Minimal non-orientable matroids in a projective plane

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Abstract: We construct a new family of minimal non-orientable matroids of rank three. Some of these matroids embed in Desarguesian projective planes. This answers a question of Ziegler: for every prime power q, find a minimal non-orientable submatroid of the projective plane over the q-element field.

1. introduction

The study of non-orientable matroids has not received very much attention compared with the study of representable matroids or oriented matroids. Proving non-orientability of a matroid is known to be a difficult problem even for small matroids of rank 3. Richter-Gebert [4] even proved that this problem is NP-complete. In general, there are only some necessary conditions (Proposition 6.6.1 of [1]).

In 1991 Ziegler [6] constructed a family of minimally non-orientable matroids of rank three which are submatroids of a projective plane over $\mathbb{F}_p$ for p a prime. These matroids are of size $3n + 2$ with $n \geq 2$ and the smallest is the Mac Lane matroid on 8 elements (the only non-orientable matroid on 8 or fewer elements). Ziegler raised this question ([1], page 337): For every prime power q, determine a minimal non-orientable submatroid of the projective plane of order q that is not a submatroid of any smaller projective plane.

We study an infinite family $\{F(n) : n \in \mathbb{N}\}$ of line arrangements in the real projective plane (where $\mathbb{N}$ is the set of positive integers). $F(n)$ consists of $2n + 1$ lines constructed by taking the infinite line together with a series of parallel lines going through two points. We give an easy criterion to decide when it is possible to extend the arrangement by a pseudoline passing through given intersection points of $F(n)$. This criterion gives a construction of a family of non-orientable matroids with $2n + 2$ elements for $n \geq 3$. Our smallest example is again the Mac Lane matroid but all others are different from Ziegler's matroids. Finally, for each prime power q, we will construct a matroid that embeds in a projective plane over the q-elements field but not in a smaller plane. This answers Ziegler's question.

[1]The work of the second author was performed while visiting the State University of New York at Binghamton.

2. Extension of pseudoline arrangements

We define a family of pseudoline arrangements $F(n)$ of size $2n+1$ in the real projective plane. We then study the possibility of extending such an $F(n)$ by a new pseudoline going through a given set of intersection points. In fact we are interested in the cases of impossibility, which will give us non-orientable matroids.

A pseudoline arrangement L is a set of simple closed curves in the real projective plane, of which each pair intersects at exactly one point, at which they cross. An arrangement is *stretchable* if it is isomorphic to an arrangement of straight lines. The *extension* of an arrangement L by a pseudoline l is the arrangement $L \cup l$ if the line l meets correctly all the lines of L. Given a finite set P of points it is always possible to draw a pseudoline going through the points of P. However, given an arrangement L and a set P of points, it may be impossible to construct an extension of L by a pseudoline going through P.

We will use the following simple case of impossible extension. Let $L = \{l_1, l_2\}$ be an arrangement of two pseudolines meeting at a point P_1. These two lines separate the real projective plane into two connected components C_1 and C_2. Let P_2 and P_3 be two points, one in each of the two connected components defined by L. Then there is no extension of L by a pseudoline going through the points P_1, P_2, P_3.

Let n be a positive integer. We adopt the notation $[n] := \{1, 2, \ldots, n\}$. Let c_0 be the line at infinity in the projective plane, and let A and B be two points not on c_0. Let $\{X_i : i \in [n]\}$ be a set of n points of c_0 that appear in the order $X_1, X_2, \ldots, X_n$ on c_0. Let us call $F(n)$ a pseudoline arrangement with $2n+1$ pseudolines a_i for $i \in [n]$, b_i for $i \in [n]$, c_0 such that

$$\bigcap_{i=1}^{n} a_i = A, \quad \bigcap_{i=1}^{n} b_i = B, \qquad \text{and} \quad a_i \cap b_i \cap c_0 = X_i, \ \forall i \in [n].$$

Let us denote by $X_{i,j}$ the intersection point of the lines a_i and b_j for two different integers i and j (in this notation the point X_i corresponds to $X_{i,i}$).

