FLIPPING IN ACYCLIC AND STRONGLY CONNECTED GRAPHS

RAUL CORDOVIL AND DAVID FORGE

ABSTRACT. A flippable edge in an acyclic digraph is an edge whose reorientation leaves the graph acyclic. Flippable edges was recently considered by K. Fukuda et al. In an acyclic digraph the set of flippable edges is spanning. We then characterize the spanning trees T of an undirected graph G such that there exists an acyclic orientation of G whose set of flippable edges is T. In particular for every edge $e \in E(G)$ we give a linear algorithm returning an acyclic orientation and a spanning tree T containing e such that T is the set of flippable edges of the digraph.

After going to oriented matroid theory and dualizing the proofs we obtain theorems concerning flippable edges in strongly connected digraphs.

1. INTRODUCTION AND NOTATIONS

The famous formula of Cayley says that the number of trees on n labelled vertices equals n^{n-2}. Nevertheless finding a spanning tree in an undirected graph is a well known polynomial problem. In this paper we consider a related problem concerning the class of directed graphs (*digraphs*, for short). We will also consider a digraph as a graph together with an orientation of its edges. An *acyclic orientation* of a graph is an orientation with no directed circuit. Given a directed graph G, the flipping of an edge $e = (a, b)$ is the directed graph, denoted ${}_{-e}G$, with the same set of vertices and with the same set of directed edges except that the edge (a, b) is replaced by the edge (b, a). A *flippable edge* in an acyclic digraph is an edge whose reorientation leaves the graph acyclic. Flippable edges were recently considered by K. Fukuda et al, see [2]. We give a simple graph theoretical proof that the set of flippable edges is spanning and determine completely the digraph. We characterize the spanning trees T of an undirected graph G such that there exists an orientation of G whose set of flippable edges is T. In particular for every edge $e \in E(G)$ we give a linear algorithm giving an acyclic orientation and a spanning tree T containing e and such that T is the set of flippable edges of the digraph. These notions are more natural and more classical in

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oriented matroid theory. Some of our results hold also for non-graphic oriented matroids. Nevertheless we opt here for a graph theoretic presentation and postpone to the end of the paper the introduction of the oriented matroid point of view. We give graphic proofs when possible even when there are more general matroid proofs. From the matroid notion of "duality", we also get a series of results concerning flippable edges in strongly connected digraphs which is the property dual to acyclic.

The paper is organized as follows. In section 2, we look at the set of flippable edges of an acyclic graph and stay entirely in a graphical context for the proofs. In section 3, we give when possible analogous results for flippable edges in strongly connected graphs. We omit the proofs which are very similar from duality in the matroid sense. In section 4, we replace our results in the more general context of oriented matroids and hyperplane arrangements. In a geometrical set up, flippable edges correspond to extremal points in an affine space or to walls of the fundamental cell in a hyperplane arrangement.

We now fix some more notations and definitions on graph theory. We will call G a graph or a digraph on a set V of vertices and a set E of edges with $|V| = n$ and $|E| = m$. A *simple graph* is a graph without loops or multiple edges. Note that a graph is simple if and only it has no cycle of size smaller than three. In a digraph G the edges of a cycle C of the graph are divided naturally, after a choice of a direction, into a positive and a negative part C^+ and C^-. The signed set $C = (C^+, C^-)$ is a *circuit* of the graph. The circuit (C^+, C^-) with $e \in C^+$ is the *circuit on the cycle C and along e*. If the negative part is empty the circuit is *positive* (a positive circuit is sometimes called in the literature simply a circuit). An orientation of a graph is acyclic if it has no positive circuit. Every acyclic orientation of the graph G determines at least one linear ordering $v_1 < \cdots < v_n$ of the vertex set $V(G) := \{v_1, \ldots, v_n\}$ such that an edge (v_i, v_j) of G has the direction (v_i, v_j) if and only if $v_i < v_j$. An ordering verifying these conditions is called in the literature a *topological ordering* of the vertices relative to the orientation. Such a topological ordering can classically be obtained in linear time by a depth first search algorithm. Reciprocally every linear ordering $v_{1'} < \cdots < v_{n'}$ of $V(G)$ is a topological ordering of an acyclic orientation and determines this orientation. A *totally cyclic orientation* of a graph is an orientation such that every edge belongs to a positive circuit. It is a well known result that a 2-connected digraph is totally cyclic iff it is strongly connected (i.e., for any two vertices a and b there exists a directed path from a to b). For a spanning tree T and an edge $e \in E(G) \setminus T$, the unique cycle [resp. circuit along e] of G contained in $T \cup e$ is the *fundamental cycle* [resp. *fundamental circuit*] and is denoted $C(e, T)$. A *cutset* $C^\star$ in a connected graph is a minimal set of edges which disconnects the vertices of the graph in two non-empty parts A and B. A graph is *k-connected* if it has no cutset of size smaller than k. In a digraph the edges of a cutset $C^\star$ are naturally separated into a positive part ${C^\star}^+$ which are the edges from

