

# A NOTE ON TUTTE POLYNOMIALS AND ORLIK–SOLOMON ALGEBRAS

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ABSTRACT. Let  $\mathcal{A}_{\mathbb{C}} = \{H_1, \dots, H_n\}$  be a (central) arrangement of hyperplanes in  $\mathbb{C}^d$  and  $\mathcal{M}(\mathcal{A}_{\mathbb{C}})$  the dependence matroid of the linear forms  $\{\theta_{H_i} \in (\mathbb{C}^d)^* : \text{Ker}(\theta_{H_i}) = H_i\}$ . The Orlik-Solomon algebra  $\text{OS}(\mathcal{M})$  of a matroid  $\mathcal{M}$  is the exterior algebra on the points modulo the ideal generated by circuit boundaries. The graded algebra  $\text{OS}(\mathcal{M}(\mathcal{A}_{\mathbb{C}}))$  is isomorphic to the cohomology algebra of the manifold  $\mathfrak{M} = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}_{\mathbb{C}}} H$ . The Tutte polynomial  $T_{\mathcal{M}}(x, y)$  is a powerful invariant of the matroid  $\mathcal{M}$ . When  $\mathcal{M}(\mathcal{A}_{\mathbb{C}})$  is a rank 3 matroid and the  $\theta_{H_i}$  are complexifications of real linear forms, we will prove that  $\text{OS}(\mathcal{M})$  determines  $T_{\mathcal{M}}(x, y)$ . This result partially solves a conjecture of M. Falk.

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## 1. INTRODUCTION

Let  $V$  be a vector space of dimension  $d$  over some field  $\mathbb{K}$ . A (central) arrangement of hyperplanes in  $V$ ,  $\mathcal{A}_{\mathbb{K}} = \{H_1, \dots, H_n\}$ , is a finite listed set of codimension one vector subspaces. Given an arrangement  $\mathcal{A}_{\mathbb{K}}$  we always suppose selected a family of linear forms  $\{\theta_{H_i} : H_i \in \mathcal{A}_{\mathbb{K}}, \theta_{H_i} \in V^*, \text{Ker}(\theta_{H_i}) = H_i\}$ . Let  $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$  be the *dependence matroid* determined by the vectors  $\mathcal{V} := \{\theta_{H_i} : H_i \in \mathcal{A}_{\mathbb{K}}\}$  : i.e., the matroid on the ground set  $[n]$  which has for its independent sets the indices of linearly independent subsets of  $\mathcal{V}$ . A matroid  $\mathcal{M}$  is said to be *realizable* over  $\mathbb{K}$  if there is an arrangement  $\mathcal{A}_{\mathbb{K}}$  such that  $\mathcal{M} = \mathcal{M}(\mathcal{A}_{\mathbb{K}})$ . The manifold  $\mathfrak{M}(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}_{\mathbb{C}}} H$  plays an important role in the Aomoto-Gelfand multivariable theory of hypergeometric functions (see [11] for a recent introduction from the point of view of arrangement theory). In [10], the determination of the cohomology algebra  $H^*(\mathfrak{M}(\mathcal{A}_{\mathbb{C}}); \mathbb{C})$  from the matroid  $\mathcal{M}(\mathcal{A}_{\mathbb{C}})$ , is accomplished by first defining the Orlik–Solomon algebra  $\text{OS}(\mathcal{A}_{\mathbb{C}})$  in terms of generators and relators which depends only on the matroid  $\mathcal{M}(\mathcal{A}_{\mathbb{C}})$ , and then by showing that this graded commutative algebra is isomorphic to  $H^*(\mathfrak{M}(\mathcal{A}_{\mathbb{C}}); \mathbb{C})$ . The Orlik-Solomon algebras have been then intensively studied. One of the central questions is to find combinatorial invariants of OS. Descriptions of the developments from the late 1980s to the end of 1999, together with the contributions of many authors, can be found in [6] or [14].

