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On reconstructing arrangements from their sets of simplices

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Abstract

Let $G_{\mathscr{H}}(S,H)$ be the bipartite graph with partition sets S and H, the set of simplices and hyperplanes of $\mathscr{H}$, where simplex $s \in S$ is *adjacent* to hyperplane $h \in H$ if one facet of s lies on h. In this paper, we give a complete characterization of $G_{\mathscr{H}}(S,H)$ when $\mathscr{H}$ is a Γ*-arrangement*. We also study $G_{\mathscr{H}}(S,H)$ when $\mathscr{H}$ is a pseudoline arrangement.

Keywords: Arrangements; Hyperplanes; Oriented matroids; Union

1. Introduction

An *Euclidean* (respectively *projective*) d*-arrangement of* n *hyperplanes* $\mathscr{H}(d,n)$ is a finite collection of hyperplanes in the Euclidean space R^d (resp. the real projective space P^d) such that no point belongs to every hyperplane of $\mathscr{H}(d,n)$. Any arrangement $\mathscr{H}(d,n)$ decomposes R^d (resp. P^d) into a d-dimensional cell complex K. We may call *cells* of $\mathscr{H}(d,n)$ the d-cells of K, and *facets* of $\mathscr{H}(d,n)$ the $(d-1)$-cells of K. A simplicial d-cell (that is, a cell with exacly $d+1$ facets) in $\mathscr{H}(d,n)$ is called *simplex*. The simplices form the simplest possible cells of full dimension, and are, therefore, of basic interest. Moreover, simplices in simple arrangements correspond to possibilities of local deformations called *switchings* (see [1, 7]).

Let $G_{\mathscr{H}}(S,H)$ be the bipartite graph with partition sets S and H, the set of simplices and hyperplanes of $\mathscr{H}$, where simplex $s \in S$ is *adjacent* to hyperplane $h \in H$ if one facet

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of s lies on h (the reader can refer to [2] for details on the graph theory definitions). We say that G is the *simplex* graph of $\mathscr{H}$.

We devote special attention to *Γ-arrangements* which are constructed from *Γ-oriented matroids* (naturally obtained as a union of uniform oriented matroids of rank 1). The class of Γ-oriented matroids contains as a very special case the well-known *alternating oriented matroid.*

In this paper, we shall give a complete characterization of $G_{\mathscr{H}}(S,H)$ when $\mathscr{H}$ is a Γ-arrangement. We will do so by using the combinatorial interpretation of arrangements in terms of oriented matroids. Indeed, Folkman and Lawrence [3] have shown that there exists a *natural* bijection between the isomorphism classes of arrangements of n hyperplanes of P^{r-1} and the reorientations classes of oriented matroids (without loops and parallel elements) of rank r on n elements.

In Section 2, we study and give general results concerning Γ-oriented matroids needed for the rest of the paper.

In Section 3, we give a method to determine all the simplices of a Γ-arrangement by using the information of its corresponding Γ-oriented matroid. We will be able to show that any Γ-arrangement can be reconstructed by using only its set of simplices. Moreover, given a r-regular bipartite graph $G=G(S,H)$ with $|S|=|H|=n$ we shall show that G is the simplex graph of a Γ-arrangement if, and only if, G admits a special labeling (called *Γ-labeling*). We strongly believe that the corresponding Γ-arrangement is unique.

Conjecture 1.1. Let $G=G(S,H)$ be a r-regular bipartite graph with $|S|=|H|=n$ having a Γ-labeling. Then, there exists a unique Γ-arrangement (up to permutation) having G as a simplex graph (notice that G may be the simplex graph of another nonΓ-arrangement). In other words, there are no two nonisomorphic Γ-arrangements $\mathscr{H}_1$ and $\mathscr{H}_2$ such that the graphs $G_{\mathscr{H}_1}$ and $G_{\mathscr{H}_2}$ are isomorphic.

The above conjecture leads us to the following problem.

Reconstruction Problem (RP): If for two arrangements $\mathscr{H}$ and $\mathscr{H}'$ the graphs $G_{\mathscr{H}}(S,H)$ and $G_{\mathscr{H}'}(S',H')$ are isomorphic, must $\mathscr{H}$ and $\mathscr{H}'$ be isomorphic? In other words, is it true that the knowledge of the set of simplices of $\mathscr{H}$ is sufficient to define uniquely $\mathscr{H}$? (see [1, exercise 6.11(b) for the 2-dimensional case]).

In Section 4, we answer affirmatively (RP) for all simple arrangements of pseudolines $\mathscr{A}_n$ with $n \leqslant 7$ and for any arrangement of $1 \leqslant n \leqslant 6$ pseudolines. We also give a recursive construction to answer (RP) negatively for arrangements with $n \geqslant 10$ pseudolines.

2. Analyzing Γ-oriented matroids

A *Γ-oriented matroid $\mathscr{M}$* of rank r on the totally ordered set $E=\{1,\dots,n\}$, $r \leqslant n$, is a uniform oriented matroid obtained by the union of r uniform oriented matroids $\mathscr{M}_1,\dots,\mathscr{M}_r$ of rank 1 on $(E,<)$ (see [7,5]).

We can also define the Γ-oriented matroids via their chirotope χ. Indeed, the chirotope χ corresponds to some Γ-oriented matroid, $\mathcal{M}_A$, if, and only if, there exists a matrix $A=(a_{i,j})$ $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$ with entries from $\{+1,-1\}$ (where the ith row corresponds to the chirotope of the oriented matroid $\mathcal{M}_i$) such that

$$\chi(B) = \prod_{i=1}^{r} a_{i,j_i} \qquad (*)$$

where B is an ordered r-tuple $j_1 \leqslant \cdots \leqslant j_r$ elements of E.

Remarks. Let $A=(a_{i,j})$ $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$ be a matrix with entries from $\{+1,-1\}$ and $\mathcal{M}_A$ its corresponding Γ-oriented matroid.

(i) The coefficients $a_{i,j}$ with $i > j$ or $j-n > i-r$ do not play any role in the definition of $\mathcal{M}_A$ (since they never appear in $(*)$). So, we may give them any arbitrary value from $\{+1,-1\}$ or ignore them completely.

(ii) An opposite chirotope $-\chi$ is obtained by reversing the sign of all the coefficients of a line of A.

(iii) The oriented matroid ${}_{\bar{c}}\mathcal{M}_A$ is obtained by reversing the sign of all the coefficients of column c in A.

(iv) If $a_{i,j}=1$ for all $i > j$ or $j-n > i-r$ then $\mathcal{M}_A$ is the *alternating oriented matroid.*

Let $A=(a_{i,j})$ $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$ be a matrix with entries from $\{+1,-1\}$. Let $A'=(a'_{i,j})$ $0 \leqslant i \leqslant r+1$, $1 \leqslant j \leqslant n$ be the matrix where $a'_{i,j}=a_{i,j}$ for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$ and $a'_{0,j}=a'_{r+1,j}=0$ for $1 \leqslant j \leqslant n$. We shall construct the *Top Travel* (TT) (and the *Bottom Travel* (BT)) on the entries of A', formed by horizontal and vertical movements according to the following procedure.

