

Linial arrangements and local binary search trees

David Forge ¹

Laboratoire de recherche en informatique UMR 8623

Bât. 650, Université Paris-Sud

91405 Orsay Cedex, France

E-mail: `forge@lri.fr`

ABSTRACT. We study the set of NBC sets (no broken circuit sets) of the *Linial* arrangement and deduce a constructive bijection to the set of local binary search trees. We then generalize this construction to two families of Linial type arrangements for which the bijections are with some k -ary labelled trees that we introduce for this purpose.

Mathematics Subject Classifications (2010): Primary 05C22; Secondary 05A19, 05C05, 05C30, 52C35.

Key words and phrases: Integral gain graph, no broken circuits, local binary search tree, Linial arrangement

1. INTRODUCTION

An *integral gain graph* is a graph whose edges are labelled invertibly by integers; that is, reversing the direction of an edge negates the label (the *gain* of the edge). The *gainic hyperplane arrangement*, $\mathcal{A}[\Phi]$, that corresponds to an integral gain graph Φ is the set of all hyperplanes in \mathbb{R}^n of the form $x_j - x_i = g$ for edges (i, j) with $i < j$ and gain g in Φ . (See [8, Section IV.4.1, pp. 270–271] or [4].) The use of graph representing “graphic” arrangements is common and we think that it should also be the case for gain graphs representing gainic arrangements.

In last ten years there has been much interest in real hyperplane arrangements of this type, such as the Shi arrangement, the Linial arrangement, and the composed-partition or Catalan arrangement. For all these families, the characteristic polynomials and the number of regions have been found. For the Shi arrangement, Athanasiadis [1] gave a bijection between the regions and the parking functions.

In [3], we started to replace the study of the regions of such arrangements by the study of the NBC sets of the corresponding gain graphs. This study works specially well in the case of the complete gain graphs with gains in intervals $[a, b]$ with $a + b = 0$ or 1. This permits us to give a bijection between the NBC sets of the braid arrangement and the increasing labelled trees and another one between the NBC sets of the Shi arrangements and the labelled trees. For the other values such that $a + b = 0$ or 1 we introduced $[a, b]$ -trees to get similar bijections.

In this paper, we do the same thing for the cases where $a + b = 2$. The first case, where $a = b = 1$, corresponds to the so called Linial arrangement. The construction for the Linial case goes to the *local binary search trees* (*LBS* for short) as proposed by Stanley et al. For the intervals of the form $[1, k + 1]$ and $[-k + 1, k + 1]$ we introduce two k -generalizations of the LBS which are $(k+1)$ -ary.

¹The research of the author is supported by the TEOMATRO project, grant number ANR-10-BLAN 0207.

2. BASIC DEFINITIONS

An *integral gain graph* $\Phi = (\Gamma, \varphi)$ consists of a graph $\Gamma = (V, E)$ and an orientable function $\varphi : E \rightarrow \mathbb{Z}$, called the *gain mapping*. Orientability means that, if (i, j) denotes an edge oriented in one direction and (j, i) the same edge with the opposite orientation, then $\varphi(j, i) = -\varphi(i, j)$. We have no loops but multiple edges are permitted. For the rest of the paper, we denote the vertex set by $V = \{1, 2, \dots, n\} =: [n]$ with $n \geq 1$. We use the notations (i, j) for an edge with endpoints i and j , oriented from i to j , and $g(i, j)$ for such an edge with gain g ; that is, $\varphi(g(i, j)) = g$. (Thus $g(i, j)$ is the same edge as $(-g)(j, i)$.) A *circle* is a connected 2-regular subgraph, or its edge set. Writing a circle C as a word $e_1 e_2 \cdots e_l$, the gain of C is $\varphi(C) := \varphi(e_1) + \varphi(e_2) + \cdots + \varphi(e_l)$; then it is well defined whether the gain is zero or non zero. A subgraph is called *balanced* if every circle in it has gain zero. We will consider most especially balanced circles.

Given a linear order $<_O$ on the set of edges E , a *broken circuit* is the set of edges obtained by deleting the smallest element in a balanced circle. A set of edges, $N \subseteq E$, is a *no-broken-circuit set* (NBC set for short) if it contains no broken circuit. This notion from matroid theory (see [2] for reference) is very important here. We denote by \mathcal{N} the set of NBC sets of the gain graph. It is well known that this set depends on the choice of the order, but its cardinality does not.

We can now transpose some ideas or problems from hyperplane arrangements to gain graphs. For any integers a, b, n , let K_n^{ab} be the gain graph built on vertices $V = [n]$ by putting on every edge (i, j) all the gains k , for $a \leq k \leq b$. These gainic arrangements are called sometimes deformations of the braid arrangement or truncated arrangements. We have four main examples coming from hyperplane arrangements. We denote by B_n the gain graph K_n^{00} and call it the *braid gain graph*, by L_n the gain graph K_n^{11} and call it the *Linial gain graph*, by S_n the gain graph K_n^{01} and call it the *Shi gain graph* and finally by C_n the gain graph K_n^{-11} and call it the *Catalan gain graph*.

3. HEIGHT FUNCTION OF A BALANCED GAIN GRAPH

We introduce the notion of height function on an integral gain graph on the vertex set $[n]$. A height function h defines two important things for the rest of the paper: the induced gain graph $\Phi[h]$ of a gain graph Φ and an order O_h on the set of vertices extended lexicographically to the set of edges.