We remark that $F(n)$ is unique up to isomorphism (this is a key remark for the following). Indeed, since the lines a_i all meet at the point A, they also cross there and nowhere else. This gives all the other crossings and their order on the lines. The points $A, X_{i,j}$ for $j \in [n]$ appear on the line a_i in the order

$$\big(A, X_{i,1}, \ldots, X_{i,n}\big)$$

and similarly the points $B, X_{i,j}$ for $i \in [n]$ appear on the line b_j in the order

$$\big(B, X_{1,j}, \ldots, X_{n,j}\big).$$

$F(n)$ is stretchable; one can just put c_0 at infinity and take for the a_i and b_i n pairs of parallel lines passing through the two given points A and B. In fact, $F(n)$ is rational, i.e., it is isomorphic to an arrangement in the real projective plane of lines defined by equations with integer coefficients. However, in the proofs we will not use the fact that $F(n)$ is stretchable or rational and for convenience in our figures we will represent $F(n)$ with pseudolines.

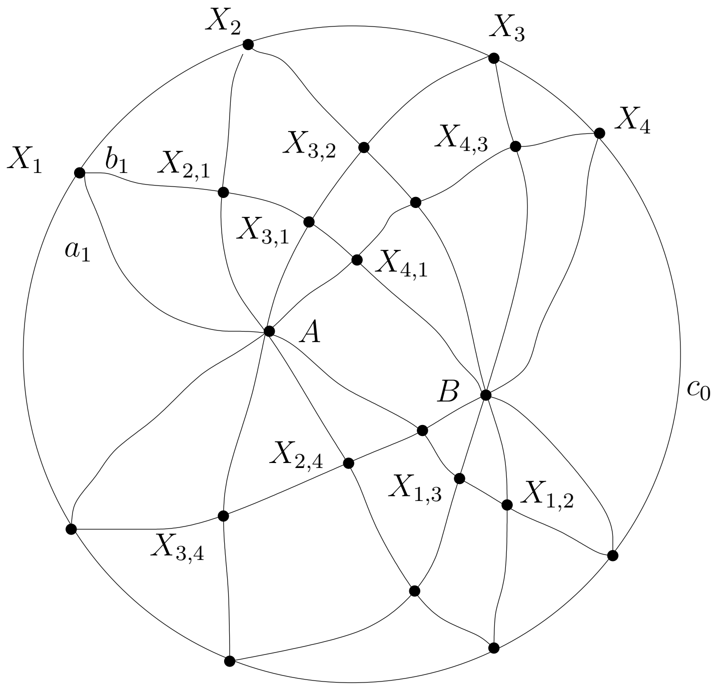

Figure 1. The pseudoline arrangement $F(4)$.

Lemma 1. *For any integer $n \geq 3$ and any three increasing integers $1 \leq i_1 < i_2 < i_3 \leq n$, there exists an extension of the arrangement $F(n)$ by a pseudoline passing through the three points X_{i_1,j_1}, X_{i_2,j_2}, and X_{i_3,j_3} if and only if $j_1 < j_2 < j_3$ or $j_1 > j_2 > j_3$.*

Proof. We know the order in which the points $A, X_{i,j}$ for $j \in [n]$ appear on the line a_i and similarly the order in which the points $B, X_{i,j}$ for $i \in [n]$ appear on the line b_j. The two lines a_i and b_j meeting at $X_{i,j}$ separate the projective plane into two connected components. Hence, point $X_{i,j}$ defines a partition of the point set $S_{i,j} = \{X_{i',j'} : i' \neq i, j' \neq j\}$ into the two parts

$$S^+_{i,j} = \big\{X_{i',j'} : (i'-i)(j'-j) > 0\big\} \text{ and } S^-_{i,j} = \big\{X_{i',j'} : (i'-i)(j'-j) < 0\big\}.$$