Procedure A

(1) TT (BT) starts at $a_{1,1}$ (at $a_{r,n}$)
(2) when TT (BT) arrives at $a_{i,j}$
If $a_{i,j}=a_{i,j+1}$ $(a_{i,j}=a_{i,j-1})$ **then**
    TT goes to $a_{i,j+1}$ (horizontally)
    (BT goes to $a_{i,j-1}$ (horizontally))
else
    TT goes to $a_{i+1,j+1}$ (horizontally and vertically)
    (BT goes to $a_{i-1,j-1}$ (horizontally and vertically))
(3) TT (BT) stops when it arrives at line $r+1$ (0) or at column n (1).

We write $\mathrm{TT}=(a_{1,1},\ldots,a_{1,t_1},a_{2,t_1},\ldots,a_{2,t_2},\ldots,a_{x,t_{x-1}},\ldots,a_{x,t_x})$, $1 \leqslant x \leqslant r+1$, where $a_{l,t_{l-1}},\ldots,a_{l,t_l}$ are the entries in line l of A' along TT with $1 \leqslant l \leqslant x \leqslant r+1$. We may also use the shorter notation

$\mathrm{TT}=[t_0,\ldots,t_1][t_1,\ldots,t_2]\cdots[t_{x-1},\ldots,t_x]$ where $[t_{l-1},\ldots,t_l]$ denote the entries $[a_{l,t_{l-1}},\ldots,a_{l,t_l}]$ with $1 \leqslant l \leqslant x \leqslant r+1$ and $t_0=1$ (similar for BT).

$$
\begin{array}{ccccccccccccc}
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
TT & + & - & - & + & - & + & + & + & - & - & + & \\
 & + & - & - & + & - & - & + & - & + & + & - & \\
 & + & + & - & + & + & + & - & + & + & + & + & \\
 & - & + & + & - & - & - & + & - & + & - & + & \\
 & - & - & + & - & + & + & - & + & + & + & + & \\
 & + & - & - & - & + & - & - & + & - & + & - & BT \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}
$$

Fig. 1.

Example 1. Let E be the following matrix:

$$
E = \begin{pmatrix}
+ & - & - & + & - & + & + & + & - & - & + \\
+ & - & - & + & - & - & + & - & + & + & - \\
+ & + & - & + & + & + & - & + & + & + & + \\
- & + & + & - & - & - & + & - & + & - & + \\
- & - & + & - & + & + & - & + & + & + & + \\
+ & - & - & - & + & - & - & + & - & + & -
\end{pmatrix}.
$$

The Top and Bottom Travels in E' are shown in Fig. 1.

Recall that an oriented matroid $\mathscr{M} = (E, \mathscr{C})$ is *acyclic* if it does not contain positive circuits (otherwise, $\mathscr{M}$ is called *cyclic*). We say that an element $e \in E$ of uniform oriented acyclic matroid is *interior* if there exists a signed circuit $C = (C^+, C^-)$ with $C^- = \{e\}$. It is equivalent to define the interior points as the elements whose reorientation gives a cyclic matroid.

Proposition 2.1. *Let* $A = (a_{i,j})$ $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$ *be a matrix with entries from* $\{+1, -1\}$ *and let* TT *and* BT *be the Top and Bottom Travels constructed on* A', *respectively. Then the following conditions are equivalent*

(a) $\mathscr{M}_A$ is cyclic,
(b) TT arrives at line $r + 1$,
(c) BT arrives at line 0.

Proof. We show only the equivalence between (a) and (b) (the equivalence between (a) and (c) is analogous). Let $C = (j_1, \ldots, j_{r+1})$, $j_1 \leqslant \cdots \leqslant j_{r+1}$ be a circuit of $\mathscr{M}_A$. Given a sign of element $j_1 = +1$ (or -1) and the chirotope χ of $\mathscr{M}_A$, we find the sign, $C(j_l)$, of element j_l, $2 \leqslant l \leqslant r + 1$ as follows: $C(j_{i+1}) = -C(j_i)\chi(C \setminus j_i)\chi(C \setminus j_{i+1})$ for $i = 1, \ldots, r$. Hence, by $(*)$ we have $C(j_{i+1}) = -C(j_i)a_{i,j_i}a_{i,j_{i+1}}$.

Without loss of generality, assume that the entry $a_{1,1} = +1$. Suppose that TT arrives at line $r+1$, then, there exist exactly r columns $j_1, \ldots, j_r$ in which TT makes a vertical movement. So, $a_{i,j_{i+1}} = -a_{i,j_i}$ for $i = 1, \ldots, r-1$ and hence, by the above formula, $C(1, j_1, \ldots, j_r)$ is a positive circuit.

Now, suppose that $\mathscr{M}_A$ is cyclic, then, there exists a positive circuit $C(j_0, j_1, \ldots, j_r)$ with $a_{i,j_{i-1}} = -a_{i,j_i}$ for $i = 1, \ldots, r$. Hence, TT must arrive at line $i + 1$ at column l for some $l \leqslant j_i$ and the result follows. In fact, if $\mathscr{M}$ is cyclic then the set of columns where TT (BT) makes a vertical movement defines the first (the last) positive circuit in the lexicographic (colexicographic) order. □

We denote by ${}_{\bar{k}}A$, the matrix obtained by reversing all the sign of column k and ${}_{\bar{k}}$TT and ${}_{\bar{k}}$BT the corresponding travels of ${}_{\bar{k}}A'$. We say that TT and BT are *parallel* at column k with $2 \leqslant k \leqslant n-1$ in A if $\mathrm{TT}=(a_{1,1},\ldots,a_{i,k-1},a_{i,k},a_{i,k+1},\ldots)$ and either

$\mathrm{BT}=(a_{r,n},\ldots,a_{i,k+1},a_{i,k},a_{i,k-1},\ldots)$ or

$\mathrm{BT}=(a_{r,n},\ldots,a_{i+1,k+1},a_{i+1,k},a_{i+1,k-1},\ldots)$, $1 \leqslant i \leqslant r$.

Lemma 2.2. *Let $A=(a_{i,j})$ $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n$ be a matrix with entries from $\{+1,-1\}$ such that $\mathscr{M}_A$ is acyclic and let* TT *and* BT *be the travels in A'. Then k is an interior element of $\mathscr{M}_A$, if and only if,*

(a) $\mathrm{BT}=(a_{r,n},\ldots,a_{1,2},a_{1,1})$ *for* $k=1$,

(b) $\mathrm{TT}=(a_{1,1},\ldots,a_{r,n-1},a_{r,n})$ *for* $k=n$,

(c) TT *and* BT *are parallel at k for* $2 \leqslant k \leqslant n-1$.

