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Straight Line Arrangements in the Real Projective Plane*

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Abstract. Let $\mathcal{A}$ be an arrangement of n pseudolines in the real projective plane and let $p_3(\mathcal{A})$ be the number of triangles of $\mathcal{A}$. Grünbaum has proposed the following question. Are there infinitely many simple arrangements of *straight* lines with $p_3(\mathcal{A}) = \frac{1}{3}n(n-1)$? In this paper we answer this question affirmatively.

1. Introduction

An *arrangement of pseudolines* is a finite collection $\mathcal{A}$ of $n \geq 3$ simple closed curves in the real projective plane $\mathbb{P}$ such that every two curves have exactly one point in common at which they cross. In the case where no point on $\mathbb{P}$ belongs to more than two lines of $\mathcal{A}$ we say that $\mathcal{A}$ is *simple,* see [2]. An arrangement of lines decomposes $\mathcal{A}$ into a two-dimensional cell complex. The number of faces with k vertices in $\mathcal{A}$ is denoted by $p_k(\mathcal{A})$. A face with three vertices is called a *triangle*. A simple combinatorial argument shows that $p_3(\mathcal{A}) \leq \frac{1}{3}n(n-1)$ for a simple arrangement $\mathcal{A}$ with $n \geq 4$ pseudolines. We say that a simple arrangement $\mathcal{A}$ is p_3-*maximal* if $p_3(\mathcal{A}) = \frac{1}{3}n(n-1)$. Grünbaum has proposed the following question (see p. 279 of [1]).

Question. Are there infinitely many p_3-maximal arrangements of *straight* lines?

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In this paper we answer the above question affirmatively. Indeed, we contruct an infinite family of p_3-maximal simple arrangements of *straight* lines.

There are known recursive methods by Roudneff [4] and Harborth [3] to construct a simple p_3-maximal arrangement with $2(n-1)$ pseudolines from a simple p_3-maximal arrangement with n pseudolines.

In Section 2 we give a recursive method to construct an infinite class of simple p_3-maximal arrangements of pseudolines. This proves the following well-known theorem.

Theorem 1.1 [3], [4]. *Let $\mathcal{A}$ be a simple p_3-maximal arrangement of $n > 4$ pseudolines. Then there exists a simple p_3-maximal arrangement $\mathcal{A}'$ with $2(n-1)$ pseudolines.*

Our method is closely related to that in [3] and it leads to a proof of the following theorem in Section 3.

Theorem 1.2. *There exists an infinite family of p_3-maximal simple arrangements of straight lines.*

2. The Method

Proof of Theorem 1.1. Let $\mathcal{A}$ be a p_3-maximal arrangement of n pseudolines $l_0, \dots, l_{n-2}$ and the pseudoline l_∞ at infinity. It is well known and easy to see that n must be even and that each l_i and l_∞ is adjacent to $n-1$ triangles. This is illustrated in Fig. 1 for l_∞ where the images of the intersection point of l_i and l_∞ are labeled with i and i', $0 \le i \le n-2$, and where the vertex (not lying on l_∞) of each triangle adjacent to l_∞ is labeled with v_j, $1 \le j \le n-1$.

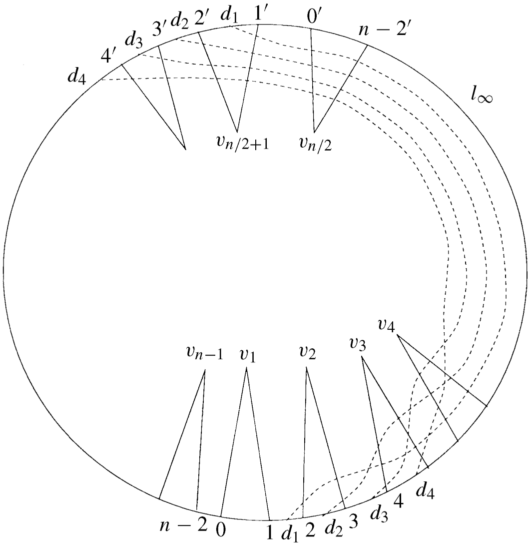


Fig. 1

We add $n-2$ new pseudolines $d_1, \ldots, d_{n-2}$ (one by one) to $\mathcal{A}$ according to the following rules:

(i) d_i starts between i and $(i+1)$ and ends between i' and $(i+1)'$ for $i = 1, \ldots, n-3$, and d_{n-2} starts between $(n-2)'$ and 0 and ends between $n-2$ and $0'$, see Fig. 1.
(ii) Each pseudoline d_i, $1 \le i \le n-2$, does not intersect the convex hull of vertices v_j, $1 \le j \le n-1$.
(iii) For each $i = 1, \ldots, n-2$, d_i crosses the pseudolines of $\mathcal{A}$ in the following order. If $i = 1$, then d_1 crosses $l_2, l_3, \ldots, l_0, l_1$. If $i \ge 2$, then d_i crosses $l_{i+1}, d_1, l_{i+2}, d_2, \ldots, l_{2i-1}, d_{i-1}, l_{2i}, l_{2i+1}, \ldots, l_{n+i-1}$ where subscripts are understood mod $(n-1)$.

We claim that the new arrangement $\mathcal{A}' = \{l_i\}_{0\le i\le n-1} \cup \{d_j\}_{1\le j\le n-2}$ is p_3-maximal. Indeed, the new lines d_i destroy $n-2$ triangles adjacent to l_∞ (all except the one formed by 0, 1, and v_1) and each line d_i, $i = 1, \ldots, n-2$, creates the following $2i+1$ triangles: d_i, l_i, l_∞, d_i, l_{i+1}, l_∞, d_i, l_{2i}, l_{2i+1} and for each $i \ge 2$ the triangles d_i, d_k, l_{k+i} and d_i, d_k, l_{k+i+1} for $k = 1, \ldots, i-1$. So, we have

$$\begin{aligned} p_3(\mathcal{A}') &= \frac{n(n-1)}{3} - (n-2) + \sum_{i=1}^{n-2}(2i+1) \\ &= \frac{n(n-1)}{3} + \sum_{i=1}^{n-2} 2i = \frac{n(n-1)}{3} + \frac{2(n-2)(n-1)}{2} \\ &= \frac{n^2 - n + 3n^2 - 9n + 6}{3} = \frac{(2n-2)(2n-3)}{3}. \end{aligned}$$

Hence, $\mathcal{A}'$ is p_3-maximal. □

We illustrate the above method with $n = 6$ in Fig. 2.

3. Straight Lines

We need the following definitions and lemma before proving Theorem 1.2. Let $S_C(l_0, \ldots, l_{n-1})$ denote the *star* formed by lines $l_0, \ldots, l_{n-1}$ all passing through a point C and such that the angle between l_i and l_j is $(\pi/n)(j-i)$, $0 \le i < j \le n-1$, see Fig. 3.

We denote by $\widehat{l_1 l_2}$ and $\widehat{P_1 P_2 P_3}$ the angles formed by lines l_1, l_2 and by the lines passing through points P_1, P_2 and P_3, P_2, respectively.

Lemma 3.1. *Let $S_C(l_0, \ldots, l_{n-1})$ be a star. Let d_1 be a line parallel to the angle bisector of $\widehat{l_0 l_1}$ which crosses l_0 above C. Let d_{i+1} be the line parallel to the angle bisector of $\widehat{l_i l_{i+1}}$ which passes through the intersection of d_1 and l_i for $i = 1, \ldots, n-1$ where the sum is modulo n. Then the intersection of lines d_i and d_j lies on l_{i+j-1} for each $0 \le i < j \le n-1$ where the sum $i + j - 1$ is modulo n.*