Definition 1. For V a finite set of \mathbb{N} , a *height function* on V is a function h from V to \mathbb{N} such that $h^{-1}(0) \neq \emptyset$. The *corner* of V defined by the height function is the smallest element of greatest height.

Definition 2. Let Φ be a connected and balanced integral gain graph on a set V of integers. The height function of the gain graph is the unique function h_Φ such that for every edge $g(i, j)$ we have $h_\Phi(j) - h_\Phi(i) = g$. (Such a function exists iff Φ is balanced.)

Definition 3. Let Φ be a gain graph also on $V = [n]$ and let h be a height function on V . We say that an edge $g(i, j)$ is *coherent with h* if $h(j) - h(i) = g$. The subgraph $\Phi[h]$ of Φ *selected by h* is the gain graph on the same vertex set V whose edges are the edges of Φ coherent with h .

Definition 4. Given a height function h on the set V , the order O_h on the set $V = [n]$ is defined by $i <_{O_h} j$ iff $h(i) > h(j)$ or $(h(i) = h(j) \text{ and } i < j)$. The order O_h is extended lexicographically to an order O_h on the set of edges coherent with the height function.

With this definition, the corner is the smallest vertex of the whole graph. The *subcorner* of V defined by the height function is the smallest neighbour of the corner in $\Phi[h]$. When $|V| = 1$, the subcorner is by definition the corner. In the Linial case, where all gains are equal to 1, the subcorner is the smallest vertex on the second level. When 0 is a gain value, the subcorner is the second vertex on the first level when it exists.

To explain a little all these definitions, a height function is just putting the vertices on different levels. An integral gain graph Φ on integers defines a height function h_Φ when it is balanced (a tree works). On the other hand, given a height function h and a gain graph Φ , we get a subgraph $\Phi[h]$ of Φ by keeping only the edges coherent with h . Finally, the height function defines the order O_h on the edges of the gain graph which we will need to define the NBC sets.

The height function will play a very central role in all the paper. The constructions for the Linial arrangement as well as for its generalisations will always work once the height function has been chosen. This property will have for consequence, once we also define a height function on the corresponding trees (the other side of the bijection), to have a finer correspondence such as fixing the root of the tree.

4. NBC SETS AND NBC TREES IN GAIN GRAPHS

An NBC set in a gain graph Φ is basically an edge set, as it arises from matroid theory. We usually assume an NBC set is a spanning subgraph, i.e., it contains all vertices. Thus, an NBC tree is a spanning tree of Φ . Sometimes we wish to have non-spanning NBC sets, such as the components of an NBC forest; then we write of NBC *subtrees*, which need not be spanning trees.

Given a height function h , a gain graph Φ and the linear order $<_{O_h}$ on the edges, they determine the set of NBC sets of the subgraph $\Phi[h]$ relative to the order $<_{O_h}$, denoted by $\mathcal{N}_O(\Phi[h])$. As always, this set depends on the choice of the order but its cardinality does not.

Lemma 5. *Given an NBC tree A of height function h with corner c , the forest $A \setminus c$ is a disjoint union of NBC subtrees of height functions h_1, \dots, h_k , and the orders O_{h_i} are restrictions of the order O_h .* \square

It is known from matroid theory that the NBC sets of the semimatroid of an affine arrangement \mathcal{A} , with respect to a given ordering $<_O$ of the edges, correspond to the regions of the arrangement [6, Section 9]. The semimatroid of $\mathcal{A}[\Phi]$ is the frame (previously “bias”) semimatroid of Φ , which consists of the balanced edge sets of the gain graph Φ ([8, Sect. II.2] or [4]). Thus, the NBC sets of that semimatroid are the spanning forests of Φ . Therefore $|\mathcal{N}_O(\Phi)|$ equals the number of regions of $\mathcal{A}[\Phi]$.

We show that the total number of NBC trees in an integral gain graph Φ equals the sum, over all height functions h , of the number of NBC trees in $\Phi[h]$.

Let Φ be connected. Then we can decompose $\mathcal{N}_O(\Phi)$ into disjoint subsets $\mathcal{N}_O(\Phi[h])$, one for each height function h that is coherent with Φ (that means that $\Phi[h]$ is also connected). We have now:

$$\mathcal{N}_O(\Phi) = \bigsqcup \{ \mathcal{N}_O(\Phi[h]) \mid h \text{ is coherent with } \Phi \}.$$

Therefore, the total number of NBC trees of all $\Phi[h]$ with respect to all possible height functions h equals the number of NBC trees of Φ .

5. COMPLETE $[a, b]$ -GAIN GRAPHS AND THEIR NBC TREES

Let a and b be two integers such that $a \leq b$. The interval $[a, b]$ is the set $\{i \in \mathbb{Z} \mid a \leq i \leq b\}$. We consider the gain graph K_n^{ab} with vertices labelled by $[n]$ and with all the edges $g(i, j)$, with $i < j$ and $g \in [a, b]$. The arrangements that correspond to these gain graphs, called deformations of the braid arrangement, have been of particular interest. The braid arrangement corresponds to the special case $a = b = 0$. Other well studied cases are $a = -b$ (extended Catalan), $a = b = 1$ (Linial) and $a = b - 1 = 0$ (Shi).

We will describe the set of NBC trees of $K_n^{ab}[h]$ for a given height function h . The idea is that, as mentioned above, the height function h defines an order O_h on a balanced subgraph. We will then be able to describe the NBC sets coherent with h for the order O_h .

