Discrete Mathematics 258 (2002) 353–359

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

Disconnected coverings for oriented matroids via simultaneous mutations

D. Forge[a,1], J.L. Ramírez Alfonsín[b,*,2], H. Yeun[b]

[a] CIO, Departamento de Estatisticae e, Investigacao Operacional, Universidade de Lisboa, Bloco C2-Piso2, Campo Grande, 1749-016 Lisboa, Portugal

[b] Equipe Combinatoire, Université Pierre et Marie Curie, Paris 6, 4 Place Jussieu, 75252 Cedex 5 Paris, France

Received 30 October 2000; received in revised form 19 March 2002; accepted 18 April 2002

Abstract

Let $\mathscr{U}_{n,r}$ be a uniform oriented matroid having as bases, $\mathscr{B}$, all r-subsets (resp. as circuits, $\mathscr{C}$, all $(r+1)$-subsets) of $\{1,\ldots,n\}$. We say that $\mathscr{C}_1 \subseteq \mathscr{C}$ is a *covering*, of $\mathscr{U}_{n,r}$, if for any base $B \in \mathscr{B}$ there is a circuit $C \in \mathscr{C}_1$ such that $B \subset C$. Let $G(\mathscr{C}_1)$ be the graph having as set of vertices the elements of $\mathscr{C}_1$ and where two vertices are joined if they have one base in common. We say that $\mathscr{C}_1 \subseteq \mathscr{C}$ is a *connected* covering if $\mathscr{C}_1$ is a covering and $G(\mathscr{C}_1)$ is connected. It is easy to show that if a covering is connected then it completely determines $\mathscr{U}_{n,r}$. In this note, we show that connectivity is not always necessary.

1. Introduction

Let n,r be positive integers with $n \geqslant r$. Let $\mathscr{U}_{n,r}$ be the *uniform oriented matroid* having as bases, $\mathscr{B}$, all r-subsets (resp. as circuits, $\mathscr{C}$, all r-subsets) of $\{1,\ldots,n\}$. Consider the following question. *What is the smallest number of circuits*, $s(n,r)$, *that is sufficient to determine* $\mathscr{U}_{n,r}$?

In [2], the first two authors achieved different upper bounds for $s(n,r)$ by analyzing the smallest number of circuits needed to determine the signs of all the basis of $\mathscr{U}_{n,r}$.

* Corresponding author.

E-mail address: ramirez@ccr.jussieu.fr (J.L. Ramirez Alfonsin).

[1] Supported by DONET (European Union contract ERBFMRX-CT98-0202).

[2] This work was done while the author was visiting the Forschungsinstitut für Diskrete Mathematik, Universität Bonn and was supported by the Alexander von Humboldt Foundation.

PII: S0012-365X(02)00536-8

To this end, it was defined a *covering* for the bases of $\mathscr{U}_{n,r}$ and noticed that if the covering is *connected* then it *determines* $\mathscr{U}_{n,r}$. The connected coverings were then related to the well-known block designs from which upper bounds for $s(n,r)$ were obtained (improving the best upper bound for $s(n,r)$, known at that time, due to Hamidoune and Las Vergnas [4]).

A natural question is whether connectivity is necessary for a covering to determine $\mathscr{U}_{n,r}$.

It turns out that connectivity is not always necessary. In this note, we shall generalize the notion of *mutation* in order to construct special *disconnected* coverings that determine $\mathscr{U}_{n,3}$ for each $n \geqslant 8$.

2. Definitions and notation

A *basis orientation* of an oriented matroid $\mathscr{M}$ is a mapping Φ of the set of ordered bases of $\mathscr{M}$ to $\{-1,1\}$ satisfying certain properties (see [1] for further details).

If $\mathscr{M}$ and $\mathscr{M}'$ are two rank r uniform oriented matroids then $\mathscr{M}$ and $\mathscr{M}'$ are called *mutants* if their basis orientation coincide except for one ordered base. In this case the base is called a *mutation* of $\mathscr{M}$ (and $\mathscr{M}'$). Las Vergnas [1] proved that every oriented matroid $\mathscr{M}$ has exactly two basis orientations and these two basis orientations are opposite, Φ and $-\Phi$.

Remark. Let C be a circuit and B a base of $\mathscr{U}_{n,r}$ with $B \subseteq C$. Given the sign of B the signature of C allows us to sign the other r basis contained in C.