Proof. By Proposition 2.1, k is an interior element of $\mathscr{M}_A$, if and only if, TT (or BT) does not arrive at line $r+1$ (or 0) in A' but ${}_{\bar{k}}$TT (or ${}_{\bar{k}}$BT) does in ${}_{\bar{k}}A'$. Since $\mathscr{M}_A$ is acyclic then TT and BT do not arrive at lines $r+1$ and 0 in A' respectively. We shall show, in each case, that either ${}_{\bar{k}}$TT (or ${}_{\bar{k}}$BT) arrives at line $r+1$ (or 0) in ${}_{\bar{k}}A'$.

Part (a) ((b)) clearly follows, since ${}_{\bar{1}}$BT (${}_{\bar{n}}$TT) arrives at line 0 ($r+1$) in ${}_{\bar{1}}A'$ (${}_{\bar{n}}A'$), if and only if, $\mathrm{BT}=(a_{r,n},\ldots,a_{1,2},a_{1,1})$ ($\mathrm{TT}=(a_{1,1},\ldots,a_{r,n-1},a_{r,n})$).

For part (c), given $\mathrm{TT}=[t_0,\ldots,t_1][t_1,\ldots,t_2]\ldots[t_{l-1},\ldots,t_l]$, with $1 \leqslant l \leqslant r$ we partition the entries of A' (not lying in TT) into sets P^1_{TT} and PP^2_{TT} where

$$P^1_{\mathrm{TT}}=\{a_{i,j} \mid j < t_{i-1} \text{ for each } i=1,\ldots,l\} \quad \text{and}$$

$$P^2_{\mathrm{TT}}=\{a_{i,j} \mid j > t_i \text{ for each } i=1,\ldots,l\}.$$

Let $\mathrm{BT}_{a_{i,k}}$ be the travel in A', starting at $a_{i,k}$ and formed according to the above rules for BT. Consider the travel ${}_{\bar{k}}\mathrm{BT}_{a_{i,k}}$ in ${}_{\bar{k}}A'$, we claim that (1) if either $a_{i,k} \in P^2_{\mathrm{TT}}$ or $a_{i,k} \in [t_{i-1},\ldots,t_i]$ with $k > t_{i-1}$ for some $1 \leqslant i \leqslant r$ then all elements of ${}_{\bar{k}}\mathrm{BT}_{a_{i,k}}$ belong to either P^2_{TT} or TT and (2) ${}_{\bar{k}}\mathrm{BT}_{a_{i,k}}$ arrives at line 0. Indeed, part (1) can be easily verified and since ${}_{\bar{k}}\mathrm{BT}_{a_{i,j}}$ arrives at $a_{1,m}$ for some $t_1 \leqslant m$ (by part (1)) and $a_{1,t_1} = -a_{1,t_1-1}$ then part (2) follows.

Let us suppose that $\mathrm{TT}=(a_{1,1},\ldots,a_{i,k-1},a_{i,k},a_{i,k+1},\ldots)$ and either $\mathrm{BT}=(a_{r,n},\ldots,a_{j,k+1},a_{j,k},a_{j,k-1},\ldots)$ or $\mathrm{BT}=(a_{r,n},\ldots,a_{j+1,k+1},a_{j+1,k},a_{j+1,k-1},\ldots)$ with $1 \leqslant i \leqslant j \leqslant r$. Now, clearly, ${}_{\bar{k}}$BT can be seen as BT from $a_{r,n}$ until $a_{i,k}$ is followed by $a_{j,k-1},a_{j-1,k-1}$ and ${}_{\bar{k}}\mathrm{BT}_{a_{j-1,k-1}}$. So, after reorienting column k, we have that $a_{j-1,k-1} \in PP^2_{\mathrm{TT}}$ and $a_{j-1,k-1} \in [t_{i-1},\ldots,t_i]$ with $k-1 > t_{i-1}$, if and only if, $j=i$ and $j=i+1$ respectively. Hence, by the above claim, k is an interior element, if and only if, TT and BT are parallel at column k for $2 \leqslant k \leqslant n-1$. □

Example 1 (*continued*). By Lemma 2.2 and Fig. 1, it is easy to check that 5 and 9 are the interior elements of the oriented matroid $\mathscr{M}_E$.

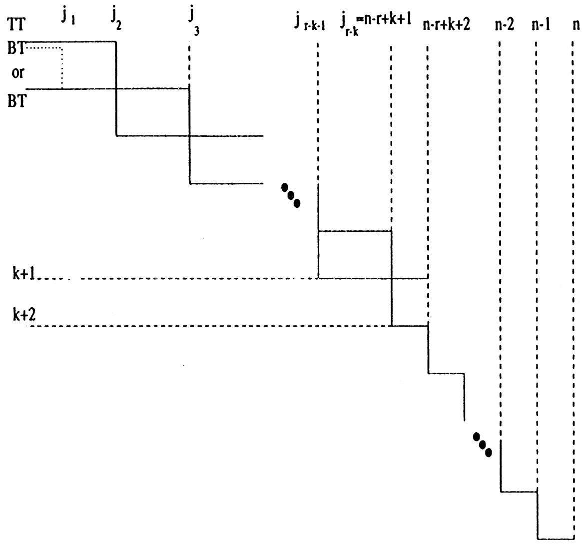

Fig. 2.

3. The simplices in Γ-arrangements

Let $\mathscr{H}=\{h_i\}_{1\leqslant i\leqslant n}$ be an arrangement of hyperplanes and $\mathscr{M}_{\mathscr{H}}$ its corresponding oriented matroid. We denote by e_i the element of $\mathscr{M}_{\mathscr{H}}$ corresponding to hyperplane h_i. It is well known [3] that an acyclic reorientation of $\mathscr{M}_{\mathscr{H}}$ having $\{e_{i_1},\dots,e_{i_l}\}$, $l\leqslant n$ as interior elements corresponds to a cell in $\mathscr{H}$ which is bounded by hyperplanes $h_j\in\{h_{i_1},\dots,h_{i_l}\}$. Thus, given a matrix $A=(a_{i,j})$ $1\leqslant i\leqslant r$, $1\leqslant j\leqslant n$ with entries from $\{+1,-1\}$, we shall give the complete list of the n simplices of the Γ-arrangement, obtained from $\mathscr{M}_A$, by finding precisely n reorientations of A each having exactly $n-r$ interior elements. In order to do this, we need the following definitions.