Proposition 6. *Let a and b be integers such that $a \leq b$. Let h be a height function of corner c and let Φ be a spanning tree of $K_n^{ab}[h]$. Suppose c is incident to the edges $g_i(c, v_i)$, $1 \leq i \leq k$, and let Φ_i be the connected component of $\Phi \setminus c$ containing c_i (that is a subtree). Then Φ is an NBC tree if and only if all the Φ_i are NBC trees and each v_i is the O_h -smallest vertex of Φ_i adjacent to c in $K_n^{ab}[h]$.*

Proof. Everything comes from the choice of the order O_h for the vertices and the edges. If we have a vertex v in Φ_i such that $v <_{O_h} v_i$ for which the edge $(c, v) \in K_n^{ab}[h]$ exists then this edge is smaller than all the edges of $\Phi_i + c$. Such an edge then closes a balanced circuit being the smallest edge of the circuit which is not possible.

In the other direction, if Φ is not an NBC tree then there is an edge (x, y) in $K_n^{ab}[h]$ closing a balanced circuit by being the smallest edge of the circuit. Since the Φ_i are by hypothesis are NBC trees the vertices x and y can not be in a same Φ_i . They can not be in two different Φ_i neither since the smallest edge would contain c necessarily. The last solution is that one of the vertices say x is c and that the other vertex y is in a Φ_i . Since the edge (c, v_i) will be in the circuit we need to have $(x, y) <_{O_h} (c, v_i)$. This implies the condition of the proposition. \square

6. LOCAL BINARY SEARCH TREES AND TWO GENERALISATIONS

The local binary search trees (LBS for short) are labelled rooted plane binary trees such that a vertex has possibly two children a left one and a right one with the property that : the value of the parent is bigger than the value of the left child and smaller than the value of the right child. The number of LBS labelled on the set $[n]$ is known to be equal to the number of regions of the Linial arrangement in dimension n .

Definition 7. A LBS is called a *left LBS* (LLBS for short) (resp. a *right LBS* (RLBS for short)) if the root has no right child (resp. left child).

Let T be a LBS tree of root $r = r_0$. Let r_1 be its right child and r_2 be r_1 's right child and so on... r_{k+1} be r_k 's right child. Let r_ℓ the last of these vertices. If we delete the edges $\{r_i, r_{i+1}\}$ we obtain $\ell + 1$ disjoint LLBS L_0, \dots, L_ℓ . We call this the *left decomposition* of an LBS.

Similarly we have the right decomposition of an LBS by taking $r = r'_0$. Let r'_1 be its left child and r'_2 be r'_1 's left child and so on... r'_{k+1} be r'_k 's left child. Let r''_ℓ the last of these

vertices. If we delete the edges $\{r'_i, r'_{i+1}\}$ we obtain $\ell' + 1$ disjoint RLBS $R_0, \dots, R_{\ell'}$. We call this the *right decomposition* of an LBS.

We introduce now a generalization of LBS which will have many similar properties. The local k -ary search trees (LkS for short) are labelled rooted plane k -ary trees such that a vertex has possibly k children numbered from 1 to k such that the value of the parent is bigger than the value of the number 1 child and smaller than the value of the number k child. Note that a vertex has any number of children from 0 to k but that the number of the child does not depend on the presence of the other children (this is what means plane k -ary). Note also that of course a LBS is a LkS for $k = 2$.

Definition 8. A LkS is called a *left LkS* (LLkS for short) if the root has no number k child.

Let T be a LkS tree of root $r = r_0$. Let r_1 be its number k child and r_2 be r_1 's number k child and so on... r_{k+1} be r_k 's number k child. Let r_ℓ the last of these vertices. If we delete the edges $\{r_i, r_{i+1}\}$ we obtain $\ell + 1$ disjoint LLkS L_0, \dots, L_ℓ . We call this the *left decomposition* of an LkS. If we start with a LLkS tree T (already left) of root and subtrees T^i for $1 \leq i \leq k - 1$, then each T^i has a left decomposition $D^i = \{L_0^i, L_1^i, \dots, L_{\ell_i}^i\}$. We call the set of the D^i the left decomposition of T .

We can define a height function of an LLkS of root r and left decompositions $D_i = \{L_0^i, L_1^i, \dots, L_{\ell_i}^i\}$ recursively by: define $h(r) = 0$ and take a height function h_j^i in each L_j^i . Then define h on all the roots of the L_j^i by:

- if r_j^i is the root of a L_j^i and is smaller than r take $h(r_j^i) = -(k - i + 1)$;
- if r_j^i is the root of a L_j^i is bigger than r take $h(r_j^i) = -(k - i + 1) + 1$.

We just need to take for the other vertices $h(x) = h_j^i(x) + h(r_j^i) - h_j^i(r_j^i)$ if $x \in L_j^i$.

Note that r is the corner of this height function. Note also that the vertices of weight 1 are in the lowest levels. This choice comes from the future bijection with the NBC sets. It could be easily reversed. Note also that the level of r_j^i the root of L_j^i has two the possible values $-k + i - 1$ or $-k + i$ depending on the fact $r_j^i < r$ or not.

We introduce now another generalization of LBS corresponding to another generalization of the Linial arrangement. The semi local k -ary search trees (SLkS for short) are labelled rooted plane k -ary trees such that a vertex has possibly k children numbered from 1 to k such that the value of the parent is bigger than the value of the first child from 1 to $k - 1$ and smaller than the value of the number k child. Note that of course a LBS is a SLkS for $k = 2$. Note also that a SLkS tree is also a LkS tree and that therefore we will not need to define the decomposition or the height function of this new family.