We say that $\mathscr{C}_1 \subseteq \mathscr{C}$ is a *covering*, of $\mathscr{U}_{n,r}$, if for any base $B \in \mathscr{B}$ there is a circuit $C \in \mathscr{C}_1$ such that $B \subset C$. Let $G(\mathscr{C}_1)$ be the graph having as set of vertices the elements of $\mathscr{C}_1$ and where two vertices are joined if they have one base in common. We say that $\mathscr{C}_1 \subseteq \mathscr{C}$ is a *connected* covering (resp. a *disconnected* covering) if $\mathscr{C}_1$ is a covering and $G(\mathscr{C}_1)$ is connected (resp. disconnected).

It is said that $\mathscr{C}_1$ *determines* $\mathscr{U}_{n,r}$ if the signature of the circuits in $\mathscr{C}_1$ are sufficient to sign the rest of the circuits in $\mathscr{U}_{n,r}$. Or equivalently, if they are sufficient to sign all the bases of $\mathscr{U}_{n,r}$. Notice that if $\mathscr{C}_1$ is not a covering then it cannot define $\mathscr{U}_{n,r}$. The following proposition follows from the above remark.

Proposition 2.1 (Forge and Ramirez [2]). *Let $\mathscr{C}_1$ be a covering of $\mathscr{U}_{n,r}$. If $\mathscr{C}_1$ is connected then it determines $\mathscr{U}_{n,r}$.*

Here, we are interested in the converse of the above proposition.

[Q1]. Let n,r be positive integers with $n \geqslant r$. Let $\mathscr{C}_1 \subseteq \mathscr{C}$ be a covering of $\mathscr{U}_{n,r}$ and suppose that $\mathscr{C}_1$ determines $\mathscr{U}_{n,r}$. Then, is $\mathscr{C}_1$ connected?

We shall answer [Q1] negatively by considering the following question.

[Q2]. Let $\mathscr{DC}$ be a disconnected covering of $\mathscr{U}_{n,r}$ having two components $\mathscr{DC}_1$ and $\mathscr{DC}_2$. Assume that $\mathscr{DC}_1$ and $\mathscr{DC}_2$ contain the set of bases $\mathscr{B}_1$ and $\mathscr{B}_2$, respectively (and so, $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$). Do there always exist a uniform oriented matroid $\mathscr{U}$ with basis orientation Φ and such that

$$\Phi'(B) = \begin{cases} \Phi(B) & \text{if } B \in \mathscr{B}_1, \\ -\Phi(B) & \text{otherwise,} \end{cases}$$

is also the basis orientation of another uniform oriented matroid $\mathscr{U}'$?

Note that [Q2] is asking for two uniform oriented matroids having as mutants a given set of basis.

Claim 2.2. *If* [Q2] *has a negative answer then* [Q1] *also does.*

Proof. If it never exist oriented matroids $\mathscr{U}$ and $\mathscr{U}'$ as in [Q2] then the signatures of the bases $\mathscr{B}_1$ and $\mathscr{B}_2$ can be uniquely obtained from $\mathscr{DC}_1$ and $\mathscr{DC}_2$, respectively. Thus, $\mathscr{DC}$ determines $\mathscr{U}_{n,r}$. □

In the next section, we construct a disconnected covering of $\mathscr{U}_{n,3}$, $n \geqslant 8$ having as components $\mathscr{DC}_1$ and $\mathscr{DC}_2$ such that there never exist two uniform oriented matroids $\mathscr{U}$ and $\mathscr{U}'$ such that the only mutants of $\mathscr{U}$ (or $\mathscr{U}'$) are the bases in $\mathscr{DC}_2$. Thus, answer negatively [Q2] and therefore, by Claim 2.2, [Q1] as well.

3. Disconnected coverings and switchings

In this section, we answer negatively [Q2] when $r=3$ and $n \geqslant 8$. To this end, we need the following definitions. An *arrangement of pseudolines* is a finite collection $\mathscr{A}$ of $n \geqslant 3$ simple closed curves in the real projective plane P^2 such that every two curves have exactly one point in common at which they cross. In the case where no point on P^2 belongs to more than two lines of $\mathscr{A}$ we say that $\mathscr{A}$ is *simple*, see [3]. A face with three vertices is called a *triangle*. A *switching* in an arrangement is the local deformation of a triangle showed in Fig. 1.