Let $1\leqslant j_1<\cdots<j_k\leqslant n$ be a k-tuple with $0\leqslant k<r-1$ and $j_{r-k}\leqslant n-k$. We say that TT and BT are *parallel* in A if TT makes vertical movements at columns $j_2,\dots,j_{r-k}$ arriving at element $a_{k+1,n-r+k+1}$ and BT makes vertical movements at columns $n-r+k+2,\dots,n-1$ arriving at element $a_{k+2,n-r+k+2}$ and from here BT makes vertical movements at each column j_i, $2\leqslant i\leqslant k$ and maybe at a column j_1 between columns 1 and j_2 (if BT does not make this last movement we let $j_1=1$) (see Fig. 2).

Let $X_i=\{i+1\leqslant j\leqslant n-r+i\mid a_{i,j}=a_{i+1,j}\}\cup\{i,n-r+i+1\}$ and $\bar{X}_i=\{i+1\leqslant j\leqslant n-r+i\mid a_{i,j}\neq a_{i+1,j}\}\cup\{i,n-r+i+1\}$ for each $i=1,\dots,r-1$. Let $G_i(A)$ be the graph which is the union of the two paths from i to $n-r+i+1$, the first going through the vertices in the set X_i in order, and the second going through those in $\bar{X}_i$ in order.

Observations. (a) Element i (resp. $n-r+i+1$) is taken in both X_i and $\bar{X}_i$ since the value of $a_{i+1,i}$ (resp. $a_{n-r+i,n-r+i+1}$) can be arbitrarily equal to either $+1$ or -1.

(b) j_i and j_{i+1} are adjacent in $G_i(A)$, if and only if, $a_{i,j}a_{i+1,j}=a_{i,j+1}a_{i+1,j+1}$ and $a_{i,k}a_{i+1,k}=-a_{i,j}a_{i+1,j}$ for all $j_i<k<j_{i+1}$.

Proposition 3.1. *Let* $A=(a_{i,j})$ $1\leqslant i\leqslant r$, $1\leqslant j\leqslant n$ *be a matrix with entries from* $\{+1,-1\}$. *Then* TT *and* BT *are parallel in* A, *if and only if*, $\mathcal{M}_A$ *has exactly* r *noninterior elements.*

Proof. It is clear that if TT and BT are parallel then A contains exactly $n-r$ interior elements. Indeed, it is enough to verify that the only noninterior elements are exactly the columns where BT makes a vertical movement (to arrive at $a_{2,l}$) plus an extra noninterior element between columns 1 and $l-1$.

Now, suppose that A contains exactly r noninterior elements. Assume that TT makes a vertical movement at columns $j_1<\cdots<j_k$ with $j_{r-k}\leqslant n-k$ arriving at element $a_{k+1,n-r+k+1}$. For any BT there are at least r noninterior points (the $r-k+1$ elements after column $n-r+k+2$ plus k elements corresponding to the vertical movements of TT plus one element lying between 1 and j_1-1). Hence, in order to avoid an extra noninterior elements (in particular for element $n-r+k+1$) BT must make vertical movements at columns $n-1,\dots,n-r+k+2$ arriving at $a_{k+2,n-r+k+2}$ then BT must go horizontally until a_{k+2,j_k} and make a vertical movement, and so on. Therefore, TT and BT are parallel. $\square$

Theorem 3.2. *Let* $A=(a_{i,j})$ $1\leqslant i\leqslant r$, $1\leqslant j\leqslant n$ *be a matrix with entries from* $\{+1,-1\}$. *Then the hyperplanes* $j_1,\dots,j_r$ *form a simplex in the arrangement obtained from* $\mathcal{M}_A$, *if and only if*, j_i *and* j_{i+1} *are adjacent in* $G_i(A)$ *for each* $i=1,\dots,r-1$.

Proof. Consider a reorientation of A, obtaining matrix $\bar{A}$, such that TT (resp. BT) makes vertical movements at columns $j_2,\dots,j_k\leqslant n-r+k$ (resp. $n-1,\dots,n-r+k+1$). By Proposition 3.1, it is enough to show that TT and BT are parallel in $\bar{A}$ if, and only if, j_i and j_{i+1} are adjacent in $G_i(A)$ for each $i=1,\dots,r-1$.

Remark. $\bar{a}_{i,j}\bar{a}_{i+1,j}=a_{i,j}a_{i+1,j}$ for all $1\leqslant i\leqslant r-1$ and $1\leqslant j\leqslant n$.

Suppose that TT and BT are parallel in $\bar{A}$. We have four cases.

(A) $i<j_i<j_{i+1}<n-r+i+1$: Since TT and BT are parallel then $\bar{a}_{i,j_i}=\bar{a}_{i,j_i+1}=\cdots=-\bar{a}_{i,j_{i+1}}$ and $-\bar{a}_{i+1,j_i}=\bar{a}_{i+1,j_i+1}=\cdots=\bar{a}_{i+1,j_{i+1}}$. Hence, $-\bar{a}_{i,j_i}\bar{a}_{i+1,j_i}=\bar{a}_{i,j_i+1}\bar{a}_{i+1,j_i+1}=\cdots=-\bar{a}_{i,j_{i+1}}\bar{a}_{i+1,j_{i+1}}$.

Then, by the above remark, j_i and j_{i+1} are adjacent in $G_i(A)$.

(B) $i<j_i<j_{i+1}=n-r+i+1$: Since TT and BT are parallel then $\bar{a}_{i,j_i}=\bar{a}_{i,j_i+1}=\cdots=\bar{a}_{i,j_{i+1}-1}$ and $-\bar{a}_{i+1,j_i}=\bar{a}_{i+1,j_i+1}=\cdots=\bar{a}_{i+1,j_{i+1}-1}$.

Hence, $-\bar{a}_{i,j_i}\bar{a}_{i+1,j_i}=\bar{a}_{i,j_i+1}\bar{a}_{i+1,j_i+1}=\cdots=\bar{a}_{i,j_{i+1}-1}\bar{a}_{i+1,j_{i+1}-1}$.

Thus, $j_i+1,\dots,j_{i+1}-1$ are all in the same set X_i or $\bar{X}_i$ and j_i is the largest element in the other; and, therefore, j_i is adjacent to $j_{i+1}=n-r+i+1$ in $G_i(A)$.

(C) $i = j_i < j_{i+1} < n - r + i + 1$: By similar argument as in case (B), we have that $j_i + 1, \ldots, j_{i+1} - 1$ are all in the same set X_i or $\bar{X}_i$ and j_{i+1} is the smallest element in the other; and, therefore, j_{i+1} is adjacent to $j_i = i$ in $G_i(A)$.

(D) $i = j_i < j_{i+1} = n - r + i + 1$: Again, by similar argument as in case (B), we have that $j_i + 1, \ldots, j_{i+1} - 1$ are all in the same set X_i or $\bar{X}_i$ and, therefore, $j_i = i$ and $j_{i+1} = n - r + i + 1$ are adjacent in $G_i(A)$.