Throughout this note  $\mathcal{M}$  denotes a (*simple*) rank  $r$  matroid without loops or parallel elements on the ground set  $[n]$ . Let  $\mathfrak{C}(\mathcal{M})$  be the set of *circuits* (i.e., minimal dependent sets with respect to inclusion) of  $\mathcal{M}$ . Let  $\mathfrak{B}(\mathcal{M})$  be the set of *bases* of the matroid. The *Tutte polynomial* of a matroid  $\mathcal{M}$ , denoted  $T_{\mathcal{M}}(x, y)$ ,

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is the unique two-variable polynomial over  $\mathbb{Z}$ , satisfying the following recursive properties for every  $\ell \in [n]$ , see [1]:

- $T_{\mathcal{M}}(x, y) = T_{\mathcal{M} \setminus \ell}(x, y) + T_{\mathcal{M}/\ell}(x, y)$  if  $\ell$  is neither a loop nor a coloop (isthmus),
- $T_{\mathcal{M}}(x, y) = x T_{\mathcal{M} \setminus \ell}(x, y)$  if  $\ell$  is a coloop,
- $T_{\mathcal{M}}(x, y) = y T_{\mathcal{M} \setminus \ell}(x, y)$  if  $\ell$  is a loop,
- $T_{\mathcal{M}}(x, y) = 1$  if the ground set of  $\mathcal{M}$  is the emptyset.

In this note it is important to recall that the Tutte polynomial of a rank 3 matroid  $\mathcal{M}$  merely records the number of rank 2 flats of each cardinality. More generally the Tutte polynomial of  $\mathcal{M}$  can be reconstructed from the multiset of isomorphism types of its hyperplanes, see [1]. We refer to [12, 13] as standard sources for matroids.

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## 2. THE MAIN RESULT

Let  $E := \{e_1, e_2, \dots, e_n\}$  be a fixed basis of the vector space  $\mathbb{C}^n$ . Consider the *graded exterior algebra* of the vector space  $\bigoplus_{i=1}^n \mathbb{C}e_i$ :

$$\mathcal{E} := \bigwedge \left( \bigoplus_{i=1}^n \mathbb{C}e_i \right) = \mathcal{E}_0 (= \mathbb{C}) \oplus \dots \oplus \mathcal{E}_\ell (= \sum \alpha_i e_{i_1} \cdots e_{i_\ell}, \alpha_i \in \mathbb{C}) \oplus \dots \oplus \mathcal{E}_n.$$

When necessary we see the subset  $\{i_1, \dots, i_\ell\} \subset [n]$  as the ordered set  $i_1 < \dots < i_\ell$ . Set  $e_X := e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_\ell}$ , in particular set  $e_{ij} := e_i \wedge e_j$ . By convention set  $e_\emptyset := 1$ . Let  $\partial : \mathcal{E} \rightarrow \mathcal{E}$  be the unique morphism such that  $\partial(e_i \wedge e_j) = \partial(e_i) \wedge e_j - e_i \wedge \partial(e_j)$  and  $\partial(e_i) = 1$ , for all  $e_i, e_j \in E$  and  $n \in \mathbb{N}$ . Let  $\mathfrak{Z}(\mathcal{M}) = \langle \partial(e_C) : C \in \mathfrak{C}(\mathcal{M}) \rangle$  be the two-sided ideal of  $\mathcal{E}$  generated by  $\{\partial(e_C) : C \in \mathfrak{C}(\mathcal{M})\}$ .

**Definition 2.1.** The quotient  $\mathcal{E}/\mathfrak{Z}(\mathcal{M})$  is termed the *Orlik-Solomon  $\mathbb{C}$ -algebra* of the matroid  $\mathcal{M}$ , denoted  $\text{OS}(\mathcal{M})$ .