7. LINIAL GAIN GRAPH

In [3], we have studied with more details the cases $a + b = 0$ and $a + b = 1$ which contain the braid and the Shi cases. Now we will consider the case $a + b = 2$ and starting by $a = b = 1$ corresponding to the Linial case.

The Linial gain graph is the first K_n^{ab} graph with $a + b = 2$ by taking $a = b = 1$. It corresponds to the Linial arrangement, whose hyperplanes have equation $x_i - x_j = 1$, with $i < j$. The number of regions (and then of NBC sets) of the Linial arrangement in dimension n is known to be equal to

$$\frac{1}{n2^{n-1}} \sum_{k=1}^n \binom{n}{k} k^{n-1}.$$

It is also known to be equal to the number of local binary search trees on n vertices. We will give now a bijection between the NBC sets of the Linial arrangement on $[n]$ and the LBS on $[n]$. We start by giving a bijection between the NBC trees of the Linial arrangement on $[n]$ and the LLBS on $[n]$.

Theorem 9. *The number of NBC of the Linial gain graph on $[n]$ is equal to the number of LBS on $[n]$. Moreover, the number of NBC of corner c of the Linial gain graph on $[n]$ is equal to the number of LBS of root c on $[n]$. And even for any height function on $[n]$, the number of NBC of height h of the Linial gain graph on $[n]$ is equal to the number of LBS of height h on $[n]$.*

Example 10. In the figure, we give on the left a LLBS and on the right the NBC tree corresponding. The special point in both sides is 3; in the left side it is the biggest number smaller than the root 4 in the right chain $1 - 3 - 5 - 7$ and in the right side it is the biggest neighbour of the corner.

From left to right, the node which are smaller than the root becomes neighbours of the corner. The two other node 5 and 7 will be corners of their subtree but are connected differently. The subtree of root 5 gives a sub NBC tree of corner 5 (always the root becomes the corner) and of sub corner 2. Since 2 is smaller than the root 4, this subtree is connected by the edge $\{4, 2\}$. The subtree of root 7 gives a sub NBC tree of corner 7 and of sub corner 6. Since 6 is bigger than the root 4, this subtree is connected by the edge $\{3, 7\}$.

In the other direction, 3 is again recognized as the special vertex and gives the decomposition in subtrees. The LLBS obtained are just making the chain of the left decomposition of the LLBS of root 4.

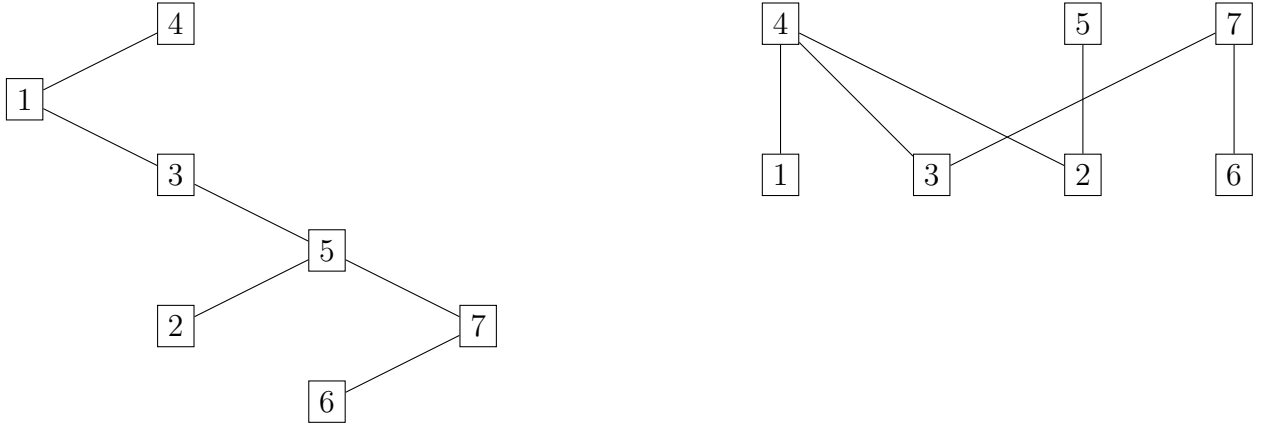


Figure 1 : Correspondance from LLBS to NBC

Proof. From the definitions of height functions, the following correspondence preserves height function in both directions and therefore corner goes to root. The fact that the height function is preserved is of course very important but it also forces the position of the vertices.

To go from LLBS to LBS we just use the left decomposition which correspond to a partition of $[n]$ in the same way as a NBC is just a union of NBC trees (we could call this a decomposition of an NBC in NBC trees!).

(From LLBS to NBC trees) Let L be a LLBS with root r and left decomposition in LLBS L_i of root r_i for $1 \leq i \leq k$. We have $r_1 < r_2 < \dots < r_k$ and let r_p be the biggest r_i smaller than r . The bijection is recursive as follows:

- Let N_i be the NBC trees corresponding to L_i for $1 \leq i \leq k$. They are of corner r_i .
- Add the edge $\{r, r_i\}$ for $1 \leq i \leq p$.
- For $p+1 \leq i \leq k$, let c_i and sc_i be the corner and subcorner of N_i . By hypothesis c_i is bigger than r and also r_p . If $sc_i < r_p$ then add the edge $\{r, sc_i\}$ and otherwise add the edge $\{r_p, c_i\}$.

