Colouring sparse graphs

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Mathematical notations

Set theory

1. Set union: $A \cup B$; set intersection: $A \cap B$; set difference: $A \setminus B$.
2. Set inclusion: $A \subseteq B$; strict set inclusion: $A \subsetneq B \equiv (A \subseteq B \text{ and } A \neq B)$.
3. Addition of an element: $A + x = A \cup \{x\}$.
4. Deletion of an element: $A - x = A \setminus \{x\}$.
5. Cartesian product: $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.
6. $\binom{X}{k}$ denotes the collection of subsets of X on k elements.
7. $[n] = \{0, \ldots, n-1\}$ denotes the set of the first n natural integers.
8. $\#A = |A|$ denotes the cardinality of A.

Logical operations

1. logical equivalence $x \equiv x'$.
2. negation: $\overline{x} \equiv \neg x$.
3. and: $x \cdot y \equiv x \wedge y$.
4. or: $x + y \equiv x \vee y$.
5. implication: $x \implies y \equiv \overline{x} + y$.
6. equivalence: $x \iff y \equiv x \cdot y + \overline{x} \cdot \overline{y}$.

Asymptotics

Let f and g be two positive functions defined on integers.

1. $f(n) = o(g(n))$ if $\limsup_{n \to \infty} \frac{f(n)}{g(n)} = 0$.
2. $f(n) = \omega(g(n))$ if $\liminf_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.

3. $f(n) = O(g(n))$ if $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$.
4. $f(n) = \Omega(g(n))$ if $\liminf_{n\to\infty} \frac{f(n)}{g(n)} > 0$.
5. $f(n) = \Theta(g(n))$ if $0 < \liminf_{n\to\infty} \frac{f(n)}{g(n)} \leq \limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$.
6. $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.
7. $f(n) \lesssim g(n)$ if $f(n) \leq (1 + o(1))g(n)$, i.e. $\limsup_{n\to\infty} \frac{f(n)}{g(n)} \leq 1$.

Arithmetic

1. $n \bmod p$ is the reminder of the Euclidian division of n by p.
2. $d \mid n$ stands for d divides n, so $n \bmod d = 0$.
3. $\lfloor x \rfloor$ is the rounding of x to the smaller integer.
4. $\lceil x \rceil$ is the rounding of x to the bigger integer.
5. $\lfloor x \rceil$ is the rounding of x to the closest integer.
6. $n! = 1 \times 2 \times \ldots \times n$ is the factorial of n.
7. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of possible subsets of k elements among a set of n elements.
8. $\binom{n}{a_1,\ldots,a_k} = \frac{n!}{\prod_{i=1}^{k} a_i!}$ is the number of ordered partitions of a set of n elements into k parts, of respective size $a_1, \ldots, a_k$. This is a multinomial coefficient.

Probabilities

1. $\mathbb{P}[A]$ is the probability of the random event A.
2. $\mathbb{E}[X] = \sum_{k\in\mathbb{N}} k \cdot \mathbb{P}[X = k]$ is the expectancy of the (non negative) random variable X.
3. $\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$ is the probability of the random event A conditioned on the random event B.
4. $\mathbb{E}[X \mid A] = \sum_{k\in\mathbb{N}} k \cdot \mathbb{P}[X = k \mid A]$ is the expectancy of the random variable X conditioned on the random event A.

Abstract

This thesis focuses on generalisations of the colouring problem in various classes of sparse graphs. Triangle-free graphs of maximum degree d are known to have independence ratio at least $(1 - o(1)) \ln d/d$ by a result of Shearer [108], and chromatic number at most $O(d/\ln d)$ by a result of Johansson [65], as $d \to \infty$. This was recently improved by Molloy [90], who showed that the chromatic number of triangle-free graphs of maximum degree d is at most $(1 + o(1))d/\ln d$ as $d \to \infty$.

While Molloy's result is expressed with a global parameter, the maximum degree of the graph, we first show that it is possible to extend it to *local colourings*. Those are list colourings where the size of the list associated to a given vertex depends only on the degree of that vertex. With a different method relying on the properties of the hard-core distribution on the independent sets of a graph, we obtain a similar result for *local fractional colourings*, with weaker assumptions. We also provide an analogous result concerning local fractional colourings of graphs where each vertex is contained in a bounded number of triangles, and a sharp bound for the occupancy fraction — the expected size of an independent set — of those graphs. In another direction, we also consider graphs of girth 7, and prove related results which improve on the previously known bounds when the maximum degree does not exceed 10^7. Finally, for d-regular graphs with $d \in \{3, 4, 5\}$, of girth g varying between 6 and 12, we provide new lower bounds on the independence ratio.

The second chapter is dedicated to distance colourings of graphs, a generalisation of strong edge-colourings. Extending the theme of the first chapter, we investigate minimal sparsity conditions in order to obtain Johansson-like results for distance colourings. While Johansson's result follows from the exclusion of triangles — or actually of cycles of any fixed length — we show that excluding cycles of even length ℓ, provided that $\ell \geq 2t + 2$, has a similar effect for the distance-t chromatic number and the distance-$(t+1)$ chromatic index. When t is odd, the same holds for the distance-t chromatic number by excluding cycles of odd length ℓ, provided that $\ell \geq 3t$. We investigate the asymptotic sharpness of our results with constructions of combinatorial, algebraic, and probabilistic natures.

In the third chapter, we are interested in the bipartite induced density of triangle-free graphs, a parameter which conceptually lies between the independence ratio and the fractional chromatic number. Motivated by a conjecture of Esperet, Kang, and Thomassé [49], which states that the bipartite induced density of a triangle-free graph of average degree d should be $\Omega(\ln d)$, we prove that the conjecture holds when d is large enough in terms of the number of vertices n, namely $d = \Omega\left(\sqrt{n \ln n}\right)$. Our result is shown to be sharp up to a multiplicative constant, with a construction relying on the triangle-free process. Our work on the bipartite induced density raises an interesting related problem, which aims at determining the maximum possible fractional chromatic number of sparse graph where the only known parameter is the number of vertices. We prove non-trivial upper bounds for triangle-free graphs, and graphs where each vertex belongs to a bounded number of triangles.

The entirety of this thesis is linked to the off-diagonal Ramsey numbers. To this date, the best-known bounds on the off-diagonal Ramsey number $R(3,t)$ come from the aforementioned result of Shearer for the upper bound, and a recent analysis of the triangle-free process [16, 52] for the lower bound, giving

$$\frac{t^2}{4\ln t} \lesssim R(3,t) \lesssim \frac{t^2}{\ln t}. \qquad (*)$$

Many of our results are best possible barring an improvement of $(*)$, which would be a breakthrough in quantitative Ramsey theory.

Abstract in French

Cette thèse étudie principalement le problème de coloration dans diverses classes de graphes épars. Un résultat de Shearer [108] indique que le ratio d'indépendance des graphes sans triangle de degré maximal d est au moins $(1-o(1))\ln d/d$, et fut suivi 13 ans plus tard par un résultat de Johansson [65] qui démontre que le nombre chromatique de ces graphes est au plus $O(d/\ln d)$ quand $d \to \infty$. Ce dernier résultat fut récemment amélioré par Molloy [90], qui montra que le nombre chromatique des graphes sans triangle de degré maximal d est au plus $(1+o(1))d/\ln d$ quand $d \to \infty$.

Tandis que le résultat de Molloy s'exprime à l'aide d'un paramètre global, le degré maximal du graphe, nous montrons en premier lieu qu'il est possible de l'étendre à la *coloration locale*. Il s'agit de la coloration par liste, où la taille de la liste associée à chaque sommet ne dépend que de son degré. Avec une méthode différente se basant sur les propriétés de la distribution hard-core sur les ensembles indépendants d'un graphe, nous obtenons un résultat similaire pour la *coloration fractionnaire locale*, avec des hypothèses plus faibles. Nous démontrons également un résultat analogue concernant la coloration fractionnaire locale des graphes où chaque sommet est contenu dans un nombre borné de triangles, et une borne principalement optimale sur le taux d'occupation — l'espérance de la taille des ensembles indépendants — de ces graphes. Dans une autre direction, nous considérons également les graphes de maille 7, et prouvons des résultats similaires qui améliorent les bornes précédemment connues quand le degré maximal du graphe est au plus 10^7. Finalement, pour les graphes d-réguliers où $d \in \{3,4,5\}$, de maille g variant entre 6 et 12, nous démontrons de nouvelles bornes inférieures sur le ratio d'indépendance.

Le second chapitre est dédié à la coloration à distance t d'un graphe, qui généralise la notion de coloration forte des arêtes. Nous étendons le thème du premier chapitre en étudiant des conditions éparses minimales permettant d'obtenir des résultats de la même nature que celui de Johansson pour la coloration à distance t. Tandis que le résultat de Johansson s'obtient par exclusion des triangles — et en fait des cycles de n'importe quelle taille préalablement fixée — nous montrons que l'exclusion des cycles de taille ℓ, pour n'importe quel $\ell \geq 2t+2$ pair, a un effet similaire sur le nombre chromatique à distance t, et sur l'indice chromatique à distance $t+1$. En outre, quand t est impair, une conclusion similaire peut se faire pour le nombre chromatique à distance t par l'exclusion des cycles de taille impaire $\ell \geq 3t$ fixée. Nous étudions l'optimalité asymptotique de ces résultats à l'aide de constructions de nature combinatoire, algébrique, et probabiliste.

Dans le troisième chapitre, nous nous intéressons à la densité bipartie induite des graphes sans triangle, un paramètre dont la difficulté se trouve conceptuellement quelque part entre celle du ratio d'indépendance et celle du nombre chromatique fractionnaire. Motivés par une conjecture de Esperet, Kang, et Thomassé [49], qui prétend que la densité bipartie induite des graphes sans triangle de degré moyen d devrait être $\Omega(\ln d)$, nous montrons que cette conjecture est vraie quand d est suffisamment grand en termes du nombre de sommets n, à savoir $d = \Omega\left(\sqrt{n \ln n}\right)$. Ce résultat ne pourrait être amélioré que par une valeur de l'ordre $\ln n$, ce que nous montrons à l'aide d'une construction reposant sur le processus sans triangle. Nos travaux sur la densité bipartie induite

soulèvent un problème associé intéressant, celui de déterminer le nombre chromatique fractionnaire maximal sur une classe de graphes épars dont seul le nombre de sommets est connu. Nous prouvons des bornes supérieures non triviales pour les graphes sans triangle, et pour les graphes dont chaque sommet appartient à un nombre borné de triangles.

L'intégralité de cette thèse est fortement reliée aux nombres de Ramsey. À ce jour, le meilleur encadrement connu sur le nombre de Ramsey $R(3,t)$ nous est donné par le sus-mentionné résultat de Shearer pour la borne supérieure, et par une analyse récente du processus sans triangle [16, 52] pour la borne inférieure, ce qui donne

$$\frac{t^2}{4\ln t} \lesssim R(3,t) \lesssim \frac{t^2}{\ln t}. \qquad (*)$$

Beaucoup de nos résultats ne pourraient être améliorés à moins d'améliorer par la même occasion $(*)$, ce qui constituerait une révolution dans la théorie de Ramsey quantitative.

Abstract in Dutch

Dit proefschrift gaat over generalisaties van het kleuringprobleem in verscheidene klassen van grafen met lage dichtheid.

Driehoekvrije grafen met begrensde maximale graad Δ vormen een belangrijke klasse van zulke grafen. Vanwege een resultaat van Shearer [108] hebben zij onafhankelijkheidsratio ten minste $(1 - o(1)) \ln \Delta/\Delta$. Bovendien is het kleuringsgetal van deze grafen ten hoogste $O(\Delta/\ln \Delta)$, voor $\Delta \to \infty$, vanwege een stelling van Johansson [65] die recent is verbeterd door Molloy [90]. Molloy heeft aangetoond dat het kleuringsgetal van driehoekvrije grafen met maximum graad Δ ten hoogste $(1 + o(1))\Delta/\ln \Delta$ is, voor $\Delta \to \infty$.

Het resultaat van Molloy is uitgedrukt in termen van een globale parameter: de maximale graad. Het is echter mogelijk om zijn stelling te generaliseren naar een bepaald soort *lokale kleuringen*, namelijk lijstkleuringen waarbij de grootte van de lijst geassocieerd met een gegeven knoop alleen afhangt van de graad van die knoop. Met behulp van een andere methode en onder zwakkere aannames verkrijgen we vergelijkbare resultaten voor *lokale fractionele kleuringen*. Deze andere methode is gebaseerd op de eigenschappen van de *hard-core verdeling* op onafhankelijke verzamelingen van een graaf. We geven ook een analoog resultaat betreffende lokale fractionele kleuringen van grafen waarin elke knoop is bevat in een begrensd aantal driehoeken, alsmede een scherpe grens op de verwachte grootte van een onafhankelijke verzameling in zulke grafen. In het bijzonder beschouwen we grafen met kleine maximale graad waarin de kleinste cykel ten minste lengte 7 heeft. Zulke grafen onderwerpen we aan een meer gedetailleerde analyse: we verbeteren de voorheen beste bekende grenzen, onder voorwaarde dat de maximum graad niet meer dan 10^7 is. Ten slotte verkrijgen we nieuwe ondergrenzen op de onafhankelijkheidsratio van d-reguliere grafen waarvoor $d \in \{3, 4, 5\}$ en de kortste cykellengte g tussen 6 en 12 ligt.

Het tweede hoofdstuk is gewijd aan afstandskleuringen van grafen, een veralgemening van sterke tak-kleuringen. Als voortzetting op het thema van het eerste hoofdstuk onderzoeken we minimale dichtheidscriteria die leiden tot Johansson-achtige resultaten voor afstandskleuringen. Johanssons stelling gaat over het verbieden van driehoeken – of eigenlijk cykels van elke gegeven lengte. Wij laten echter zien dat het verbieden van cykels met even lengte ℓ, met $\ell \geq 2t + 2$, een zelfde effect heeft op het afstand-t kleuringsgetal en de afstand-t kleuringsindex. Indien t oneven is dan geldt het zelfde voor het afstand-t kleuringsgetal van grafen zonder cykel van oneven lengte $\ell \geq 3t$. We onderzoeken de scherpheid van onze resultaten aan de hand van combinatorische, algebraïsche en probabilistische constructies.

In het derde hoofdstuk zijn we geïnteresseerd in de bipartiete geïnduceerde dichtheid van driehoekvrije grafen. Deze parameter ligt tussen de onafhankelijkheidsratio en het fractionele kleuringsgetal. We zijn gemotiveerd door een vermoeden van Esperet, Kang, and Thomassé [49], volgens welke de bipartiete geïnduceerde dichtheid van een driehoekvrije graaf met gemiddelde graad d van orde $\Omega(\ln d)$ zou moeten zijn. We bewijzen dat dit vermoeden waar is voor d groot genoeg, namelijk voor $d = \Omega\left(\sqrt{n \ln n}\right)$. Ons resultaat is scherp op een multiplicatieve constante na,

hetgeen we hebben aangetoond met een probabilistische constructie gebaseerd op het zogenaamde driehoekvrije proces. Ons werk aan de bipartiete geïnduceerde dichtheid leidt tot een ander interessant probleem, waarin het doel is om het fractionele kleurgetal van boven te begrenzen in termen van enkel het aantal knopen van de graaf. We bewijzen niet-triviale bovengrenzen voor driehoekvrije grafen en, algemener, voor grafen waarin iedere knoop onderdeel is van slechts een begrensd aantal driehoeken.

Dit gehele proefschrift is nauw gerelateerd aan de buitendiagonale Ramseygetallen $R(3,t)$. Tot op heden worden de beste bekende bovengrenzen voor $R(3,t)$ gegeven door het eerdergenoemde resultaat van Shearer. De beste bekende ondergrenzen zijn afkomstig van een recente analyse van het driehoekvrije proces [16, 52]. Samen geeft dit

$$\frac{t^2}{4\ln t} \lesssim R(3,t) \lesssim \frac{t^2}{\ln t}. \qquad (*)$$

Veel van onze resultaten zijn best mogelijk, modulo een hypothetische verbetering van $(*)$, hetgeen een doorbraak in quantitatieve Ramseytheorie zou betekenen.

Summary

Graph colouring

In 1852, the young student Francis Guthrie noticed while trying to colour the map of counties of England that he could do it with only 4 colours, all the while respecting the rule that two counties sharing a border should have different colours — we call this a *proper colouring*. He wondered if this could be done with any map one could imagine, and proposed this problem to Augustus DeMorgan through his brother, one of his students at University College London. The problem rapidly aroused the curiosity of many mathematicians, until Alfred Kempe proposed a proof of the 4-colour theorem in 1879, followed by another proof by Peter Guthrie Tait in 1880. However, both proofs turned out to be false after 11 years each. It took more than a century before a valid proof of the 4-colour theorem was proposed by Kenneth Appel and Wolfgang Haken in 1976, who used a framework which had been developed during the previous decade known as the *discharging method*. This did not however settle the 4-colour theorem in the mind of everyone; their proof relies on a computer program, which would check thousand of small statements. It is so long that it is inconceivable to complete it by hand. To this date, the proof has known several simplifications, but a human-checkable proof has yet to be found.

The 4-colour theorem has been a challenge for mathematicians, but this is only one problem in a much larger framework, *graph colouring*. A graph consists of an abstract set of *vertices*, together with a set of *edges* which connect some pairs of those vertices. The graph corresponding to a given map would have a vertex representing each region, and there would be an edge connecting every two regions sharing a border. In such a graph, the edges do not cross themselves; we say that the graph is *planar*. While it is possible to properly colour planar graphs with only 4 colours, this is not the case for every graphs. Finding a proper colouring of a given graph with as few colours as possible is known as the *colouring problem*, and the number of colours in such an optimal proper colouring is called the *chromatic number* of the graph. The colouring problem has an impressive range of applications in many branches of science, not only mathematics. The efficiency of a computer processor, the number of tracks needed in a railway station, the number of frequencies that our smartphones should be able to capture, all those seemingly unrelated topics depend on a colouring problem.

The main issue with the colouring problem is that it is NP-hard to solve, or even to approach with a non optimal solution. This implies that it is highly unlikely that we will ever be able to come up with an efficient way of finding an optimal colouring of any given graph. If we assume that $P \neq NP$, as it is widely believed in the mathematical community, such a goal is even impossible to achieve. For this reason, a large proportion of research in graph colouring is devoted to finding large classes of graphs for which either the colouring problem is easier, or the chromatic number is significantly lower. This is the case of *sparse graphs*, graphs with few edges.

Sparse graphs and random colourings

A sparse graph is a graph with few edges, either globally speaking, or at a local level. The most notorious class of sparse graphs is the one of *triangle-free graphs*, that is graphs where no triplet of vertices are pairwise connected. One could also think of graphs of large girth (graphs which do not contain a small cycle), graphs of bounded clique number (no more than a bounded number of vertices can be all pairwise connected), graphs of small average degree (each vertex is incident with a small average number of edges), graphs with few triangles, and many more. Sparse graphs share the property than they can be properly coloured with significantly fewer colours than what could be expected in general. The most general upper bound which holds for the chromatic number of a graph depends on its *maximum degree*. Indeed, a graph with maximum degree Δ can be coloured using a greedy algorithm with never more than $\Delta + 1$ colours.

This first bound can be reduced by 1 for almost every graph; the only exceptions are complete graphs and odd cycles. This is known as Brook's theorem. This improved bound is still far from capturing the general behaviour of a graph with maximum degree Δ, since most of them can actually be coloured with $(1 + o(1))\Delta/(2 \ln \Delta)$ colours as $\Delta \to \infty$. Indeed, this holds for binomial random graphs with probability tending to 1 as the number of vertices grows to infinity. So, given any graph of maximum degree Δ, it is most likely possible to colour it with $(1 + o(1))\Delta/(2 \ln \Delta)$ colours; however we have no characterisation of the graphs which do not have this property. What we do know is that triangle-free graphs are not much harder to colour in general than random graphs. A first hint of this fact was provided by Shearer in 1983 [108], improving on [3]; he proved that the independence ratio of triangle-free graphs of average degree d is at least $(1+o(1)) \ln d/d$ as $d \to \infty$. The extension of this result to the chromatic number of triangle-free graphs of maximum degree Δ was proved by Johansson in 1996 [65] up to some multiplicative constant, and recently Molloy [90] improved this upper bound to $(1 + o(1))\Delta/ \ln \Delta$, as $\Delta \to \infty$. This fact on triangle-free graphs is central in this thesis. We will see on several aspects how the random graphs provide insights on the behaviour of triangle-free graphs, and of other classes of sparse graphs. This comes mainly from the fact that random processes tend to be as efficient on triangle-free graphs as on random graphs, for the emptiness of the neighbourhoods ensures a sufficient relative independence between the different random events at play. Indeed, the proofs of Johansson and Molloy both rely on random colouring procedures.

Hard-core distribution

During the first chapter, we make a great use of the hard-core distribution in order to derive our results. This distribution is widely studied in statistical physics, and it turns out that its special properties are particularly useful in the context of graph colouring when it is applied on the independent sets of a graph. Writing $\mathcal{I}(G)$ for the collection of independent sets of a given graph G, a random independent set $\mathbf{I}$ drawn according to the *hard-core distribution at fugacity* λ on $\mathcal{I}(G)$ satisfies that, for every independent set $I \in \mathcal{I}(G)$,

$$\Pr[\mathbf{I} = I] = \frac{\lambda^{|I|}}{Z_\lambda(G)}, \quad \text{where } Z_\lambda(G) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$$

is a normalising factor, called the *independence polynomial* of G.

The hard-core distribution has a very useful spatial Markov property. This informally means

that the hard-core distribution behaves independently from the outside at a local level. More formally, given any subset of vertices X, if we condition on the fact that $\mathbf{I} \setminus X = J$, for any possible realisation J of $\mathbf{I} \setminus X$, then the variable $\mathbf{I} \cap X$ follows a hard-core distribution at fugacity λ on the independent sets of $G[X \setminus N(J)]$. A consequence of this property is that it is possible to analyse an independent set $\mathbf{I}$ drawn according to the hard-core distribution at a local level. Such a local analysis [34] demonstrates that $\mathbb{E}[|\mathbf{I}|] \geq (1-\varepsilon)n \ln d/d$ on any n-vertex triangle-free graph of maximum degree d large enough, by fixing $\lambda = 1/\ln d \ll 1$. This implies that the average size of an independent set in such a graph is at least $(1-\varepsilon)n \ln d/d$.