The residue class in  $\text{OS}(\mathcal{M})$  determined by the element  $e_X \in \mathcal{E}$  is denoted by  $e_X$  as well. The algebra  $\text{OS}(\mathcal{M})$  has the natural graduation

$$\text{OS}(\mathcal{M}) = \mathbb{C} \oplus \text{OS}_1 (= \mathcal{E}_1) \oplus \dots \oplus \text{OS}_\ell (= \mathcal{E}_\ell / \mathfrak{Z}(\mathcal{M}) \cap \mathcal{E}_\ell) \oplus \dots \oplus \text{OS}_r.$$

We start by recalling some definitions and results from Falk [5, 6] about the resonance varieties of the  $\mathbb{C}$ -algebra  $\text{OS}(\mathcal{M})$ . We identify  $\text{OS}_1$  and  $\mathbb{C}^n$ . Fix an element  $e_\lambda = \sum_{i=1}^n \lambda_i e_i$  in  $\text{OS}_1$ . Then left multiplication by  $e_\lambda$  defines a map  $\text{OS}_p \rightarrow \text{OS}_{p+1}$ , which squares to zero. Thus it defines a cochain complex whose cohomology determines a stratification of the parameter space  $\mathbb{C}^n$ . The  $p^{\text{th}}$  resonance variety of  $\text{OS}$  is defined by

$$\mathcal{R}_p(\text{OS}) := \{\lambda \in \mathbb{C}^n \mid H^p(\text{OS}, e_\lambda) \neq 0\}.$$

The  $\mathcal{R}_p(\text{OS})$  are invariants of  $\text{OS}$ . They are the unions of subspaces of  $\mathbb{C}^n$ , see Corollary 3.8 in [2]. More is known about  $\mathcal{R}_1(\text{OS})$ : it is a union of two kinds of linear subspaces of  $\mathbb{C}^n$  with trivial intersection. The first are the *local* components which correspond to flats of rank 2 of  $k, k \geq 3$ , elements and are of dimension  $k - 1$ . The second are the *non-local* components which correspond to some particular colourations of the elements of submatroids of  $\mathcal{M}$ , see [5] for details. A very interesting application of the variety  $\mathcal{R}_1(\text{OS}(\mathcal{M}))$  has been shown by M. Falk:

**Theorem 2.2** (Theorem 3.16 in [5]). *Let  $\mathcal{M}$  be a rank 3 matroid. Suppose every non-local component of  $\mathcal{R}_1(\text{OS}(\mathcal{M}))$  has dimension two. Then  $\text{OS}(\mathcal{M})$  determines the Tutte polynomial of  $\mathcal{M}$ .*  $\square$

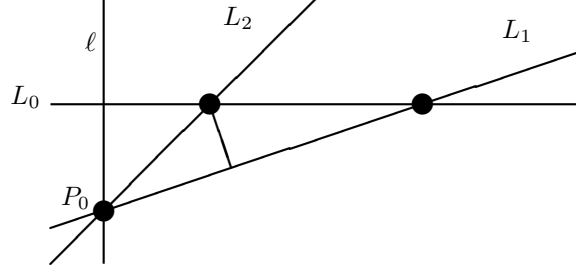


FIGURE 1. The Herzog and Kelly line arrangement.

In relation to this theorem Falk conjectured that if  $\mathcal{M}$  has rank 3 then  $\text{OS}(\mathcal{M})$  determines the Tutte polynomial of  $\mathcal{M}$ . We prove this conjecture for matroids realizable over  $\mathbb{R}$ . The proof is obtained simply by combining Theorem 2.2 with a recent result of Libgober and Yuzvinsky and an old colouration result of Herzog and Kelly. The next theorem of Libgober and Yuzvinsky, which is fundamental in this note, makes more precise previous results of Falk, see [5]. A *regular* colouration of  $\mathcal{M}([n])$  is a partition  $\Pi$  of  $[n]$  such that  $\#\{\pi \in \Pi \mid \pi \cap X \neq \emptyset\}$  is equal to 1 or  $|\Pi|$  for all rank 2 flats  $X$  of  $\mathcal{M}$ .