A to B and a negative part ${C^\star}^-$ which are the edges from B to A. The signed set $C^\star = ({C^\star}^+, {C^\star}^-)$ is a cocircuit of the digraph. If an edge e is in ${C^\star}^+$, the cocircuit $C^\star$ is called the *cocircuit on the cutset $C^\star$ and along e*. If the negative part of a cocircuit is empty, the cocircuit is *positive*. The circuits and cocircuits of a directed graph G are precisely the circuits and the cocircuits of the oriented matroid $M(G)$. For a spanning tree T and an edge $e \in T$, the unique cutset [resp. cocircuit along e] of G contained in $(E(G) \setminus T) \cup e$ is the *fundamental cutset* [resp. *fundamental cocircuit*] and is denoted $C^\star(e, T)$. A cycle C [resp. circuit $C = (C^+, C^-]$ of G corresponds to a vector in $\mathbb{R}^{E(G)}$ with 1 on C [resp. 1 on C^+ and -1 on C^-] and 0 on $E(G) \setminus C$. Similarly, a cutset $C^\star$ [resp. cocircuit $C^\star = ({C^\star}^+, {C^\star}^-)$] of G corresponds to a vector in $\mathbb{R}^{E(G)}$ with 1 on $C^\star$ [resp. 1 on ${C^\star}^+$ and -1 on ${C^\star}^-$] and 0 on $E(G) \setminus C$. Let $\boldsymbol{\mathcal{C}}(G)$ [resp. $\boldsymbol{\mathcal{C}}^\star(G)$] be the the vector subspace of $\mathbb{R}^{E(G)}$ spanned by the vectors corresponding to the circuits [resp. cocircuits].

Remark that the graphic oriented matroid $M(G)$, of a connected digraph G, is acyclic if and only if the oriented dual matroid $M^\star(G)$ is totally cyclic, see [1] for details. Applying this duality K. Fukuda et al [2] obtained some interesting results. In particular we recall that, given a graph G, an orientation of G is determined by the graphic oriented matroid determined by G and the orientation of an edge in every connected component.

2. Flippable edges in acyclic digraphs

The following lemma is a known result of oriented matroid theory. Proposition 2.2 is also partially known.

Lemma 2.1. *An edge $e = (a, b)$ is not flippable in an acyclic digraph G iff there exists a path from a to b in the digraph $G \setminus e$.*

Proof. If there exists a path P from a to b in the digraph $G \setminus e$ there will be a directed circuit in ${}_{-e}G$ by completing P by the arc (b, a). Reciprocally if there exists a directed circuit C in ${}_{-e}G$, it necessarily contains the edge $\bar{e} = (b, a)$. The set $C \setminus \bar{e}$ is a path from a to b in G. □

We say that a set of edges $E' \subseteq E(G)$ *preserves* the directed connectivity of the graph if for any two vertices $a, b \in V(G)$ the following two conditions are equivalent:

(i) There is a directed path from a to b in G;
(ii) There is a directed path from a to b in $G' = (V, E')$.

Proposition 2.2. *In a digraph G of order n the set of flippable edges preserves the directed connectivity of G. Consequently, if the graph is connected then there are at least $n-1$ flippable edges and if there is exactly $n-1$ flippable edges the set is a tree.*

Proof. Let a and b be two vertices connected by a directed path in G. Let P^{ab}_{max} be a maximal path connecting these two vertices (maximal with respect to cardinality). All the edges of P^{ab}_{max} are flippable since otherwise by the previous lemma we could replace an edge of P^{ab}_{max} by a path, which contradicts the maximally of P^{ab}_{max}. If now the graph is connected, the set of flippable edges is spanning which implies that it has at least $n-1$ edges. □

From the simple preceding results we can see that the set of flippable edges with their orientation determines the directions of all the edges in the digraph. Indeed let e be a non-flippable edge of G. Necessarily from the lemma it belongs to a circuit C of G where it is the only positive edge. From this remark, one can also deduce that two different acyclic orientations of the same graph differ at least on one flippable edge, which is exactly Lemma 1.1 of [2].