Throughout Chapter 1, we show how the hard-core distribution can be used in order to derive similar results for fractional colourings. To this end, we extend a greedy fractional colouring algorithm first introduced in a book by Molloy and Reed [93], which relies on an input probability distribution on the independent sets of a graph. The algorithm constructs a fractional colouring of a given graph G which relies on a weighting function $\hat{w}\colon \mathcal{I}(G) \to [0,1]$ on the independent sets of G, such that every vertex belongs to independent sets of total weight at least 1. We write $\hat{w}(G)$ for the sum of the weights of all the independent sets of G. It is then possible to assign to every independent set I a measurable subset $w(I)$ of the interval $[0, \hat{w}(G)]$, of measure $\mu(w(I)) = \hat{w}(I)$; this assignment is what we call the fractional colouring of G of weight $\hat{w}(G)$. The algorithm works as follows: at each step, it increases the weight of every independent set of the graph by a value proportional to its probability in the input probability distribution, in such a way that the weight induced on every vertex is at most 1, and is exactly 1 for some vertices. It then repeats this operation on the graph obtained after removing the vertices on which the induced weight is 1, until reaching the empty graph, which means that every vertex has been entirely coloured.

This algorithm lets us extend the local occupancy of the probability distribution into a fractional colouring. Namely, if in any induced subgraph H of a given graph G, for every vertex v and a random independent set $\mathbf{I}_H$ drawn according to the input probability distribution on the independent sets of H, it holds that

$$\sum_{i=0}^{r} \alpha_i(v) \mathbb{E}\left[\left|\mathbf{I}_H \cap N_H^i(v)\right|\right] \geq 1, \tag{1}$$

for a well-chosen set of parameters $(\alpha_i(v))_{i=0}^r$, then there is a fractional colouring of G where every vertex v is coloured with a measurable subset of the interval $\left[0, \sum_{i=0}^r \alpha_i(v)\,|N_G^i(v)|\right]$. Fix $\varepsilon > 0$. If G is a graph of maximum degree d, the greedy fractional colouring algorithm with the hard-core distribution at fugacity λ produces a fractional colouring of G of weight at most

(i) $(1+\varepsilon)d/\ln d$ if G is triangle-free by fixing $\lambda = \varepsilon/2$, provided that $d \geq g_\varepsilon$ for some d_ε which depends only on ε;

(ii) $(1+\varepsilon)d/\ln(d/\sqrt{T})$ if every vertex of G is contained in at most $T \geq 1$ triangles, by fixing $\lambda = \min\{\varepsilon, 1/\sqrt{T}\}$, provided that $d \geq d_\varepsilon \sqrt{T}$ for some d_ε which depends only on ε;

(iii) $\min\limits_{k \in \mathbb{Z}_{\geq 4}} \left(2d + 2^{k-3} + k\right)\big/\, k$ if G is of girth at least 7, by fixing $\lambda = 4$ and restricting the hard-core distribution to *maximal* independent sets of G.

We also prove the counterparts of (i) and (ii) for the occupancy fraction of G, that is the expected size of an independent set of G. These are mainly sharp, as illustrated by random constructions.

In order to obtain the above fractional colouring results on a graph G, we need that every induced subgraph of G satisfies (1). For this reason, increasing the depth r cannot improve our

results with this method, since all the constraints present at depth $r' < r$ would also be present at depth r, and thus increasing the girth of the class of graphs considered does not yield better bounds in this setting. However, it is possible to use a similar method without the need to consider induced subgraphs, in order to derive lower bounds on the independence ratio of graphs. Now, using the hard-core distribution with $\lambda = \infty$, that is the uniform distribution on maximum independent sets, we are able to improve the lower bounds for the independence ratio of d-regular graph with $d \in \{3, 4, 5\}$, of girth g varying between 6 and 12.

d \ g	6	7	8	9	10	11	12
3	$30/11 \approx 2.727272$	$30/11$	2.625224	2.604167	2.557176	2.539132	2.510378
4	$41/13 \approx 3.153846$	$41/13$	3.038497	3.017382	3		
5	$69/19 \approx 3.631579$	3.6	3.5				

Table 1: Upper bounds on the inverse of the independence ratio.

Local colourings

The results of Johansson and Molloy actually hold for a stronger form of the colouring problem, the list colouring problem. In this setting, each vertex has a private list of available colours which it may be coloured with. The colouring problem is a special case of the list colouring problem, but in general a proper list colouring may require lists of size much more than the chromatic number of a given graph. The minimum size of the lists always ensuring the existence of a proper list colouring is called the *choosability* of the graph. For instance, *bipartite graphs* of minimum degree d have choosability at least $(1 - o(1)) \ln d$.

In most real-life applications of list colourings, the choosability of the graph corresponds to the number of resources that must be available for each vertex of the graph in order to ensure that they can all be coloured. In the setting of list colourings, all vertices are treated equally, even though the graph may be highly asymmetric, and some of its vertices more constrained than the others. Now, what if we wish to assign lists of different sizes to the vertices, which depend on the level of constraints of each vertex? This kind of question was first considered in one of the originating papers for list colouring [41], where there is a characterisation of degree-choosable graphs, graphs which can be properly L-coloured for any list assignment L such that $|L(v)| = \deg(v)$ for every vertex v. The graphs which are not degree-choosable are exactly the *Gallai forests*, graphs where every block is either a clique or an odd cycle.

In the first chapter of this thesis, we are interested in *local colourings*. A local list colouring is a proper list colouring from any list assignment which satisfies the property that the list assigned to a given vertex should have a size which depends only on the local properties of that vertex. We first prove a generalisation of Molloy's result to local list colourings. Given any $\varepsilon > 0$, every triangle-free graph G of maximum degree Δ can be properly L-coloured from any list assignment L where $|L(v)| = (1+\varepsilon)d/\ln d$ for every vertex v of degree $d \geq d_\varepsilon = (72 \ln \Delta)^{2/\varepsilon}$. Vertices of degree $d \leq d_\varepsilon$ should be assigned a list of size $\min\{(1+\varepsilon)d_\varepsilon/\ln d_\varepsilon, d+1\}$. We actually prove this statement for DP-colourings, a generalisation of list colourings. We additionally provide a construction of

a bipartite graph where this special treatment of vertices of small degree is necessary in order to find a local list colouring, where the value of d_ε is indeed necessarily increasing with that of Δ.

We also consider *local fractional colourings*, which are fractional colourings c satisfying that the colours assigned to every vertex v is a measurable subset of the interval $[0, \gamma(v)]$, where $\gamma(v)$ depends only on the local properties of v, namely its degree and the density of its neighbourhood. All the results stated in the previous paragraph on fractional colouring also hold in the local setting. While the fractional chromatic number equals the fractional choosability of any given graph, it is not clear whether the existence of a local fractional colouring would ensure the existence of a local fractional list colouring for any list assignment of corresponding measure.

Distance colouring

The *strong chromatic index* of a graph is the smallest size of a partition of its edges into induced matchings — we call such a partition a *strong edge-colouring*. The strong chromatic index of a graph G can be equivalently defined as the chromatic number of $L(G)^2$, the square of the line graph of G. One possible application of strong edge-colourings lies in the channel allocation schemes of wireless networks. In 1985, Erdős and Nešetřil asked for the maximum value of the strong chromatic index of any graph of maximum degree Δ. They conjecture that this is $5/4 \cdot \Delta^2$, reached by the blown-up 5-cycle, and in particular asked whether it is possible to obtain an upper bound of the form $(2-\varepsilon)\Delta^2$ for some $\varepsilon > 0$, which would improve the trivial upper bound of $2\Delta^2$ corresponding to the maximum degree of the square of the line graph. This latter question was first answered in the affirmative by Molloy and Reed [91] in 1997, who showed that $1.998\Delta^2$ is a valid upper bound, provided that Δ is large enough. They obtained this result by showing that the square of a line graph is relatively sparse, namely the number of edges in any of its neighbourhoods is no more than $(1-\delta)\binom{2\Delta^2}{2}$, and they prove it with $\delta = 1/36$. A graph with such a sparsity condition can be coloured with a reduced number of colours through a random procedure. Their method has since been improved; notably the sparsity result has been improved up to asymptotic optimality with $\delta = 1/4 + o(1)$ as $\Delta \to \infty$ [25]. To this date, the best known general upper bound is $1.835\Delta^2$, provided that Δ is large enough [20]. On the other hand, a Johansson-like result due to Mahdian [87] states that the strong chromatic index of graphs of maximum degree Δ with excluded C_4 is at most $(2+\varepsilon)\Delta^2/\ln\Delta$, for any $\varepsilon > 0$, provided that Δ is large enough.

It is possible to generalise the notion of strong chromatic index to distance-t chromatic index of a graph G, which is the chromatic number of $L(G)^t$, the t-th power of the line graph of G. Likewise, we define the distance-t chromatic number as the chromatic number of G^t, the t-th power of the graph G. In Chapter 2, we analyse the framework of distance colourings, and in particular seek for a generalisation of the result of Mahdian to any distance t. As a direct application of the sparse graphs colouring results discussed in Chapter 1, we are able to demonstrate that the distance-t chromatic index of C_{2k}-free graphs of maximum degree Δ, for any $k \geq t$, is at most $(8+\varepsilon)\Delta^t/\ln\Delta$, provided that Δ is large enough as a function of t and ε. When $t \in \{2, 3, 4, 6\}$, there exist bipartite graphs of maximum degree Δ, girth $2t$, and distance-t chromatic index $(1-o(1))\Delta^t$ as $\Delta \to \infty$, for infinitely many values of Δ. Those are the incidence graphs of projective geometries. Concerning distance-t vertex-colourings, we show that the distance-t chromatic number of a C_ℓ-free graph of maximum degree Δ, for any even $\ell \geq 2t+2$, or any odd $\ell \geq 3t$ provided that t is odd, is at most $(4+\varepsilon)\Delta^t/\ln\Delta$, for any $\varepsilon > 0$, provided that Δ is large enough as a function of t and ε. When $t \geq 11$, we provide constructions of even maximum degree Δ, girth 8, and distance-t chromatic number $(\Delta/2)^t$. Moreover, these constructions are bipartite if t is even. When t is odd, we provide

constructions of maximum degree Δ, odd-girth $3t$, and distance-t chromatic number $(\Delta/3)^t$. All the Johansson-like results in this chapter are sharp up to a multiplicative constant because of random graphs.

Bipartite induced density

The *bipartite induced density* $\mathrm{bid}(G)$ of a graph G is defined as the maximum average degree of a bipartite induced subgraph of G. This parameter was introduced by Esperet, Kang, Thomassé in the study of separation choosability [49]. They conjectured that the bipartite induced density of a triangle-free graph of average degree d should be at least $C \ln d$ for some constant $C > 0$. The conjecture remains open in its general formulation, but a marginally weaker lower bound of the form $C_r \ln d / \ln \ln d$ has been shown to hold for the bipartite induced density of K_r-free graphs, for any fixed $r \geq 3$ [83].

Finding an induced bipartite subgraph of a given graph G consists in finding two independent sets of G, not necessarily disjoint. In addition, it is required that the subgraph induced by two two independent sets is as dense as possible; for this reason one can think of the problem of determining the bipartite induced density of a graph as conceptually harder than that of computing the independence ratio. On the other hand, we show that the two independent sets inducing the densest bipartite subgraph among the independent sets used in a fractional colouring of weight k of a graph G of average degree d certify that the bipartite induced density of G is at least d/k. So the bipartite induced density problem is conceptually easier to analyse than the fractional colouring problem. In particular, the fractional colouring counterpart of the conjecture on the bipartite induced density is harder to solve — its statement is that any triangle-free graph of maximum average degree d has fractional chromatic number at most $Cd/\ln d$, for some absolute constant C; this was posed by Harris [59].

We prove the bipartite induced density conjecture when d is sufficiently large in terms of the number of vertices n of the graph, namely $d = \Omega\Big(\sqrt{n \ln n}\Big)$. Actually, in this regime, the bipartite induced density of triangle-free graphs is always at least $d^2/n = \Omega(\ln d)$, and we demonstrate that this is sharp up to a $O(\ln n)$ factor, by a pseudo-random construction. Solving the conjecture in this setting reduces to a nice fractional colouring problem, that of determining the maximum possible fractional chromatic number of a triangle-free graph on n vertices. We demonstrate that this is at most $(2 + o(1))\sqrt{n/\ln n}$ as $n \to \infty$, and once again random graphs provide the best known lower bound for this question. The triangle-free process constructs a triangle-free graph on a given number n of vertices by drawing every possible pair of vertices in a random order, and adding such a pair as an edge to the graph under the condition that it does not create a triangle. The random graph obtained by the triangle-free process is a maximal triangle-free graph. With high probability, it is of maximum degree $(1 + o(1))\sqrt{n \ln n/2}$; its independence number is $(1+o(1))\sqrt{2n \ln n}$, and its fractional chromatic number is $(1/\sqrt{2} - o(1))\sqrt{n/\ln n}$, as $n \to \infty$. This demonstrates that our result is sharp up to a $2\sqrt{2}$ asymptotic factor.

Close links to off-diagonal Ramsey numbers

It is always possible to find regular structures in large enough graphs. A formalisation of this conceptual statement was shown by Ramsey in 1930 [103]; Ramsey's theorem states that given any integers s and t, there exists an integer $R(s,t)$ called the *Ramsey number* such that any

graph on at least $R(s,t)$ vertices contains either a clique of size s or an independent set of size t, while there are graphs on $R(s,t)-1$ vertices avoiding both. This theorem is fundamental in combinatorics, and led to the emergence of Ramsey theory, which captures many problems that search for various kinds of structural properties in large graphs.

Determining the exact value of $R(s,t)$ is a notorious problem; there is still a multiplicative factor of 4 between the best known lower and upper bounds of the logarithm of the diagonal Ramsey number $\ln R(s,s)$. There is no known explicit construction of graphs demonstrating an exponential lower bound on $R(s,s)$; only random graphs certify that it has an exponential growth in our current knowledge. Particular attention has been paid to *off-diagonal Ramsey numbers* $R(s,t)$ where s is fixed and t grows. The value of $R(3,t)$ is directly related to the independence ratio of triangle-free graphs, so its best estimations provide a standard for the sharpness of many of our results in this thesis. By a direct application of Shearer's result on the independence ratio of triangle-free graphs, and a recent analysis of the triangle-free process [16, 52], we know that

$$\frac{t^2}{4\ln t} \lesssim R(3,t) \lesssim \frac{t^2}{\ln t}. \qquad (*)$$

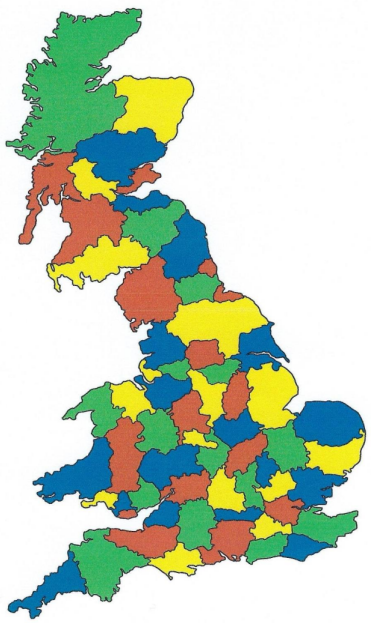
The whole content of this thesis is linked to the off-diagonal Ramsey numbers. It extends the study of off-diagonal Ramsey numbers in many directions, by excluding cycles rather than cliques, by bounding the number of triangles rather than excluding them, and by adding information on the independent sets, both qualitative (a colouring covers all the vertices of the graph with independent sets) and quantitative (we sometimes have information on the average size of independent sets). Many of these extensions are best possible barring an improvement of $(*)$, which would be a breakthrough in Ramsey theory.

Résumé substantiel en français

Coloration de graphes

En 1852, alors qu'il était encore étudiant, Francis Guthrie remarqua qu'il était possible de colorier les comtés anglais avec seulement 4 couleurs différentes, tout en s'assurant de suivre la règle élémentaire que deux comtés ayant une frontière commune doivent être coloriés avec deux couleurs différentes — c'est ce que l'on appelle une *coloration propre*. Il se demanda alors si cela pouvait se faire avec n'importe quelle carte concevable. Il proposa ce problème à Augustus DeMorgan par l'intermédiaire de son frère, qui était un de ses étudiants à l'University College de Londres. Le problème attira rapidement l'attention de bon nombre de mathématiciens, jusqu'à ce qu'Alfred Kempe propose une preuve du théorème des 4 couleurs en 1879, suivi d'une autre preuve par Peter Guthrie Tait l'année suivante. Toutefois, les deux preuves finirent par se révéler fausses, après avoir résisté à toute forme de contradiction pendant 11 ans chacune. Il fallut plus d'un siècle avant qu'une preuve valide du théorème des 4 couleurs ne fût proposée par Kennet Appel et Wolfgang Haken en 1976, qui usèrent d'une méthode développée pendant la décennie précédente, désormais bien connue sous le nom de *méthode de déchargement*. Leur preuve ne fit cependant pas l'unanimité au sein de la communauté mathématique, notamment car elle reposait sur un programme informatique, qui vérifiait la validité de plusieurs milliers de résultats intermédiaires. Cette preuve est si longue qu'il n'est pas concevable de l'écrire à la main, ni même de la lire dans son entièreté. À ce jour, de nombreuses simplifications y ont été apportées, mais aucune preuve humainement accessible n'a encore été trouvée, malgré une recherche encore active.

Le théorème des 4 couleurs fut un défi remarquable pour les mathématiciens, et pourtant il ne s'agit que d'un problème isolé au sein d'une très riche famille de problèmes liés à la *coloration de graphes*. Un graphe est représenté par deux ensembles abstraits ; le premier contient ses *sommets*, et le second regroupe un certain nombre d'*arêtes*, qui relient chacune une paire de sommets. Le graphe associé à une carte représente chaque région par un sommet, et contient une arête entre chaque paire de sommets représentant deux régions avec une frontière commune. Il est possible de dessiner un tel graphe de sorte que ses arêtes ne se croisent pas ; on dit que le graphe est *planaire*. Tandis qu'il est possible de colorier proprement n'importe quel graphe planaire avec seulement 4 couleurs, c'est loin d'être le cas pour tous les graphes. Le *problème de coloration* consiste alors à déterminer le nombre de couleurs minimal nécessaire à toute coloration propre d'un graphe donné, c'est ce qu'on appelle le *nombre chromatique* du graphe. La plupart du temps, on cherche évidemment également à exhiber une coloration propre optimale qui utilise exactement ce nombre de couleurs. Le problème de coloration a un nombre impressionnant d'applications dans beaucoup de domaines scientifiques, et pas uniquement en mathématiques. L'efficacité d'un processeur, le nombre de voies nécessaires dans une gare, le nombre de fréquences que nos smartphones doivent être capables de recevoir, tous ces domaines en apparence sans rapport se ramènent à un problème de coloration.



Une coloration possible avec 4 couleurs des comtés anglais.

Au grand dam des scientifiques, le problème de coloration est NP-dur à résoudre, et il en va de même pour trouver une solution non optimale au problème de coloration. Cela signifie qu'il est fortement improbable que l'on parvienne à trouver une méthode efficace permettant d'établir le nombre chromatique de n'importe quel graphe, et d'en trouver une coloration optimale. Quitte à supposer que $P \neq NP$, ce que la grande majorité des mathématiciens croient être vrai, une telle méthode ne peut purement et simplement pas exister. C'est pour cela que la recherche en coloration de graphe s'évertue principalement à établir de grandes classes de graphes pour lesquelles ou bien le problème de coloration est relativement plus simple à résoudre, ou bien le nombre chromatique est relativement plus petit que dans le cas général. C'est notamment le cas des *graphes épars*, les graphes dont la proportion d'arêtes est relativement faible.

Graphes épars et coloration aléatoire

Un graphe épars est un graphe contenant peu d'arêtes, soit globalement, soit à une échelle locale. La classe de graphes épars la plus étudiées est certainement celle des *graphes sans triangle*, autrement dit les graphes dont aucun ensemble de trois sommets n'est entièrement connecté par des arêtes. Mais il existe de nombreuses autres classes de graphes épars très classiques, la classe des graphes de grande maille (les graphes ne contenant pas de petit cycle), celle des graphes de clique bornée (seul un nombre borné de sommets peuvent être entièrement connectés deux-à-deux), celle des graphes de petit degré moyen (en moyenne, chaque sommet est incident à un petit nombre d'arêtes), celle des graphes contenant un nombre borné de triangles, et tant d'autres. Les graphes épars sont remarquables par le fait qu'ils ont tous une coloration propre qui utilise un nombre de couleurs très inférieur à ce qui peut être attendu dans le pire des cas. En particulier, il est toujours possible de majorer le nombre chromatique d'un graphe en fonction de son *degré maximal*. En effet, il est possible de trouver une coloration propre d'un graphe de degré maximal Δ n'utilisant que $\Delta + 1$ couleurs, à l'aide d'un algorithme glouton.