Now, suppose that j_i and j_{i+1} are adjacent in $G_i(A)$ for each $i = 1, \ldots, r - 1$. Since j_i and j_{i+1} are adjacent in $G_i(A)$, $i = 1, \ldots, r - 1$ then, by observation (b), TT and BT are parallel in $\bar{A}$. □

Roudneff and Sturmfels [7] have given an inductive proof to show that Γ-arrangements with n hyperplanes have exactly n simplices, see [8] for the special case of *cyclic arrangements* (obtained from the alternating oriented matroid). We give a constructive proof of this property.

Theorem 3.3. *Let* $A = (a_{i,j})$ $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$ *be a matrix with entries from* $\{+1, -1\}$. *Then the arrangement obtained from* $\mathscr{M}_A$ *contains exactly* n *simplices.*

Proof. Consider a reorientation of A, obtaining matrix $\bar{A}$, having exactly r noninterior elements, say $j_1 < \cdots < j_r$.

(a) If $j_1 = 1$ then j_1 is adjacent to both elements 2 and $k_1 > 2$ in $G_1(A)$. if $j_2 = k_1$ then, by Theorem 3.2, there exists a unique element j_{i+1} adjacent to j_i in $G_i(A)$ for each $i = 2, \ldots, r-1$ (and thus, obtaining a simplex, say S_1). If $j_2 = 2$ then j_2 is adjacent to both elements 3 and $k_2 > 3$ in $G_2(A)$. Now, if $j_3 = k_2$ then, by Theorem 3.2, there exists a unique element j_{i+1} adjacent to j_i in $G_i(A)$ for each $i = 3, \ldots, r - 1$ (and thus, obtaining a simplex, say S_2). If $j_3 = 3$ then j_3 is adjacent to both elements 4 and $k_3 > 4$ in $G_3(A)$, and so on. Hence, we obtain on this way r simplices, say $S_1 = \{1, k_1, \ldots\}$, $S_2 = \{1, 2, k_2, \ldots\}, \ldots, S_r = \{1, \ldots, r\}$.

(b) if $1 < j_1 \leqslant n - r + 1$ then, by Theorem 3.2, there exists a unique element j_{i+1} adjacent to j_i in $G_i(A)$ for each $i = 1, \ldots, r-1$. Hence, we obtain another $n-r$ simplices, say $S_{r+1} = \{2, \ldots\}, S_{r+2} = \{3, \ldots\}, \ldots, S_n = \{n - r + 1, \ldots, n\}$. □

Example 2. Consider the matrix E of Example 1. Then the 11 simplices of the arrangement obtained from $\mathscr{M}_E$ are

$$S_1 = \{1,2,3,4,5,6\}, \quad S_2 = \{1,2,3,4,5,7\}, \quad S_3 = \{1,2,3,4,9,11\},$$

$$S_4 = \{1,2,3,9,10,11\}, \quad S_5 = \{1,2,5,6,7,8\}, \quad S_6 = \{1,6,7,8,10,11\},$$

$$S_7 = \{2,3,4,5,6,9\}, \quad S_8 = \{3,4,8,9,10,11\}, \quad S_9 = \{4,5,6,7,8,10\},$$

$$S_{10} = \{5,7,8,9,10,11\}, \quad S_{11} = \{6,7,8,9,10,11\}.$$

Proposition 3.4. *Let* $A = (a_{i,j})$ $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$ *be a matrix with entries from* $\{+1, -1\}$. *Then there exists a unique simplex* S_l, $1 \leqslant l \leqslant n$ (*with elements ordered* *lexicographically) in the arrangement obtained from $\mathcal{M}_A$ having element j placed in its ith position for each $1 \leqslant i \leqslant r$ and each $i+1 \leqslant j \leqslant n-r+i-1$.*

Proof. We claim that there exists a unique set of r elements $k_1, \ldots, k_r$ such that $j = k_i$. Indeed, by Theorem 3.2, there exists a unique element k_{i+1} (resp. k_{i-1}) adjacent to k_i in $G_i(A)$ if $i < r$, otherwise, we ignore this part (resp. $G_{i-1}(A)$ if $i > 1$, otherwise, we ignore this part). Now, again by Theorem 3.2, there exists a unique element k_{i+2} (resp. k_{i-2}) adjacent to k_{i+1} in $G_{i+1}(A)$ if $i+1 < r$ otherwise, we ignore this part (resp. adjacent to k_{i-1} in $G_{i-2}(A)$ if $i-1 > 1$ otherwise, we ignore this part) and so on. $\square$

We invite the reader to check the property of Proposition 3.4 in the set of simplices given in Example 2. We are now able to show that, given the set of simplices of a Γ-arrangement $\mathcal{H}$, one can construct (only using the information of its simplices) a unique matrix A with entries from $\{-1, +1\}$ such that $\mathcal{M}_A$ gives rises $\mathcal{H}$.

Theorem 3.5. *Let $A = (a_{i,j})$ $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$ be a matrix with entries from $\{+1, -1\}$. Then the arrangement obtained from $\mathcal{M}_A$ can be uniquely reconstructed from its set of simplices.*

Proof. We may find the sign of each $a_{i,j}$ with $i \leqslant j \leqslant n+i-r$. Without lose of generality, we assume that $a_{1,j} = +1$ for all $1 \leqslant j \leqslant n$, $a_{i,i} = +1$ for all $1 \leqslant i \leqslant r$ and $a_{i,j} = +1$ for all $1 \leqslant i \leqslant r$ and $j = n+i-r$. This can be done by an inversion of signs of the appropriate columns and rows of A. We first inverse the signs of the appropriate columns (to have $+1$'s in the first row) then we inverse the signs of the appropriate rows (to have $+1$'s in the first diagonal) and finally we inverse again the signs of the appropriate columns between $n-r+2$ and n (to have $+1$'s in the second diagonal). Notice that we end with a matrix $\hat{A}$ (said to be in the *standard form*) such that $\mathcal{M}_A$ and $\mathcal{M}_{\hat{A}}$ are in the same orientation class. Thus, $\mathcal{M}_A$ and $\mathcal{M}_{\hat{A}}$ give rise to the same arrangement of hyperplanes.

We find the signs recursively (from top to bottom and from left to right). Suppose that we want to find the sign of $a_{i,j}$ and that we already know the signs of $a_{i-1,j}$, $a_{i-1,j-1}$ and $a_{i,j-1}$. By Proposition 3.4, there exists a unique simplex $S = \{k_1, \ldots, k_r\}$ with $k_i = j$ and by Theorem 3.2, $j-1$ belongs to S, if and only if, $j-1$ and j are adjacent in $G_{i-1}(A)$. Hence, if $j-1$ belongs (resp. does not belong) to S then $a_{i,j} = a_{i-1,j}a_{i-1,j-1}a_{i,j-1}$ (resp. $a_{i,j} = -a_{i-1,j}a_{i-1,j-1}a_{i,j-1}$). $\square$

Observation 1. Let S_i^j be the simplex containing element j placed in its ith position. Then, by the above theorem,

(a) $j-1 \in S_j^i$ if, and only if, $a_{i,j} = a_{i-1,j}a_{i-1,j-1}a_{i,j-1}$, if and only if, $j \in S_{j-1}^{i-1} = S_j^i$ and

(b) $j-1 \in S_j^i$ if, and only if, $a_{i,j} = -a_{i-1,j}a_{i-1,j-1}a_{i,j-1}$, if and only if, $j \in S_{j-1}^{i-1} \neq S_j^i$.