**Theorem 2.3** (Theorem 3.9 and Remark 3.10 in [9]). *Let  $\Pi$  be a  $k$ -colouration,  $k \geq 2$ , of  $\mathcal{M}$  which determines a non-local component of  $\mathcal{R}_1(\text{OS}(\mathcal{M}))$ . Then  $k \geq 3$ ,  $\Pi$  is regular and the dimension of the non-local component is  $k - 1$ .  $\square$*

We recall the following ‘‘Sylvester type’’ result of Herzog and Kelly:

**Lemma 2.4** (Theorem 4.1 of [8]). *Let  $\mathcal{M}$  be a rank 3 matroid realizable over  $\mathbb{R}$ . Then it has no regular colouration with four or more colours.*

*Proof.* Let  $\mathcal{A}$  be an arrangement of lines in  $\mathbb{R}^2$ . Let suppose that there exists a regular colouration  $\Pi$  of the matroid  $\mathcal{M}(\mathcal{A})$  with  $k, k \geq 4$ , colours. The lines [resp. points] in  $\mathcal{A}$  corresponding to elements [resp. rank 2 flats] of  $\mathcal{M}(\mathcal{A})$ . Then all multi-coloured points of intersection of  $\mathcal{A}$  are intersection of at least  $k$  lines of different colour. Let  $P_0$  be a multi-coloured point and  $L_0 \in \mathcal{A}, P_0 \notin L_0$ , a line such that  $d(P_0, L_0)$ , the euclidean distance between  $P_0$  and  $L_0$ , is minimum. Let  $L_1, L_2, \dots, L_k$  be  $k$  lines through  $P_0$  of different colours. At least three of them are not of the colour of  $L_0$  and at least two of these three are not separated by line  $\ell$  perpendicular from  $P_0$  to the line  $L_0$ . Let w.l.o.g  $P_0, L_0, L_1, L_2$  be as in Fig. 1. Then we have  $d(L_0 \cap L_2, L_1) < d(P_0, L_0)$  a contradiction.  $\square$

Note that Lemma 2.4 is not true for matroids realizable over a finite field. Indeed consider the affine plane  $\text{AG}(2, p^n)$ ,  $p^n \geq 4$ . Choose a (maximal) set  $L$  of  $p^n$  ‘‘parallel’’ (i.e., not intersecting two-by-two) lines of  $\text{AG}(2, p^n)$ . Colour the points of every line of  $L$  with the same colour and two different lines with two different colours. So, every line of the affine plane not in  $L$  is  $p^n$ -coloured and the colouration is regular. (See Example 4.5 in [5], where a similar regular 3-colouration of the affine plane  $\text{AG}(2, 3)$  over  $\mathbb{Z}_3$  is given.) We do not know if the lemma remains true for line arrangements in  $\mathbb{C}^2$ . Theorem 2.5 below partially solves Conjecture 4.8 in [5]. It remains open the important case of matroids realizable over the complexes but not realizable over the reals.

**Theorem 2.5.** *If  $\mathcal{M}$  is a matroid of rank at most 3, realizable over  $\mathbb{R}$ , then the Orlik-Solomon algebra  $\text{OS}(\mathcal{M})$  determines the Tutte polynomial  $T_{\mathcal{M}}(x, y)$ .*

*Proof.* The Theorem is a consequence of Theorems 2.2, 2.3 and Lemma 2.4.  $\square$

## 3. APPENDIX

From Lemma 2.4 and Theorem 2.3 we know that in the rank 3 real case the non-local components come only from regular 3-colourations, see Theorem 3.10 in [5] for details. In [3, 7] we can find the following regular 3-colourable real matroids  $\mathcal{M}(\mathcal{A})$  defined by line arrangements:

- There is an infinite family of arrangements of  $3k, k \geq 3$ , lines. Take a regular  $2k$ -gon with its  $k$  longest diagonals. colour the lines of the  $2k$ -gon alternatively in two colours and the diagonals with a third colour, see [3, 7].
- There is also the following simplicial arrangement,  $\mathcal{A}_1(12)$ , of 12 lines. Take the regular 6-gon together with its three longest and its three shortest diagonals. Colour the lines of the 6-gon cyclically in three colours as well as the diagonals longest such that the vertices of the 6-gon are 3-coloured. Then there is an unique colouration of the three shortest diagonals such the 3-colouration is regular, see [7].