There exists a pseudoline passing through X_{i_2,j_2} and the two other points X_{i_1,j_1} and X_{i_3,j_3} if and only if X_{i_1,j_1} and X_{i_3,j_3} belong to the same part of the partition defined by X_{i_2,j_2}. Since we know that $i_1 < i_2 < i_3$, the last statement is equivalent to the conclusion. □

Lemma 2. *For any integer n and any injective function $f : D \to [n]$ where $D \subseteq [n]$, there exists an extension of the arrangement $F(n)$ by a pseudoline c_1 passing through the points $X_{i,f(i)}$, $i \in D$, if and only if the function f is increasing or decreasing.*

Proof. The preceding lemma implies the conclusion. □

Lemma 3. *For any integer $n \geq 2$ and for any cyclic permutation α of $[n]$, there exists an extension of the arrangement $F(n)$ by a pseudoline passing through the points $X_{\alpha^{i-1}(1),\alpha^i(1)}$, $i \in [n]$, if and only if $n = 2$ and $\alpha = (1\ 2)$.*

Proof. If $n \geq 3$ then Lemma 2 applied to α implies that the bijection α is increasing or decreasing. But a cyclic permutation on more than two elements cannot be increasing or decreasing. If $n = 2$ then the only cyclic permutation is $\alpha(1) = 2$ and $\alpha(2) = 1$. And clearly one can find a pseudoline passing through the two points $X_{1,2}$ and $X_{2,1}$ (in fact, any two points). □

3. Orientability of matroids

The orientability of a rank-3 matroid is known to be equivalent to its representability by a pseudoline arrangement in the projective plane. The arrangement will be called a *realization* of the matroid. In this section we will define a family of minimal non-orientable matroids using Lemma 3.

Let $A = \{a_i : i \in [n]\}$, $B = \{b_i : i \in [n]\}$ and $\{c_0\}$ be disjoint sets. For $i \in [n]$, let us call X_i the set $\{a_i, b_i, c_0\}$. Let $M'(n)$ be the simple rank-3 matroid on the ground set $E := A \cup B \cup \{c_0\}$ defined by the $n + 2$ non-trivial rank-2 flats: A, B and the n sets X_i, $i \in [n]$.

Let τ be a permutation of $[n]$. We denote by $F(n, \tau)$ the pseudoline arrangement obtained from $F(n)$ by relabelling the points X_i to $X_{\tau(i)}$.

Lemma 4. *Up to isomorphism, the realizations of $M'(n)$ are the pseudoline arrangements $F(n, \tau)$ where τ is a permutation of $[n]$.*

Proof. The permutation τ fixes the order of the points X_i on the line c_0. Once this order is fixed, every thing else is determined by the fact that the lines a_i, for $i \in [n]$, go through the points X_i and A and that the lines b_i, for $i \in [n]$, go through the points X_i and B. □

Let σ be a permutation of $[n]$ without fixed elements. We denote by $M(n, \sigma)$ the matroid extension of $M'(n)$ by an element c_1 such that the sets $\{a_i, b_{\sigma(i)}, c_1\}$, for $i \in [n]$ are the additional non-trivial rank-2 flats. This means that in $M(n, \sigma)$, the new element c_1 is the intersection of the lines $\mathrm{cl}(a_i, b_{\sigma(i)})$, $i \in [n]$. To the permutation σ corresponds naturally the bipartite graph G_σ with vertex set $A \cup B$ and with edge set

$$\big\{\{a_i, b_i\} : i \in [n]\big\} \cup \big\{\{a_i, b_{\sigma(i)}\} : i \in [n]\big\}.$$

In the graph G_σ, two vertices a_i and b_j form an edge $\{a_i, b_j\}$ if and only if they both belong to some 3-point line with c_0 or c_1. The graph G_σ is clearly 2-regular, which implies that it is a union of disjoint cycles.