Cette première borne peut être réduite de 1 pour tous les graphes, à l'exception des graphes

Algorithm 1 Algorithme glouton de coloration

Require: Un graphe G sur des sommets $\{v_1, \ldots, v_n\}$
Ensure: c est une coloration propre de G utilisant k couleurs
  $k \leftarrow 1, i \leftarrow 1$
  while $i \leq n$ **do**
    if Toutes les couleurs apparaissent sur les sommets adjacents à v_i **then**
      $k \leftarrow k + 1$
      $c(v_i) \leftarrow k + 1$
    else
      $c(v_i) \leftarrow a$, où a est la couleur minimale n'apparaissant pas parmi les sommets adjacents à v_i
    end if
    $i \leftarrow i + 1$
  end while
  Return c

complets et des cycles impairs. Il s'agit du théorème de Brook. Cette borne améliorée est toutefois encore très loin de décrire le comportement typique des graphes de degré maximal Δ, puisque la plupart ont nombre chromatique au plus $(1 + o(1))\Delta/(2 \ln \Delta)$ quand $\Delta \to \infty$. En effet, cette propriété est vraie pour les graphes aléatoires binomiaux avec probabilité tendant vers 1 tandis que le nombre de sommets tend vers l'infini. Ainsi, étant donné un graphe de degré maximal Δ, il est très improbable que son nombre chromatique soit supérieur à $(1+o(1))\Delta/(2 \ln \Delta)$. Cependant, il n'existe pas de caractérisation précise des graphes qui n'ont pas cette propriété. En revanche, les graphes sans triangles peuvent tous être proprement coloriés avec un nombre de couleurs proche de ce qui est attendu en général pour les graphes aléatoires. En 1983, Shearer [108] propose un premier indice de cette propriété, en prouvant que le *ratio d'indépendance* des graphes sans triangle de degré moyen d est au moins $(1 - o(1)) \ln d/d$ quand $d \to \infty$, améliorant ainsi le résultat de [3]. L'extension de ce résultat au nombre chromatique des graphes sans triangle de degré maximal Δ a été prouvée par Johansson en 1996 [65] à une constante multiplicative près, puis récemment améliorée par Molloy [90] qui démontra la borne de $(1 + o(1))\Delta/ \ln \Delta$, quand $\Delta \to \infty$. Cette propriété des graphes sans triangle est cruciale dans cette thèse. Nous aurons l'occasion de constater que les graphes aléatoires offrent de nombreuses intuitions sur le comportement des graphes sans triangle, et d'autres classes de graphes épars. Cela est principalement dû au fait que les processus aléatoires ont tendance à avoir une efficacité comparable sur les graphes sans triangle et sur les graphes aléatoires, de par le fait que, chaque voisinage de sommet étant vide, une relative indépendance entre les événements aléatoires entrant en jeu est garantie. Et en effet, les preuves de Johansson et de Molloy reposent toutes deux sur une procédure de coloration aléatoire, qui est démontrée suffisamment efficace.

Distribution hard-core

Au cours du premier chapitre, nous utilisons principalement la distribution hard-core afin d'en déduire plusieurs de nos résultats. L'utilisation de cette distribution de probabilité est très répandue dans le domaine de la physique statistique, et il se trouve que ses propriétés très spéciales se révèlent particulièrement utiles à l'étude de la coloration de graphe, en appliquant cette distribu-

tion aux ensembles indépendants d'un graphe. Étant donné un graphe G, notons $\mathcal{I}(G)$ l'ensemble des ensembles indépendants de G. Un ensemble indépendant aléatoire $\mathbf{I}$ tiré selon la *distribution hard-core de fugacité* λ sur $\mathcal{I}(G)$ est telle que, pour tout ensemble indépendant $I \in \mathcal{I}(G)$,

$$\Pr[\mathbf{I} = I] = \frac{\lambda^{|I|}}{Z_\lambda(G)}, \quad \text{où } Z_\lambda(G) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$$

est un facteur de normalisation, appelé le *polynôme d'indépendance* de G.

La distribution hard-core a une propriété spatiale de Markov très utile. De façon informelle, cela signifie que la distribution hard-core se comporte indépendamment de l'extérieur à une échelle locale. Plus formellement, étant donné un sous-ensemble de sommets X, si l'on conditionne sur le fait que $\mathbf{I} \setminus X = J$, pour n'importe quelle réalisation possible J de $\mathbf{I} \setminus X$, alors la variable $\mathbf{I} \cap X$ suit une distribution hard-core de fugacité λ sur les ensembles indépendants de $G[X \setminus N(J)]$. Cette propriété a pour conséquence qu'il est possible d'analyser un ensemble indépendant $\mathbf{I}$ tiré selon la distribution hard-core à une échelle locale. Une telle analyse locale [34] permet de démontrer que

$$\frac{1}{n}\mathbb{E}\left[|\mathbf{I}|\right] \geq (1-\varepsilon)\frac{\ln d}{d}$$

sur tout graphe sans triangle à n sommets de degré d suffisamment grand, en fixant $\lambda = 1/\ln d \ll 1$. Cela implique que la taille moyenne d'un ensemble indépendant dans un tel graphe est au moins $(1-\varepsilon)n \ln d/d$.

Tout au long du Chapitre 1, nous montrons qu'il est possible d'utiliser la distribution hard-core afin d'obtenir des résultats similaires sur les colorations fractionnaires. Pour cela, nous étendons un algorithme glouton de coloration fractionnaire qui a été introduit dans un livre de Molloy et Reed [93], et qui repose sur une distribution de probabilité sur les ensembles indépendants d'un graphe. Étant donné un graphe G, cet algorithme construit une coloration fractionnaire de G à partir d'une fonction de pondération $\hat{w}\colon \mathcal{I}(G) \to [0,1]$ définie sur les ensembles indépendants de G, de sorte que chaque sommet appartient à des ensembles indépendants de poids total 1. On note $\hat{w}(G)$ la somme des poids des ensembles indépendants de G. Il est alors possible d'assigner à chaque ensemble indépendant I un sous-ensemble mesurable $w(I)$ de l'intervalle $[0, \hat{w}(G)]$, de mesure $\mu(w(I)) = \hat{w}(I)$; cette assignation est ce que l'on appelle une coloration fractionnaire de G de poids $\hat{w}(G)$.

L'algorithme fonctionne de la sorte : à chaque étape, il incrémente le poids de chaque ensemble indépendant du graphe par une valeur proportionnelle à sa probabilité dans la distribution de probabilité donnée en paramètre de l'algorithme, de sorte que le poids induit en chacun des sommets est au plus 1, et atteint exactement 1 pour au moins un des sommets. Cette opération est alors répétée sur le graphe obtenu après avoir retiré l'ensemble des sommets pour lesquels le poids induit était 1, jusqu'à ce que le graphe obtenu soit vide, ce qui signifie que tous les sommets ont été entièrement coloriés.

Cet algorithme nous permet d'étendre l'occupation locale de la distribution de probabilité en une coloration fractionnaire. Plus explicitement, si étant donné un graphe G, pour tout sous-graphe induit H de G, et pour tout sommet v de H, étant donné un ensemble indépendant aléatoire $\mathbf{I}_H$ tiré selon la distribution de probabilité sur les ensembles indépendants de H donnée en paramètre, on a

$$\sum_{i=0}^{r} \alpha_i(v)\mathbb{E}\left[\left|\mathbf{I}_H \cap N_H^i(v)\right|\right] \geq 1, \tag{2}$$

pour un ensemble de paramètres $(\alpha_i(v))_{i=0}^r$ préalablement fixé, alors il existe une coloration fractionnaire de G où chaque sommet est colorié avec un sous-ensemble mesurable de l'intervalle

$$\left[0, \sum_{i=0}^{r} \alpha_i(v)\left|N_G^i(v)\right|\right].$$

En fixant un certain $\varepsilon > 0$, si G est un graphe de degré maximal d, l'algorithme glouton de coloration fractionnaire exécuté avec la distribution hard-core de fugacité λ en paramètre renvoie une coloration fractionnaire de G de poids au plus

(i) $(1+\varepsilon)d/\ln d$ si G est sans triangle, en fixant $\lambda = \varepsilon/2$, à la condition que $d \geq d_\varepsilon$ pour un certain d_ε qui ne dépend que de ε;

(ii) $(1+\varepsilon)d/\ln(d/\sqrt{T})$ si tout sommet de G est contenu dans au plus $T \geq 1$ triangles, en fixant $\lambda = \min\{\varepsilon, 1/\sqrt{T}\}$, à la condition que $d \geq d_\varepsilon\sqrt{T}$ pour un certain d_ε qui ne dépend que de ε;

(iii) $\min_{k\in\mathbb{Z}_{\geq 4}} (2d + 2^{k-3} + k)/k$ si G est de maille au moins 7, en fixant $\lambda = 4$ et en restreignant la distribution hard-core aux indépendants *maximaux* du graphe G.

Nous établissons également les résultats analogues à (i) et (ii) pour le taux d'occupation de G, c'est-à-dire l'espérance de la taille d'un ensemble indépendant de G. Ces résultats sont principalement optimaux, ce que l'on illustre par des constructions aléatoires.

Afin d'obtenir les résultats sur la coloration fractionnaire d'un graphe G, il est nécessaire que tout sous-graphe induit de G satisfasse (2). À cause de cela, on ne peut obtenir de meilleurs résultats avec notre méthode en augmentant la profondeur r, car toutes les contraintes présentes pour une certaine profondeur $r' < r$ sont également présentes à la profondeur r. Ainsi, augmenter la maille de la classe de graphes étudiée ne nous permet pas d'obtenir de meilleures bornes dans ce contexte. Cependant, il est possible d'utiliser une méthode similaire sans avoir besoin de considérer les sous-graphes induits, afin d'obtenir des bornes inférieures sur le ratio d'indépendance des graphes. Alors, en utilisant la distribution hard-core de fugacité $\lambda = \infty$, c'est-à-dire la distribution uniforme sur les ensemble indépendants maximums, on obtient des bornes inférieures sur le ratio d'indépendance des graphes d-réguliers où $d \in \{3, 4, 5\}$, de maille g variant entre 6 et 12, qui améliorent les bornes précédemment connues.

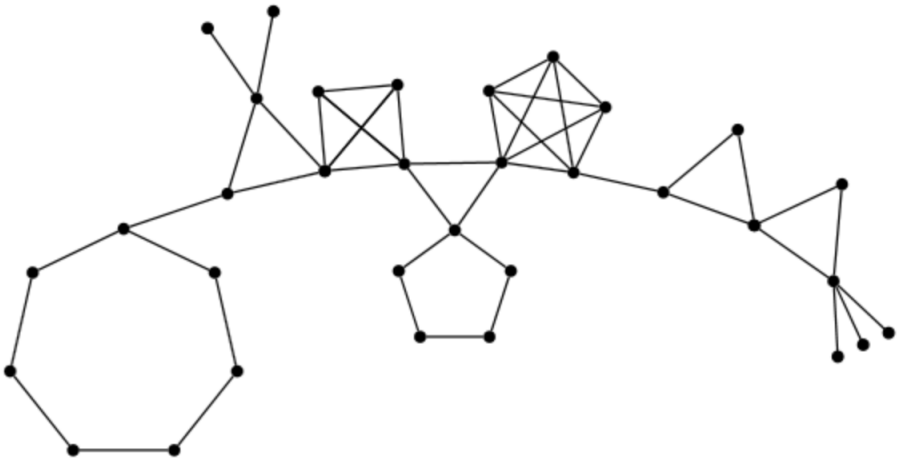
d \ g	6	7	8	9	10	11	12
3	2.727272		2.625224	2.604167	2.557176	2.539132	2.510378
4	3.153846		3.038497	3.017382	3		
5		3.6	3.5				

Table 2: Bornes supérieures sur l'inverse du ratio d'indépendance.

Colorations locales

Les résultats de Johansson et Molloy sont en fait valables pour une version plus forte du problème de coloration, le problème de coloration par liste. Dans le contexte de la coloration par liste, on attribue au préalable à chaque sommet une liste de couleurs qui lui est propre, et on n'autorise le sommet à être colorié qu'avec une des couleurs de sa liste. Tandis que le problème de coloration correspond à une instance particulière du problème de coloration par liste, où toutes les listes sont identiques, la réciproque est fausse puisqu'une coloration propre par liste aura potentiellement besoin de listes bien plus grandes que le nombre chromatique du graphe. La taille minimale requise sur les listes afin d'assurer qu'il existe toujours une coloration propre par liste pour un graphe donné est appelé sa *choisissabilité*. Par exemple, les *graphes bipartis* de degré minimal d ont tous une choisissabilité supérieure à $(1 - o(1)) \ln d$ quand $d \to \infty$.

Dans la plupart des applications de la vie courante, la choisissabilité d'un graphe correspond au nombre de ressources que l'on doit rendre disponibles pour chacun des sommets du graphe afin d'assurer que tous puissent être coloriés. Dans le cadre de la coloration par liste, tous les sommets sont traités de la même manière, bien que le graphe puisse être hautement asymétrique, et ainsi certains de ses sommets bien plus contraints que d'autres. Et si nous souhaitions maintenant assigner à chaque sommet une liste dont la taille ne dépend que des propriétés locales du sommet en question ? Ce type de question a été considéré pour la première fois dans un des articles fondateurs de la coloration par liste [41], qui propose une caractérisation des graphes dits *degré-choisissables*, c'est-à-dire les graphes pouvant être proprement L-coloriés pour toute assignation de listes L vérifiant que $L(v) = \deg(v)$ pour tout sommet v. Ainsi, les graphes qui ne sont pas degré-choisissables sont exactement les *forêts de Gallai*, à savoir les graphes dont chacun des blocs est soit une clique, soit un cycle impair.



Un arbre de Gallai à 15 blocs (illustration empruntée de [29])

Dans le premier chapitre de cette thèse, on s'intéresse aux *colorations locales*. Une coloration locale par liste est une coloration propre par liste étant donnée n'importe quelle assignation de listes qui satisfasse que la taille de la liste assignée à un sommet donné dépende uniquement des propriétés locales de ce sommet. Nous prouvons en premier lieu une généralisation du résultat de Molloy à la coloration locale par liste. Étant donné $\varepsilon > 0$, tout graphe sans triangle G de degré maximum Δ peut être proprement colorié à partir de n'importe quelle assignation de listes L dès

lors que

$$|L(v)| = (1+\varepsilon)\frac{d}{\ln d}$$

pour tout sommet v de degré $d \geq d_\varepsilon = (72 \ln \Delta)^{2/\varepsilon}$. Les sommets de degré $d \leq d_\varepsilon$ doivent être traités différemment; on requiert que leur liste soit de taille

$$\min\left\{(1+\varepsilon)\frac{d_\varepsilon}{\ln d_\varepsilon}\ ,\ d+1\right\}.$$

Notre résultat est en fait valide pour la DP-coloration, une version plus forte de la coloration par liste. Nous exhibons en outre une construction de graphes bipartis où le traitement spécial des sommets de petit degré est nécessaire afin d'assurer l'existence d'une coloration locale par liste, où la valeur de d_ε est effectivement nécessairement croissante en termes de Δ.

Nous considérons également la coloration fractionnaire locale, qui consiste à trouver une coloration fractionnaire c qui soit telle que les couleurs assignées à chaque sommet v forment un sous-ensemble mesurable de l'intervalle $[0, \gamma(v)]$, où $\gamma(v)$ dépend uniquement des propriétés locales de v, en l'occurrence son degré et la densité de son voisinage. Tous les résultats énoncés dans le paragraphe précédent concernant la coloration fractionnaire s'étendent à ce contexte local. Tandis que le nombre chromatique fractionnaire de n'importe quel graphe est toujours égal à sa choisissabilité fractionnaire, cette équivalence n'est absolument pas claire dans le contexte local. Ainsi, rien n'indique que l'existence d'une coloration fractionnaire locale assure l'existence d'une coloration fractionnaire locale par liste, étant donné une assignation de listes de mesures adéquates.

Coloration à distance t

L'*indice chromatique fort* d'un graphe est la taille minimale d'une partition de ses arêtes en des couplages induits — on désigne une telle partition par le nom de *coloration forte des arêtes*. L'indice chromatique fort d'un graphe G peut être défini de manière équivalente comme le nombre chromatique de $L(G)^2$, le carré du line graph de G. Il est possible de trouver une application aux colorations fortes des arêtes dans le cadre de l'allocation de fréquences dans les réseaux sans fil. En 1985, Erdős et Nešetřil posent le problème de la détermination de la valeur maximale possible de l'indice chromatique fort d'un graphe de degré maximal Δ. Ils conjecturent que cette valeur est $\frac{5}{4}\Delta^2$, atteinte en considérant un cycle de longueur 5 dont chaque sommet a été explosé en un ensemble indépendant de taille $\Delta/2$. Ils demandent en particulier s'il est possible d'obtenir une borne supérieure sur cette valeur de la forme $(2-\varepsilon)\Delta^2$, pour un certain $\varepsilon > 0$, ce qui améliorerait la borne triviale de $2\Delta^2$ qui correspond au degré maximal du carré d'un line graph. Une réponse positive à cette dernière question est fournie en 1997 par Molloy et Reed [91], puisqu'ils démontrent que $1.998\Delta^2$ est une borne supérieure valide, à la condition que Δ soit suffisamment grand. Ils obtiennent ce résultat en montrant que le carré d'un line graph est toujours relativement épars, à savoir que le nombre d'arêtes apparaissant dans chacun de ses voisinages est toujours au plus $(1-\delta)\binom{2\Delta^2}{2}$, ce qu'ils montrent avec la valeur $\delta = 1/36$. Un graphe satisfaisant une telle condition peut être proprement colorié avec un nombre réduit de couleurs, à l'aide d'une procédure de coloration aléatoire. Depuis, leur méthode a été améliorée ; en particulier le résultat sur les voisinages épars a été amélioré jusqu'à l'obtention d'une borne asymptotiquement optimale avec $\delta = 1/4 + o(1)$ quand $\Delta \to \infty$ [25]. À ce jour, la meilleure borne générale connue est $1.835\Delta^2$, à la condition que Δ soit suffisamment grand [20]. Dans une autre direction, un résultat de la même

nature que celui de Johansson a été proposé par Mahdian [87]. Ainsi, l'indice chromatique fort des graphes de degré maximal Δ ne contenant pas de cycle de longueur 4 est au plus $(2+\varepsilon)\Delta^2/\ln\Delta$, pour tout $\varepsilon > 0$, à la condition que Δ soit suffisamment grand.

Il est possible de généraliser la notion d'indice chromatique fort à celle d'indice chromatique à distance t d'un graphe G, qui correspond au nombre chromatique de $L(G)^t$, la puissance t du line graph de G. On définit similairement le nombre chromatique à distance t comme étant le nombre chromatique de G^t, la puissance t du graphe G. Dans le Chapitre 2, nous procédons à une analyse des colorations à distance t, et cherchons tout particulièrement à généraliser le résultat de Mahdian à toute distance t. Une application directe des résultats sur la coloration des graphes épars tels que ceux présentés dans le Chapitre 1 nous permet de démontrer que l'indice chromatique à distance t de tout graphe ne contenant pas de cycle de longueur $2k$, pour n'importe quel $k \geq t$ fixé, est au plus $(8+\varepsilon)\Delta^t/\ln\Delta$, à la condition que Δ soit suffisamment grand en termes de t et ε. Quand $t \in \{2,3,4,6\}$, il existe des graphes bipartis de degré maximal Δ, maille $2t$, et dont l'indice chromatique à distance t vaut $(1-o(1))\Delta^t$ quand $\Delta \to \infty$, pour une infinité de valeurs de Δ. Ce sont les graphes d'incidence de géométries projectives. En ce qui concerne les colorations de sommets à distance t, on montre que le nombre chromatique à distance t d'un graphe de degré maximal Δ ne contenant pas de cycle de longueur ℓ, pour n'importe quel $\ell \geq 2t+2$ pair fixé, ou bien pour n'importe quel $\ell \geq 3t$ impair fixé à la condition que t soit impair, est au plus $(4+\varepsilon)\Delta^t/\ln\Delta$, pour tout $\varepsilon > 0$, à la condition que Δ soit suffisamment grand en termes de t et ε. Quand $t \geq 11$, on exhibe des constructions de degré maximal Δ, maille 8, et dont le nombre chromatique à distance t vaut $(\Delta/2)^t$. De plus, ces constructions sont biparties quand t est pair. Quand t est impair, on exhibe une construction de degré maximal Δ, maille impaire $3t$, et dont le nombre chromatique à distance t vaut $(\Delta/3)^t$. Tous les résultats de la même nature que celui de Johansson sont optimaux à une constante multiplicative près, ce qui est illustré par les graphes aléatoires.

La densité bipartie induite

La *densité bipartie induite* d'un graphe G est définie comme étant le degré moyen maximal d'un sous-graphe biparti induit de G. Ce paramètre a été introduit par Esperet, Kang, Thomassé alors qu'ils étudiaient la choisissabilité avec séparation [49]. Ils conjecturèrent que la densité bipartie induite des graphes sans triangle de degré moyen d est au moins $C\ln d$ pour une certaine constante $C > 0$. Cette conjecture reste ouverte dans sa version générale, mais une borne inférieure légèrement plus faible, de la forme $C_r\frac{\ln d}{\ln\ln d}$, a été montrée comme valide concernant la densité bipartie induite des graphes ne contenant pas de clique de taille r, pour tout $r \geq 3$ fixé [83].

Trouver un sous-graphe biparti induit dans un graphe donné G consiste à trouver deux ensembles indépendants de G, pas nécessairement disjoints. Il est de plus demandé que le sous-graphe induit par ces deux ensembles indépendants soit aussi dense que possible; cette condition structurelle supplémentaire fait que l'on peut penser le problème de la détermination de la densité bipartie induite d'un graphe donné comme étant conceptuellement plus dur que celui de calculer le ratio d'indépendance. D'un autre côté, on montre que les deux ensembles indépendants induisant le graphe biparti de densité maximale parmi les ensembles indépendants intervenant dans une coloration fractionnaire de poids k d'un graphe G de degré maximal d certifient que la densité bipartie induite de G est au moins d/k. Ainsi, le problème de la densité bipartie induite est conceptuellement plus simple à analyser que celui de la coloration fractionnaire. En particulier, la version de la conjecture sur la densité bipartie induite des graphes sans triangle étendue à la coloration

fractionnaire est plus dure à résoudre. Cette conjecture a été proposée par Harris [59] et déclare que tout graphe sans triangle de degré moyen maximal d a un nombre chromatique fractionnaire inférieur à $Cd/\ln d$, pour une certaine constante C.