(From NBC trees to LLBS) Let T be a NBC tree of height function h_T and so for the order O_{h_T} . Let c be its corner and v_j^i its neighbours and T_j^i the subtrees obtained by deleting the vertex c (by notation the vertex v_j^i uses gain i). Note that the vertex v_j^i is not necessarily the corner of the subtree T_j^i but that it can also be the subcorner. The vertex p which is the smallest for O_h neighbour of the corner plays a special role. The vertex p is the biggest vertex using gain k . We need to recognize the pieces which were attached to N_p in the preceding construction. The vertex p by the order and the fact that T is an NBC tree has at most one neighbour v_1' bigger and (p, v_1') has gain 1. The vertex v_1' is not necessarily the corner of $T \setminus N_p$ but can have a neighbour v_2' with edge of gain 0 and $v_2' < v_1'$. Let $T'_0, T'_1, \dots, T'_\ell$ the subtrees of T_p obtained by deleting the edges $\{v_i', v_{i+1}'\}$. The subtree T'_0 is the subtree containing p and the subtree T'_i is the subtree containing v_i' . The bijection is recursive:

- Let L_i be the LLBS corresponding to T_i for $1 \leq i \leq k-1$ of corner c_i
- Let L'_j be the LLBS corresponding to T'_j for $0 \leq j \leq \ell$ of corner c_i
- Let relabel the L_i and L'_i to some L''_i along their roots r_i . That is we have $k+\ell$ LLBS L''_i such that $r_1 < r_2 < \dots < r_{k+\ell}$.
- Add the edge $\{c, r_1\}$ (a left edge) and all the right edges $\{r_i, r_{i+1}\}$ for $1 \leq i \leq k+\ell-1$.

□

We give now a more surprising correspondence between RLBS of root r and NBC trees of subcorner r . Clearly we can go by symmetry from the RLBS to LLBS so the surprise is that that there is also a correspondence between LLBS of corner c and of subcorner $n - c + 1$.

Theorem 11. *The number of NBC of the Linial gain graph on $[n]$ is equal to the number of LBS on $[n]$. Moreover, the number of NBC of subcorner c of the Linial gain graph on $[n]$ is equal to the number of RBS of root c on $[n]$.*

Proof. **(From NBC trees to RLBS)** Let N be a NBC tree of height function h_N and so for the order O_{h_N} . Let sc be its subcorner and $v_1 < v_2 < \dots < v_k$ its neighbours and N_1, \dots, N_k the subtrees obtained by deleting the vertex sc . Note that the vertex v_i is whether the corner c_i or the subcorner sc_i of the subtree N_i . From the fact that sc is the subcorner of N , we deduce that there is at least one of the N_i of subcorner sc_i bigger than sc . The bijection is recursive:

- Let R_i be the RLBS corresponding to N_i for $1 \leq i \leq k$ of root sc_i .
- Let relabel the R_i in such a way that $sc_1 > sc_2 > \dots > sc_k$.
- Add the edge $\{sc, sc_1\}$ (a right edge) and all the left edges $\{sc_i, sc_{i+1}\}$ for $1 \leq i \leq k-1$.

(From RLBS to NBC trees) Let R be a RLBS with root r and right decomposition in RLBS R_i of root r_i for $1 \leq i \leq k$. We have $r_1 > r_2 > \dots > r_k$ and let r_p be the biggest r_i smaller than r . The bijection is recursive as follows:

- Let N_i be the NBC tree corresponding to R_i for $1 \leq i \leq k$. They are of subcorner sc_i and of corner c_i with $c_i \geq sc_i$ (recall that the subcorner in a one vertex tree is the corner).
- If $sc_i > sc$ then add the edge $\{sc, c_i\}$ for $1 \leq i \leq p$.
- If $sc > c_i$ then add the edge $\{sc, c_i\}$ for $1 \leq i \leq p$.
- If $c_i > sc > sc_i$ then add the edge $\{sc, sc_i\}$ for $1 \leq i \leq p$.

□

Here is now a third decomposition of NBC trees by its corner but the pieces are given by their subcorner. There is a fourth decomposition of a RLBS tree into LLBS which is missing.

(From NBC trees to LLBS) Let N be a NBC tree of height function h_N and so for the order O_{h_N} . Let c be its corner and $v_1 < v_2 < \dots < v_k$ its neighbours and N_1, \dots, N_k the subtrees obtained by deleting the vertex c . Note that the vertex v_i is whether the corner c_i or the subcorner sc_i of the subtree N_i . From the fact that c is the corner of N , we deduce that all the N_i have a subcorner sc_i smaller than c . The bijection is recursive, we build R of root c by:

- Let R_i be the RLBS corresponding to N_i for $1 \leq i \leq k$ of root sc_i .
- Let relabel the R_i in such a way that $sc_1 > sc_2 > \dots > sc_k$.
- Add the edge $\{c, sc_1\}$ (a left edge) and all the left edges $\{sc_i, sc_{i+1}\}$ for $1 \leq i \leq k-1$.

(From LLBS to NBC trees) Let L be a LLBS with root r and right decomposition in RLBS R_i of root r_i for $1 \leq i \leq k$. We have $r_1 > r_2 > \dots > r_k$. The bijection is recursive, we build N of corner r by::

- Let N_i be the NBC tree corresponding to R_i for $1 \leq i \leq k$. They are of subcorner r_i and of corner c_i with $c_i \geq r_i$ (recall that the subcorner in a one vertex tree is the corner). We have for all N_i that the subcorner is smaller then r the corner of the full NBC tree.
- If $c > c_i$ then add the edge $\{c, c_i\}$ for $1 \leq i \leq k$.
- If $c < c_i$ then add the edge $\{c, sc_i\}$ for $1 \leq i \leq k$.