Theorem 5. *Let $n \geq 2$ and let σ be a permutation of $[n]$ without fixed elements. The matroid $M(n, \sigma)$ is orientable if and only if the graph G_σ has no cycle of length greater than four. Moreover, if for some $k \geq 3$, the graph G_σ contains a cycle of length $2k$, say on the vertex set*

$$C = \big\{a_i, a_{\sigma(i)}, \ldots, a_{\sigma^{k-1}(i)}, b_i, b_{\sigma(i)}, \ldots, b_{\sigma^{k-1}(i)}\big\},$$

then the restriction $M(n, \sigma)|\,(C \cup \{c_0, c_1\})$ is a minimal non-orientable matroid.

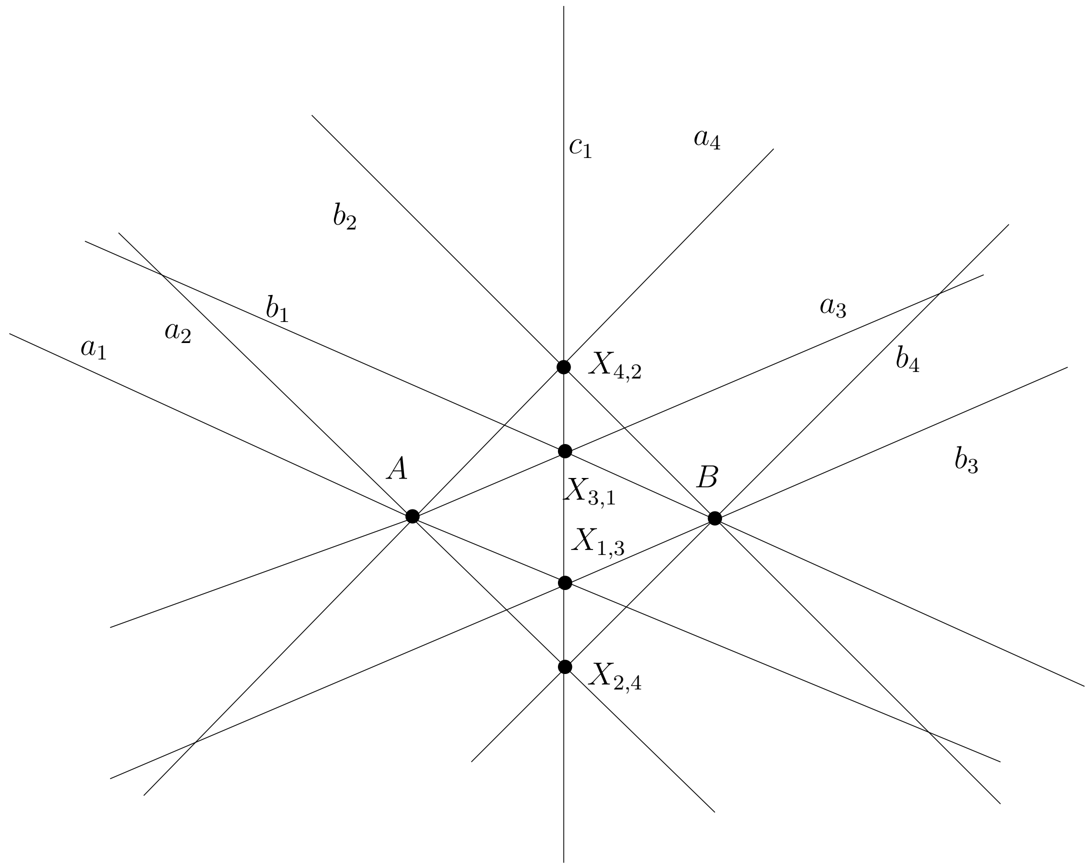

FIGURE 2. A linear realization of $M(4, (1\ 3)(2\ 4))$

Proof. If the graph G_σ has a decomposition into cycles of length only 4 (hence n must be even), we give an explicit realization (see Figure 2). We first relabel the elements using a permutation τ defining the position of the points X_i at infinity. This permutation is defined by the following algorithm:

Start with $k = 1$ and $S = [n]$. While $S \neq \emptyset$ do:

a) let i be the smallest element of S and set $\tau(i) \leftarrow k$ and $\tau(\sigma(i)) \leftarrow n + 1 - k$;

b) put $k \leftarrow k + 1$ and $S \leftarrow S \setminus \{i, \sigma(i)\}$.