Nous prouvons la conjecture sur la densité bipartie induite dans le cas où d est suffisamment grand en termes du nombre de sommets n du graphe, à savoir $d = \Omega\Big(\sqrt{n \ln n}\Big)$. La borne que l'on démontre est en fait plus forte dans ce régime, puisque l'on montre que la densité bipartie induite des graphes sans triangle de degré minimal d est toujours au moins $d^2/n = \Omega(\ln d)$, qui est également valide si d désigne le degré moyen, quitte à remplacer d par $d/2$. Nous montrons également que cette borne ne pourrait être améliorée que par un facteur multiplicatif de l'ordre de $\ln n$, à l'aide d'une construction pseudo-aléatoire. La résolution de la conjecture dans ce contexte se réduit à un élégant problème de coloration fractionnaire, qui consiste à déterminer la valeur maximale possible du nombre chromatique fractionnaire d'un graphe sans triangle à n sommets. Nous démontrons que cette valeur est au plus $(2+o(1))\sqrt{n/\ln n}$ quand $n \to \infty$, et une fois de plus les graphes aléatoires nous donnent la meilleure borne inférieure sur cette valeur. Le *processus sans triangle* construit un graphe sans triangle sur un nombre fixé de sommets n en tirant successivement toutes les paires possibles de sommets dans un ordre aléatoire, et en ajoutant l'arête correspondante au graphe à la condition qu'elle ne forme pas de triangle. Le graphe aléatoire obtenu à l'issue du processus sans triangle est un graphe sans triangle maximal. Avec forte probabilité, il est de degré maximal $(1 + o(1))\sqrt{n \ln n/2}$; son nombre d'indépendance est $(1 + o(1))\sqrt{2n \ln n}$, et son nombre chromatique fractionnaire est $(1/\sqrt{2} - o(1))\sqrt{n/\ln n}$, quand $n \to \infty$. Cela démontre que notre résultat est optimal à un facteur asymptotique $2\sqrt{2}$ près.

Liens étroits avec les nombres de Ramsey

Il est toujours possible de trouver des structures régulières dans les graphes suffisamment grands. Une formalisation de cet énoncé conceptuel a été démontrée en 1930 [103] ; le théorème de Ramsey établit que, étant donnés deux entiers s et t, il existe un entier $R(s, t)$ appelé *nombre de Ramsey* tel que tout graphe contenant au moins $R(s, t)$ sommets contient soit une clique de taille s, soit un ensemble indépendant de taille t, tandis qu'il existe des graphes à $R(s, t) - 1$ sommets ne contenant aucun des deux. Ce théorème est fondamental en combinatoire, et a conduit à l'émergence de la théorie de Ramsey, qui regroupe de nombreux problèmes cherchant à démontrer l'existence de propriétés structurelles particulières dans les graphes suffisamment grands.

La détermination de la valeur exacte de $R(s, t)$ est un problème majeur et d'une grande difficulté; il y a encore un écart multiplicatif de 4 entre les meilleures bornes inférieure et suppérieure connues à ce jour du logarithme du nombre de Ramsey diagonal $\ln R(s, s)$. Nous ne connaissons pas de construction explicite de graphes démontrant une borne inférieure exponentielle sur $R(s, s)$; seuls les graphes aléatoires nous permettent de certifier sa croissance exponentielle selon l'état actuel de nos connaissances. Une attention toute particulière est portée aux nombres de Ramsey asymétriques, c'est-à-dire au comportement de $R(s, t)$ quand s est fixé et t croît. La valeur de $R(3, t)$ est directement liée au ratio d'indépendance des graphes sans triangle, donc ses meilleures estimations nous fournissent une référence concernant l'optimalité de nombreux de nos résultats dans cette thèse. Par une application directe du résultat de Shearer sur le ratio d'indépendance des graphes sans triangle, et une analyse récente du processus sans triangle [16, 52], on sait désormais que

$$\frac{t^2}{4\ln t} \lesssim R(3,t) \lesssim \frac{t^2}{\ln t}. \qquad (*)$$

Tout le contenu de cette thèse est relié aux nombres de Ramsey asymétriques. Nous étendons l'étude des nombres de Ramsey asymétriques en de nombreuses directions, par l'exclusion de cycles plutôt que de cliques, par une borne sur le nombre de triangles plutôt qu'une stricte exclusion, et par un ajout d'informations sur les ensembles indépendants, aussi bien qualitatif (une coloration couvre tous les sommets du graphe avec des ensembles indépendants) que quantitatif (nous avons parfois des données concernant la taille moyenne des ensembles indépendants). Beaucoup de ces extensions ne pourraient être améliorées qu'à la condition d'améliorer par la même occasion $(*)$, ce qui constituerait une révolution dans la théorie de Ramsey.

Chapter 0

Introduction

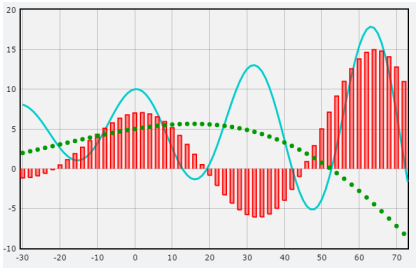
— *Oh, you're doing a PhD! So, tell me, what's your subject?*

Every PhD student has dealt with this question. They would then proceed to recite a title far too long and too technical for anybody to comprehend. And, actually, no one asking for the subject of one's thesis expects to understand it; the mere purpose of this question is often simply to have some fun hearing about an obscure branch of science that merely a handful of people in the world might know of. But then I would tell about my subject, *Graph Colouring*. And my questioners would be confused: those two words are not obscure at all to them, on the contrary they even has some vague idea of what they might refer to. They would then become truly curious, and would want to know more about this. How can someone spend three years of his life — if not his entire career — working on *colourings*? Is this guy really spending office hours with some coloured pencils, colouring some *graphs* on a piece of paper? There must be something more to this.

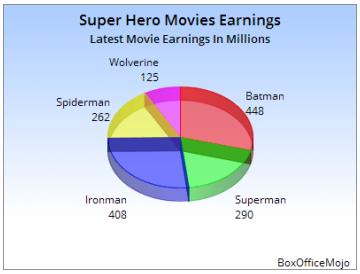
And indeed there is. Let me begin this thesis with an involved answer to this question which has been asked to me more often than I could count.

0.1 What is a graph?

But what on earth is a graph? Most have already heard this name, or used it, in various situations. Many objects can be referred to as graphs, legitimately or by mistake. Some of those is the graph of a mathematical function, which high school students are familiar with. In an abuse of language, any graphical representation of data might be referred to as a graph, which yields a second kind of occurrence of the name.



(a) Graph of a function (a trap)



(b) Pie chart (not a graph)

(c) Graffiti (not even close)

Figure 0.1.1: Throughout this thesis, a graph will *never* denote any of the above.

Let us now clarify what we will refer to as graphs all along this thesis, while entering the fascinating world of graph theory. There will be a lot of drawings, and obviously maths will be involved. We will give a precise mathematical definition to the word *graph*, which denotes a highly generic combinatorial object appearing in so many situations that anyone might be confronted to one of them, not once, but several times in their life.

0.1.1 An abstract structure to represent them all

Throughout all this work, a *graph* will denote a set of connections within some set of elements. There are several possible ways of representing a graph, each having its pros and cons. The preferred one, for it is the most visual and most often quick and handy, is through a *drawing* in the plane; each element corresponds to some point, and each connection is represented by a line — not necessarily a straight one — joining two points together (see Section 0.1.4.4 for a more formal definition of graph drawing). There are infinitely many drawings possible of any fixed graph, depending on where you decide to draw the points in the plane, and how you curve the lines joining them together. Actually, graph drawing represents a whole field of research on its own; how to draw some graph while respecting some constraints is far from being an easy task.

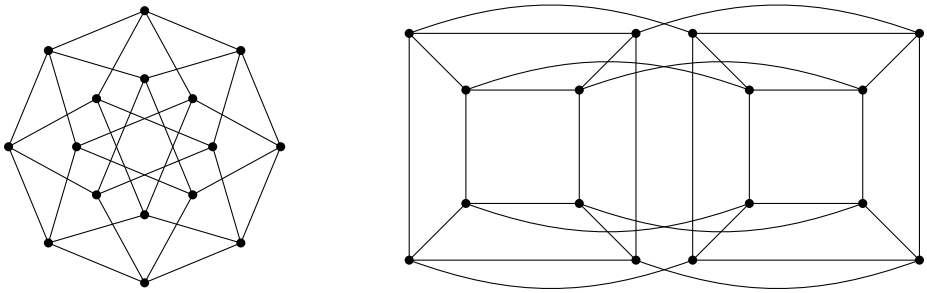
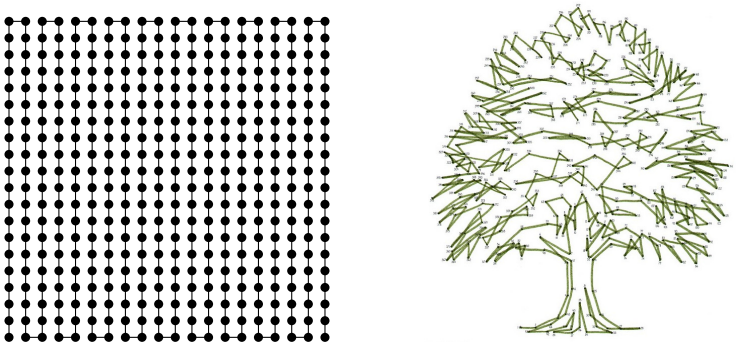


Figure 0.1.2: Two different drawings of a same graph, the tesseract Q_4



(a) A path with 400 vertices (b) The same path

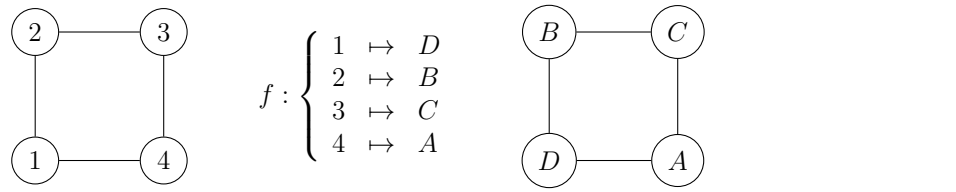
Figure 0.1.3: Sometimes the drawing is what matters!

In a more formal mathematical representation, a graph G is defined by a finite set of elements V which we call the *vertices*, or the *nodes* of the graph, and some binary relation $E \subseteq \binom{V}{2}$ between them, which is described by a set of unordered pairs of vertices; these pairs are called the *edges* of the graph. In a drawing of G, the points represent the set V, and the lines represent the set E. Graphs defined in this way are called *simple graphs*, in contrast with *multigraphs* which may

contain *loops* (an edge joining a vertex with itself) and/or *multiple edges* (several edges joining the same pair of vertices). This thesis focuses only on simple graphs.

It is often convenient, if not necessary, to label the n vertices of the graph in order to be able to refer to them; the subset $[n]$ of the first n integers being the preferred set of labels. Note that this labelling might be implicit; this is the case in the drawing of a graph in the plane: each point is labelled with its position in the plane, so with its coordinates. We do not make any distinction between two differently labelled graphs when their underlying graph structure is the same, i.e. when it is possible to obtain the second graph from the first through a relabelling of its vertices. Such a relabelling operation is called a *graph isomorphism*, and any two graphs which can be transformed one into each other through a graph isomorphism are said to be *isomorphic*. For non mathematicians, isomorphic is just a fancy word to say that two graphs are essentially the same, but potentially not represented in the same manner. Figures 0.1.2, 0.1.3, and 0.1.4 depict pairs of isomorphic graphs.

Remark 0.1.1. The fact that two graphs G and G' are isomorphic is denoted $G \cong G'$.



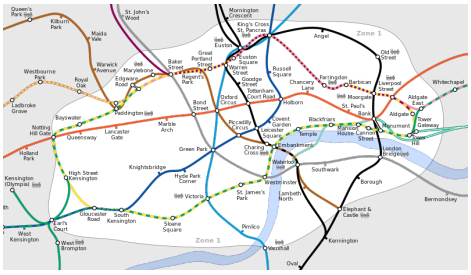
$G = (V, E)$, where
$V = \{1, 2, 3, 4\}$ and
$E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$

$G' = (V', E')$, where
$V' = \{A, B, C, D\}$ and
$E' = \{\{D, B\}, \{D, A\}, \{B, C\}, \{C, A\}\}$

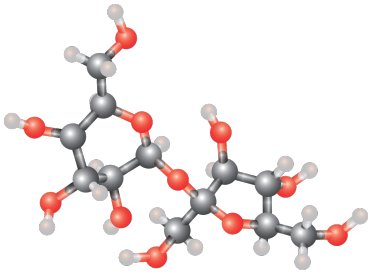
Figure 0.1.4: G and G' are isomorphic through the graph isomorphism f

Graphs are the abstract mathematical structure which embodies all possible networks which we might encounter in real life — and we are surrounded by them. In a non-exhaustive list of domains in which we deal with networks, we can quote: anatomy (neural circuit), biology (protein interaction network), chemistry (crystal structures), computer sciences (web, peer to peer networks), artificial intelligence (artificial neural network), statistics (Bayesian network), electricity (electrical grid), telecom (telecommunication network), transportation (road network, rail network), urbanism (gas network, water distribution network).

(a) Atomium



(b) London undergound



(c) Saccharose molecule

Figure 0.1.5: Some real-life encounters of graphs

A graph needs not exist as a concrete, tangible object in order to be interesting. It can be used as an abstract structure to represent any set of elements which might pairwise interact — it was actually defined exactly to fulfil this purpose. This is the case of social networks, where the underlying graph contains the set of persons in the network as vertices, and contains an edge between every two persons knowing each other (the graph of Facebook), or sharing some attribute (two actors who played together in a movie, two readers who liked the same novel, two consumers who bought the same item, ...). Studying those graphs is an efficient way of inferring some behaviour prediction. Any recommendation algorithm, should it be targeted advertising on Google, video suggestions on Youtube, or any other, works through the analysis of such structures.

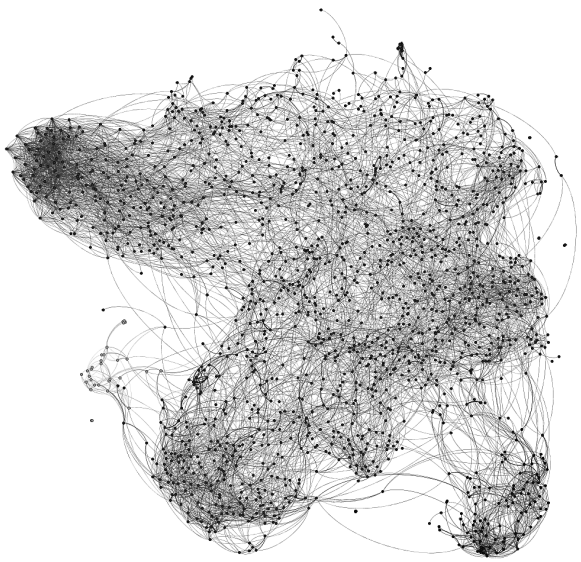


Figure 0.1.6: The social network graphs can be quite monstrous

0.1.2 Some essential graphs

The number of graphs on n vertices is bigger than $\frac{2^{\binom{n}{2}}}{n!}$, which means that there are already more than a hundred billion graphs on 12 vertices. Inside this astronomic bestiary lies a handful of graphs whose number of occurrences in the literature is quite unfair compared to the others. The reason for this surrepresentation is often due to some really specific and rare properties, which make them more interesting or useful than the others, both in an objective and subjective sense.

Each of the following graphs is interesting as a whole, but some are also widely looked upon as subgraphs of a larger graph. This is described in section 0.1.3.5.

Complete graphs. A *complete graph* is a graph containing all possible edges. The complete graph on n vertices is denoted K_n. It embodies the structure of the n-dimensional simplex.

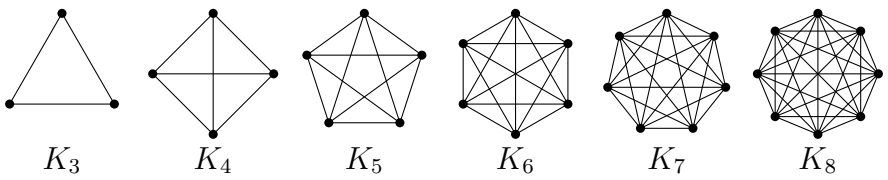


Figure 0.1.7: A depiction of the complete graphs of size 3 to 8

Cycles and paths. A *path* is a sequence of vertices which are successively adjacent to each other. A path on n vertices is denoted P_n, and its *length* is the number of edges it contains, which is $n-1$. The first and last vertex of the sequence are called the *extremities* of the path. A *cycle* on n vertices is obtained by adding an edge between the extremities of the path P_n, thus closing it. Such a cycle is denoted C_n, and its length is n.

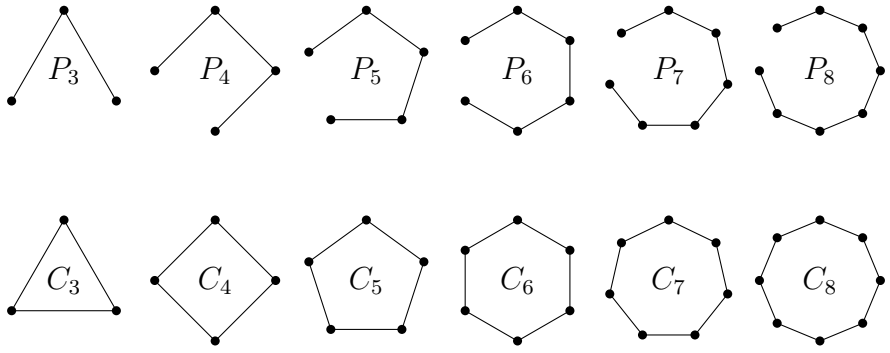


Figure 0.1.8: A depiction of the paths and cycles of length 3 to 8

The paths are the structure behind a bus line (each stop being a vertex). Each cycle C_n embodies the structure of a polygon with n sides, and is therefore often referred to with the name of that polygon.

Remark 0.1.2. The graphs C_3 and K_3 are isomorphic, and referred to as *triangle*.

Complete multipartite graphs. A *complete r-partite* graph is a graph whose vertex set V can be partitionned into $(V_1, \dots, V_r)$ for some $r \geq 2$, such that it contains no edges within V_i for any i (we say that V_i is an *independent set*), and all possible edges between V_i and V_j, for any $j \neq i$. Let n_i denote $|V_i|$ for every i, then the complete r-partite graph associated to $(V_1, \dots, V_r)$ is denoted $K_{n_1,\dots,n_r}$. When all the values n_i are equal to some n, we say that the complete r-partite graph is *balanced*, and may denote it K_{r*n}. The special case when $r = 2$ is called the *complete bipartite graph*.

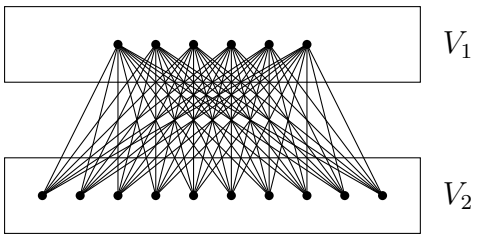


Figure 0.1.9: The complete bipartite graph $K_{6,10}$

Remark 0.1.3. The order of the parts in a complete multipartite graph does not matter;

$$\forall \sigma \in \mathfrak{S}_r, \quad K_{n_1,\dots,n_r} \cong K_{\sigma(n_1),\dots,\sigma(n_r)}.$$

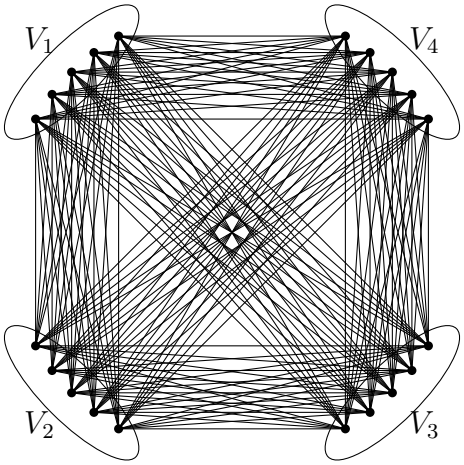


Figure 0.1.10: The complete (balanced) 4-partite graph $K_{5,5,5,5} = K_{4*5}$

Therefore, a monotone order is usually preferred.

For some fixed n, letting $n = ar+b$ be the Euclidean division of n by r, the complete multipartite graph $K_{n_1,\ldots,n_r}$, where $n_1 = \ldots = n_b = a + 1$ and $n_{b+1} = \ldots = n_r = a$, is the densest possible graph on n vertices avoiding K_{r+1} as a subgraph — which means that there exists no graph on n vertices with that property and more edges. Those are called the *Turan graphs*; they are essential in Turán-type problems, which consist in finding densest graphs avoiding one or several subgraphs (subgraphs are properly defined in Section 0.1.3.5).

Grids. We count three different regular grids on the plane. Those are the square grid, the triangular grid, and the hexagonal grid, the two later being the dual of each other (the dual operation is properly defined in Section 0.1.4.4). The square grid of dimension $n \times m$ is the dual graph of a $n \times m$ chessboard. More formally, it is the Cartesian product (defined in Section 0.1.4.6) of two paths: $P_n \square P_m$. The triangular grid of dimension $n \times m$ is obtained by adding a diagonal line within every square (copy of C_4 as a subgraph) of the square grid of dimension $n \times m$.

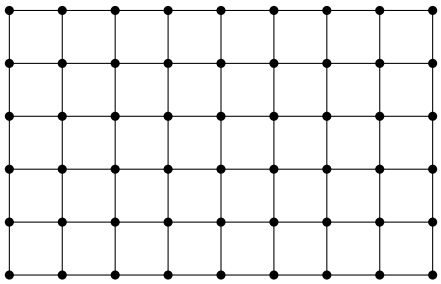


Figure 0.1.11: The square grid of dimension 6×9.

It is possible to generalise square grids to higher dimensions. In three dimension, a $n \times m \times p$ square grid is the Cartesian product of three paths $P_n \square P_m \square P_p$.