It is well-known that simple arrangements are in one-to-one correspondance with a reorientation class of uniform oriented matroids of rank 3. Moreover, the set of all mutuations of $\mathscr{U}_{n,3}$ correspond to the set of all possible switchings of the corresponding

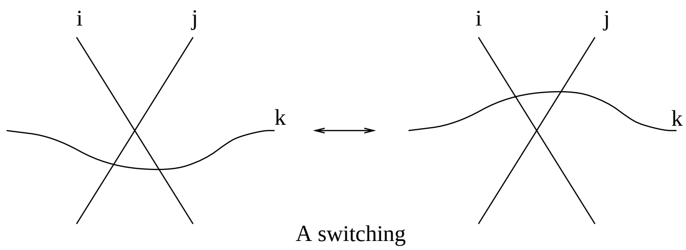

Fig. 1.

arrangement of n pseudolines. More precisely, a base (i,j,k) is a mutation of $\mathscr{U}_{n,3}$ if and only if there is a switching in the triangle formed by pseudolines i,j and k in the corresponding arrangement.

Theorem 3.1. *Let $\mathscr{DC}$ be the disconnected covering of $\mathscr{U}_{n,3}$, $n\geqslant 8$ having as components*

$$\mathscr{DC}_1 = \{(1,2,3,4),(1,2,3,5),(1,2,4,6),(1,3,5,6)\}$$

and

$$\mathscr{DC}_2 = S_2 \cup \{(1,4,5,7),(2,3,6,7),(2,4,5,7),(2,5,6,7),$$
$$(3,4,5,7),(3,4,6,7),(4,5,6,7)\}$$

where $S_2=\{(i_1,i_2,j+1,j+2)$ with $1\leqslant i_1<i_2\leqslant j$ and $6\leqslant j\leqslant n-2\}$. Then, there not exist two uniform oriented matroids $\mathscr{U}$ and $\mathscr{U}'$ such that the only mutants of $\mathscr{U}$ (or $\mathscr{U}'$) are the bases in $\mathscr{DC}_2$.

Proof. It can be checked that $\mathscr{DC}_1$ and $\mathscr{DC}_2$ are connected components each, disjoint from each other and that they contain all the bases of $\mathscr{U}_{n,3}$ (and thus, $\mathscr{DC}_1$ and $\mathscr{DC}_2$ form a disconnected covering of $\mathscr{U}_{n,3}$, $n\geqslant 8$ indeed). Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be the set of bases in $\mathscr{DC}_1$ and $\mathscr{DC}_2$, respectively. Note that $R=\{(1,4,5),(2,3,6),(2,4,5),(2,5,6),(3,4,5),(3,4,6),(4,5,6)\}$ are bases belonging to $\mathscr{B}_2$ and $\mathscr{B}_1=\{$all 3-subsets of $\{1,\ldots,6\}\}\backslash R$.

Let $\mathscr{A}$ be an arrangement of $n\geqslant 8$ pseudolines. We shall show that if $\mathscr{A}$ has as switchings the triples given by R then $\mathscr{A}$ is forced also to have a switching i',j',k' where the triple i',j',k' is a base in $\mathscr{B}_1$ (and therefore, $\mathscr{A}$ cannot have only switchings formed by triples in $\mathscr{B}_2$). W.l.o.g. suppose that pseudoline 1 is the line at infinity in $\mathscr{A}$. Now, the intersections of pseudolines $2,\ldots,5$ in $\mathscr{A}$ must look as one of the arrangements given in Figs. 2(a)–(f).

We claim that no matter how line 6 is added to any of the arrangements of Figs. 2(b), (c), (e) or (f) the switchings $(2,3,6)$, $(2,5,6)$, $(3,4,6)$ and $(4,5,6)$ cannot be achieved without making at least another switching which corresponding base belongs to $\mathscr{B}_1$. To see this, notice that if pseudoline 6 crosses (while doing a switching) the intersections of pseudolines $(2,3)$, $(2,5)$, $(3,4)$ and $(4,5)$ then it is also forced to cross the intersection of pseudolines $(2,4)$ (that is, it is forced to make the swithching $(2,4,6)$ which correspond to a base in $\mathscr{B}_1$).

On the other hand, in order to be able to make only switchings $(2,3,6)$, $(2,5,6)$, $(3,4,6)$ and $(4,5,6)$ in Figs. 2(a) and (d), pseudoline 6 must be added as it is shown in Figure 3(a) and 3(b) respectively (dotted and thick curves represent pseudoline 6 before and after doing the switchings respectively).