Theorem 2.3. *Let T be a spanning tree of a digraph G. Then the following three conditions are equivalent:*

- (i) *The digraph G is acyclic and T is the set of flippable edges of G.*
- (ii) *For every edge $e \in E(G) \setminus T$, e is the unique positive element in the fundamental circuit $C(e,T)$.*
- (iii) *For every edge $e \in T$ the fundamental cocircuit $C^\star(e,T)$ is positive.*

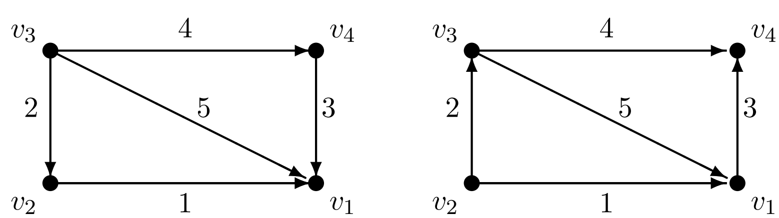
Proof. $(i) \Longrightarrow (ii)$ is a consequence of Lemma 2.1.
$(ii) \Longrightarrow (iii)$ is a consequence of the orthogonality of circuits and cocircuits of a digraph: i.e., given a circuit $C = (C^+, C^-)$ and a cocircuit $C^\star = ({C^\star}^+, {C^\star}^-)$ in a digraph such that $|C \cap C^\star| = 2$ then $|(C^+ \cap {C^\star}^+) \cup (C^- \cap {C^\star}^-)| = 1$ and $|(C^+ \cap {C^\star}^-) \cup (C^- \cap {C^\star}^+)| = 1$.
$(iii) \Longrightarrow (i)$. Suppose that G has a directed cycle C. For an edge e of T, let A_e and B_e denote the components of G that are separated by $C^\star(e,T)$. We claim that there exists an edge e of T such that $C \cap A_e$ and $C \cap B_e$ are both non-empty. Since $C^\star(e,T)$ is positive, we see that the acyclicity of G follows from this claim. To prove the claim, pick an edge (a,b) of $C \setminus T$. Let p be the undirected path from a to b that is contained in T, and let e be any edge of p. If we cut the edge e of T, then a and b are separated in T. Therefore, a and b are separated in G by $C^\star(e,T)$, proving the claim, and hence the acyclicity of G. Next, for any $e \notin T$, pick an arbitrary element $a \in C(e,T) \setminus e$. From the orthogonality of the circuit $C(e,T)$ and the cocircuit $C^\star(a,T)$ we conclude that the element e is not flippable. The minimum number of flippable elements in the acyclic graph G is $|T|$, see Proposition 2.2. So every element of T is flippable. □

Given a spanning tree T of a digraph G it is well known that the set $\{C(a,T) : a \in E(G) \setminus T\}$ is a basis of the space of circuits $\boldsymbol{\mathcal{C}}(G)$. By matroid duality the set $\{C^\star(a,T) : a \in T\}$ is a basis of the space of cocircuits $\boldsymbol{\mathcal{C}}^\star(G)$ over $\mathbb{R}$. Therefore, we can immediately deduce the following:

Corollary 2.4. *Let G be a 2-connected digraph and T be a spanning tree of G. Then the following propositions are equivalent:*

(i) *G is acyclic and T is the set of flippable edges of G.*
(ii) *All the cocircuits $\{C^\star(a,T) : a \in T\}$ are positive and form a basis of the cocircuit space $\boldsymbol{\mathcal{C}}^\star(G)$.*

Remark 2.5. The existence of a basis of positive cocircuits does not guarantee the fact that the flippable edges form a tree. This can be seen in the following example. Let G be the graph on 4 vertices and with 5 edges oriented like in Figure (a). The edges 1, 2, 3, and 4 are flippable and the positive cocircuits $\{2,4,5\}$, $\{1,4,5\}$ and $\{1,3,5\}$ form a basis of the cocircuit space $\boldsymbol{\mathcal{C}}^\star(G)$. In the digraph of figure (b), the flippable edges are 2, 3 and 5 and they form a tree.