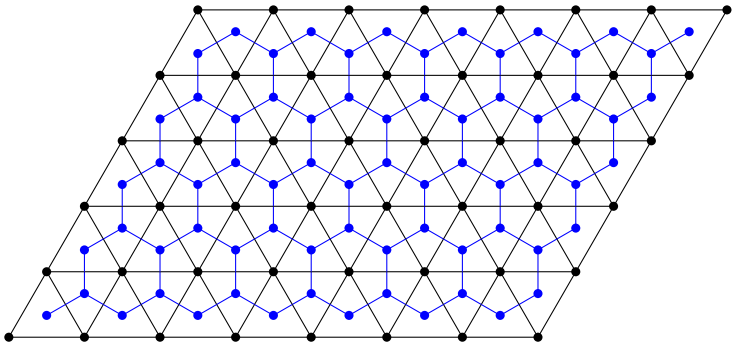


Figure 0.1.12: The triangular grid (black) and its dual hexagonal grid (blue)

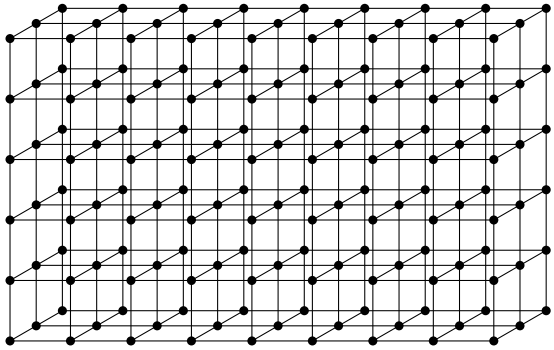
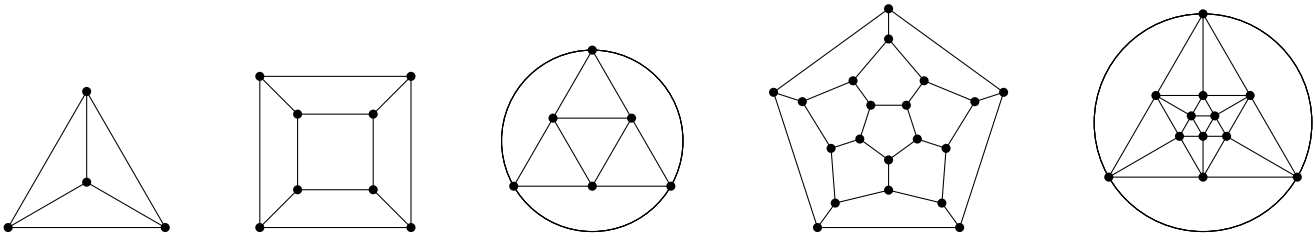


Figure 0.1.13: The square grid of dimension $6 \times 9 \times 3$

Grids are essential in game theory, since a wide range of combinatorial games are defined on grids (chess, go, minesweeper, hex, tic-tac-toe, connect four, rikudo, sudoku, ...). They also are fundamental to statistical physics, for most crystals and pseudo-crystals organise themselves into grid structures.

Platonic solids. There are five regular tridimensional polyhedra, known as the Platonic solids. These are the tetrahedron (4 triangular faces, self-dual), the cube (6 square faces) and the octahedron (8 triangular faces) which are dual of each other, and the dodecahedron (12 pentagonal faces) and the icosahedron (20 triangular faces) which are dual of each other.



(a) Tetrahedron (b) Cube (c) Octahedron (d) Dodecahedron (e) Icosahedron

Figure 0.1.14: The five Platonic solids

Remark 0.1.4. The tetrahedron graph is isomorphic to K_4, and the octahedron graph is isomorphic to $K_{2,2,2}$.

Kneser graphs. A *Kneser graph* is defined through two parameter n and k, and related to set theory. Each vertex of the Kneser graph $KG_{n,k}$ corresponds to one of the $\binom{n}{k}$ k-element subsets of a set of n elements. There is an edge between two vertices in $KG_{n,k}$ whenever the corresponding two subsets are disjoint.

Whe call the subset of Kneser graphs when $n = 2k + 1$ the *odd graphs*. So $KG_{2k+1,k}$ is referred to as the odd graph O_{k+1}.

All odd graphs O_n when $n \geq 4$ are known to have a Hamiltonian cycle (a cycle going through all its vertices, see section 0.1.3.5), while this is not the case of O_3, though it contains a Hamiltonian path (a path going through all its vertices). Moreover, their odd girth (the size of a smallest cycle of odd length, see Section 0.1.3.5) is $2n - 1$.

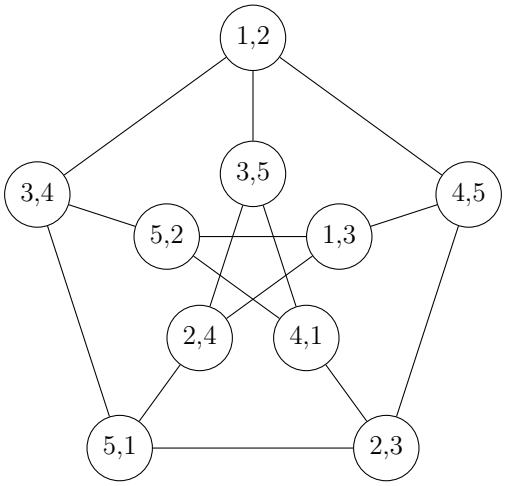


Figure 0.1.15: $KG_{5,2} = O_3$, also known as the Petersen graph

Remark 0.1.5. When $k = 1$ the Kneser Graph $KG_{n,1}$ is isomorphic to the complete graph K_n.

Remark 0.1.6. All graphs presented in this section exhibit some kind of symmetry, some more obviously than the others. We will define a precise notion of symmetry for graphs in Section 0.1.3.7, which a lot of them satisfy. The reason for this is that symmetry is interesting both from a theoretical point of view — it eases the analysis when every vertex behaves in the same way — and an applied one — each node should be treated equitably in a computer network for instance. Last but not least, symmetry occurs everywhere in nature; this is notably the case in crystal structures, so the graphs representing them must be symmetric.

0.1.3 The anatomy of graphs

As you may have noticed by now, although all sharing a common very simple definition, graphs can take a large variety of shapes. Some are well structured, some are rather messy. We are now going to gather all the tools needed in order to dissect them, and eventually to understand their various behaviours.

0.1.3.1 Incidence in graphs

When a given edge e contains a vertex u, we say that u and e are *incident*. The two vertices incident to some edge are called its *extremities*, or *end-vertices*. Two edges having a common extremity are said to be incident, or *adjacent*.

More generally, we might consider bigger structures on graphs (paths, cycles, faces ...). We generalise the notion of incidence by saying that two objects are incident whenever one is contained in the other. The notion of adjacency is more versatile when we try to generalise it; it concerns two objects having a non-empty intersection, with possible additional requirements on the type of this intersection. It can for instance be required that the intersection contains at least an edge, as it is the case for the adjacency of faces in plane graphs (see Section 0.1.4.4).

0.1.3.2 Notations and basic operations

From now on, the set of vertices of any graph G is denoted $V(G)$ and its size $n(G)$. The set of edges of G is denoted $E(G)$, and its size $e(G)$. For two given vertices u and v, the edge linking u and v is denoted uv, or vu indifferently.

We describe some basic elementary operations which might be performed on a graph.

Definition 0.1.1. Let G be a graph.

1. **Vertex deletion:** Given a vertex $v \in V(G)$, the *vertex deletion* of v from G returns the graph $G - v$, where $V(G - v) = V(G) - v$ and $E(G - v) = E(G) \setminus \big(\{v\} \times V(G)\big)$. In other words, the vertex v is removed from the set of vertices of G, and every edge incident to v is removed from the set of edges of G.

2. **Edge addition:** Given an edge $e \notin E(G)$, the *edge addition* of e to G returns the graph $G + e$, where $V(G + e) = V(G)$ and $E(G + e) = E(G) + e$. In other words, the edge e is added to the set of edges of G.

3. **Edge deletion:** Given an edge $e \in E(G)$, the *edge deletion* of e from G returns the graph $G - e$ (also denoted $G \setminus e$), where $V(G - e) = V(G)$ and $E(G - e) = E(G) - e$. In other words, the edge e is removed from the set of edges of G.

4. **Edge contraction:** Given an edge $e = uv \in E(G)$, the *edge contraction* of e in G returns the graph G/e, where $V(G/e) = V(G) - u - v + w$ for some new vertex $w \notin V(G)$, and $E(G/e) = E(G) \cap \binom{V(G/e)}{2} \cup \{wx \mid ux \in E(G) \text{ or } vx \in E(G)\}$. In other words, the extremities of e are merged together into a new vertex, all the while keeping the incidence with the relevant edges.

0.1.3.3 Neighbourhood and degree

Definition 0.1.2. Let G be a graph.

1. When there is an edge in G between two vertices u and v, we say that u is a *neighbour* of v in G, and symmetrically v is a neighbour of u in G. We also say that u and v are *adjacent*, which we denote $u \sim_G v$. The set of all neighbours of a given vertex v is called the *neighbourhood* of v, and denoted $N_G(v)$ — the subscript might be omitted when there is no

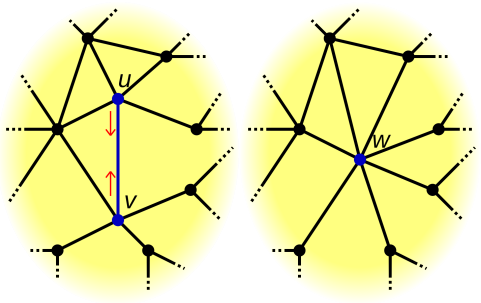


Figure 0.1.16: An illustration of edge contraction.

ambiguity on the relevant graph. The size of the neighbourhood $N_G(v)$, which corresponds to the number of edges incident to v, is called the *degree* of v, and denoted $\deg_G(v)$;

$$\begin{aligned} u \sim_G v &\iff uv \in E(G), \\ N_G(v) &= \{u \in V(G) \mid uv \in E(G)\}, \\ \deg_G(v) &= |N_G(v)| = \#\{e \in E(G) \mid v \in e\}. \end{aligned}$$

2. The *closed neighbourhood* of some vertex v in G, denoted $N_G[v]$, is obtained by adding v to its neighbourhood in G;
$$N_G[v] = N_G(v) + v.$$

3. We extend the neighbourhood notion to subset of vertices;
$$\forall X \subseteq V(G), \quad N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X, \text{ and}$$
$$N_G[X] = N_G(X) \cup X.$$

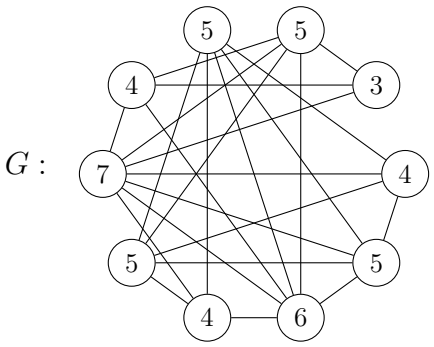
4. The *maximum degree* of G, denoted $\Delta(G)$, is the maximum of all the degrees of the vertices in $V(G)$;
$$\Delta(G) = \max_{v \in V(G)} \deg_G(v).$$
The *minimum degree* $\delta(G)$ is defined similarly;
$$\delta(G) = \min_{v \in V(G)} \deg_G(v).$$

5. A *d-regular* graph is a graph where every vertex is of degree d.

6. The *average degree* of G, denoted $\mathrm{ad}(G)$, is the average of all the degrees of the vertices in $V(G)$;
$$\mathrm{ad}(G) = \frac{1}{n(G)} \sum_{v \in V(G)} \deg_G(v) = \frac{2e(G)}{n(G)}.$$

Remark 0.1.7. The sum of the degrees in a graph G equals twice its number of edges;

$$\sum_{v \in V(G)} \deg_G(v) = 2e(G).$$



$$\begin{aligned} \delta(G) &= 3 \\ \Delta(G) &= 7 \\ \mathrm{ad}(G) &= 4.8 \end{aligned}$$

Figure 0.1.17: Each vertex is labelled with its degree.

Indeed, summing all the degrees is tantamount to counting each edge $uv \in E(G)$ twice, once in $\deg_G(u)$, and once in $\deg_G(v)$. This well-known result is known as the *handshaking lemma*. A direct consequence is that any d-regular graph with d odd must have an even number of vertices.

0.1.3.4 Connectivity in graphs

Definition 0.1.3. Let G be a graph.

1. Given two vertices $u, v \in V(G)$, let us denote $u \sim_G^* v$ whenever there exists a path in G between u and v. It is routine to show that $\sim_G^*$ is an equivalence relation on $V(G)$, which is the transitive closure of the adjacency relation $\sim_G$. The equivalence classes induced by $\sim_G^*$ are called the *connected components* of G. In other words, a connected component of G is a maximal subset of its vertices which can be pairwise joined by some path. If G contains only one connected component, we say that G is *connected*, otherwise we say that G is *disconnected*.

2. When G is connected, a *cut* in G is a subset of edges $X \subseteq E(G)$, the removal of which creates two or more connected components. Alternatively, a cut may be described by a bipartition of the vertices of G; it then consists of all the edges lying between the two parts of the partition. A *vertex-cut* in G is a subset of vertices $X \subseteq V(G)$ such that the subgraph induced by $V(G) \setminus X$ is disconnected. A cut of size 1 is called a *bridge*, and a vertex-cut of size 1 is called a *cut-vertex*.

3. For any $k \geq 1$, we say that G is *k-edge-connected* if all cuts of G have size at least k. We say that G is *k-connected* if G contains at least $k+1$ vertices, and all vertex-cuts of G have size at least k.

4. A *block* of G, or 2-connected component, is a maximal (therefore induced) 2-connected subgraph of G. When G is connected, it can be decomposed into a tree-like structure of blocks (trees are properly defined in Section 0.1.5.1).

0.1.3.5 Subgraphs

A primordial notion on graphs is the one of subgraphs. A first way to handle the structure of some large graph is to consider which small subgraphs appear within its structure, on a regular basis, on some occasional occurrences, or never at all.

Definition 0.1.4. Let G be a graph.

1. A graph H is a *subgraph* of G whenever, up to some relabelling of the vertices of H, $E(H) \subseteq E(G)$. We usually denote this $H \subseteq G$, and say that G is a *supergraph* of H. When moreover $V(H) = V(G)$, we say that H is a *spanning subgraph* of G.

2. For some subset of vertices $X \subseteq V(G)$ of G, the *induced subgraph* $G[X]$ is the graph with X as set of vertices, and $E(G) \cap \binom{X}{2}$ as set of edges. The graph $G[X]$ is obtained by performing a vertex deletion of every vertex $v \notin X$ on G. We say that H is an *induced subgraph* of G whenever there exists some subset $X \subseteq V(G)$ of vertices of G such that H is isomorphic to $G[X]$. In other words, H is a maximal subgraph of G on a fixed subset of vertices.

3. Given a fixed graph H, we say that G is H-free if H is not a subgraph of G. Depending on the context, only induced copies of H might be forbidden in G. This concept generalises to $\mathscr{F}$-free graphs, where $\mathscr{F}$ is a given family of graphs.

4. A given property on graphs is said to be *hereditary* whenever, if true for some graph G, it remains true for every induced subgraph H of G. For instance, having maximum degree at most Δ is a hereditary property, while having minimum degree at least δ or average degree at least d is not.

Now that the concept of subgraph has been introduced, we can define the maximum average degree and degeneracy of a graph, which are the hereditary versions of the average degree and minimum degree in a graph.

Definition 0.1.5. Let G be a graph.

1. The *maximum average degree* of G, denoted $\mathrm{mad}(G)$, is defined as the maximum over all subgraphs H of G of the average degree of H;

$$\mathrm{mad}(G) = \max_{H \subseteq G} \mathrm{ad}(H) = \max_{X \subseteq V(G)} \frac{2e(G[X])}{|X|}.$$

2. The *degeneracy* of G, denoted $\delta^*(G)$, is defined as the maximum over all subgraph H of G of the minimum degree of H;

$$\delta^*(G) = \max_{H \subseteq G} \delta(H).$$

When $\delta^*(G) = d$, we say that G is *d-degenerate*.

There is actually a strong correlation between the maximum average degree and degeneracy of a graph, which can differ by at most a multiplicative factor of 2, as the following lemma states.

Lemma 0.1.1. *Every graph G contains a subgraph H of minimum degree at least* $\mathrm{mad}(G)/2$. *So*

$$\left\lceil \frac{\mathrm{mad}(G)}{2} \right\rceil \leq \delta^*(G) \leq \lfloor \mathrm{mad}(G) \rfloor .$$

Proof. We show that, given a graph H of average degree d and minimum degree $\delta < d/2$ on n vertices and m edges, it is possible to find a subgraph of H of larger average degree. To this end, let v be a vertex of minimum degree in H, we show that the average degree of $H - v$ is greater than d;

$$\mathrm{ad}(H - v) = \frac{2(m - \delta)}{n - 1} > \frac{2m}{n} = d.$$

Indeed,

$$2m(n-1) = 2mn - dn < 2(m-\delta)n = 2mn - 2\delta n.$$

Now given any graph G, choose H to be a subgraph of G whose average degree reaches the maximum average degree of G. H contains no vertex v of degree less than $\mathrm{mad}(d)/2$, for otherwise $H - v$ would be of higher average degree, a contradiction. □

Cliques and independent sets. Given a graph, a classical problem with many possible applications consists in finding subsets of vertices all interacting together, or on the contrary not interacting at all together. Those are the cliques and independent sets of the graph.

Definition 0.1.6. Let G be a graph.

1. An *independent set*, or *stable set* of G is a subset $I \subseteq V(G)$ of its vertices inducing no edge;

   $$E(G[I]) = \varnothing.$$

   It is *maximal* if there exists no independent set I' of G such that $I \subsetneq I'$. It is *maximum* if there exists no independent set I' of G such that $|I| < |I'|$. The *independence number*, or *stability number* of G is the size of a maximum stable set in G; it is denoted $\alpha(G)$. We denote respectively $\mathcal{I}(G)$, $\mathcal{I}_{\max}(G)$, and $\mathcal{I}_{\alpha}(G)$ the set of independent sets, maximal independent sets, and maximum independent sets of G.

2. Complementarily, a *clique* of G is a subset $W \subseteq V(G)$ of its vertices inducing all possible edges;

   $$E(G[W]) = \binom{W}{2}.$$

   It is said to be *maximal* if there is no clique in G strictly containing it, and *maximum* if there is no clique in G of larger cardinality. The *clique number* of G is the size of a maximum clique in G; it is denoted $\omega(G)$.

Remark 0.1.8. A *clique subgraph* is the subgraph induced by a clique.

1. A clique subgraph is always induced, since it already contains all possible edges spanning a subset of the vertices of the supergaph.

2. Every graph H appears as a subgraph in the complete graph $K_{n(H)}$.

A classical algorithmic problem in graph theory consists in establishing the values $\alpha(G)$ and $\omega(G)$ of a given graph G. This problem is NP-hard (see Section 0.2.4), and is subject to an extensive amount of research.

Cliques and independent sets are the heart of Ramsey theory, an active branch of research in Combinatorics, which was initiated by a fundamental theorem by Ramsey. Its statement can be informally formulated as the fact that in any large enough graph, there must be some neat regular structures. Therefore, no graph can be *messy* everywhere.

Theorem 0.1.2 (Ramsey, 1930 [103])**.** *For every integers r and s, there exists an integer $R(r,s)$ such that any graph on at least $R(r,s)$ vertices either contains a clique of size r or an independent set of size s.*

The optimal value of $R(r, s)$ is called the *Ramsey number* with parameters r and s. When $r = s$, the term *diagonal Ramsey numer* is used, while when one of r or s is fixed, and the other varies, the term *off-diagonal Ramsey number* is used. Computing the exact value or $R(r, s)$ for various values of r and s is a hard theoretical problem; the diagonal Ramsey numbers are only known to grow exponentially, with a multiplicative constant of 4 in the exponent between the best known lower and upper bounds.

Paths. A lot of applications of graph theory consist in finding a path with some given properties as a (non induced) subgraph of a graph. For instance, the role of a GPS is to find a shortest possible path between a starting point and a finishing point in the graph formed by the roads of the world.

The paths in graphs also let us define the notion of (discrete) distance. This notion can be generalised to graphs with weighted edges. The length of a path in a weighted graph is the sum of the weights of its edges. This is needed in the case of the graph of the roads of the world; each edge representing a portion of rode between two cities has a weight corresponding to the length of the portion (in kilometres).

Definition 0.1.7. Let G be a graph.

1. A *shortest path* between two vertices $u, v \in V(G)$ is a path of extremities u and v of minimal length, among all possible paths between u and v in G.

2. The *distance* $d_G(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between u and v. The *eccentricity* $\epsilon(v)$ of a given vertex $v \in V(G)$ is its maximal distance to any other vertex in G;
$$\epsilon(v) = \max_{u \in V(G)} \text{dist}_G(u, v).$$

3. The *diameter* of G, denoted $\text{diam}(G)$, is the maximal eccentricity among all vertices of G, while the *radius* of G, denoted $\text{rad}(G)$, is the minimal eccentricity;
$$\text{rad}(G) = \min_{v \in V(G)} \epsilon(v), \text{ and}$$
$$\text{diam}(G) = \max_{v \in V(G)} \epsilon(v).$$

Remark 0.1.9. It is very well possible that in some graph G, there exists no path between two vertices u, v. In this case, we say that u and v are disconnected in G, and by convention we set $\text{dist}_G(u, v) = \infty$.

In a given real-life network G, a vertex v_0 with minimum eccentricity ($\epsilon(v_0) = \text{rad}(G)$) is of interest whenever we wish to propagate some kind of information through the connections between the nodes. Such a node would indeed be the perfect choice as a first emitter, for this would minimise the time of propagation through all the nodes.

Instead of looking for shortest paths in a graph, we might be interested in looking for longest paths (recall that a path must have distinct vertices; otherwise we call that a *walk*). While there exist efficient algorithms to compute all shortest paths in a given graph (in quadratic time), the problem of finding one longest path in a given graph is NP-complete (defined in Section 0.2.4), and so no sub-exponential time algorithm is known to complete this task.

Definition 0.1.8. A *Hamiltonian path* of a given graph G is a path subgraph of G spanning all vertices of G, so isomorphic to $P_{n(G)}$.

Cycles. Among the other subgraphs of importance lie the cycles, the most notable one being arguably the triangle.

Since the graphs that we consider in this thesis are simple and loopless, it implies that the shortest possible cycle is the triangle, of length 3.

Definition 0.1.9. Let G be a graph.

1. The *girth* of G, denoted $\text{girth}(G)$, is the size of a smallest cycle subgraph in G. The *even-girth* of G, denoted $g_e(G)$ is the length of a smallest cycle of even length in G. The *odd-girth* $g_o(G)$ is defined similarly. In particular, $\text{girth}(G) = \min(g_e(G), g_o(G))$.

2. A *Hamiltonian cycle* of G is a (non induced) subgraph of G isomorphic to $C_{n(G)}$, so a cycle spanning all vertices of G. When G contains a Hamiltonian cycle, we say that G is *Hamiltonian*.