8. OTHER $[a, b]$ GAIN GRAPH WITH $a + b = 2$

We are now considering the complete gain graph $K_n^{2-k,k}$ for any integer $k \geq 1$. The case $k = 1$ is the preceding Linial case.

Theorem 12. *The number of NBC of the $K_n^{2-k,k}$ gain graph on $[n]$ is equal to the number of LkS on $[n]$. Moreover, the number of NBC of corner c of the $K_n^{2-k,k}$ gain graph on $[n]$ is equal to the number of LkS of root c on $[n]$. And even for any height function h on $[n]$, the number of NBC of height h of the $K_n^{2-k,k}$ gain graph on $[n]$ is equal to the number of LkS of height h on $[n]$.*

Proof. The following correspondence preserves height function in both directions and therefore corner goes to root. The fact that the height function is preserved is of course very important but it also forces the position of the vertices. When we give the gain of an edge there is then no more choice. In the other direction the weight of the edge is also without

choice and comes directly from the height function. There is a special vertex p which plays a very central role. In one direction it permits to connect all the pieces even those which can not be connected directly to the root. In the other direction, it permits to recognize all the pieces.

To go from LLkS to LkS we just use the left decomposition which correspond to a partition of $[n]$ is the same way as a NBC is just a union of NBC trees. So we just need to go make the correspondence between LLkS and NBC trees.

(From LLkS to NBC trees) Let L be a LLkS with root r and left decomposition in LLkS $D^i = \{L_0^i, L_1^i, \dots, L_{\ell_i}^i\}$ each L_j^i being of root r_j^i . Let p be the biggest r_j^1 smaller than r . The bijection is recursive as follows:

- Let N_j^i be the NBC trees corresponding to L_j^i . They are of corner r_j^i .
- For $j > 1$, every N_j^i can be uniquely connected with an edge to (r, r_j^i) . If $r_j^i < r$ add the edge (r, r_j^i) of gain $k - i + 1$ and if $r_j^i > r$ add the edge (r, r_j^i) of gain $i - k$.
- For $j = 1$ we are in a case similar to the Linial case. Let sc_j^i be the subcorner of N_j^i .
 - Add the edge $\{r, r_j^i\}$ for $r_j^i \leq p$ with gain k .
 - Add the edge $\{r, r_j^i\}$ for $r_j^i > p$ but $sc_j^i <_{O_h} p$ with gain k . All the other N_j^i will be connected to p but not necessarily directly.
 - For $r_j^i > p$ and $sc_j^i >_{O_h} p$, by hypothesis c_j^i is bigger than r and also than p . If $sc_j^i < p$ then add the edge (r, sc_j^i) with gain k . For the N_j^i such that $sc_j^i > p$ taken in increasing order, we add the edge (p, r_j^i) with gain 1 for the biggest r_j^i . Then we add the edge $(r_{j'}, r_j^i)$ with gain 0 ($r_{j'}$ being the corner of the preceding such N_j^i taking the r_j^i in decreasing order). The reason of this construction apart from keeping the height function is to be able to recognize the pieces in the other direction construction.

(From NBC trees to LLkS) The case $k = 1$ is Linial and there is nothing to prove. In the case $k \geq 2$, we will use that the value 0 is a possible gain. The construction is very similar to the Linial case. The construction is just done on the neighbours with gain k . The other vertices can be directly attached to the corner in any case.

Let N be a NBC tree of height function h_N and so for the order O_{h_N} . Let c be its corner and $v_j^i < c$ its neighbours and N_j^i the subtrees obtained by deleting the vertex c and the corresponding edge of gain i by notation. From Proposition 6, the vertex v_j^i is not necessarily the corner of the subtree N_j^i . In fact this is true only if the gain is k . For all other gain, v_j^i is necessarily the corner of N_j^i . The reason comes from the choice of the order O_h . The vertex p which is the biggest neighbour of the corner with gain k plays a special role and let N_p be its component. From the choice of the order, the vertex p has at most one neighbour v'_1 bigger using gain 1. This vertex v'_1 must verify $v'_1 > r$. The vertex v'_1 can have only one neighbour v'_2 smaller using gain 0. Similarly v'_2 can also have only one smaller neighbour using gain 0. After deleting all the edges (v'_i, v'_{i+1}) we get NBC subtrees N'_i whose corner is v'_i . The subtree N'_0 is as well the subtree containing p and is of corner p . The bijection is recursive :

- Let L_j^i be the LLkS corresponding to N_j^i of corner v_j^i . We have $h(c) - h(v_j^i) = i$ and $v_j^i < c$.
- Let L'_i be the LLkS corresponding to N'_i of corner v'_i . We have $h(c) - h(v'_i) = k - 1 + i$ and $v'_i > c$.

- To every L_j^i such that $i > 1$ and $v_j^i < c$ put weight $k - i + 1$.
- To every L_j^i such that $i > 1$ and $v_j^i > c$ put weight $k - i$.
- To every L_j^i such that $i = 1$ put weight 1.
- To every L_i' put weight 1.
- The LLkS tree is given by its root c and its left decomposition D_w where D_w is just the set of sub LLkS trees to which we have put weight w .