Finally, we shall see that any simplex $S = \{k_1, \ldots, k_r\}$ (different form $\{1, \ldots, r\}$ and

$\{n-r+1,\dots,n\}$) is used to construct the matrix as in Theorem 3.5. Let i be the smallest integer such that $i \neq k_i$. So,

(a) if $i=1$ then $a_{1,k_1}=a_{2,k_1}a_{2,k_1+1}a_{1,k_1+1}$, if and only if, $k_2=k_1+1$, if and only if, $k_1 \in S^2_{k_1+1}$ if and only if $k_1+1 \in S$ and

(b) if $i \neq 1$ then $a_{i,k_i}=a_{i-1,k_i}a_{i-1,k_i-1}a_{i,k_i-1}$, if and only if, $k_i-1 \in S$ (by definition of i we always have that $k_i-1 \in S$ and therefore $a_{i,k_i}=-a_{i-1,k_i}a_{i-1,k_i-1}a_{i,k_i-1}$).

Example 2 (*continued*). We may illustrate Theorem 3.5 by finding the sign of elements $\hat{e}_{3,6}$ and $\hat{e}_{4,5}$ in $\hat{E}$ (the matrix in the standard form, obtained from matrix E of Example 1) given below.

$$\hat{E}=\begin{pmatrix} + & + & + & + & + & + & + & + & + & + & + \\ + & + & + & + & + & - & + & - & - & - & - \\ + & - & + & + & - & + & - & + & - & - & + \\ + & + & + & + & - & + & - & + & + & - & - \\ + & - & + & + & + & - & + & - & + & + & - \\ - & - & - & + & + & + & + & - & - & + & + \end{pmatrix}.$$

We have that the unique simplex containing element 6 (resp. 5) in the 3rd (resp. 4th) place is S_9 (resp. S_5). Since S_9 contains element 5 (resp. S_5 does not contain element 4) then $\hat{e}_{3,6}=+1$ (resp. $\hat{e}_{4,5}=-1$).

Let $G=G(S,H)$ be a r-regular bipartite graph with $|S|=|H|=n$. We say that G has a *Γ-labeling*, if there exists a mapping $f:\{h_1,\dots,h_n\}\to\{1,\dots,n\}$, such that the set of neighbours $N(s_1),\dots,N(s_n)$ of $s_1,\dots,s_n$ (ordered lexicographically) satisfy the following.

(1) If $1 \leqslant i \leqslant r$ then the element i appears exactly one time, in some $N(s_k)$, in position $1,\dots,i-1$ and $r-i+1$ times in position i.

(2) If $r \leqslant i \leqslant n-r+1$ then the element i appears exactly one time, in some $N(s_k)$, in position $1,\dots,r$.

(3) If $n-r+1 \leqslant i \leqslant n$ then the element i appears exactly one time, in some $N(s_k)$, in position $r,\dots,r-(n-i)+1$ and $r-(n-i)$ times in position $r-(n-i)$.

Observation 2. If G has a *Γ-labeling* then this clearly implies that there are two elements of S with neigborhoods $\{1,\dots,r\}$ and $\{n-r+1,\dots,n\}$.

Theorem 3.6. *Let $G=G(S,H)$ be a r-regular bipartite graph with $|S|=|H|=n$. Then G is the simplex graph of a Γ-arrangement, if and only if, G has a Γ-labeling.*

Proof. The necessity follows from Theorem 3.3. The sufficiency follows from Proposition 3.4, Theorem 3.5 and Observations 1 and 2. □

Corollary 3.7. *Let $A=(a_{i,j})$ $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$ be a matrix with entries from $\{+1,-1\}$ and let $\mathscr{H}_A$ be the arrangement obtained from $\mathscr{M}_A$. Then, $G_{\mathscr{H}_A}(S,H)$ is connected.*

Proof. It clearly follows from Proposition 3.4. □

We end this section by discussing a way to prove Conjecture 1.1. Assume that $G(S,H)$ has a Γ-labeling (say, f) and, therefore, $G(S,H)$ is the simplex graph of a Γ-oriented matroid (say, $\mathcal{M}_1$). Now, suppose that there exists a permutation σ, of f such that σ is also a Γ-labeling of $G(S,H)$. Then, we claim that σ gives rises a Γ-oriented matroid $\mathcal{M}_2$ where $\mathcal{M}_2 = \sigma(\mathcal{M}_1)$ (and therefore, the corresponding arrangements would be isomorphic up to a permutation of elements). In other words, we claim the following: let $G(S,H)$ be a bipartite graph having a Γ-labeling (say, f) and, therefore, $G(S,H)$ the simplex graph of a Γ-oriented matroid, say $\mathcal{M}$. Then $\Sigma_1 = \Sigma_2$ where $\Sigma_1 = \{\sigma(f) \mid \sigma(f)$ is a Γ-labeling of $G(S,H)\}$ and $\Sigma_2 = \{\sigma(\mathcal{M}) \mid \sigma(\mathcal{M})$ is a Γ-oriented matroid$\}$.

Unfortunately, we do not know much about the set Σ_2. We know that $\sigma^* \in \Sigma_2$ where σ^* is the symmetric permutation (that is, $\sigma^*(i) = n - i + 1$ where n is the number of elements of $\mathcal{M}$) since $\sigma^*(\mathcal{M})$ can be obtained from matrix a $A^* = (a^*_{i,j})$ where A^* is constructed from the matrix $A = (a_{i,j})$ of $\mathcal{M}$ by $a^*_{i,j} = a_{r-i+1,n-j+1}$.

It can be checked that $\sigma^* \in \Sigma_1$. We illustrate the latter by applying σ^* to the set of simplices of Example 2.

$$\begin{gathered}
S_1^* = \{1,2,3,4,5,6\}, \quad S_2^* = \{1,2,3,4,5,7\}, \quad S_3^* = \{1,2,3,4,8,9\},\\
S_4^* = \{1,2,3,9,10,11\}, \quad S_5^* = \{1,2,4,5,6,11\}, \quad S_6^* = \{1,3,8,9,10,11\},\\
S_7^* = \{2,4,5,6,7,8\}, \quad S_8^* = \{3,6,7,8,9,10\}, \quad S_9^* = \{4,5,6,7,10,11\},\\
S_{10}^* = \{5,7,8,9,10,11\}, \quad S_{11}^* = \{6,7,8,9,10,11\}.
\end{gathered}$$

One can verify that this new set of simplices yields a different Γ-labeling of the corresponding simplex graph. Note that it is not necessary that $S_i = S^*_{\sigma(i)}$.