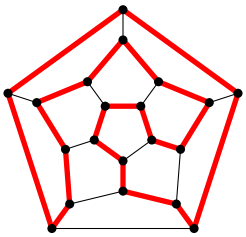


Figure 0.1.18: A Hamiltonian cycle (in red) of the dodecahedron

Remark 0.1.10. The smallest cycle subgraph of any given graph is always induced, since otherwise it would contain a *chord* (an edge between two non-consecutive vertices of the cycle), which would create two smaller cycles. So $\text{girth}(G)$ is the length of a smallest induced cycle in G. Moreover, it is always at least 3 if G is a simple loopless graph.

The problem of finding a Hamiltonian cycle in a given graph, or in one of its induced subgraphs, and more particularly a shortest Hamiltonian cycle in a weighted graph (should there exist at least one), is known as the Travelling Salesman Problem. This notorious optimisation problem consists in finding an optimal route going through a given set of cities and returning to the origin city. Depending on the formulation of the problem, it might be allowed or not to visit the same city twice.

Matchings and f-factors. We now introduce subgraphs which are both important as real-life structures, and powerful tools at the origin of many results in graph theory.

Definition 0.1.10. Let G be a graph.

1. Given a function $f\colon V(G) \to \mathbb{N}$, a *$f$-factor* of G is a spanning subgraph H of G satisfying

$$\forall v \in V(G), \quad \deg_H(v) = f(v).$$

2. Given two function $f, g\colon V(G) \to \mathbb{N}$, a *$(g, f)$-factor* of G is a spanning subgraph H of G satisfying

$$\forall v \in V(G), \quad g(v) \leq \deg_H(v) \leq f(v).$$

3. A *matching* of G is a set of pairwise non incident edges of G; in particular it induces a $(0,1)$-factor of G. A *perfect matching* of G is a spanning matching of G, which therefore induces a 1-factor of G.

A notable fact, rare enough in the domain to be highlighted, is that maximum matchings can be found efficiently (in polynomial time, proportional to $n(G)^2 \times e(G)$) in any graph G. There is a strong duality between matchings and independent sets in graphs, the former being at the core of many edge-related problems, and the latter at the core of the corresponding vertex-related problems. As a consequence, the edge version of a problem is often easier than its vertex version, because of the hardness of finding maximum independent sets in graphs.

It is also worth to mention a result from Kaino and Saito which provides a sufficient condition for finding a (g,f)-factor in a graph.

Theorem 0.1.3 (Kano, Saito, 1983 [72])**.** *Let G be a graph and $\theta \in [0,1]$. Let $g, f : V(G) \to \mathbb{N}$ be two integer functions satisfying*

$$\forall v \in V(G), \quad g(v) \leq \theta \deg_G(v) \leq f(v) \text{ and } g(v) < f(v).$$

Then G has a (g,f)-factor.

Spanning trees. There is no need to explain the interests of analysing the possible explorations and traversals of a given graph; the applications for this are countless. For this task, one subgraph is of paramount importance; this is the *spanning tree*.

Definition 0.1.11. Let G be a given connected graph. A *tree* is any connected acyclic graph, and a *spanning tree* of G is a tree subgraph of G spanning all its vertices.

Remark 0.1.11. A Hamiltonian path is a particular example of a spanning tree.

We are often looking for a *Minimum Spanning Tree (MST)* in a given weighted graph G, that is a spanning tree which minimises the sum of the weights of its edges. This is the subgraph of minimum weight which still ensures connectivity within the set of vertices $V(G)$. There are numerous efficient algorithms to find a MST of G, such as Kruskal's, which works in $O(e(G) \ln n(G))$ time.

Minimum spanning trees are fundamental in many optimisation problems related to connectivity in graphs. For instance, in the variant of the Travelling Salesman Problem where every city might be visited several times, and the distance between two cities satisfies the triangular inequality, a minimum spanning tree provides a 2-approximation of the optimal route, that is a route which is no more than twice the length of an optimal one.

In the main content of this thesis, we will use on several occasions another kind of spanning trees, the *Breadth First Search Trees*. A tree T is a breadth first search tree in a given graph G if it is a spanning subgraph of G rooted in some vertex v, with the property that the distance from the root v to any vertex u in T is equal to $\text{dist}_G(u,v)$. These trees are powerful tools in the context of structural analysis, especially when one is interested in the traversals of G.

0.1.3.6 How to represent a graph in a computer

We have seen two ways of representing a graph, through a drawing, or through a list of its edges, given some arbitrary labelling of its vertices. The first one is obviously a human-oriented representation, which is absolutely unfit for computers to handle. The second one could be grasped by computers, but is quite impractical to use. Therefore, there is a need for a better computer-oriented representation of graphs.

Adjacency matrix. The most simple and convenient way of manipulating a graph on a computer is through its adjacency matrix.

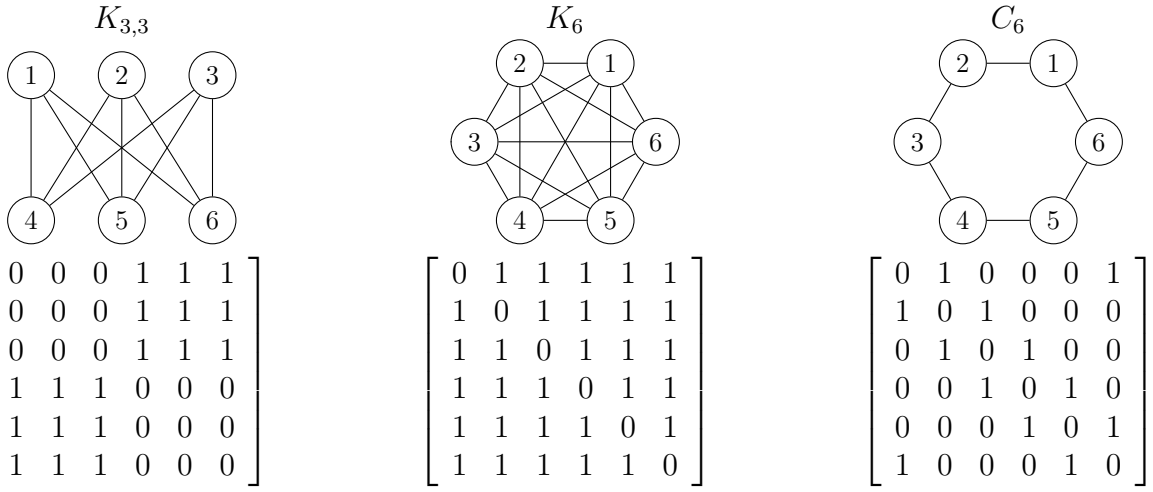
Definition 0.1.12. Let G be a graph with vertices labelled $1, \dots, n$. The *adjacency matrix* $A_G = [a_{i,j}]_{(i,j)\in[n]^2}$ of G is the $n \times n$-matrix defined by

$$\forall (i,j) \in [n]^2, \quad a_{i,j} = \begin{cases} 1 & \text{if there is an edge between } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 0.1.12.

1. The fact that the graphs we consider are simple, loopless, and *unoriented* (the adjacency relation is symmetric) implies that the element $a_{i,j}$ denotes the number of edges connecting i and j, all diagonal elements have a zero value: $\forall i \in [n], a_{i,i} = 0$, and the adjacency matrix is symmetric: $A_G^\top = A_G$.

2. Since the adjacency matrix is defined through some arbitrary labelling of the graph, there are different adjacency matrices representing the same graph. They can be obtained by permutation of the rows and columns.



$$\begin{bmatrix} 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \end{bmatrix} \qquad \begin{bmatrix} 0&1&1&1&1&1 \\ 1&0&1&1&1&1 \\ 1&1&0&1&1&1 \\ 1&1&1&0&1&1 \\ 1&1&1&1&0&1 \\ 1&1&1&1&1&0 \end{bmatrix} \qquad \begin{bmatrix} 0&1&0&0&0&1 \\ 1&0&1&0&0&0 \\ 0&1&0&1&0&0 \\ 0&0&1&0&1&0 \\ 0&0&0&1&0&1 \\ 1&0&0&0&1&0 \end{bmatrix}$$

Figure 0.1.19: Examples of adjacency matrices of some graphs

The adjacency matrix representation of graph has several advantages. Checking the presence or absence of a given edge is immediate, and there exist a lot of algebra tools which can help manipulating those adjacency matrices in an efficient way.

The main drawbacks appear when only a small fraction of all the $\binom{n}{2}$ possible edges are present in the graph. In this case, this representation of the graph occupies more memory than needed. Moreover, having access to the neighbourhood of one vertex will always need a whole exploration of one row of the adjacency matrix, which is also a loss of time when the number of neighbours is a small fraction of the total number of vertices.

Neighbourhood list. A good way to represent a *sparse* graph is through a list (actually an array) of all its neighbourhoods (which might be represented with lists or sets). This solves the two main drawbacks of the adjacency matrix representation, since now the memory occupation

is always of the order of the number of edges of the graph, and we have direct access on the neighbourhood of a given vertex. However, this is at the cost of having a non-immediate check of the presence or absence of a given edge — this is however not a problem when we need to check the presence or absence of several edges incident to a same vertex.

```
module Graph = struct
    type graph = {n: int; mutable edges: int list array}
    let empty n = {n = n; edges = Array.make n []}
    let size g = g.n
    let is_edge g i j = List.mem j g.edges.(i)
    let neighbours g i = g.edges.(i)
    let degree g i = List.length g.edges.(i)
    let add_edge g u v =
        if u <> v && not (is_edge g u v) then begin
            g.edges.(u) <- v :: g.edges.(u);
            g.edges.(v) <- u :: g.edges.(v);
        end
end
```

Remark 0.1.13. In this thesis, many results are obtained with the help of computer calculus. To this extent, we needed a way to represent graphs in a programming language within an efficient structure, in order to be able to perform powerful computations. These computations, by nature, rely on a high amount of recursion, often both on a structural level and an algorithmic level, which is quite common in mathematically oriented programming. For this reason, the chosen programming language was OCaml, for its efficient management of recursion, and for its rigour regarding compilation, which helps at ensuring the correctness of our programs. Throughout this manuscript, there will be some illustrations of code which are directly extracted from the various programs which we used in our work. Nevertheless, for convenience of the reader, every algorithm whose understanding is crucial in the context of this thesis will be written in high level pseudo-code.

0.1.3.7 Homomorphisms and symmetries

Definition 0.1.13.

1. A *graph homorphism* from a graph G to a graph H is a mapping $\phi\colon V(G) \to V(H)$ which preserves the edges;
$$uv \in E(G) \implies \phi(u)\phi(v) \in E(H).$$

2. If ϕ is bijective, and ϕ^{-1} is also a graph homomorphism, then ϕ is a *graph isomorphism*;
$$uv \in E(G) \iff \phi(u)\phi(v) \in E(H).$$
A graph isomorphism from a graph to itself is called a *graph automorphism*.

3. We say that a given graph G is *vertex-transitive* if for every pair of vertices $u, v \in V(G)$, there exists some automorphism f of G such that $f(u) = v$. Informally, this means that the vertices of the graph are indistinguishable one from each other.

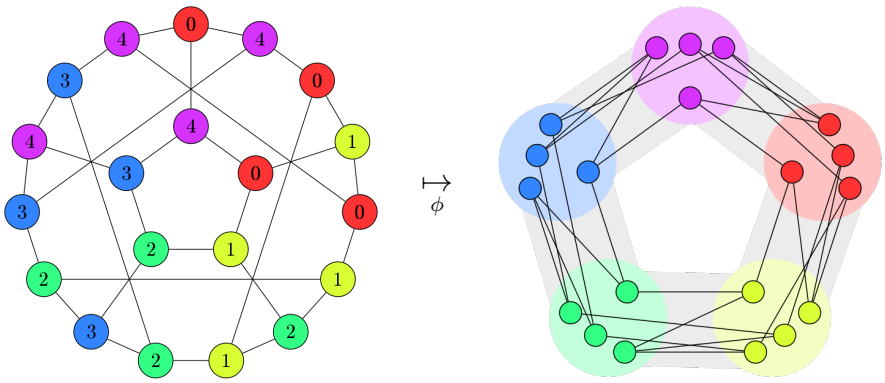


Figure 0.1.20: Some homomorphism ϕ into the graph C_5

From the essential graphs listed in Section 0.1.2, the vertex-transitive ones are the complete graphs, cycles, balanced multipartite graphs, Platonic solids, and Kneser graphs. In general, most graphs obtained from algebraic structures are highly symmetric, and so vertex-transitive.

0.1.4 Constructions

We are now going to see some convenient operations which let us construct a graph from another, two others, or a structure from some other domain (geometry, algebra, ...).

0.1.4.1 Complement graph

Definition 0.1.14. Given a graph G on n vertices, the *complement* of G, denoted $\overline{G}$, is obtained by removing all the edges of G from K_n;

$$V\left(\overline{G}\right) = V(G), \text{ and}$$
$$E\left(\overline{G}\right) = \overline{E(G)} = \binom{V(G)}{2} \setminus E(G).$$

0.1.4.2 Line graph

Most concepts defined on graphs focus on the vertices, with some constraints added by the edges spanning them. One might want to focus on the edges of the graph instead, with constraints on incident edges. In order to avoid having to give an alternative definition of any graph concept with a focus on the edges, we consider line graphs.

Definition 0.1.15.

1. Given a finite collection $\mathscr{F} = (X_i)_{i\in[n]}$ of sets, the *intersection graph* $G_{\mathscr{F}}$ associated to $\mathscr{F}$ is defined by

$$V(G_{\mathscr{F}}) = [n], \text{ and}$$
$$E(G_{\mathscr{F}}) = \left\{ ij \in \binom{[n]}{2} \;\middle|\; X_i \cap X_j \neq \varnothing \right\}.$$

2. Given a graph G, the *line graph* of G, denoted $L(G)$, is the intersection graph of the edges of G. Every pair of incident edges in G is linked by an edge in $L(G)$;

$$V(L(G)) = E(G), \text{ and}$$
$$E(L(G)) = \left\{ \{e, e'\} \in \binom{E(G)}{2} \;\middle|\; e \cap e' \neq \varnothing \right\}.$$

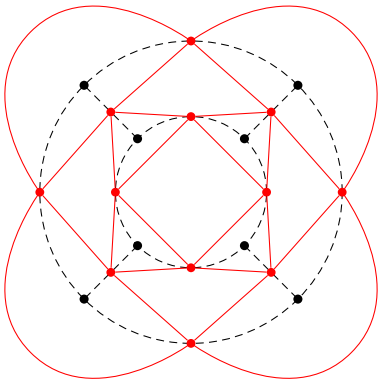


Figure 0.1.21: The line graph of the cube, drawn in red over the cube

Line graphs have a lot of nice properties, and so numerous hard problems on graphs become easy on line graphs.

Remark 0.1.14. Let G be a *simple* connected graph of maximum degree $\Delta(G) \geq 3$. Then

$$\omega(L(G)) = \Delta(G).$$

Indeed, the set of edges incident to any vertex $v \in V(G)$ induces a clique subgraph in $L(G)$, and if a clique W_L in $L(G)$ contains at least 3 edges incident to some vertex $v \in V(G)$, then all the edges in W_L must be incident to v. Therefore, a clique subgraph in $L(G)$ is either a subgraph of G of maximum degree 2, and so necessarily a triangle, or a star, the biggest possible one being $K_{1,\Delta(G)}$.

0.1.4.3 Graph powers

Definition 0.1.16. Given a graph G, the *t-th power* G^t of G is obtained from G by adding edges between all pairs of vertices at distance at most t in G;

$$V\big(G^t\big) = V(G), \text{ and}$$
$$uv \in E\big(G^t\big) \iff \operatorname{dist}_G(u, v) \leq t.$$

Remark 0.1.15. Let G be a connected graph on n vertices of maximum degree Δ.

1. $G^{\operatorname{diam}(G)}$ is isomorphic to K_n.

2. $\Delta(G^t) \leq \sum_{i=1}^{t} \Delta(\Delta - 1)^{i-1} \leq \Delta^t$.

3. The adjacency matrix A_{G^t} of G^t can be easily obtained from the adjacency matrix A_G of G;

$$A_{G^t} + I_n = (A_G + I_n)^t,$$

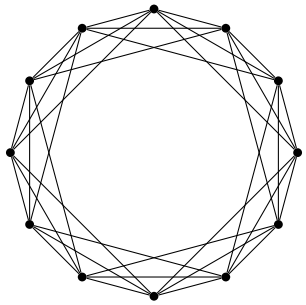


Figure 0.1.22: The distance-3 graph of the dodecagon C_{12}

where the operation $(A_G + I_n)^t$ is performed in a Boolean algebra.

0.1.4.4 Dual graph

We define here the notion of dual graph, which is an operation defined on *plane graphs*. A plane graph is the drawing of a graph in the plane such that all the intersecting points of the lines of the drawing represent a vertex. So the lines do not cross except in the vertices of the graph.

More formally, plane graphs rely on an *embedding* on a surface, which we hereby define.

Definition 0.1.17. Let G be a graph.

1. Given a surface Σ, an *embedding* of G on Σ is a mapping from $V(G)$ to a subset of points P of Σ, and from $E(G)$ to a subset of lines L of Σ, where a line is an homeomorphic image of the interval $[0, 1]$. Moreover, each line $\ell \in L$ must contain exactly two points $p_1, p_2 \in P$, which lie on its extremities (the images of 0 and 1), and every intersecting point $p \in \ell_1 \cap \ell_2$ between two lines $\ell_1, \ell_2 \in L$ must be a point of P.

2. The graph G is *planar* if there exists an embedding of G on the Euclidian plane $\mathbb{R}^2$. Such an embedding is called a *plane graph*.

The motivations behind plane graphs are multiple. Being able to represent a graph with no crossings tends to enhance the readability of the drawing, without mentioning the numerous real-life applications which can be thought of. For instance, in electronics, the integrated circuit designs must satisfy the property that two paths never cross.

Graphs are identified by two kinds of objects, their vertices and their edges. For plane graphs, a third kind of object appears, those are the *faces* of the graph.

Definition 0.1.18. A fundamental (and surprisingly highly non trivial) result from topology, known as the Jordan curve theorem, states that any *Jordan curve* — a non-self-intersecting continuous loop in the plane — separates the plane into exactly two regions, its interior (bounded) and its exterior (unbounded).

1. Given a plane graph G with point set P and line set L, the *faces* of G are the (arc-wise) connected regions of $\mathbb{R}^2 \setminus L$ (including the unbounded exterior one). We denote $F(G)$ the set of faces of G.

2. Each face $f \in F(G)$ is delimited by a Jordan curve, which we call the *boundary* of f. The lines and points belonging to the boundary of f are said to be *incident* to f. We denote respectively $E(f)$ and $V(f)$ the set of lines and points incident to f.

3. Note that, given a line ℓ of a plane graph, there is an incident face on each of its two sides. When the same face f is incident to ℓ on each sides, we say that ℓ is incident to f with multiplicity 2. The *degree* $\deg_G(f)$ of a face f in a plane graph G is the number of lines of G incident to f, counted with their multiplicity. Two faces are *adjacent* when they have a common incident line.

Remark 0.1.16. In a given connected plane graph G, if a line is incident to a face with multiplicity 2, then it is a bridge — its removal disconnects G. In a 2-edge-connected plane graph G, any face $f \in F(G)$ is an induced cycle. However, not all induced cycles in a plane graph form a face.

Definition 0.1.19. Given a 3-edge-connected plane graph G, the *dual graph* of G, denoted G^*, is obtained by considering the faces of G as the vertices in G^*, and adding an edge in G^* between all pairs of adjacent faces in G. This is the intersection graph of $\{E(f) \mid f \in F(G)\}$;

$$V(G^*) = F(G), \text{ and}$$
$$E(G^*) = \left\{ \{f, f'\} \in \binom{F(G)}{2} \;\middle|\; E(f) \cap E(f') \neq \varnothing \right\}.$$

Given two 3-edge-connected planar graphs G and H, we say that H is *dual* to G if there exists an embedding $\widetilde{G}$ of G on $\mathbb{R}^2$ such that $H = \widetilde{G}^*$.

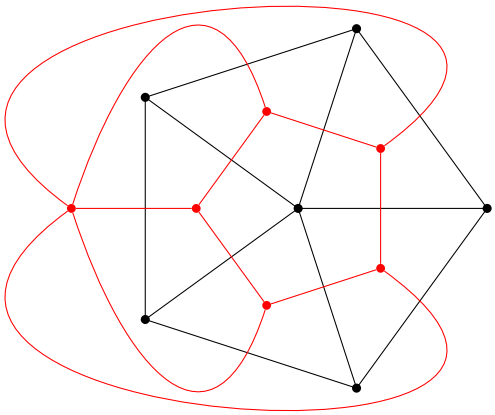


Figure 0.1.23: The wheel is its own dual.

Remark 0.1.17.

1. The dual of a graph depends on its embedding. So the dual operation is properly defined only on plane graphs, and not on planar graphs.

2. The dual relation is a symmetric relation on the class of 3-edge-connected planar graphs.

3. We need to restrict the dual operation to 3-edge-connected planar graphs in order to remain within the class of simple loopless graphs. Indeed, every bridge of a planar graph G would produce a loop in G^*, and every cut of size 2 of G would produce a multiple edge in G^*.

0.1.4.5 Graph additions

Given a set of graphs, we might be interested in gluing them together to form one larger graph. There are several notable operations which perform this.

Disjoint union. The *disjoint union* of two graph G_1 and G_2 is denoted $G_1 \cup G_2$. After a potential relabelling of $V(G_2)$ in such a way that $V(G_1)$ and $V(G_2)$ are disjoint, it is defined by

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2),$$
$$E(G_1 \cup G_2) = E(G_1) \cup E(G_2).$$

So the disjoint union of two graphs is obtained simply by putting a drawing of G_1 and a drawing of G_2 side by side.

Remark 0.1.18. Any graph G is the disjoint union of its connected components.

The iterated disjoint union of a graph G with itself is denoted nG for $n-1$ iterations.

Join. The *join* of two graphs G_1 and G_2 is the complementary operation to the disjoint union, and denoted $G_1 \wedge G_2$. It is obtained by adding all edges between $V(G_1)$ and $V(G_2)$ in the disjoint union of G_1 and G_2. So, once $V(G_1)$ is disjoint from $V(G_2)$,

$$V(G_1 \wedge G_2) = V(G_1) \cup V(G_2),$$
$$E(G_1 \wedge G_2) = E(G_1) \cup E(G_2) \cup V(G_1) \times V(G_2).$$

Remark 0.1.19.