(a) **(b)**

Corollary 2.6. *Let G be a 2-connected graph and T be a spanning tree of G. If T is the set of flippable edges of an acyclic orientation of G then it is the set of flippable edges of exactly two opposite acyclic orientations of G.*

Proof. Suppose that T is the set of flippable edges of a digraph G. From the equivalence of (i) and (ii) of Proposition 2.2 we know a basis of the space of circuits $\{C(e,T) : e \in E(G) \setminus T\}$. From the 2-connectivity of the graph, we know that the two opposite orientations of an edge of G give two opposite orientations of the graph G. □

From a classical result of R. W. Shannon [4] (on hyperplane arrangements), there exist at least $2m$ different acyclic orientations having a spanning tree of flippable edges. We present here a linear algorithm, based on a depth first search, to find for every edge $e \in E(G)$ two acyclic orientations and a spanning tree T corresponding to the edge e.

Theorem 2.7. *Let $G = (V,E)$ be a 2-connected graph. There exist at least $2m$ acyclic orientations with exactly $n-1$ flippable edges.*

Proof. The proof is based on the following algorithm for numbering the vertices based on a depth first search with postfix numbering starting with a specific edge. Indeed, we can see that each edge e gives two different correct orientations, one for each orientation of e.

Algorithm

Input: A graph G, two vertices a and b forming an edge (and a numbering of the vertices where a is first and b second to determine the search).
Output: A numbering α of the vertices with $\alpha(a) = n$, $\alpha(b) = n - 1$ and such that the associated orientation has exactly $n - 1$ flippable edges.
Procedure Search(v: vertex)
$S \leftarrow S \setminus v$; For every neighbor v' of v do if $v' \in S$ then Search(v');
$\alpha(v) \leftarrow k$; $k \leftarrow k + 1$;
Main
$k \leftarrow 1$; $S \leftarrow V$; Search(a); return α.

□

The result of the preceding algorithm is a numbering α of the vertices. This numbering induces two orientations: the increasing orientation if the edges $\{a, b\}$ are oriented from a to b if $\alpha(a) < \alpha(b)$ and the decreasing orientation if the edges $\{a, b\}$ are oriented from a to b if $\alpha(a) > \alpha(b)$. In these orientations, the set of flipping edges is exactly the search tree. Since a and b are the two first vertices in the order used for the search they will be numbered n and $n - 1$ respectively.

From the preceding discussion, we deduce the next corollary.

Corollary 2.8. *Let G be a 2-connected graph with $2m$ acyclic orientations with a tree of flippable edges. Then the tree T of flippable edges in one of these orientations has a unique source or sink, which has also a unique neighbor.* □

Theorem 2.7 suggests the following open question:

Problem 2.9. *Given a spanning subset $X \subseteq E(G)$ of edges of a connected graph, find a polynomial algorithm to decide if there is an orientation of G whose set of flippable edges is exactly X.*

Following definitions of oriented matroids, a 2-connected graph G on n vertices is *simplicial* if every acyclic orientation of G has exactly $n - 1$ flippable edges (see Section 4 for a geometric explanation of this definition).

Proposition 2.10. *In the complete graph K_n a tree is flippable iff it is a path. A 2-connected graph G on n vertices is simplicial iff it is the complete graph K_n.*

Proof. Any acyclic orientation of the complete graph K_n, which corresponds to a numbering of the vertices, contains a directed path of length $n-1$. Every other edge outside this path are clearly non-flippable.
For a graph G which is not complete, let a and b be two non-adjacent vertices.

Let C be a circuit of G containing a and b, and of minimum length k. Such a circuit exists from the connectivity of G. Let us first remark that k, the length of C, must be at least four. Consider now an acyclic orientation of G defined by an ordering of the vertices such that 1 and k are the neighbors (for consistency) of a, 2 is a, the others vertices of C are numbered in the order with labels $3, 4, \dots, k-1$ from vertex 1 to vertex k going by b and finally the other vertices outside of C are numbered indifferently from $k+1$ to n. By the minimality of C, in such an acyclic orientation all the edges of C are flippable which with Proposition 2.2 gives that there are at least n of them. □

Note that the number of flippable trees in the complete graph K_n is $n!$, a small number compared with the total number of trees, the Cayley number n^{n-2}.