Finally, we also know that $\sigma', \sigma'' \in \Sigma_2$ where $\sigma'(1) = 2$, $\sigma'(2) = 1$ and $\sigma'(k) = k$ for all $k \neq 1,2$ and $\sigma''(n) = n-1$, $\sigma''(n-1) = n$ and $\sigma''(k) = k$ for all $k \neq n-1, n$ (we leave it to the reader to check this and that $\sigma', \sigma'' \in \Sigma_1$).

4. Arrangements of pseudolines

An *arrangement of pseudolines* is a finite collection $\mathcal{A}$ of $n \geqslant 3$ simple closed curves in the real projective plane P^2 such that every two curves have exactly one point in common at which they cross. In the case where no point on P^2 belongs to more than two lines of $\mathcal{A}$ we say that $\mathcal{A}$ is *simple* (see [4]). A face with three vertices is called a *triangle*.

A simple combinatorial argument shows that $n \leqslant p_3(\mathcal{A}) \leqslant 13n(n-1)$ for a simple arrangement $\mathcal{A}$ with $n \geqslant 10$ pseudolines. An arrangement $\mathcal{A}_n$ is p_3-*maximal* (resp. p_3-*minimal*), if $p_3(\mathcal{A}_n) = 13n(n-1)$ (resp. $p_3(\mathcal{A}_n) = n$).

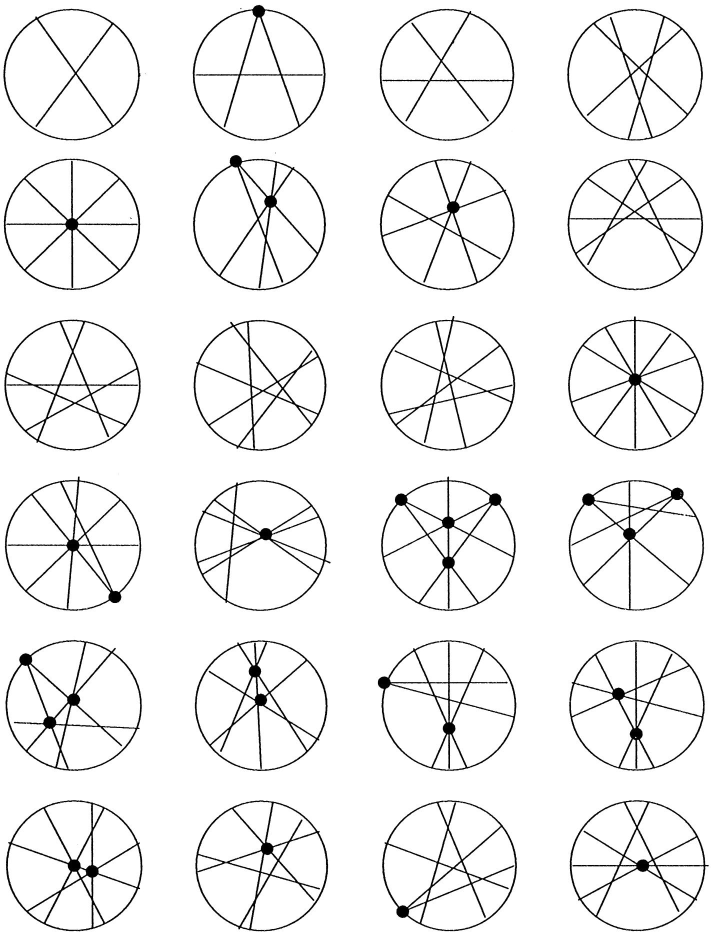

Fig. 3. Vertices belonging to 3 or more lines are indicated by black circles.

Theorem 4.1. *Let $\mathscr{A}_n$ and $\mathscr{A}'_n$ be two non isomorphic arrangements. Then $G_{\mathscr{A}}$ is not isomorphic to $G'_{\mathscr{A}}$ if either $3 \leqslant n \leqslant 6$ or $\mathscr{A}_n$ and $\mathscr{A}'_n$ are simple with $3 \leqslant n \leqslant 7$.*

Proof. Consider all nonisomorphic arrangements $\mathscr{A}_n$ with $3 \leqslant n \leqslant 6$, see Fig. 3 (obtained from [4, p. 5]). Also, consider all nonisomorphic simple arrangements $\mathscr{A}_7$, see Fig. 4 (obtained by using a representative *chirotope* from each of the 11 orientation classes arising from all uniform oriented matroids of rank 3 on 7 elements, see Table 1).

The result follows by checking that the corresponding bipartite graphs of any two arrangements as above, are not isomorphic. $\square$

Lemma 4.2. *There always exist two non isomorphic arrangements $\mathscr{A}_n$ and $\mathscr{A}'_n$ such that $G_{\mathscr{A}_n}(T,L)$ is isomorphic to $G_{\mathscr{A}'_n}(T',L')$ for each $n \geqslant 10$.*

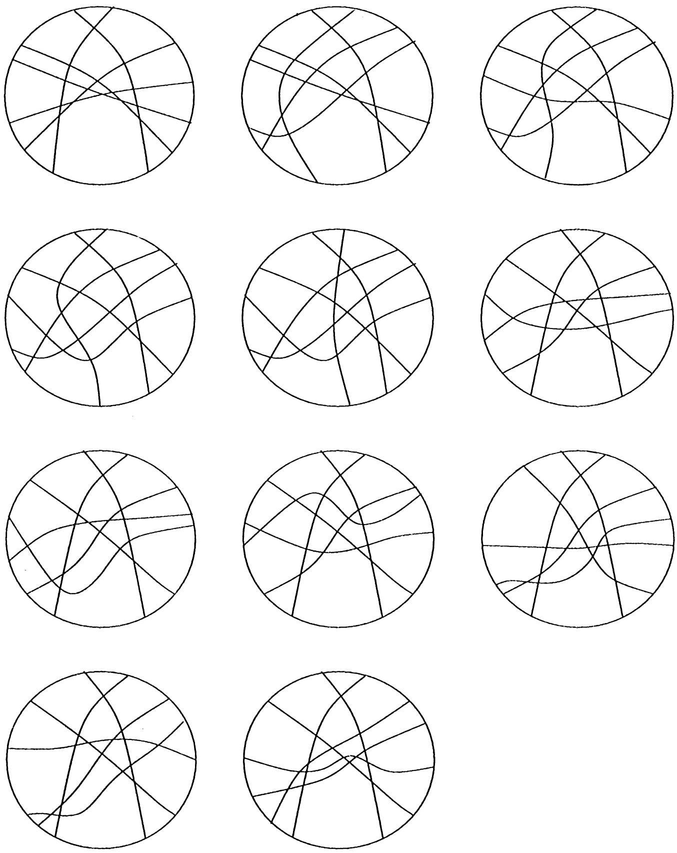

Fig. 4.