1. $\forall G_1, G_2, \quad G_1 \wedge G_2 \cong \overline{\overline{G_1} \cup \overline{G_2}}$.
2. The complete biparte graph $K_{n,m}$ is the join of two independent sets of size n and m.
3. $\forall n \geq 1, \quad K_{n+1} \cong K_n \wedge K_1$.

0.1.4.6 Graph products

While an addition of two graphs G_1 and G_2 on n_1 and n_2 vertices returns a graph on $n_1 + n_2$ vertices, a product returns a graph on $n_1 \times n_2$ vertices. Actually, the vertex set of any graph product of G_1 and G_2 is the Cartesian product of $V(G_1)$ and $V(G_2)$, and so its edge set is included in $\binom{V(G_1) \times V(G_2)}{2}$.

Graph products are often interesting in order to obtain, from one graph with an interesting property, an infinite family of graphs with the desired property. We present here the four more widely used graph products — though there exist several other ones with nice properties —, the *Cartesian product*, the *strong product*, the *tensor product*, and the *lexicographical product*.

Given two graphs G_1 and G_2, a graph product on G_1 and G_2 is entirely defined through its adjacency relation between two pairs vertices $(u_1, u_2), (v_1, v_2) \in V(G_1) \times V(G_2)$.

Cartesian product. $G_1 \square G_2$

$$(u_1, u_2) \sim_{G_1 \square G_2} (v_1, v_2) \iff \begin{array}{ll} & (u_1 = v_1 \text{ and } u_2 \sim_{G_2} v_2) \\ \text{or} & (u_1 \sim_{G_1} v_1 \text{ and } u_2 = v_2) \end{array}$$

As mentioned in Section 0.1.2, bidimensional square grids are defined as the Cartesian product of two paths, and $(n+1)$-dimensional grids as the Cartesian product of an n-dimensional grid with a path. So the hypercube Q_n, which is no more than an n-dimensional square grid of dimension $2 \times \ldots \times 2$, is the Cartesian product of n edges;

$$Q_n = \underbrace{K_2 \square \ldots \square K_2}_{n \text{ factors}} = K_2^{\square n}.$$

Remark 0.1.20. The line graph of a complete bipartite graph is the Cartesian product of two complete graphs;

$$\forall n, m, \quad L(K_{m,n}) = K_m \square K_n.$$

Strong product. $G_1 \boxtimes G_2$

$$(u_1, u_2) \sim_{G_1 \boxtimes G_2} (v_1, v_2) \iff \begin{array}{ll} & (u_1 = v_1 \text{ and } u_2 \sim_{G_2} v_2) \\ \text{or} & (u_1 \sim_{G_1} v_1 \text{ and } u_2 = v_2) \\ \text{or} & (u_1 \sim_{G_1} v_1 \text{ and } u_2 \sim_{G_2} v_2) \end{array}$$

Remark 0.1.21. The balanced complete k-partite graph K_{k*n} is the strong product of the complete graph K_k with an independent set of size n.

Tensor product. (or *categorical product*) $G_1 \times G_2$

$$(u_1, u_2) \sim_{G_1 \times G_2} (v_1, v_2) \iff u_1 \sim_{G_1} v_1 \text{ and } u_2 \sim_{G_2} v_2$$

Remark 0.1.22. The symbol for these three first products actually corresponds to a drawing of the graph obtained by the product of two edges;

$$\begin{aligned} K_2 \square K_2 &= C_4, \\ K_2 \boxtimes K_2 &= K_4, \\ K_2 \times K_2 &= 2K_2. \end{aligned}$$

Lexicographical product. $G_1 \cdot G_2$ or $G_1[G_2]$

$$(u_1, u_2) \sim_{G_1[G_2]} (v_1, v_2) \iff \begin{array}{ll} & u_1 \sim_{G_1} v_1 \\ \text{or} & (u_1 = v_1 \text{ and } u_2 \sim_{G_2} v_2) \end{array}$$

Remark 0.1.23. This product is not commutative, contrary to the previous ones. This is why the asymmetric notation $G_1[G_2]$ is sometimes used. It is however associative, which justifies the use of the notation $G_1 \cdot G_2$.

0.1.5 Graph classes

Let me end with the biological metaphor on graphs by introducing the main families of notable graphs. In biology, a lot of animal species remain unknown. Likewise in graph theory, an overwhelming majority of graphs are unknown from the community, some of which might have properties which are widely and unsuccessfully looked for. As previously mentioned, the number of different graphs grows so quickly (in a quadratic exponential) that it is impossible to study them all. Instead, a solution is to gather all graphs sharing some common attributes into families of graphs, and then studying all the graphs from such a family at once. What is of importance is therefore to characterise which attributes are determinant when studying some parameter or problem on graphs, in order to extract meaningful related families of graphs.

0.1.5.1 Trees and forests

We begin with the family of graphs which is arguably the easiest to consider, namely the forests. A lot of really hard problems on graphs become easy to solve on forests, and an almost systematic first step in the direction of solving a problem on graphs is to solve it for forests.

Definition 0.1.20.

1. A *forest* F is an *acyclic* graph, so a graph which does not contain any subgraph isomorphic to C_n, for any length n.

2. A *tree* T is a connected forest, so a forest is a disjoint union of trees.

3. In a tree, a *leaf* is any vertex of degree 1.

4. A *subtree* of a tree T is a connected subgraph of T.

5. A *rooted tree* T is a tree given with a special vertex $r \in V(T)$, the *root* of T. In a rooted tree T, the vertex set $V(T)$ is usually considered together with a partition $(V_0, \dots, V_p)$, where $V_0 = \{r\}$, and V_i is the set of vertices at distance i from r in T, for any $i \geq 1$. Each V_i is called the *layer* at depth i of T. The *depth* p of the tree is the maximum depth of its layers, which corresponds to the eccentricity of its root r.

6. Given a vertex $v \in V_i(T)$ at depth i in a rooted tree T, the *children* of v are its neighbours at depth $i+1$, that is $N(v) \cap V_{i+1}$. When $i > 0$, the vertex v has exactly one other neighbour, which lies at depth $i-1$; we call it the *parent* of v.

7. Given a rooted tree T and one of its nodes x, the set of *descendants* of x is either $\{x\}$ when x is a leaf, or the union of the descendants of all the children of x, together with x. The set of *ancestors* of x is either $\{x\}$ if x is the root, or the set of ancestors of the parent of x, together with x.

8. In a rooted tree, a *branch* is a path from the root to some leaf.

Remark 0.1.24.

1. A tree T on n vertices contains exactly $n-1$ edges. As a consequence, a forest F on n vertices, composed of r connected components, contains exactly $n-r$ edges.

2. Every tree on at least two vertices contains at least two leaves.

3. In a rooted tree T, the ancestors of any node x induce a path from the root to x. The descendants of x induce a subtree of T rooted in x.

Rooted trees are inescapable in the domain of data structures, where we store data into the nodes. Indeed, they have a really basic inductive definition which makes them really convenient to use in most programming languages.

```
type 'a tree = Node of 'a * 'a tree list
```

Moreover, most classical hard problems on graphs can be solved on trees in linear time.

Remark 0.1.25. Every tree T contains a maximum independent set containing all its leaves.

Indeed, in any maximum (and therefore maximal) independent set I, given some leaf v, either $v \in I$ or $N(v) \cap I \neq \varnothing$. But since v is a leaf, it has only one neighbour, its father u. So, assuming that $v \notin I$, the set $I - u + v$ is also a maximum independent set of T. Doing this for every leaf, if we denote $\mathcal{L}(T)$ the set of leaves of T, we infer that $I \setminus N(\mathcal{L}(T)) \cup \mathcal{L}(T)$ is a maximum independent set of T, containing all its leaves. We can then use this result in order to compute the independence number of any tree in linear time.

```
let independence_number t =
    let rec aux = function
    | Node (x,children) -> let (child_taken,res) = List.fold_left
            (fun (b,r) t' -> let (b',r') = aux t' in (b||b',r+r'))
            (false,0) children in
        if child_taken then (false,res)
        else (true,res+1) in
    snd (aux t)
```

In the domain of recursive programming, trees are also the structure behind the recursive calls of recursive functions.

0.1.5.2 Bipartite graphs

We next consider the class of bipartite graphs, which contains all the forests.

Definition 0.1.21. A graph G of n vertices is *bipartite* if it appears as a subgraph of $K_{n,n}$. Equivalently, its vertex set $V(G)$ can be partitioned into two independent sets. A bipartite graph is usually denoted $G = (U, V, E)$, where (U, V) is the bipartition of its vertices into two independent sets, and $E \subseteq U \times V$ is its set of edges.

Remark 0.1.26. Bipartite graph are easy to recognise.

1. Bipartite graphs are exactly the class of graphs not containing any cycle of odd length as a subgraph. They can be recognised in linear time.

2. Trees and forests are bipartite graphs, since they are acyclic.

Lemma 0.1.4. *Every graph G with m edges contains a bipartite subgraph H with at least $\lceil \frac{m}{2} \rceil$ edges.*

Proof. Let $H = (X, Y, E)$ be a bipartite subgraph of G, such that $X \cup Y = V(G)$, and $E(H) = (X \times Y) \cap E(G)$ is of maximum size.

Claim.

$$\forall v \in V(H), \quad \deg_H(v) \geq \frac{\deg_G(V)}{2}$$

This claim implies the desired result through an application of the handshaking lemma;

$$e(H) = \frac{1}{2} \sum_{v \in V(H)} \deg_H(v) \geq \frac{1}{2} \sum_{v \in V(G)} \frac{\deg_G(V)}{2} = \frac{m}{2}.$$

There remains to prove the claim. Let us assume otherwise that there exists some $v \in V(H)$ such that $\deg_H(v) < \deg_G(v)/2$ — so in particular $\deg_G(v) > 0$. Without loss of generality, we might assume that $v \in X$.

Let $X' = X - v, Y' = Y + v$, and $H' = (X', Y', (X' \times Y') \cap E(G))$. We observe that

$$e(H') = e(H) + \deg_G(v) - 2\deg_H(v) > e(H),$$

which contradicts the choice of H. □

An application of lemma 0.1.4 together with lemma 0.1.1 implies that every graph G of average degree d contains a bipartite subgraph of minimum degree at least $\lceil \frac{d}{4} \rceil$.

0.1.5.3 Planar and plane graphs

We have already defined planar and plane graphs in Section 0.1.4.4. As mentioned in that section, planar graphs are essential in many applications, and therefore constitute an appealing family of graphs to analyse. They also have really specific properties, which mainly arise from a theorem of Euler which exhibits a correlation between their number of vertices, edges, and faces.

Theorem 0.1.5 (Euler's formula)**.** *Let G be a connected plane graph on n vertices, m edges, and f faces. Then*

$$n + f = m + 2. \tag{1}$$

This formula has numerous consequences. The general method which exploits this formula in order to prove results on planar graphs is called the discharging method, which we illustrate in Section 0.3.4. One of the most immediate consequences of Euler's formula is that the average degree of a planar graph is bounded.

Corollary 0.1.5.1. *The average degree of a planar graph of girth at least g is less than $\frac{2g}{g-2}$.*

Proof. Let G be a simple loopless plane graph on n vertices, m edges, and f faces. First note that the handshaking lemma can be generalised to hold also when summing the degrees of the faces of a plane graph;

$$\sum_{f \in F(G)} \deg_G(f) = 2m.$$

Indeed, each edge is either incident to two faces f and f', and counted once in $\deg_G(f)$ and once in $\deg_G(f')$, or it is incident to only one face f with multiplicity 2, and counted twice in $\deg_G(f)$.

Now, every face is of degree at least $\text{girth}(G) \geq g$. So

$$\sum_{f \in F(G)} \deg_G(f) \geq gf.$$

It follows that $f \leq 2m/g$. Substituting this into Euler's formula (1), we obtain

$$\begin{aligned} m + 2 &\leq n + \frac{2}{g}m \\ \frac{g-2}{g}m + 2 &\leq n \\ \mathrm{ad}(G) = \frac{2m}{n} &\leq \frac{2g}{g-2} - \frac{4}{n} < \frac{2g}{g-2}. \end{aligned}$$

□

A consequence is that every planar graph contains a vertex of degree at most 5. Since every subgraph of a planar graph is also a planar graph (being planar is a hereditary property), the degeneracy of a planar graph is therefore always at most 5.

Theorem 0.1.6. *For every planar graph G,*

$$\delta^*(G) \leq 5,$$

and if moreover G is triangle-free (and so of girth $g \geq 4$),

$$\delta^*(G) \leq 3.$$

0.1.5.4 H-free graphs

Given a fixed graph H, the class of H-free graphs is the set of graphs which do not contain H as a subgraph, or as an induced subgraph depending on the context. This also generalises to $\mathscr{F}$-free graphs, where $\mathscr{F}$ is a (possibly infinite) family of graphs.

Remark 0.1.27. A lot of graph families can be expressed through H-free graphs.

1. Forests are the same as $\{C_n \mid n \geq 3\}$-free graphs.
2. Bipartite graphs are the same as $\{C_{2k+1} \mid k \geq 1\}$-free graphs.
3. Graphs of girth at least g are the same as $\{C_n \mid n < g\}$-free graphs.
4. Graphs of maximum degree at most Δ are the same as $K_{1,\Delta+1}$-free graphs.
5. Graphs on n vertices of minimum degree at least δ are the same as *complements* of $K_{1,n-\delta+1}$-free graphs on n vertices.
6. Line graphs are the same as induced F_L-free graphs, for some family F_L of 9 minimal graphs which was identified by Beineke in 1968 [12].

0.1.5.5 Random graphs

The last class of graphs I wish to highlight is the one of *random graphs*, and more specifically the widely used Erdős-Rényi model $G(n, p)$.

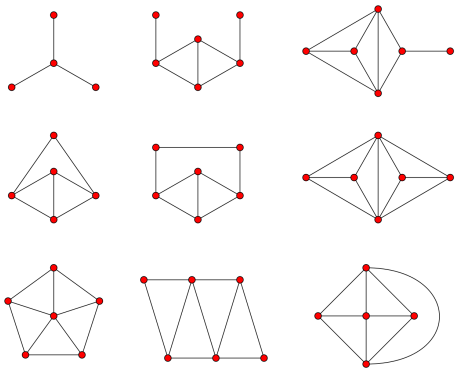


Figure 0.1.24: The 9 forbidden induced subgraphs in line graphs

Definition 0.1.22. Given an integer n, and some parameter $p \in (0,1)$, the *Erdős-Rényi model* $G(n,p)$ returns a random graph on n vertices, where each pair of vertices (u,v) induces an edge with probability p. Given $(u,v) \in \binom{[n]}{2}$, let X_{uv} be the random variable counting the number of edges between u and v (so either $X_{uv} = 0$ or $X_{uv} = 1$). All variables X_{uv} are independent and identically distributed random variables following a Bernoulli distribution of parameter p.

By definition, any graph on n vertices can be drawn from the Erdős-Rényi model, moreover with equal probability if $p = 1/2$. What interest could a class containing all graphs possibly have?

When considering random graphs from $G(n,1/2)$, we are not seeking for universal properties which would hold for all outcomes of the model, but instead we are interested in properties which would hold *asymptotically almost surely* for a graph drawn from this model, so with probability tending to 1 as n grows to infinity. By studying this model, we are able to find a surprisingly large amount of properties which hold for all but a negligible fraction of graphs on n vertices. Through the so-called *probabilistic method*, random graphs help us also prove the existence of graphs with certain peculiar properties which we have no idea how to construct.

When p is a positive constant independent of n, graphs drawn from $G(n,p)$ are called *dense graphs*, in the sense that their number of edges is asymptotically almost surely a constant fraction of the maximum possible number of edges in a graph on n vertices. But it can also be of interest to consider graphs where p is a decreasing function of n, for instance $p = d/n$ for some constant d. Graphs drawn from $G(n,p)$ when $p = o_{n\to\infty}(1)$ are called *sparse graphs*, in the sense that asymptotically almost surely only a negligible fraction of all possible edges are part of their edge set.

Remark 0.1.28. Let $\varepsilon > 0$, and G be drawn from $G(n,d/n)$. The degrees in G all follow the binomial distribution $\mathrm{Bin}(n-1,p)$. So asymptotically almost surely,

$$\delta(G) \geq (1-\varepsilon)d \quad \text{and} \quad \Delta(G) \leq (1+\varepsilon)d.$$

0.2 Graph Colouring

The previous section was just a glimpse of what graph theory is about. It covers almost the whole range of mathematical domains, from algebra to topology, and can be considered in so many different contexts that no one can decently claim to be a specialist of the whole domain of graph theory.

Researchers tend to focus mainly on two kinds of graphs. The first kind are graphs which capture the combinatorial structure of some existing network, and whose properties directly translate into properties of the network. The analysis of those graphs is always done with the applications on the network in mind, in order to understand its behaviour, to target its weaknesses, or to solve related algorithmic problems. The second kind are graphs which are artificially constructed as tools which describe combinatorial problem in an unified structure, therefore more convenient to use. All the known results on graphs can now be applied in order to dissect the problem, and sometimes to solve it in an unexpected and elegant way.

This thesis focuses on the second kind of graphs. And on these graphs, we focus on an innocent looking problem — it could even be considered a game — which consists in colouring their vertices following the simplest possible rule. The kind of game you could imagine asking a child to play.

0.2.1 Presentation and definition

Definition 0.2.1. Let G be a graph.

1. A *k-colouring* of G is a function $c\colon V(G) \to [k]$. This can be seen as a function which associates to every vertex $v \in V(G)$ a colour among a palette of k possible ones. A *partial colouring* of G is a colouring of a proper induced subgraph of G.

2. A colouring c of G is said to be *proper* whenever it follows the rule that no edge in G has two extremities of the same colour;

$$\forall uv \in E(G), \quad c(u) \neq c(v).$$

   When there exists a proper k-colouring of G, we say that G is *k-colourable*.

3. The minimum k such that there exists a proper k-colouring of G is called the *chromatic number* of G, denoted $\chi(G)$.

4. For every colour $i \in [k]$ used in some k-colouring c of G, the subset of vertices $c^{-1}(\{i\})$ coloured with colour i is called a *colour class*, or *monochromatic class* of c. The colour classes of c yield a partition of $V(G)$, and if c is proper, every colour class is an independent set of G.

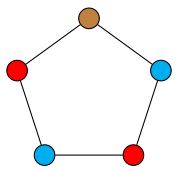


Figure 0.2.1: A proper 3-colouring of C_5.

Remark 0.2.1. There are several equivalent possible definitions of a proper k-colouring of a given graph G on n vertices. A proper k-colouring of G is isomorphic to

(i) a partition of $V(G)$ into k independent sets,

(ii) a graph homomorphism from G to the complete graph K_k,

(iii) a (maximum) independent set of size n of the Cartesian product $G \square K_k$.

Note also that G is k-colourable if and only if it is a subgraph of the complete k-partite graph K_{k*n}. In particular, the class of bipartite graphs is exactly the class of 2-colourable graphs.

Remark 0.2.2. Being k-colourable is a hereditary property;

$$\forall H \subseteq G, \quad \chi(H) \leq \chi(G);$$

0.2.2 What for?

To anyone not familiar with graph colouring, the concept might appear somewhat arbitrary. Indeed, the formal definition could give the picture of a combinatorial game, like many others, which is only worth as a mind entertaining problem.

Sudoku. To be entirely honest, the entertaining aspect of solving some colouration problem by hand has already been demonstrated. This is what solving a Sudoku grid consists in! A Sudoku grid has an underlying graph on 81 vertices — each vertex is a cell of the Sudoku grid — such that all 9 cells lying on a common line, column, or 3×3 square form a clique. This underlying graph is depicted in Figure 0.2.2. Solving a Sudoku grid consists in, given some partial 9-colouring of its underlying graph, completing it into a proper 9-colouring of the whole graph.

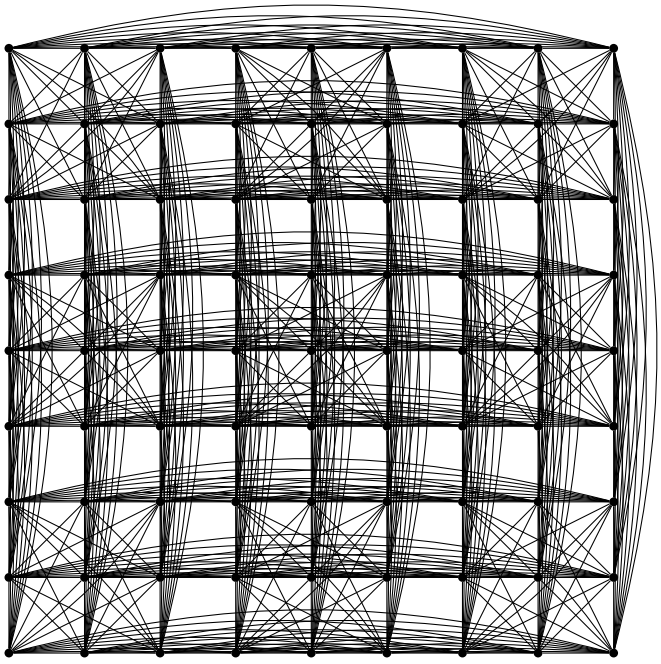


Figure 0.2.2: The underlying graph of a Sudoku grid

Real life problem. It is now time to illustrate the real strength of the k-colouring problem. We are going to see with an example — which is one among many others — how being able to solve the k-colouring problem on graphs can yield a solution to a real life situation.

Suppose that you are a professor willing to give an assignment to your students. You know that your students have a history of sharing their results together, among friends. In order to avoid that, you intend to write several subjects, so that any two friends are assigned a different one. You are therefore wondering how many different subjects you will need to write at least, and how you should assign them to your students. We will assume that you have a plain knowledge of the friendship relation among your students, for instance through a thorough analysis of their activities on social media.

Let G be the friendship graph of your students, that is a graph where each vertex is labeled with one of your students' name, and two vertices share an edge whenever the two corresponding students are friends on some social network. The solution to your problem is the chromatic number $\chi(G)$ of G. A proper k-colouring yields a way to assign one among k subjects to each of your students, so that two friends are assigned a diffent subject. The chromatic number of G is the minimum such k, so the number of different subjects that you need to write.