The algorithm stops when the permutation τ has been completely defined (i.e., when S is finally empty, which will happen after $n/2$ steps). Put the points A and B at $(-1, 0)$ and $(1, 0)$ respectively. Using the permutation τ, the following realization works:

(a) the line $a_{\tau^{-1}(i)}$ has equation $y = -ix - i$, for $i \leq n/2$,

(b) the line $b_{\tau^{-1}(i)}$ has equation $y = -ix + i$, for $i \leq n/2$,

(c) the line $a_{\tau^{-1}(n-i+1)}$ has equation $y = ix + i$, for $i \leq n/2$,

(d) the line $b_{\tau^{-1}(n-i+1)}$ has equation $y = ix - i$, for $i \leq n/2$,

(e) the line c_0 is at infinity,

(f) the line c_1 has equation $x = 0$.

If G_σ contains a cycle C of length $2k \geq 6$ then the matroid $M(n, \sigma) \mid (C \cup \{c_0, c_1\})$ is an extension of $M'(k)$ by the element c_1. By Lemma 4, a representation of $M'(k)$ is a pseudoline arrangement $F(k, \tau)$ for a permutation τ. Then a representation of $M(n, \sigma) \mid (C \cup \{c_0, c_1\})$ is an extension of $M'(k)$ by a pseudoline c_1 going through the points $X_{\tau(i),\tau(\sigma(i))}$, $i \in [n]$. By Lemma 3, this is impossible.

Let us now prove the minimality of $M(n, \sigma) \mid (C \cup \{c_0, c_1\})$ as a non-orientable matroid. If one of the c_i is deleted we get a matroid isomorphic to $M'(n)$, which is orientable. If we delete one of the a_i or one of the b_i (say a_1) then the matroid $M(n, \sigma) \mid C \setminus a_1$ is realized by the following line arrangement in the real projective plane: put the points A and B at $(-1, 0)$ and $(0, 0)$ respectively and

(a) the line a_i has equation $x = (i - 1)y - 1$, for $2 \leq i \leq k$,

(b) the line b_i has equation $x = (i - 1)y$, for $1 \leq i \leq k$,

(c) the line c_0 is at infinity,

(d) the line c_1 has equation $y = 1$. □

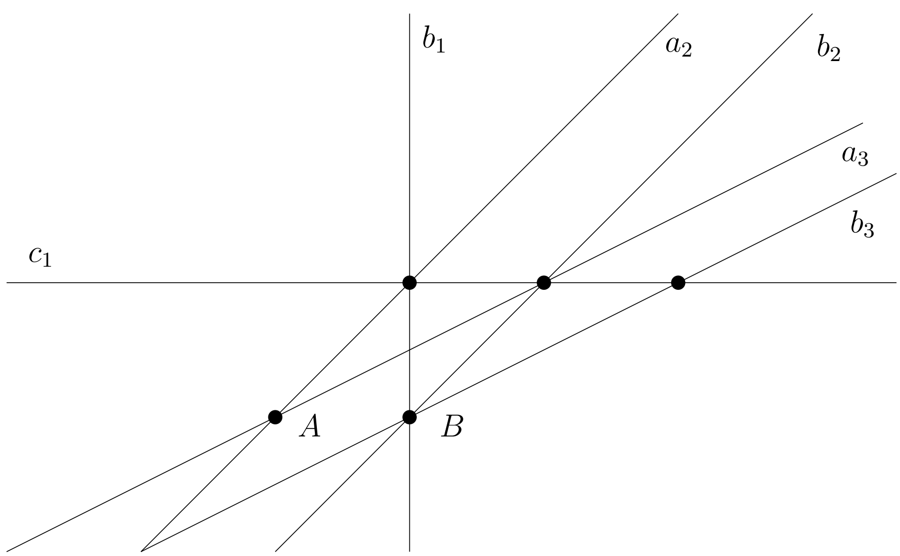

FIGURE 3. A linear realization of $M(3, (1\ 2\ 3)) \setminus a_1$.