Table 1
The chirotopes are lexicographically ordered

+ +
+ −
+ − −
+ + + + + + + + + + + + + + + + + + + − + + + + + + + + + − + + + − +
+ + + + + + + + + + + + + + + + + + + − + + + + + + + + + − + + − − −
+ + + + + + + + + + + + + + + + + + + − + + + + + + + + + − − + + + +
+ + + + + + + + + + + + + + + + + + − − + + + + + + + + + − − + + + +
+ + + + + + + + + + + + + + + + + + − − + + + + + + + + − − + + + + +
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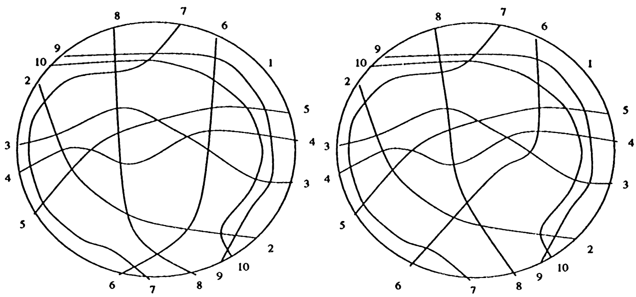

Fig. 5.

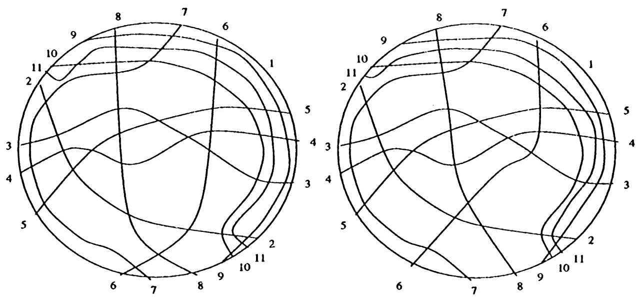

Fig. 6.

Proof. Consider the arrangements $\mathscr{A}_{10}$ and $\mathscr{A}'_{10}$ given in Fig. 5.

Notice that $\mathscr{A}'_{10}$ is obtained from $\mathscr{A}_{10}$ by *switching* the triangle adjacent to l_2, l_6 and l_8. It is clear that $G_{\mathscr{A}_{10}}$ and $G_{\mathscr{A}'_{10}}$ are isomorphic. However, $\mathscr{A}_{10}$ and $\mathscr{A}'_{10}$ are not isomorphic since $p_6(\mathscr{A}_{10}) - 1 = p_6(\mathscr{A}'_{10})$, $p_5(\mathscr{A}_{10}) + 2 = p_5(\mathscr{A}'_{10})$ and $p_4(\mathscr{A}_{10}) - 1 = p_4(\mathscr{A}'_{10})$.

Now, we construct two arrangements $\mathscr{A}_{11}$ and $\mathscr{A}'_{11}$ having 11 pseudolines by adding a new pseudoline l_{11} to $\mathscr{A}_{10}$ and $\mathscr{A}'_{10}$ as shown in Fig. 6.

Again, $G_{\mathscr{A}_{11}}$ and $G_{\mathscr{A}'_{11}}$ are isomorphic and $\mathscr{A}_{11}$ and $\mathscr{A}'_{11}$ cannot be isomorphic (for the same counting argument as above). In fact, we may recursively construct two arrangements $\mathscr{A}_{i+1}$ and $\mathscr{A}'_{i+1}$ having i pseudolines by adding a new pseudoline l_{i+1} to $\mathscr{A}_i$ and $\mathscr{A}'_i$ such that l_{i+1} is parallel to line l_i for $i \geqslant 11$.

It is easy to verify that $G_{\mathscr{A}_{i+1}}$ and $G_{\mathscr{A}'_{i+1}}$ remain isomorphic and $\mathscr{A}_{i+1}$ is not isomorphic to $\mathscr{A}'_{i+1}$. Fig. 4 illustrates the recursive construction in the case when $i = 10$. □

Lemma 4.3. *Let $\mathscr{A}_n$ be a p_3-maximal simple arrangement. Then $\mathscr{A}_n$ can be uniquely constructed from its set of triangles.*

Proof. Let $\mathscr{A}_n$ be a p_3-maximal arrangement, we have that

(1) exactly $n-1$ triangles lie on line l_i for each $i = 1, \ldots, n$

(2) $p_4(\mathscr{A}_n) = 0$ if $n > 4$, (see [6, Lemma 2.1]) and

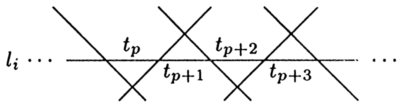

Fig. 7.

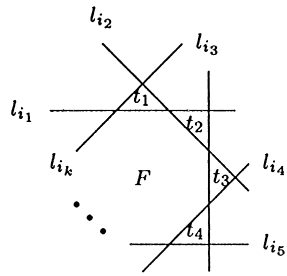

Fig. 8.

(3) If $t_1,\ldots,t_{n-1}$ are the triangles lying on line l_i (ordered in some direction) then t_j and t_{j+1}, $j=1,\ldots,n-2$ have two lines in common (with l_i one of those) and they are on opposite sides of l_i (see Fig. 7).

We show, that the set of triangles of $\mathscr{A}_n$ uniquely determines all its faces with k vertices $k \geqslant 5$ (by Remark (2)).

Let t_1 and t_2 be triangles of $\mathscr{A}_n$ such that t_1 and t_2 lie on $l_{i_k}, l_{i_1}, l_{i_2}$ and $l_{i_1}, l_{i_2}, l_{i_3}$, $1 \leqslant i_j \leqslant n$, respectively. Suppose that $l_{i_k}, l_{i_1}, l_{i_2}, l_{i_3}$ are four sides of a face F. We may find all the sides of F by applying recursively the following argument. From Remark (3) there exists a unique triangle t_3 that lies on $l_{i_2}, l_{i_3}, l_{i_4}$ where l_{i_4} is a side of F, and so on (see Fig. 8). □

We close this section by posing some questions.

(Q1) Is Lemma 4.2 true with $7 \leqslant n < 10$ and/or $\mathscr{A}_n$ and $\mathscr{A}'_n$ realizable?
(Q2) Is [RP] true for any p_3-minimal arrangement of pseudolines?
(Q3) Is $G_{\mathscr{A}}(T,L)$ connected? or perhaps is $G_{\mathscr{A}}(T,L)$ 2-connected?

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