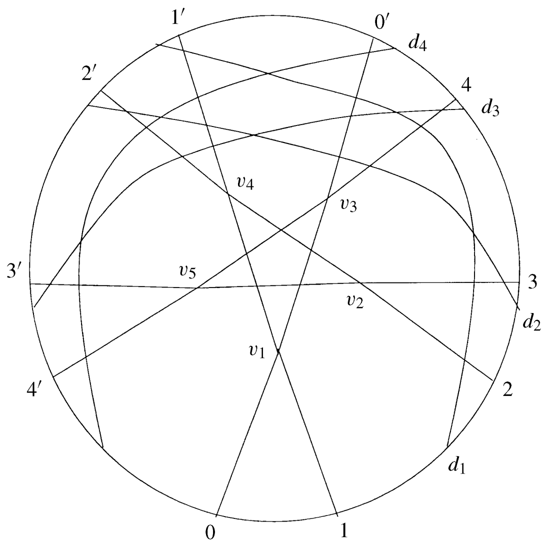


Fig. 2

Proof. Let P_i be the intersection of lines d_1 and l_i for $0 \le i \le n-1$ and let $Q_{i,j}$ be the intersection of lines d_i and d_j for $0 \le i < j \le n-1$. We show that $Q_{i,j} \in l_{i+j-1}$ where the sum $i+j-1$ is modulo n. Consider Fig. 4.

We have that $\widehat{P_i C P_j} = \widehat{l_i l_j} = (\pi/n)(j-i)$ and $\widehat{P_i Q_{i,j} P_j} = \widehat{d_i d_j} = (\pi/n)(j-i)$. So, the points C, P_i, P_j, and $Q_{i,j}$ are on a circle; hence, $\widehat{P_j C Q_{i,j}} = \widehat{P_j P_i Q_{i,j}} = \widehat{d_1 d_j} = (\pi/n)(j-1)$. Therefore, $Q_{i,j} \in l_{i+j-1}$.