4. Kung conjecture

The *Reid cycle matroid* $R_{cycle}[k]$ for $k \geq 3$ is a single-extension of the restriction matroid $M(n, \sigma)| (C \cup \{c_0, c_1\})$ (given in the Theorem 5) by one element w such that the sets

$\{a_i, a_{\sigma(i)}, \ldots, a_{\sigma^{k-1}(i)}, w\}$, $\{b_i, b_{\sigma(i)}, \ldots, b_{\sigma^{k-1}(i)}, w\}$, and $\{c_0, c_1, w\}$ are the additional non-trivial rank-2 flats.

Joseph Kung [2, page 52] state the conjecture: For $k \geq 3$ the matroid $R_{cycle}[k]$ is non-orientable matroid. The following theorem show that this conjecture is true.

Theorem 6. *For $k \geq 3$ the matroid $R_{cycle}[k]$ is non-orientable.*

Proof. Follows by Theorem 5. □

5. MINIMAL NON-ORIENTABLE MATROIDS CONTAINED IN A PROJECTIVE PLANE

In this section we will define a simple matroid $M(\mathfrak{G}, g_0, g_1)$ where the definition of the lines depends on a given group $\mathfrak{G}$ and two fixed elements of $\mathfrak{G}$. We will see that this matroid is a particular case of $M(n, \sigma)$. The special case $M(\mathbb{Z}_n, 0, 1)$ is a submatroid of a non-orientable matroid given by McNulty [3]. If a finite field F contains $\mathfrak{G}$ as a multiplicative or an additive subgroup then $M(\mathfrak{G}, g_0, g_1)$ embeds in the projective plane coordinatized by F. In Lemma 8 (which follows by Theorems 2.1 and 4.1 in [5], because $M(\mathfrak{G}, g_0, g_1)$ is a bias matroid of a gain graph) we prove this fact for finite fields. With this lemma and Theorems 7 and 9 we will answer Ziegler's question.

Let p^t be a prime power and let $\mathbb{F}_{p^t}$ be a Galois field. We will denote by Π_{p^t} the projective plane coordinatized by $\mathbb{F}_{p^t}$. The points and lines of Π_{p^t} will be denoted by $[x, y, z]$ and $\langle a, b, c\rangle := \{[x, y, z] : ax + by + cz = 0\}$ for a, b, c in $\mathbb{F}_{p^t}$, not all of which are zero.

Let $\mathfrak{G}$ be a finite group of order n and let g_0, g_1 be two of its elements. Let $A = \{a_g : g \in \mathfrak{G}\}$, $B = \{b_g : g \in \mathfrak{G}\}$ and $\{c_{g_0}, c_{g_1}\}$ be disjoint sets. Let $M(\mathfrak{G}, g_0, g_1)$ be the simple matroid of rank 3 on the ground set $E := A \cup B \cup \{c_{g_0}, c_{g_1}\}$ defined by the $2n + 2$ non-trivial rank-2 flats A, B, and the $2n$ sets $\{a_g, b_{g\cdot g_0}, c_{g_0}\}$, $g \in \mathfrak{G}$, and $\{a_g, b_{g\cdot g_1}, c_{g_1}\}$, $g \in \mathfrak{G}$.

Theorem 7. *Let g_0 and g_1 be two different elements of a finite abelian group $\mathfrak{G}$. Let r be the order of $g_0 \cdot g_1^{-1}$. Then $M(\mathfrak{G}, g_0, g_1)$ is non-orientable if and only if $r \geq 3$.*

Proof. Let n be the order of $\mathfrak{G}$. Let us first note that $M(\mathfrak{G}, g_0, g_1)$ is isomorphic to an $M(n, \sigma)$. Let α be a bijection from $[n]$ to $\mathfrak{G}$. Let β be the bijection from $[n]$ to $\mathfrak{G}$ defined by $\beta(i) = \alpha(i) \cdot g_0$. Let σ be the permutation on $[n]$ defined by $\sigma(i) = \beta^{-1}\big(\alpha(i) \cdot g_1\big)$. The permutation σ is clearly without fixed elements. We now have an isomorphism ϕ between $M(n, \sigma)$ and $M(\mathfrak{G}, g_0, g_1)$ given by $\phi(c_0) = c_{g_0}$, $\phi(c_1) = c_{g_1}$, . $\phi(a_i) = a_{\alpha(i)}$ and $\phi(b_i) = b_{\beta(i)}$.