But now, switchings $(2,4,5)$ and $(3,4,5)$ cannot be achieved without making either switching $(2,3,4)$ or switching $(2,3,5)$ (both corresponding to bases belonging to $\mathscr{B}_1$). $\square$

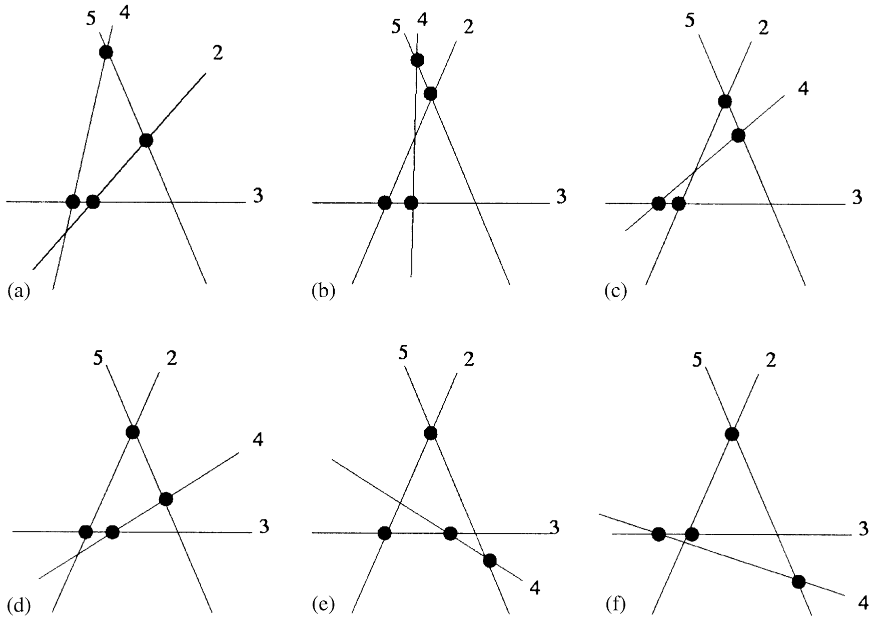

Fig. 2. Possible intersections in $\mathscr{A}$.

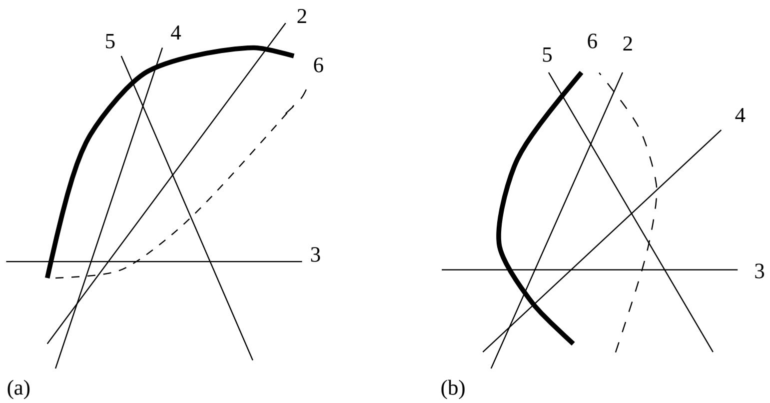

Fig. 3. Switching line 6.

Notice that the disconnected coverings given in Theorem 3.1 do not improve the upper bounds for $s(n,3)$ given in [2].

Problem. Is there a disconnected covering which improves the upper bounds for $s(n,3)$ with $n \geqslant 8$?

We finally present a result for disconnected coverings that cannot define $\mathscr{U}_{n,3}$.

Lemma 3.2. *Let $\mathscr{DC}$ be a disconnected covering of $\mathscr{U}_{n,3}$ having as components $\mathscr{DC}_1$ and $\mathscr{DC}_2$. Then, there exist two uniform oriented matroids having as mutants the set of bases $\mathscr{B}_1$ (in $\mathscr{DC}_1$) if either*

(*a*) $\bigcap_{C\in\mathscr{DC}_1} C=\{i,j,k\}$ *with* $1\leqslant i<j<k\leqslant n$ *or*
(*b*) $\mathscr{B}_1$ *are all the* 3*-subsets elements of a set* $E'\subseteq\{1,\ldots,n\}$ (*this case can be considered as a generalization of a simple mutation*).

Proof. In each case, it can be found an appropriate arrangement having only the desired switchings. □

Example. We illustrate Lemma 3.2. In case (a) we take $n=8$ with

$$\mathscr{DC}_1=\{(1,2,3,4),(1,2,3,5),(1,2,3,6)\}$$

and

$$\mathscr{DC}_2=\{(1,2,7,8),(1,3,7,8),(1,4,5,6),(1,4,7,8),(1,5,6,7),$$
$$(1,5,7,8),(1,6,7,8),(2,3,7,8),(2,4,5,6),(2,4,7,8),$$
$$(2,5,6,7),(2,5,7,8),(2,6,7,8),(3,4,5,6),(3,4,7,8),$$
$$(3,5,6,7),(3,5,7,8),(3,6,7,8),(4,5,6,7),(4,5,7,8),$$
$$(4,6,7,8),(5,6,7,8)\}.$$