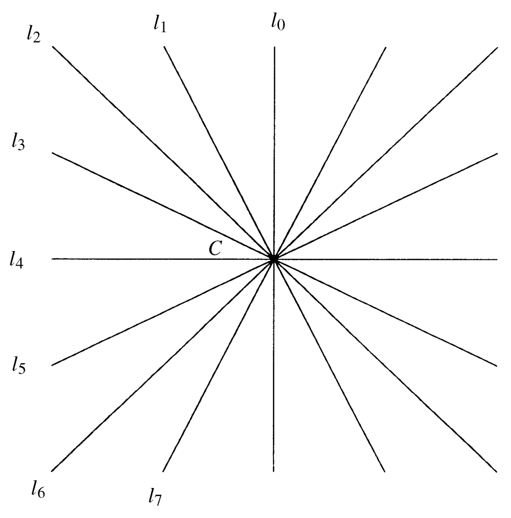


Fig. 3

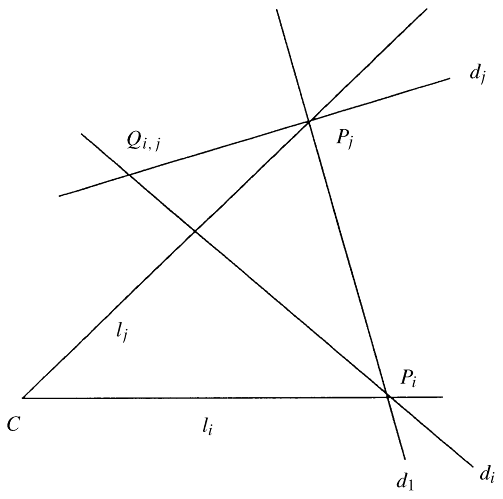


Fig. 4

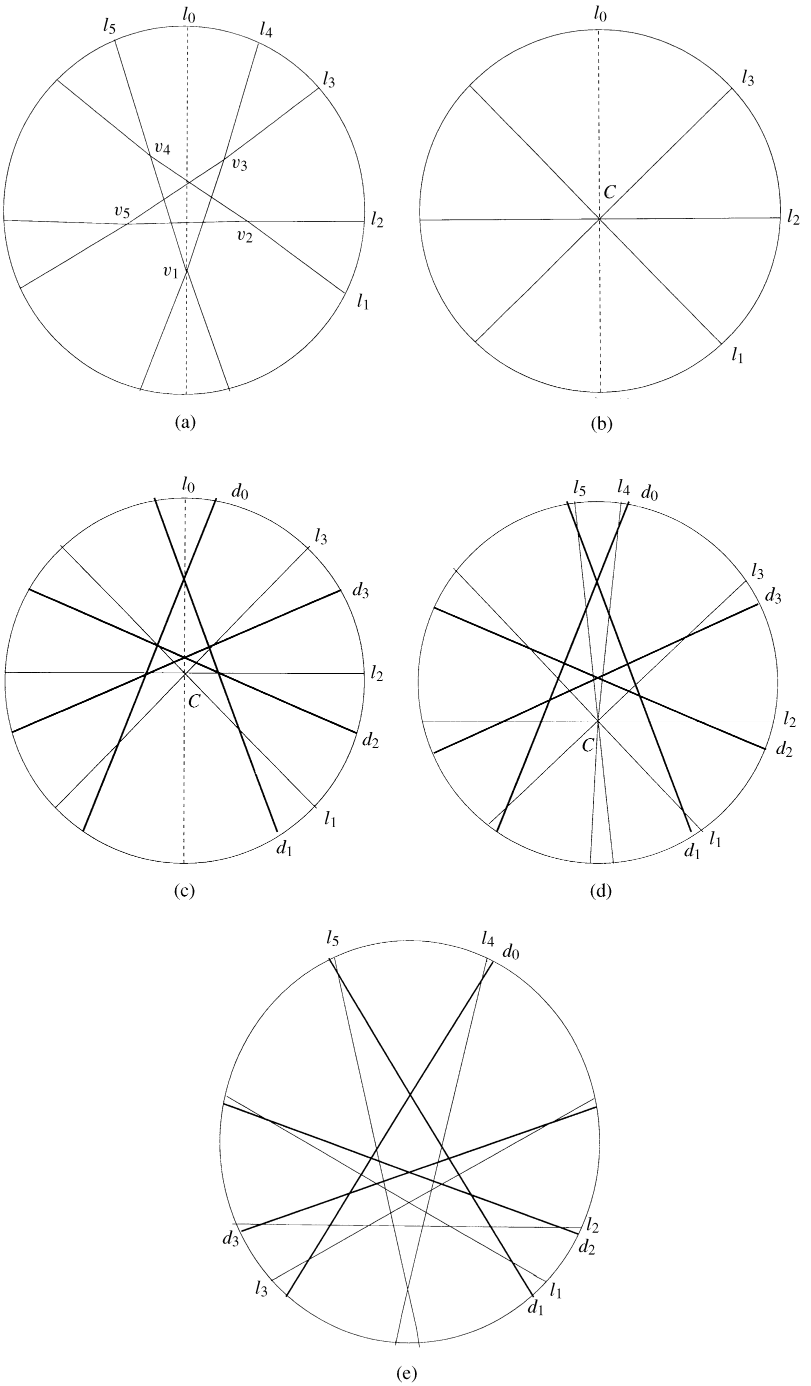
In particular the lines d_i and d_{n-i+1} cross at line l_0 with $\widehat{l_0 d_i} = -\pi/2n + i(\pi/n)$ and $\widehat{l_0 d_{n-i+1}} = -(\pi/2n + (i-1)(\pi/n))$. So, d_i and d_{n-i+1} are symmetric with respect to l_0. $\square$

We now prove Theorem 1.2.

Proof of Theorem 1.2. We recursively construct an arrangement $\mathcal{A}$ such that $\mathcal{A}\backslash\{l_\infty, l_{n-1}, l_{n-2}\}$ is obtained by a translation of a star where $l_\infty, l_{n-1}, l_{n-2}$ are special lines in $\mathcal{A}$.