Let G be the graph with vertex set $\{a_g : g \in \mathfrak{G}\} \cup \{b_g : g \in \mathfrak{G}\}$ and edges $\{a_g, b_{g'}\}$ such that $\{a_g, b_{g'}, c_{g_0}\}$ or $\{a_g, b_{g'}, c_{g_1}\}$ is a line of $M(\mathfrak{G}, g_0, g_1)$. This graph is the graph G_σ for the corresponding permutation σ.

A cycle of G has the form

$$\big\{a_g, b_{g\cdot g_0}, a_{g\cdot g_0\cdot g_1^{-1}}, b_{g\cdot g_0\cdot(g_0\cdot g_1^{-1})}, a_{g\cdot(g_0\cdot g_1^{-1})^2}, \ldots, a_{g\cdot(g_0\cdot g_1^{-1})^{r-1}}, b_{g\cdot g_0\cdot(g_0\cdot g_1^{-1})^{r-1}}\big\}.$$

Therefore the length of a cycle of G is $2r$. So, Theorem 5 implies that $M(\mathfrak{G}, g_0, g_1)$ is non-orientable if and only if $r \geq 3$. □

Lemma 8. *Let p be a prime number and let $m \geq 2$ and $t \geq 1$ be two integers.*

(i) $M(\mathbb{Z}_p, 0, 1)$ embeds in Π_p.

(ii) If m divides $p^t - 1$, then $M(\mathbb{Z}_m, 0, 1)$ embeds in Π_{p^t}.

Proof of (i). Let ψ be the map from the ground set of $M(\mathbb{Z}_p, 0, 1)$ into the point set of Π_p defined as follows:

$$\psi(a_i) = [0, i, 1],\ \psi(b_i) = [1, i, 1],\ \psi(c_0) = [1, 0, 0],\ \psi(c_1) = [1, 1, 0] \text{ for } i \in \mathbb{Z}_p.$$

By the definition of the incidence relation between points and lines in Π_p,

$\big\{[0, i, 1] : i \in \mathbb{Z}_p\big\} \subseteq \langle -1, 0, 0\rangle$,

$\big\{[1, i, 1] : i \in \mathbb{Z}_p\big\} \subseteq \langle -1, 0, 1\rangle$, and

$\big\{[1, 0, 0], [1, 1, 0]\big\} \subseteq \langle 0, 0, 1\rangle$.

Now, for fixed $i, j \in \mathbb{Z}_p$ and fixed $k \in \{0, 1\}$, it is easy to verify that $\psi(\{a_i, b_j, c_k\})$ is collinear in Π_p if and only if $j = i + k$. □

Proof of (ii). Let ϕ be an isomorphism between the group $\mathbb{Z}_{p^t-1}$ and the multiplicative group $\mathbb{F}^*_{p^t}$. Since m divides $p^t - 1$, $\phi(\mathbb{Z}_m)$ is a subgroup of $\mathbb{F}^*_{p^t}$. Let ψ be a map from the ground set of $M(\mathbb{Z}_m, 0, 1)$ into the point set of Π_{p^t} defined as follows:

$$\psi(a_i) = [\phi(i), 0, 1],\ \psi(b_i) = [0, -\phi(i), 1],$$
$$\psi(c_0) = [1, \phi(0), 0],\ \psi(c_1) = [1, \phi(1), 0] \text{ for } i \in \mathbb{Z}_m.$$