Theorem 2.5 does not hold for matroids of rank at least 4. Infinite families of counter-examples to Theorem 2.5 appeared in [4]. We improve these counter-examples to the class of “connected matroids”. We make use of the Proposition 3.1 below. We say that the matroid  $\mathcal{M}'([n+1])$ , is a *single-element free extension* of its restriction  $\mathcal{M}'([n+1])|_{[n]} := \mathcal{M}([n])$ , if

$$\mathfrak{C}(\mathcal{M}([n+1])) = \mathfrak{C}(\mathcal{M}([n])) \cup \{B \cup \{n+1\} : B \in \mathfrak{B}(\mathcal{M})\}.$$

We set  $\mathcal{M}([n]) \hookrightarrow \mathcal{M}'([n+1])$  if  $\mathcal{M}'([n+1])$  is free extension of  $\mathcal{M}([n])$ .

**Proposition 3.1.** *Consider the two free extension  $\mathcal{M}_1([n]) \hookrightarrow \mathcal{M}'_1([n+1])$  and  $\mathcal{M}_2([n]) \hookrightarrow \mathcal{M}'_2([n+1])$  and suppose that there is the graded isomorphism  $\Phi : \text{OS}(\mathcal{M}_1([n])) \rightarrow \text{OS}(\mathcal{M}_2([n]))$ . Then there is a graded isomorphism*

$$\Phi' : \text{OS}(\mathcal{M}_1([n+1])) \rightarrow \text{OS}(\mathcal{M}_2([n+1]))$$

*extending  $\Phi$  and such that  $\Phi'(e_{n+1}) = e_{n+1}$ .*

*Proof.* Set  $\mathcal{E}^* := \bigwedge(\bigoplus_{i=1}^{n+1} \mathbb{C}e_i)$ . We claim that:

$$(3.1) \quad \mathfrak{Z}(\mathcal{M}'_1) = (\mathfrak{Z}(\mathcal{M}_1) \oplus \mathfrak{Z}(\mathcal{M}_1) \wedge e_{n+1}) + \langle \partial(\mathcal{E}_{r+1}^*) \rangle$$

and similarly for  $\mathfrak{Z}(\mathcal{M}'_2)$ . Indeed consider the ideal  $\Delta_{r+1}$  of  $\mathcal{E}^{n+1}$

$$\Delta_{r+1} := \left\langle \left\{ \partial(e_Y) : Y \in \binom{[n+1]}{r+1}, \text{ and } (Y \setminus \{n+1\}) \in \mathfrak{B}(\mathcal{M}_1) \right\} \right\rangle.$$

It is clear that

$$\mathfrak{Z}(\mathcal{M}_1) = (\mathfrak{Z}(\mathcal{M}_1) \oplus \mathfrak{Z}(\mathcal{M}_1) \wedge e_{n+1}) + \Delta_{r+1}.$$

Consider a subset  $X' \in \binom{[n+1]}{r+1}$ , and suppose that  $X' \setminus \{n+1\} = X'' \notin \mathfrak{B}(\mathcal{M}_1)$ . Then we have  $\partial(e_{X'}) \in \mathfrak{Z}(\mathcal{M}_1) \oplus \mathfrak{Z}(\mathcal{M}_1) \wedge e_{n+1}$  and  $\partial(e_{X''}) \in \mathfrak{Z}(\mathcal{M}_1)$ . So

$$(\mathfrak{Z}(\mathcal{M}_1) \oplus \mathfrak{Z}(\mathcal{M}_1) \wedge e_{n+1}) + \langle \partial(\mathcal{E}_{r+1}^*) \rangle \subset (\mathfrak{Z}(\mathcal{M}_1) \oplus \mathfrak{Z}(\mathcal{M}_1) \wedge e_{n+1}) + \Delta_{r+1}.$$