□

9. $[1, k]$ GAIN GRAPH

We are now considering the complete gain graph $K_n^{1,k}$ for k any integer bigger than 1. The case $k = 1$ is the preceding Linial case. The bijection is based on the same idea: to connect all the blocks to the root or a special vertex. Here it becomes a little more complicated because of the different possible gains.

Theorem 13. *The number of NBC of the Linial gain graph on $[n]$ is equal to the number of SLkS on $[n]$. Moreover, the number of NBC of corner c of the Linial gain graph on $[n]$ is equal to the number of SLkS of root c on $[n]$. And even for any height function on $[n]$, the number of NBC of height h of the Linial gain graph on $[n]$ is equal to the number of SLkS of height h on $[n]$.*

Proof. The following correspondence preserves height function in both directions and therefore corner goes to root. To go from LLBS to LBS we just use the left decomposition which correspond to a partition of $[n]$ is the same way as a NBC is just a union of NBC trees. So we just need to make the correspondence between SLLkS and NBC trees.

(From LLkS to NBC trees) Let L be a LLkS with root r and left decomposition in LLkS $D^i = \{L_0^i, L_1^i, \dots, L_{\ell_i}^i\}$ each L_j^i being of root r_j^i . Let p be the biggest r_j^1 smaller than r . The height function h of L will be needed. The bijection is recursive as follows:

- Let N_j^i be the NBC trees corresponding to L_j^i . They are of corner r_j^i . Let x_j^i be, if it exists, the smallest vertex (for order O_h) such that an edge (r, x_j^i) exists in $\Phi[h]$. Previously, this vertex could be only the corner or the subcorner. Now it can be on any level between 1 and $h(r) - h(r_j^i)$. However it is the smallest element of its level.
- We will connect the N_j^i to r by taking them in the decreasing order for O_h of their corner r_j^i . The first such N_i^j is then N_p of corner p . Some of the N_j^i will be connected directly to the corner r . The other N_j^i will be connected to N_p making a special component which will be growing during the construction: we call this component T which is a name of variable.
- Add the edge (r, p) of gain $k - \ell + 1$.
 - If $x_j^i <_h p$ then add the edge the edge (x_j^i, p) . For th gain there are two cases : if $r_j^i < r$ then the gain is $k - i + 1$ and else of gain $k - i + h(r_j^i) - h(x_j^i)$ (the $+1$ disappeared to conserve the height function). This is the case where N_j^i can be connected directly to the corner r . The other case will go to T .
 - If $x_j^i >_h p$ or also x_j^i does not exist now we connect to T . We just need to find in T the vertex t_j^i to add edge (t_j^i, r_j^i) . The vertex t_j^i is simply the biggest vertex of T for O_h such that the edge (t_j^i, r_j^i) exists. So we add the edge (r_j^i, t_j^i) with gain $h(r_j^i) - h(t_j^i)$.

To be complete we must check that the sign graph is indeed a NBC tree of height function h and such that p is the biggest neighbour of r . First, the tree T is at every moment a NBC tree by the choice of t_j^i . Also the condition $x_j^i >_h p$ makes that the vertex p is always the biggest vertex of T for O_h which can be connected to r . So by Proposition 6 we obtain an NBC tree. The vertex p is then the biggest neighbour of r since every other neighbour is a x_j^i verifying $x_j^i <_h p$. Finally the height function is well preserved by the choices of the gains.

(From NBC trees to LLkS) The case $k = 1$ is Linial and there is nothing to prove. In the case $k \geq 2$, we will use that the value 0 is a possible gain.

Let N be a NBC tree of height function h_N and so for the order O_{h_N} . Let c be its corner and $v_j^i < c$ its neighbours, N_j^i the subtrees obtained by deleting the vertex c , c_j^i the corner of N_j^i . By notation we have that $h(c) - h(c_j^i) = i$ by notation. If we want the gain of the edge (c, v_j^i) is $h(c) - h(v_j^i)$. From Proposition 6, the vertex v_j^i is not necessarily the corner of the subtree N_j^i but the smallest vertex for O_h which can be connected in N_j^i . The vertex p which is the biggest neighbour of the corner for O_h plays a special role and let N_p be its component.

We need now to cut N_p to get back the blocks in the preceding construction (N_p corresponds to the final T). The algorithm to get back the pieces is simply a kind of depth first search starting at p . Each time we found an edge (x, y) such that $y <_h p$ we know that the vertex y is not in T_p . In fact we know also that y is the corner of a new NBC which we will obtain by continuing the search and cutting now edges (x', y') where $y' <_h y$. Finally we get subtrees N'_i of corner y_i when we put indices at each vertex y found. The subtree N'_0 is as well the subtree containing p and is of corner p . The bijection is recursive :

- Let L_j^i be the LLkS corresponding to N_j^i of corner c_j^i . We have $h(c) - h(c_j^i) = i$.
- Let L'_i be the LLkS corresponding to N'_i of corner y_i .
- To every L_j^i such that $i > 1$ and $c_j^i < c$ put weight $k - i + 1$.
- To every L_j^i such that $i > 1$ and $c_j^i > c$ put weight $k - i$.
- To L'_0 put weight $k - (h(c) - h(p)) + 1$. Recall that p is the corner of L'_0 and that $p <_h c$.
- To every L'_i put weight $k - (h(c) - h(y_i))$. Now we have that y_i is the corner of N'_i and that $y_i >_h c$.
- The LLkS tree is given by its root c and its left decomposition D_w where D_w is just the set of sub LLkS trees to which we have put weight w .