By the definition of the incidence relation between points and lines in Π_{p^t},

$$\big\{[\phi(i), 0, 1] : i \in \mathbb{Z}_m\big\} \subseteq \langle 0, 1, 0\rangle,$$
$$\big\{[0, -\phi(i), 1] : i \in \mathbb{Z}_m\big\} \subseteq \langle 1, 0, 0\rangle, \text{ and}$$
$$\big\{[1, \phi(0), 0], [1, \phi(1), 0]\big\} \subseteq \langle 0, 0, 1\rangle.$$

Now, for fixed $i, j \in \mathbb{Z}_m$ and fixed $k \in \{0, 1\}$ it is easy to verify that $\psi(\{a_i, b_j, c_k\})$ is collinear in Π_{p^t} if and only if $j = i + k$. □

Theorem 9. *Let $p \geq 3$ be a prime number and let $m \geq 3$ and $t \geq 1$ be two integers.*

(i) $M(\mathbb{Z}_p, 0, 1)$ is a minimal non-orientable matroid that embeds in Π_p.

(ii) If m is a divisor of $p^t - 1$ then $M(\mathbb{Z}_m, 0, 1)$ is a minimal non-orientable matroid that embeds in Π_{p^t}.

(iii) $M(\mathbb{Z}_{p^t-1}, 0, 1)$ is a minimal non-orientable matroid that embeds in Π_{p^t} and in none of the Π_{p^k} for $k < t$.

Proof. Parts (i) and (ii) follow by Theorem 7 and Lemma 8.

As a consequence of part (ii) $M(\mathbb{Z}_{p^t-1}, 0, 1)$ is a minimal non-orientable matroid in Π_{p^t}. Since $M(\mathbb{Z}_{p^t-1}, 0, 1)$ has a line with $p^t - 1$ points, $M(\mathbb{Z}_{p^t-1}, 0, 1)$ does not embed in Π_{p^k} for $k < t$. □

The matroids given in parts (i), (ii), and (iii) of the previous theorem are new minimal non-orientable matroids embeddable in projective planes, except for $M(\mathbb{Z}_3, 0, 1)$, which is the Mac Lane matroid. Part (iii) answers Ziegler's question.

6. Concluding remarks

At no moment in the previous sections did we really need to have a finite set of points. We could have considered infinite rank-3 matroids and infinite pseudoline arrangements. For a permutation on $\mathbb{N}$ without fixed elements, we can define the rank-3 infinite matroid $M(\mathbb{N}, \sigma)$ on the set $\{a_i : i \in \mathbb{N}\} \cup \{b_i : i \in \mathbb{N}\} \cup \{c_0, c_1\}$ by taking for its non-trivial rank-2 flats $A = \{a_i : i \in \mathbb{N}\}$, $B = \{b_i : i \in \mathbb{N}\}$, $X_i = \{a_i, b_i, c_0\}$, $i \in \mathbb{N}$, and $\{a_i, b_{\sigma(i)}, c_1\}$, $i \in \mathbb{N}$. The permutation σ, as in the finite case, also defines a graph G_σ on the vertex set $A \cup B$. This graph is infinite but still of degree 2. This implies that G_σ is a union of cycles and infinite 2-way paths. We then have the following results, which are similar to Theorems 5 and 7:

Theorem 10. *Let σ be a permutation of $\mathbb{N}$ without fixed elements. The matroid $M(\mathbb{N}, \sigma)$ is orientable if and only if the graph G_σ has no cycle of length greater than four.*

Theorem 11. *Suppose that $\mathfrak{G}$ is a finitely generated abelian group. Then $M(\mathfrak{G}, g_0, g_1)$ is non-orientable if and only if the order of $g_0 \cdot g_1^{-1}$ is finite and greater than 2.*

We want to remark also that $M(\mathbb{Z}_n, 0, 1)$ is linearly representable over the complex numbers $\mathbb{C}$ (It follows by [5, Theorem 2.1]). Therefore, $M(\mathbb{Z}_n, 0, 1)$ embeds in the projective plane coordinatized by $\mathbb{C}$.

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