As the other inclusion is trivial Equality (3.1) follows. The restriction of the automorphism  $\Phi$  to  $\text{OS}_1(\mathcal{M}_1) (= \mathcal{E}_1)$  fixes a graded automorphism  $\tilde{\Phi} : \mathcal{E} \rightarrow \mathcal{E}$ , so  $\tilde{\Phi}(\mathfrak{Z}(\mathcal{M}_1)) = \mathfrak{Z}(\mathcal{M}_2)$ . Consider now the graded automorphism  $\Xi : \mathcal{E}^* \rightarrow \mathcal{E}^*$ , such that  $\Xi|_{\mathcal{E}} = \tilde{\Phi}$  and  $\Xi(e_{n+1}) = e_{n+1}$ . It is clear that:

$$\begin{aligned} \Xi(\mathfrak{Z}(\mathcal{M}'_1)) &= (\mathfrak{Z}(\mathcal{M}_2) \oplus \mathfrak{Z}(\mathcal{M}_2) \wedge e_{n+1}) + \langle \Xi(\partial(\mathcal{E}^*)) \rangle = \\ &= (\mathfrak{Z}(\mathcal{M}_2) \oplus \mathfrak{Z}(\mathcal{M}_2) \wedge e_{n+1}) + \langle \partial(\Xi(\mathcal{E}^*)) \rangle = \mathfrak{Z}(\mathcal{M}'_2). \end{aligned}$$

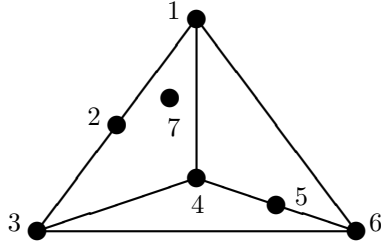
□

**Remark 3.2.** If  $\mathcal{M}([n]) \hookrightarrow \mathcal{M}'([n+1])$  the matroid  $\mathcal{M}'([n+1])/n+1$  is called the *truncation* of  $\mathcal{M}([n])$ . So we deduce from Proposition 3.1 that isomorphism of Orlik–Solomon algebras preserves truncation, see Theorem 3.11 in [6]. Conversely we know that free extension by a point is the same as adding a coloop and truncation. It is clear that isomorphism of Orlik–Solomon algebras is preserved under directed sum with a coloop. So Proposition 3.1 could be also deduced from Theorem 3.11 in [6] and these two results are equivalent. We leave the details to the reader.

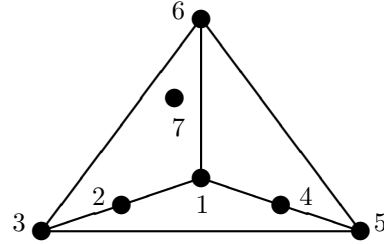
Consider now the two rank 4 matroids  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , given by the affine dependencies of the seven points in  $\mathbb{R}^3$ , as indicated in Fig. 2 and 3 and characterized by the following properties:

- The point 7 is in general position in both matroids.
- Both matroids have exactly two lines with 3 points and these lines are the unique lines with at least three elements.
- The union of the two lines with three elements has rank 4 in  $\mathcal{M}_1$ .
- The union of the two lines with three elements has rank 3 in  $\mathcal{M}_2$ .

The number of bases of the first matroid is  $T_{\mathcal{M}_1}(1, 1) = 27$  and of the second matroid is  $T_{\mathcal{M}_2}(1, 1) = 26$ . So we have  $T_{\mathcal{M}_1}(x, y) \neq T_{\mathcal{M}_2}(x, y)$ . We claim that  $\text{OS}(\mathcal{M}_1) \cong \text{OS}(\mathcal{M}_2)$ . From Proposition 3.1 it is enough to prove that  $\text{OS}(\mathcal{M}_1 \setminus 7) \cong \text{OS}(\mathcal{M}_2 \setminus 7)$ . Note that the matroids  $\mathcal{M}_1 \setminus 7$  and  $\mathcal{M}_2 \setminus 7$  are particular cases of the graphic matroids considered in [4]. (Make  $\mathcal{M}_0$  and  $C_n$  both equal to triangles, in Fig. 1 of [4].) So the isomorphism follows from Theorem 1.1 of [4].



**Figure 2:** The matroid  $\mathcal{M}_1([7])$ .



**Figure 3:** The matroid  $\mathcal{M}_2([7])$ .

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