Proposition 2.11. *Let $G = (V, E)$ be a 2-connected non-Hamiltonian graph $G = (V, E)$. There is an acyclic orientation of the edges such that every edge of G is on a positive cocircuit and outside another positive cocircuit.*

Proof. By Theorem 2.7, there exists a tree T and an acyclic orientation of G, such that T is the set of flippable edges of this orientation. By Theorem 2.3, every edge $e \in E \setminus T$ is the only positive edge in the fundamental circuit $C(e, T)$.

Note that there are exactly $n-1$ positive cocircuits and for every element $x \in T$ there is exactly one positive cocircuit $C^\star_x$ containing x, the fundamental cocircuit $C^\star(x, T)$. Every edge $x \in T$ is then on a positive cocircuit and outside another positive cocircuit.

Since G is non-Hamiltonian, for every edge edge $b \notin T$, we know that $T \not\subseteq C(b, T)$. By the orthogonality of circuits and cocircuits, we have that $x \in C(b, T) \iff b \in C^\star(x, T)$. We can deduce then that b is in all the fundamental cocircuits $C^\star(x, T)$ for $x \in C(b, T) \setminus b$ and in none of the fundamental cocircuits $C^\star(x, T)$ for $x \in T \setminus C(b, T)$, which concludes the proof. □

3. Flippings preserving strong connectivity

Similarly to the acyclic case, an edge $e \in E(G)$ in a strongly connected digraph G is *flippable* if the digraph $_{-e}G$ is also strongly connected. A graph is 3-edge connected if it has no cutset of size smaller than three. Note that in the matroid sense, the notions of “strongly connected” and “3-edge connected” are dual, respectively, to the notions of “acyclic” and “simple”. These matroid notions will be developed largely in the next section. In this section, the dual results of the previous section are given without proof. Dualizing Lemma 2.1, Proposition 2.2 and Theorem 2.3 we have:

Lemma 3.1. *An edge $e = (a, b)$ is not flippable in a strongly connected digraph G iff e is the only edge in the negative part of a cutset.* □

Proposition 3.2. *In a 3-edge connected and strongly connected directed graph G of order n the set of non-flippable edges does not contain a circuit. Consequently, there are at least $m-n+1$ flippable edges.* □

Theorem 3.3. *Let T be a spanning tree of a 3-connected digraph G. Then the following three conditions are equivalent:*

(i) *The directed graph G is strongly connected and T is the set of non-flippable edges of G.*

(ii) *Every edge $e \in T$ is the unique positive element in the fundamental cutset $C^\star(e,T)$.*

(iii) *For every edge $e \in E(G) \setminus T$ the fundamental circuit $C(e,T)$ is positive.* □

Proposition 3.4. *A 2-connected graph $G = (V, E)$ admits a strongly connected orientation such that all the edges are flippable iff it is 4-connected.*

Proof. This can be deduced almost directly from the theorem of C. St. J. A. Nash-Williams [3] which says that: "A graph admits a 2 edge-strongly connected orientation iff it is 4 edge-connected." Indeed the difficult part is the if part and in the Nash-Williams's result the edge can even be deleted and then clearly flipped. The only if part is immediate after one notes that if there is a 3-edge cutset then one of the edges of the cutset in any orientation will be not flippable. □

The following theorem can be obtained by dualizing Theorem 2.7.

Theorem 3.5. *Let $G = (V, E)$ be a 3-connected graph. There exist at least $2m$ strongly connected orientations with exactly $m-n+1$ flippable edges.* □

The cycle with n vertices is the only graph with exactly $2m$ acyclic orientations with $n-1$ flippable edges. In contrast, all $n!$ acyclic orientations of the complete graph on n vertices have $n-1$ flippable edges.