So, $\bigcap_{C\in\mathscr{DC}_1} C=(1,2,3)$ and

$$\mathscr{B}_1=\{(1,2,3),(1,2,4),(1,2,5),(1,2,6),(1,3,4),(1,3,5),(1,3,6),(2,3,4),$$
$$(2,3,5),(2,3,6)\}.$$

The corresponding arrangements are given in Fig. 4(a). In case (b) we take $n=7$ with $E'=\{1,2,3,4,5\}$,

$$\mathscr{DC}_1=\{(1,2,3,4),(1,2,3,5),(1,2,4,5),(1,3,4,5)\}$$

and

$$\mathscr{DC}_2=\{(1,2,6,7),(1,3,6,7),(1,4,6,7),(1,5,6,7),(2,3,6,7),$$
$$(2,4,6,7),(2,5,6,7),(3,4,6,7),(3,5,6,7),(4,5,6,7)\}.$$

So, $\mathscr{B}_1=\{(1,2,3),(1,2,4),(1,2,5),(1,3,4),(1,3,5),(1,4,5),(2,3,4),(2,3,5),(2,4,5),(3,4,5)\}$. The corresponding arrangements are given in Fig. 4(b).

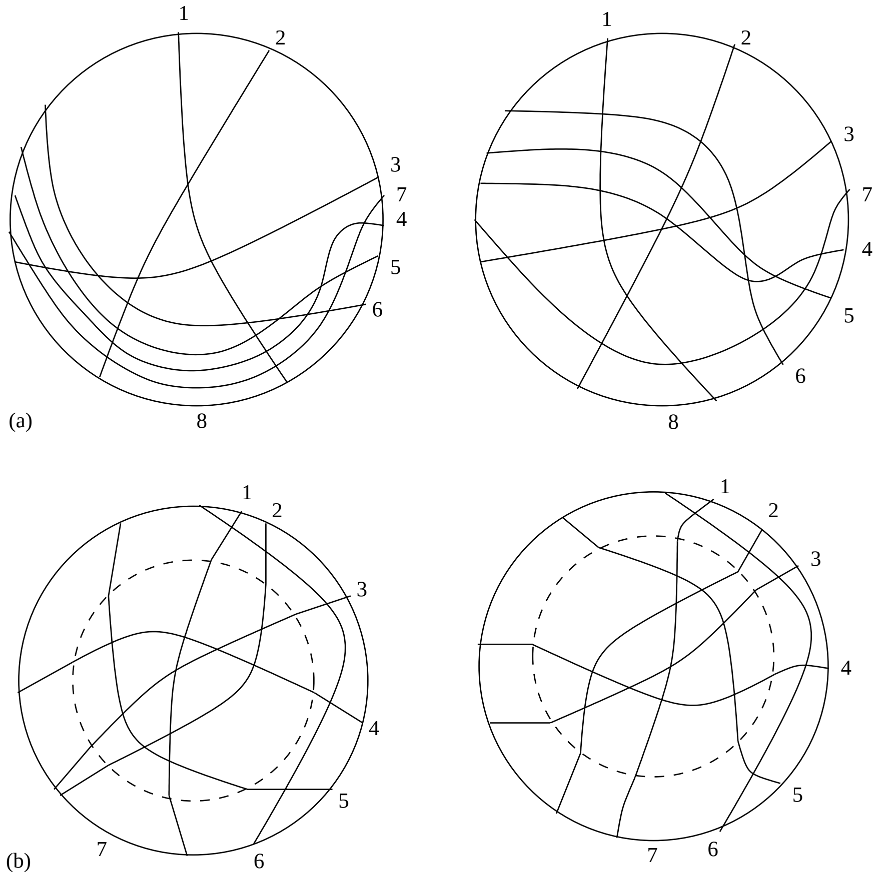

Fig. 4. Examples for Lemma 3.2.

References

[1] B. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, Oriented Matroids, 2nd Edition, Encyclopedia of Mathematics and its Applications, Vol. 46, Cambridge University Press, Cambridge, 1999.

[2] D. Forge, J.L. Ramírez Alfonsín, Connected coverings and an application to oriented matroids, Discrete Math. 187 (1998) 109–121.

[3] B. Grünbaum, Arrangements and Spreads, Regional Conference Series in Mathematics, Vol. 10, American Mathematical Society, Providence, RI, 1972.

[4] Y.O. Hamidoune, M. Las Vergnas, Directed switching games on graphs and matroids, J. Combin. Theory Ser. B 40 (1986) 237–269.