# The Directed Switching Game on Lawrence Oriented Matroids

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**Abstract**

The main content of the note is a proof of the conjecture of Hamidoune-Las Vergnas on the directed switching game in the case of Lawrence oriented matroids.

C.E. Shannon has introduced the switching game for graphs circa 1960. It has been generalized and solved for matroids by A. Lehman [4]. A switching game on graphs and oriented matroids was introduced by Y. O. Hamidoune and M. Las Vergnas [3]. They have solved it for graphic and cographic oriented matroids. They have stated as a conjecture that the classification of the oriented game is identical to the classification of its non oriented version. This conjecture holds for the graphic and cographic cases, but remains open for more general classes of oriented matroids. In this note, we show that it holds for the class of Lawrence oriented matroids.

**Definition 1** (**Directed switching game on an oriented matroid**). *Let $\mathcal{M}$ be an oriented matroid and $e$ one of its elements. In the directed switching game on $\mathcal{M}$, Maker and Breaker alternately play on $\mathcal{M}$ and choose an unplayed element different from $e$, Maker signs it and Breaker deletes it. Maker wins the game if the final orientation of $\mathcal{M}$ contains a positive circuit containing $e$.*

By the results of [3], in order to prove that the classification of the directed switching game is identical to the classification of the undirected game, it suffices to prove

**Conjecture 1.** *[3] If $\mathcal{M}$ is the union of two disjoint bases, then the directed switching game on oM is winning for Maker playing first.*

**Definition 2.** *The* Lawrence oriented matroid *defined by an $n \times r$ matrix $A = (a_{ij})$ with coefficients in $\{-1, 1\}$ is the uniform oriented matroid of rank $r$ on $n$ elements such that the sign of an ordered basis $(i_1, \dots, i_r)_<$ is given by:*

$$\chi(i_1, \dots, i_r) = \prod_{j=1}^{r} a_{j,i_j}.$$

For more details, we refer the reader to Section 3.5 of [2] for chirotopes and Section 7.6 for Lawrence matroids (where they are called $\Gamma$). Lawrence matroids form a special class of oriented matroids that are uniform and vectorial. The special case where all the coefficients of the matrix are 1 gives the alternating matroid (the oriented matroid of the cyclic polytope).

For the convenience of the reader, we recall the relation between the chirotope (basis orientation) and the signature of a circuit in a uniform oriented matroid. Let $C = \{i_1, i_2, \cdots, i_{r+1}\}_<$ be a circuit of a uniform matroid. For every element $i$ of the circuit, the set $C \setminus i$ is an ordered basis of the matroid. The relation

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between the sign in the circuit of two consecutive elements $i_j$, $i_{j+1}$ and the signs of the basis $B_j = C \setminus i_j$, $B_{j+1} = C \setminus i_{j+1}$ is:

$$\chi(B_j) \cdot \chi(B_{j+1}) = -C(i_j) \cdot C(i_{j+1}).$$

where the sign of an element $i$ in $C$ is denoted $C(i)$.

It follows that in a Lawrence matroid a signature of $C$ is given by $C(i_1) = +$ and recursively by $C(i_{j+1}) = -C(i_j) \cdot a_{j,i_j} \cdot a_{j,i_{j+1}}$ for $1 \leq j \leq r$.

**Theorem 1.** *The directed switching game on a Lawrence matroid of rank $r$ and of order $n$ is winning for Maker playing first if and only if $n \geq 2r$.*

*Proof.* If $n < 2r$ then Maker do not sign enough elements to create a circuit and then he loses.

Suppose that $n = 2r$ and let $k$ be the initial element. A winning strategy for Maker will be to play $a = \lceil \frac{k-1}{2} \rceil$ elements smaller than $k$ and $b = \lceil \frac{n-k}{2} \rceil$ elements bigger than $k$. Note that the relation $a+b+1 = r+1$ is verified and that $k$ in the end will corresponds to the element $c_{a+1}$ in the constructed circuit $C = \{c_1, \ldots, c_{r+1}\}$.

Maker plays on the side (left or right) of $k$ where an odd number, say $2j+1$, of elements is left. On this side, he chooses $i$, the closest element to $k$. In the case where the chosen element is smaller than $k$, this element will be $c_{j+1}$. In the other case, the element $i$ will be $c_{r+1-j}$.

This permits Maker to sign $i$ by $-a_{j+1,i} \cdot a_{j+1,i'}$ in the first case and by $-a_{r-j,i} \cdot a_{r-j,i'}$, where $i'$ is the previous element played on this side (possibly $k$ if none). Of course these rules are used to have at the end $C(i) = C(i')$ which implies that after $r$ moves of Maker, the set of selected signed elements forms a positive circuit.

In the case $n > 2r$, Maker can use fictitious moves like in [3]. Maker will first select from the all set a subset of $2r$ elements containing $k$. Then he applies the previous strategy on this subset by choosing himself an element for Breaker in the case where Breaker plays outside the subset (these are the fictitious moves). $\square$

# References


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