ be a χ-algebra on $\mathcal{M}(\mathcal{A})$. We know that

$$\boldsymbol{nbc}_3=\{e_{124},e_{125},e_{126},e_{134},e_{135},e_{136}\}$$

together with $\sigma_{124}=\sigma_{125}=\sigma_{134}=\sigma_{135}=\sigma_{136}=\sigma_{156}=\mathrm{id}$ is a diagonal basis of $\mathbb{A}_3$, from Corollary 2.9. Directly from the Definition 2.7 we see that $\mathfrak{B}_3=\{e_{124},e_{125},e_{134},e_{135},e_{136},e_{156}\}$ with $\sigma_{124}=\sigma_{134}=\sigma_{135}=\sigma_{136}=\sigma_{156}=\mathrm{id}$ and $\sigma_{125}=(132)$ is also a diagonal basis of $\mathbb{A}_3$. We will look at expression on the basis $\boldsymbol{nbc}_3$ (resp. $\mathfrak{B}_3$) of the vector space $\mathbb{A}_3$, of some elements of the type e_B, B basis of $\mathcal{M}(\mathcal{A})$, for the three χ-algebras of Example 2.1. Especially, one can verify as stated in Remark 2.2 that $\mathfrak{p}_{125^{\mathrm{id}}}(e_{235})=\mathfrak{p}_{125^{(132)}}(e_{235})$. Let us also point out that for the Orlik-Terao algebra, we have $\sum_{I\in\mathfrak{B}}\mathfrak{p}_{I^\sigma}(e_J)=1$ as proved in Proposition 2.10.

- Consider the basis $\boldsymbol{nbc}_3$ of the $\mathbb{K}$-vector space $\mathbb{A}_3$. So we have:

$$e_{235}=\mathrm{sgn}(325)e_{125}+\mathrm{sgn}(235)e_{135}=-e_{125}+e_{135}\ \text{ in }\ \mathrm{OS}(\mathcal{M}(\mathcal{A})),$$

$$e_{235} = \frac{\det(125)}{\det(325)} e_{125} + \frac{\det(135)}{\det(235)} e_{135} = -e_{125} + 2e_{135} \quad \text{in} \quad \mathrm{OT}(\mathcal{A}),$$

$$e_{235} = \chi(125)\chi(325)e_{125} + \chi(135)\chi(235)e_{135} = -e_{125} + e_{135} \quad \text{in} \quad \mathbb{A}(\boldsymbol{\mathcal{M}}(\mathcal{A})),$$

$$e_{156} = \mathrm{sgn}(165)e_{125} + \mathrm{sgn}(156)e_{126} = -e_{125} + e_{126} \quad \text{in} \quad \mathrm{OS}(\mathcal{M}(\mathcal{A})),$$

$$e_{156} = \frac{\det(125)}{\det(165)} e_{125} + \frac{\det(126)}{\det(156)} e_{126} = \frac{3}{2} e_{125} - \frac{1}{2} e_{126} \quad \text{in} \quad \mathrm{OT}(\mathcal{A}),$$

$$e_{156} = \chi(125)\chi(165)e_{125} + \chi(126)\chi(156)e_{126} = e_{125} - e_{126} \quad \text{in} \quad \mathbb{A}(\boldsymbol{\mathcal{M}}(\mathcal{A})).$$

- Consider now the basis $\mathfrak{B}_3$ of the $\mathbb{K}$-vector space $\mathbb{A}_3$. So we have:

$$e_{235} = \mathrm{sgn}(152)\mathrm{sgn}(352)e_{125} + \mathrm{sgn}(235)e_{135} = -e_{125} + e_{135} \quad \text{in} \quad \mathrm{OS}(\mathcal{M}(\mathcal{A})),$$

$$e_{235} = \frac{\det(152)}{\det(352)} e_{125} + \frac{\det(135)}{\det(235)} e_{135} = -e_{125} + 2e_{135} \quad \text{in} \quad \mathrm{OT}(\mathcal{A}),$$

$$e_{235} = \chi(152)\chi(352)e_{125} + \chi(135)\chi(235)e_{135} = -e_{125} + e_{135} \quad \text{in} \quad \mathbb{A}(\boldsymbol{\mathcal{M}}(\mathcal{A})),$$

$$e_{126} = \mathrm{sgn}(162)\mathrm{sgn}(152)e_{125} + \mathrm{sgn}(126)e_{156} = e_{125} + e_{156} \quad \text{in} \quad \mathrm{OS}(\mathcal{M}(\mathcal{A})),$$

$$e_{126} = \frac{\det(152)}{\det(162)} e_{125} + \frac{\det(156)}{\det(126)} e_{156} = 3e_{125} - 2e_{156} \quad \text{in} \quad \mathrm{OT}(\mathcal{A}),$$

$$e_{126} = \chi(152)\chi(162)e_{125} + \chi(156)\chi(126)e_{156} = e_{125} - e_{156} \quad \text{in} \quad \mathbb{A}(\boldsymbol{\mathcal{M}}(\mathcal{A})).$$

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