Conjecture 3.6. *A 2-connected graph $G = (V, E)$ admits an acyclic orientation such that all the edges are flippable iff it has no triangle.*

4. Concluding remarks: oriented matroids

In oriented matroid theory, the notion of acyclic reorientation is very important and has a geometric flavor. In the particular case of an oriented matroid M defined by a hyperplane arrangement H, there is a bijection between the acyclic reorientations of the matroid and the cells of the hyperplane arrangement, see [1]. All the results of this paper have a nice geometric (or in the language of oriented matroid) interpretation. We present now a very short and partial survey for the convenience of the non-specialist reader. More details can be found in [1]. To every 2-connected simple graph $G = (V, E)$ on n vertices $V := \{v_1, \dots, v_n\}$ and m edges

$E := \{e_1, \dots, e_i := \{v_k, v_\ell\}, \dots, e_m\}$ we attach a hyperplane arrangement $\mathcal{A}_G := \{H_{e_1}, \dots, H_{e_i}, \dots, H_{e_m}\}$ where H_{e_i} is the hyperplanes through the origin defined by

$$H_{e_i} := \{x \in \mathbb{R}^V : x_{v_k} = x_{v_\ell}\}.$$

The complement of the arrangement $\mathbb{R}^V \setminus \bigcup_{j=1}^m H_{e_j}$ is the union of n-dimensional open polyhedral cones called the *n-cells* of the arrangement. A hyperplane arrangement is said to be *simplicial* if the polyhedral cones are simplicial. With the definition of simplicial graph given in Section 2, the hyperplane arrangement $\mathcal{A}_G$ is simplicial if and only if the attached graph G is simplicial. If $e_i = (v_k, v_\ell)$ is a directed edge from v_k to v_ℓ we say that $H_{e_i}^+ := \{x \in \mathbb{R}^V : x_{v_k} < x_{v_\ell}\}$ and $H_{e_i}^- := \{x \in \mathbb{R}^V : x_{v_k} > x_{v_\ell}\}$ are respectively the *positive* and *negative* regions of $\mathbb{R}^V \setminus H_{e_i}$ determined by the directed edge $(v_k v_\ell)$. Suppose now that G is a directed graph. The position of a generic point $x \in \mathbb{R}^V$ relatively to the hyperplanes and its positive and negative regions can be encoded by the symbolic vector $\mathrm{sg}(x) \in \{0, +, -\}^E$ with,

$$\mathrm{sg}(x)_{e_i} = \begin{cases} 0, & \text{if } x \in H_{e_i} \\ +, & \text{if } x \in H_{e_i}^+ \\ -, & \text{if } x \in H_{e_i}^-. \end{cases}$$

All these vectors are the *covectors of the oriented matroid determined by the digraph G.* The n-cells of the arrangement correspond to covectors of the oriented matroid with all the coordinates different from zero. From a well known result of Stanley, the digraph G is acyclic if and only if there is one (and exactly one) symbolic covector with all the coordinates equal to $+$. The corresponding n-cell is called the *fundamental* cell of the arrangement $\mathcal{A}_G$. The hyperplanes bounding the positive cell correspond to the flippable edges of the graph. There is a polytope in $\mathbb{R}^n$ whose lattice of faces is the same as the lattice of faces of the positive n-cell. The polar of this polytope is known as the *Las Vergnas polytope* determined by the acyclic digraph G (i.e., the oriented matroid attached to the acyclic digraph). Note that the bounding hyperplanes correspond to faces of dimension 0 of this polytope. So Proposition 2.11 can be translated in a more geometric language into:

Proposition 4.1. *Let $G = (V, E)$ be a 2-connected non-Hamiltonian graph $G = (V, E)$. There is an acyclic orientation of the edges such that all the edges are over facets of the associated Las Vergnas matroid polytope.*

It is an easy exercise to translate also the other results of Section 2 in geometric theorems. One of the attractions of the theory of oriented matroids is certainly the capacity to encode different points of view in a small axiomatic. In particular there is a natural notion of oriented matroid duality easily applicable. This notion encodes the duality between circuits and oriented cutsets (cocircuits) in digraphs. The geometric interpretation of duality is not so elementary as in our cases and the interested reader can consult [1] for details. The results of Section 3 can be obtained by matroid duality from the results of Section 2, we leave the details to the reader.

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Raul Cordovil
Departamento de Matemática,
Instituto Superior Técnico
Av. Rovisco Pais - 1049-001 Lisboa - Portugal
E-mail address: cordovil@math.ist.utl.pt
URL: http://www.math.ist.utl.pt/~rcordov

David Forge
Laboratoire de Recherche en Informatique UMR 8623
Batiment 490 Université Paris Sud
91405 Orsay Cedex - France
E-mail address: forge@lri.fr
URL: http://www.lri.fr/~forge