Let $\mathcal{A}$ be a simple p_3-maximal arrangement of n straight lines $l_1, \ldots, l_{n-1}$ and l_∞ where the l_i's are labeled in the order of their appearence along l_∞. Suppose that $\mathcal{A}$ verifies the following properties: there exists a straight line l_0 (without loss of generality assume that l_0 is on the Y axis) such that (a) $\widehat{l_i l_j} = (\pi(n-2))(j-i)$ for all $0 \le i < j \le n-3$, (b) $\widehat{l_0 l_{n-1}} = \widehat{l_0 l_{n-2}} < \pi(n-2)$ and (c) no intersection point $l_i \cap l_j$, $1 \le i < j \le n-3$, lies in the interior of the cone borded by l_{n-1}, l_{n-2} and containing l_0 except maybe on l_0.

We give a procedure to construct an arrangement $\mathcal{A}'$ with $2(n-1)$ lines from $\mathcal{A}$. First, replace l_{n-1} and l_{n-2} by l_0 (note that, by property (c), l_{n-1} and l_{n-2} can be brought arbitrarily close to l_0). Let T_j be the triangles adjacent to l_∞ with $j = 1, \ldots, n-1$ and let v_j be the vertex in T_j not lying in l_∞. By continuity, we may assume that vertices v_j are *shrunk* into one point C (we can do this by extending the ends of line l_i far enough from the set of v_j's). So, $\mathcal{A}\backslash\{l_\infty\}$ can be identified with the star $S_C(l_0, \ldots, l_{n-3})$, see Fig. 5(b). Let $\mathcal{S} = S_C(l_0, \ldots, l_{n-3}) \cup \{d_i\}_{0 \le i \le n-3}$ be the arrangement given as in Lemma 3.1. We can see $\mathcal{S}$ as an arrangement formed by overlapping the two *slide-arrangements*



l_0
l_5
l_4
l_3
v_4
v_3
v_5
l_2
v_2
v_1
l_1
(a)
l_0
l_3
C
l_2
l_1
(b)
l_0
d_0
l_3
d_3
l_2
C
d_2
l_1
d_1
(c)
l_5
l_4
d_0
l_3
d_3
l_2
C
d_2
l_1
d_1
(d)
l_5
l_4
d_0
l_2
d_3
d_2
l_3
l_1
d_1
(e)

Fig. 5

$\mathcal{S}_1 = S_C(l_0, \ldots, l_{n-3})$ and $\mathcal{S}_2 = \{d_i\}_{0 \le i \le n-3}$, see Fig. 5(c). We form arrangement $\mathcal{A}'$ from $\mathcal{S}$ as follows:

(1) Fix slide $\mathcal{S}_1$ and translate slide $\mathcal{S}_2$ (in direction to the positive part of axe Y) a distance of $\varepsilon/2$ where ε is the minimum positive distance such that when moving $\mathcal{S}_2$ as above then a point $Q_{i,j}$ touches a line l_k for some k.
(2) Replace l_0 by l_{n-1} and l_{n-2} which cross at C with $\widehat{l_0 l_k} = \beta/2$ for $k = n-1, n-2$ where β is the angle between l_0 and the line passing through C and $Q_{n/2, n/2-2}$, see Fig. 5(d).

It is easy to check that $\mathcal{A}'$ satisfies rules (i)–(iii) in proof of Theorem 1.1. Hence, $\mathcal{A}'$ is a simple p_3-maximal arrangement of $2(n-1)$ straight lines. Moreover, by Lemma 3.1 and by construction, $\mathcal{A}'$ verifies properties (a)–(c). Therefore, we may apply recursively the above procedure starting with the six lines arrangement (that verifies the above properties) drawn in Fig. 5(a). □

Example. We illustrate the above procedure for the case with $n = 6$ in Fig. 5.

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