□

10. LkS AND SLkS AS COLOURED TREES AND FORESTS

We kept the definition of LBS and gave their two similar generalisations because our first goal was to go from NBC to LBS. However we already made the remark that the right decomposition means that LBS are like forests of LLBS. Similarly the LkS and SLkS are also forests of left parts. We will give in this section a definition which would have been more suitable for our constructions and that is in some sense more natural.

We consider Tn, k the set of rooted coloured labelled trees that is rooted labelled trees on $[n]$ where the edges have k possible colours in $[k]$. On $[n]$ the number of rooted labelled trees is known to be n^{n-1} and so the cardinality of Tn, k is $2^{n-1}n^{n-1}$.

Definition 14. An inner vertex x is said to be a *descent* if it has no child of colour 1 or if it has a child y of colour 1 such that $y < x$.

An inner vertex x is said to be a *S-descent* if its smallest child of smallest caller y is such that $y < x$.

Theorem 15. *The LLkS are in bijection with the rooted coloured labelled trees such that all inner vertices are descents. The SLLkS are in bijection with the rooted coloured labelled trees such that all inner vertices are S-descents.*

Proof. From SLLkS to labelled trees. Let L be a LLkS, r be its root, r_i be r 's number i child for $1 \leq i \leq k-1$ (it has no number k child). The vertex r_i is the root of an LkS L^i . Let take the left decomposition of L^i into the set of LLkS $L_0^i, \dots, L_{\ell_i}^i$ with corresponding roots $r_0^i, \dots, r_{\ell_i}^i$. We obtain recursively the bijection just by replacing each LLkS L_i by its corresponding rooted tree and adding the edges $\{r, r_j^i\}$ for $0 \leq j \leq \ell_i$ with weight i .

From labelled trees to LLkS. Let T be a rooted labelled tree with the property and r be its root. Let T_j^i be the rooted labelled trees obtain by deleting the root r where i is the value of the weight of the deleted edge. The tree T is not a rooted plane tree so there is special order on these subtrees. Any way they have root r_j^i which can be uniquely ordered by their label and let us suppose then that $r_0^i < r_1^i \dots < r_{\ell_i}^i$. That means we work on each weight separately. By hypothesis we have $r_0 < r$. So recursively again, we obtain the LLkS corresponding to T by replacing each T_i by its corresponding LLkS and joining them to the root by adding the edge $\{r, r_0^i\}$ (r_0^i is the number i child of r) and all the edges $\{r_{j-1}^i, r_j^i\}$ (r_j^i is the number i child of r_{j-1}^i) for $1 \leq j \leq \ell_i$. \square

Remark 16. (1) Note that LBS are both LkS and SLkS for $k = 1$. Of course this comes from the fact that the two different definitions are the same when $k = 1$.

(2) Note also that we could make a new definition with two different set of weights $W_1 = [k_1]$ and $W_2 = [k_2]$. A W_1 -descent would be a inner vertex x such that y its smallest child of smallest weight in W_1 is such that $y < x$. Then the descent definition corresponds to the case $k_1 = 1$ and $k_2 = n-1$ and the S-descent definition corresponds to the case $k_1 = n$ and $k_2 = 0$. Other values of $k_1 \geq 1$ and $k_2 \geq 0$ are possible and will correspond in our construction to the complete gain graphs $K_n^{[1-k_2, k_1+k_2]}$.

REFERENCES

- [1] C. A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields. *Adv. in Math.* 122 (1996), 193–233. MR 97k:52012. Zbl 872.52006.
- [2] Anders Björner, The homology and shellability of matroids and geometric lattices. In Neil White, ed., *Matroid Applications*, Ch. 7, pp. 226–283, Encyc. Math. Appl., Vol. 40, Cambridge Univ. Press, Cambridge, U.K., 1992. MR 94a:52030. Zbl. 772.05027.
- [3] S. Corteel, D. Forge, V. Ventos, Bijections between truncated affine arrangements and valued graphs, to appear in *Eur. J. of Comb.*, (arXiv:1403.2573 [math.CO]).
- [4] David Forge and Thomas Zaslavsky, Lattice point counts for the Shi arrangement and other affino-graphic hyperplane arrangements. *J. Combin. Theory Ser. A* 114(1) (2007), 97–109. MR 2007i:52026. Zbl 1105.52014.
- [5] Emeric Gioan and Michel Las Vergnas, The active bijection between regions and simplices in supersolvable arrangements of hyperplanes. *Electronic J. Combin.* (Stanley Festschrift) 11(2) (2006), #R30, 39pp.
- [6] A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements. *J. Combin. Theory Ser. A* 91 (2000), 544–597. MR 2002g:52032. Zbl 962.05004.
- [7] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Wadsworth & Brooks-Cole, Belmont, CA, 1986; reprinted by Cambridge Univ. Press, Cambridge, U.K., 1997.

- [8] Thomas Zaslavsky, Biased graphs. I. Bias, balance, and gains. II. The three matroids. III. Chromatic and dichromatic invariants. IV. Geometrical realizations. *J. Combin. Theory Ser. B* **47** (1989), 32–52; **51** (1991), 46–72; **64** (1995), 17–88; **89** (2003), 231–297. MR 90k:05138; 91m:05056; 96g:05139, 2005b:05057. Zbl. 714.05057; 763.05096; 950.25778; 1031.05034.