It often happens that the graphs considered in these situations are well-structured. Namely, they can sometimes be constructed through a combination of the operations defined in Section 0.1.4. Let us consider an illustration of this hereafter.

After some successful assignments, you might notice that the students are beginning to share their answers once again. Aware of your trick, they have organised together so that every student is helping each of its friends finding someone with the same subject. As a consequence, every two students having a common friend are able to share their answers whenever they are assigned the same subject. In order to avoid that, you need to assign different subjects to all students having a common friend; these are the vertices linked by a path of length 2 in the friendship graph G. The new solution to your problem is the chromatic number $\chi(G^2)$ of the distance-2 graph of G.

Let us now consider a slightly different situation. All your students have to make d different assignments, which they can choose freely among n different ones. Every assignment needs the use of a very special machine, and you dispose of only one of each. Each machine can be used by only one student at a time. You will need to make several sessions, during each of which every student works on a different machine. You wonder how many sessions you will need to organise.

Let $G = (U, V, E)$ be the bipartite graph where each vertex $u \in U$ is labelled with one student, each vertex $v \in V$ is labelled with one machine, an there is an edge $uv \in E$ if the student u needs to work with the machine v. Note that the degree of every vertex $u \in U$ is $\deg_G(u) = d$. The solution to your new problem is the chromatic number $\chi(L(G))$ of the line graph of G. Indeed, the d edges incident to some student correspond to the d assignments he needs to perform. A student can perform only one assignment per session, and a machine can be use by only one student per session, therefore every two incident edges cannot be part of the same session, so they cannot be of the same colour.

Those are merely two illustrative examples; one might easily imagine other practical scenarios in other contexts — other examples will be presented later in this thesis. Actually, any given assignment problem with constraints is very likely to be expressible into the colouring problem on an associated graph. The occasions where you might be confronted to a problem equivalent to a graph colouring problem may arise at any time, in any situation. Having some knowledge, expertise, and efficient tools to deal with graph colouring problems is of importance; this thesis intends to be one more building block in this domain.

0.2.3 Computing the chromatic number

Backtracking algorithm. A first way to compute a colouring of a graph is through a backtracking algorithm.

```
exception Uncolourable
(* lists.(v) is the list of available colours for vertex v *)
let rec colour lists graph =
    let n = Graph.size graph in

    (* returns the vertex with smallest list of size > 1 *)
    let next lists =
        let rec aux min v_min v =
            if i = n then v_min
            else let size = List.length lists.(v) in
            if size < min && size > 1 then aux size v (v+1)
            else aux min v_min (v+1) in
        aux max_int (-1) 0 in

    (* Deletes the colour of v from the lists of all its neighbours *)
    let rec propagation lists v =
        let c = List.hd lists.(v) in
        List.iter
            (fun u ->
                lists.(u) <- List.filter (<> c) lists.(u);
                let size = List.length lists.(u) in
                if size = 0 then raise Uncolourable
                else if size = 1 then propagation lists u
            ) (Graph.neighbours graph v) in

    (* Finds a possible colour for vertex v *)
    let rec affectation lists v =
        let l0 = Array.copy lists in
        let c = List.hd lists.(v) in
        try
            lists.(i) <- [c];
            propagation lists v;
            colour lists graph
        with Uncolorable ->
            l0.(v) <- List.tl l0.(v);
            affectation l0 v in

    let v = next lists in
    if v = -1 then Array.map List.hd lists
    else affectation lists v
```

Linear program. The chromatic number $\chi(G)$ of a graph G is naturally defined as the solution of an integer linear program with one variable w_I for every of its maximal independent sets $I \in \mathcal{I}_{\max}(G)$.

$$\begin{aligned} \chi(G) &= \min \sum_{I \in \mathcal{I}_{\max}(G)} w_I, \\ \text{such that } & \begin{cases} \forall v \in V(G), & \displaystyle\sum_{\substack{I \in \mathcal{I}_{\max}(G) \\ v \in I}} w_I \geq 1, \\ \forall I \in \mathcal{I}_{\max}(G), & w_I \in \{0,1\}. \end{cases} \end{aligned} \tag{2}$$

Consider the definition of the chromatic number of G as a partition of its vertex set $V(G)$ into as few independent sets as possible. The variable w_I associated to a given maximal independent set $I \in \mathcal{I}(G)$ can have two possible values, 1 if I is a part of the aforementioned partition, and 0 otherwise. So every assignment $I \in \mathcal{I}_{\max}(G) \mapsto w_I \in \{0,1\}$ corresponds to a collection of maximal independent sets of G, and if every constraint of the Linear Program (2) is satisfied, this collection covers the set of vertices of G. The size of this collection is given by the objective function, which is required to be minimised. In this collection, some vertex $v \in V(G)$ might be part of several independent sets; removing every such v from all but one of these independent sets yields a partition of $V(G)$ into (non maximal) independent sets, thus a proper colouring of G.

The number of constraints of the Linear Program (2) is $n(G)$, and its number of variables is $|\mathcal{I}_{\max}(G)|$, which might be exponential in $n(G)$ — the number of maximal independent sets of a graph G on n vertices can be as large as $3^{n/3}$, which is reached when G is a disjoint union of $n/3$ triangles [94].

Logical formulation. It is possible to express the k-colouring problem for a given graph G through the following logical formula.

$$\left(\bigwedge_{v \in V(G)} \bigvee_{i \in [k]} x_{v,i} \right) \wedge \left(\bigwedge_{uv \in E(G)} \bigwedge_{i \in [k]} \overline{x_{u,i}} \vee \overline{x_{v,i}} \right) \tag{3}$$

For every $v \in V(G)$ and $i \in [k]$, the boolean variable $x_{v,i}$ is true if the vertex v is coloured with colour i, and false otherwise. The first half of formula (3) is true only when every vertex in $V(G)$ is coloured with some colour in $[k]$, and the second half of the formula is true only when no edge has the same colour appearing on both its extremities.

There exist several solvers in order to attack such a logical formula. These algorithms are intensively worked on, with increasingly efficient heuristics, so they often provide a cheap way of having reasonably fast programs in order to solve the k-colouring problem.

0.2.4 A few words on NP-complete problems

All the methods presented in Section 0.2.3 in order to compute the chromatic number of a graph have an exponential complexity in the size of the graph in the general setting. There is almost no hope of finding any algorithm of polynomial complexity computing the chromatic number of any graph, since it is strongly believed by the scientific community that P $\neq$ NP, while the k-colouring problem is NP-complete.

Definition 0.2.2. We define the two fundamental complexity classes P and NP.

1. The complexity class P contains all decision problems which can be solved in *polynomial time*, so in time less than n^c for some constant $c \in \mathbb{R}$ on any instance of size n.

2. A given decision problem p is in NP if, on any instance x of size n, it holds that

   (a) The return value of $p(x)$ is either 0 or 1.

   (b) If $p(x) = 1$, there exists a *certificate* y of this answer such that, given y, it can be checked in polynomial time that $p(x) = 1$ is a valid answer.

3. A given problem p is *polynomially reducible* to a problem q if, given any instance x of size n of p, it is possible to compute in polynomial time an instance y of size less than n^c (for some constant $c \in \mathbb{R}$) such that $p(x) = q(y)$. We denote this fact $p \leq_{\mathrm{P}} q$.

4. A given problem $p \in \mathrm{NP}$ is NP-*complete* if it is maximal with respect to $\leq_{\mathrm{P}}$;

$$\forall q \in \mathrm{NP}, \quad q \leq_{\mathrm{P}} p.$$

5. A given problem p is NP-*hard* if some NP-complete problem is polynomially reducible to it.

So NP-complete problems are the hardest to solve among the ones in NP. Being able to solve one efficiently would imply that every problem in NP can be solved efficiently. In particular, if there exists a polynomial-time algorithm solving some NP-complete problem, then P = NP.

The first problem proven to be NP-complete is the SAT problem, which decides whether a given boolean formula can be true on some instance of its boolean variables. The k-colouring problem, which consists in deciding, given a graph G, whether or not $\chi(G) \leq k$, is also an NP-complete problem when $k \geq 3$ — the certificate of the k-colouring problem is a proper k-colouring of G, whose validity can be checked easily in time linear in the number of edges of G. So computing the chromatic number of a graph is an NP-hard problem.

With this in mind, one could be pessimistic about the graph colouring problem. Since it is NP-complete, and therefore likely impossible to solve in polyomial time, then it is impractical to even consider it for graphs on more than a few dozen vertices. A theoretical researcher has the opposite opinion; this is in order to circumvent those seemingly unsolvable issues that research work is needed. The NP-completeness of the colouring problem is — among other things — what makes the content of this thesis relevant.

0.2.5 Approaching the chromatic number

Since computing the exact value of the chromatic number of a given graph G seems out of reach in the general case, a first step would be to bound it in terms of other parameters of the graph G.

0.2.5.1 Naive lower bounds

In a proper colouring of the complete graph K_n, every vertex must have a colour distinct from the others, since they are all pairwise adjacent. So $\chi(K_n) = n$, and any graph G having K_n as a subgraph needs at least n colours in a proper colouring;

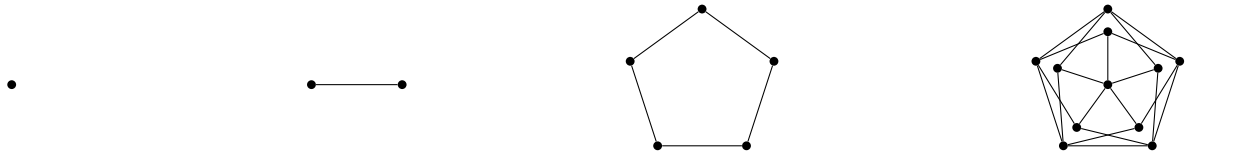
$$\forall G, \quad \chi(G) \geq \omega(G).$$

A proper k-colouring of G yields a partition of $V(G)$ into independent sets. Since each independent set is of size at most $\alpha(G)$, it means that $k \times \alpha(G) \geq |V(G)|$;

$$\forall G, \quad \chi(G) \geq \frac{n(G)}{\alpha(G)}.$$

These two lower bounds are sufficient in many applications in order to give a good estimate of the chromatic number of classical graphs. However, the chromatic number can be arbitrarily larger than both of them.

Mycielski graphs. Given a graph G, the *Mycielskian* of G, denoted $M(G)$, is obtained by adding all the edges of G inside one of the copies of $V(G)$ in $G \times K_2$, and joining all vertices of the other copy to some new vertex. The *Mycielski graph* M_k is defined inductively by $M_1 = K_1$, $M_2 = K_2$, and $M_{n+1} = M(M_n)$.



(a) $M_1 = K_1$ (b) $M_2 = K_2$ (c) $M_3 = C_5$ (d) M_4: Grötzsch graph

Figure 0.2.3: The first steps of the Mycielskian construction

The following lemma is helpful in order to analyse the chromatic number of the Mycielski graphs.

Lemma 0.2.1. *Let G be a graph of chromatic number k, and c be a proper k-colouring of G. For every colour $i \in [k]$, there exists some vertex $v_i \in V(G)$ such that $c(v_i) = i$ and the set of colours appearing in $N(v_i)$ is $c(N(v_i)) = [k] - i$.*

Proof. Let us assume otherwise that, for some colour i, every vertex v such that $c(v) = i$ misses a colour $a_v \in [k] - i$ in its neighbourhood. Let us recolour every such vertex v with colour a_v, thus creating some colouring c' of G. Since the recoloured vertices where the ones coloured with i by c, they form an independent set, and so after the recolouring process the colours in their neighbourhood remain unchanged. So c' is a proper colouring of G, and c' uses colours only from $[k] - i$, so $k - 1$ different ones. This contradicts the fact that $\chi(G) = k$. □

Lemma 0.2.2. *The Mycielskian construction satisfies two useful properties.*

1. *If G is triangle-free, then so is $M(G)$.*
2. *$\chi(M(G)) = \chi(G) + 1$*

As a consequence, M_k is a triangle-free graph of chromatic number k, for every $k \geq 1$.

Proof. Let G be a triangle-free graph of chromatic number k, and $H = M(G)$.

1. Assume for the sake of contradiction that H contains a triangle $T = (u, v, w)$. Let x be the new vertex added to the vertex set of $G \times K_2$ in order to obtain $V(H)$, and let V_0 and V_1 be the two copies of $V(G)$ in $G \times K_2$. Assume that V_0 is the one inducing G, and V_1 the one

inducing an independent set. By construction, the neighbourhood of v is the independent set V_1, and therefore v is contained in no triangle. Moreover, since G is triangle-free, at least one vertex from T lies outside of V_0, and since V_1 is an independent set, at most one vertex from T lies in V_1. So exactly two vertices of T (say u and v) lie in V_0, and one (say w) in V_1. Let u', v', w' be the corresponding copies of those vertices in G. By definition of the tensor product $\times$, it means that u' and v' are neighbours of w' in G, and $u'v' \in E(G)$. This yields a triangle $T' = (u', v', w')$ in G, a contradiction.

2. The fact that $\chi(H) \leq \chi(G)+1$ is straightforward ; it suffices, given a proper k-colouring c of G, to use it to colour the two copies of $V(G)$ is H, and to use a new colour for the last vertex x. In order to prove the other side of the inequality, assume for the sake of contradiction that there exists a proper k-colouring c of H. c induces a proper colouring of G on V_0. By applying Lemma 0.2.1 on the graph $H[V_0]$ and the colour $c(x)$, there exists some vertex $v_0 \in V_0$ such that $c(v_0) = c(x)$ and all other colours appear in $N_H(v_0) \cap V_0$. Let $v_1 \in V_1$ be the corresponding copy of v_0 in V_1. By construction, $N_H(v_0) \cap V_0 = N_H(v_1) \cap V_0$, and since moreover $x \in N_H(v_1)$, all colours from c appear in $N_H(v_1)$, a contradiction.

□

We have just seen that there exist graphs with clique number 2, and arbitrarily large chromatic number. Before going further, let us mention this other remarkable property of the Mycielski graph.

Theorem 0.2.3 (Cropper, Gyárfás, Lehel, 2006 [30]). *Every connected triangle-free graph on n vertices is contained as an induced subgraph in M_n.*

Kneser graphs. For a given graph G, it is possible to capture both lower bounds $\omega(G)$ and $n(G)/\alpha(G)$ of $\chi(G)$ in a stronger lower bound, the *Hall ratio* of G, denoted $\rho(G)$, and defined by

$$\rho(G) = \max_{H \subseteq G} \frac{n(H)}{\alpha(H)} \geq \max\left(\frac{n(G)}{\alpha(G)}, \omega(G)\right).$$

Because being k-colourable is a hereditary property — we can also say that the chromatic number is monotone — it holds that

$$\forall G, \quad \chi(G) \geq \rho(G),$$

but again the chromatic number of a graph can be arbitrarily larger than its Hall ratio. This is the case for Kneser graphs.

Theorem 0.2.4 (Lovász, 1978 [86]). *For every $k \geq 1$ and $n \geq 2k-1$, the chromatic number of the Kneser graph $KG_{n,k}$ is $n-2k+2$.*

Theorem 0.2.5 (Erdős-Ko-Rado, 1961 [46]). *Let $A \subseteq \binom{[n]}{k}$ be a family of pairwise intersecting subsets on k elements of $[n]$. Then*

$$|A| \leq \binom{n-1}{k-1},$$

with equality only when A consists of all subsets containing a common element $x \in [n]$, if $n > 2k$.

Note that an independent set in $KG_{n,k}$ is precisely a family of pairwise intersecting subsets, so a consequence of Lemma 0.2.5 is that

$$\alpha(KG_{n,k}) = \binom{n-1}{k-1},$$

and so

$$\frac{n(KG_{n,k})}{\alpha(KG_{n,k})} = \frac{\binom{n}{k}}{\binom{n-1}{k-1}} = \frac{n}{k}.$$

Now, since every family of sets containing a common element $x \in [n]$ forms a maximum independent set in $KG_{n,k}$, in order to decrease the independence number of $KG_{n,k}$ by some constant i, one needs to remove i sets containing x from $V(KG_{n,k})$, for every $x \in [n]$. Consequently, the multiset obtained as the union of all the removed sets should have cardinality at least $i \times n$. It means that at least $\left\lceil \frac{in}{k} \right\rceil$ vertices must have been removed from $KG_{n,k}$;

$$\rho(KG_{n,k}) \leq \max_{i \in [n]} \frac{n - \left\lceil \frac{in}{k} \right\rceil}{k-i} = \frac{n}{k}.$$

For every $k \geq 1$, the Kneser graph $KG_{3k-1,k}$ has Hall ratio $3 - 1/k$, and chromatic number $k+1$. This is another example of a triangle-free graph of arbitrarily large chromatic number, and moreover with bounded Hall ratio.

0.2.5.2 Naive upper bounds

It is easy to bound from above the chromatic number of a graph G in terms of its maximum degree by considering a greedy colouring algorithm. The idea is to sequentially colour the vertices of G, assigning the smaller colour which does not appear in the neighbourhood of the considered vertex at each step when there exists one, or adding a new colour to the colouring otherwise.

Algorithm 2 Greedy colouring

Require: Some graph G with $V(G) = \{v_1, \ldots, v_n\}$
Ensure: c is a proper k-colouring of G
  $k \leftarrow 1, i \leftarrow 1$
  while $i \leq n$ **do**
    if $c(N_G(v_i)) = [k]$ **then**
      $k \leftarrow k + 1$
      $c(v_i) \leftarrow k + 1$
    else
      $c(v_i) \leftarrow \min \quad [k] \setminus c(N_G(v_i))$
    end if
    $i \leftarrow i + 1$
  end while
  Return c

Lemma 0.2.6. *The proper colouring returned by the greedy colouring applied on any given graph G uses at most $\Delta(G) + 1$ colours.*

Proof. The variable k counting the number of colours used by c during the execution of the algorithm is incremented only when $c(N(v)) = [k]$, for some vertex $v \in V(G)$. When this happens, in particular, $\Delta(G) \geq |N(v)| \geq k$. So, if k reaches the value $\Delta(G) + 1$, it cannot be further incremented. □

Note that the quality of the colouring returned by the greedy algorithm depends highly on the order in which the vertices of the graph G are considered. In particular, for some optimal colouring c of G, if the vertices are ordered such that their assigned colour is non-decreasing, then the algorithm returns the optimal colouring c.

Let us construct an ordering $v_1, \ldots, v_n$ of the vertices of a given graph G by sequentially deleting a vertex of minimum degree from G. We denote $N^+(v_i) = \{v_j \in N_G(v_i) \mid j > i\}$ and $\deg^+(v_i) = |N^+(v_i)|$. When G is a d-degenerate graph, it holds that $\deg^+(v_i) \leq d$ for every $v_i \in V(G)$. By greedily colouring the vertices of G in the reversed order, we obtain a colouring using at most $d + 1$ colours.

$$\forall G, \quad \chi(G) \leq \delta^*(G) + 1 \leq \Delta(G) + 1$$

We finish this section by stating an upper bound on the chromatic number of a graph G as a function of the Hall ratio of G and the number of vertices of G. Recall that no upper bound exists as a function of the Hall ratio alone.

Theorem 0.2.7 (Chvátal, 1975; Johnson, 1974; Lovász, 1975)**.** *Let G be a graph on $n \geq 2$ vertices of Hall ratio ρ. Then*

$$\chi(G) \leq \rho \ln n + 1.$$

Proof. We prove this result by induction on n. If $n = 1$, then G is K_1 and so $\chi(G) = 1$.

Let us now assume that $n > 1$. Let I be any maximum independent set of G; by definition of ρ it is of size at least $\left\lceil \frac{n}{\rho} \right\rceil$. We colour I with one colour, then apply induction on the graph G' obtained after deleting the vertices of I from G. The number of vertices of G' is at most $\left\lfloor \left(1 - \frac{1}{\rho}\right) n \right\rfloor < n$, and the Hall ratio of G' is at most ρ since G' is an induced subgraph of G. So by induction,

$$\begin{aligned}
\chi(G') &\leq \rho \ln\left(\left(1 - \frac{1}{\rho}\right) n\right) + 1 \\
&\leq \rho \ln\left(1 - \frac{1}{\rho}\right) + \rho \ln n + 1 \\
&\leq \rho \ln n.
\end{aligned}$$

By adding the new colour of I to the colouring of G', we obtain a colouring of G using at most $\rho \ln n + 1$ colours, as desired. □

0.2.5.3 Hardness results

When a given problem is NP-hard, there is still some hope that it might be possible to give a good approximation of its solution in polynomial time. In the case of the colouring problem, we might be satisfied with some polytime algorithm which could yield a $C\chi(G)$-colouring of any graph G, for some constant $C > 1$. Unfortunately, many results prove that such an algorithm cannot exist, under the hypothesis that P $\neq$ NP.

Theorem 0.2.8 (Khanna, Linial, Safra, 2000 [76]). *There exists some $\varepsilon > 0$, such that it is* NP*-hard to compute an n^{ε}-approximation of the chromatic number.*

Theorem 0.2.9 (Emden-Weinert, Hougardy, Kreuter, 1998 [38]). *The k-colouring problem is* NP*-complete on the class of graph of maximum degree Δ, when $3 \leq k \leq \Delta - \sqrt{\Delta} + 1$.*

0.2.6 Main classical results on graph colouring

Regarding the hardness of computing even an approximate value of the chromatic number, and all its variants, a classic goal in graph colouring is to provide some sufficient conditions in order to have a significantly better upper bound on the chromatic number than the naive ones. There has been a lot of research in this direction in the last decades; hereafter is a list of some of the most important results in the area.

Theorem 0.2.10 (4-colour theorem [105]). *For every planar graph G,*

$$\chi(G) \leq 4.$$

Theorem 0.2.11 (Grötzsch's theorem [57]). *For every planar triangle-free graph G,*

$$\chi(G) \leq 3.$$

Theorem 0.2.12 (Bipartite graphs characterisation). *For every graph G,*

$$\chi(G) \leq 2 \quad \Longleftrightarrow \quad G \text{ contains no odd cycle.}$$

Theorem 0.2.13 (Brook's theorem [24]). *For every connected graph G, either G is a complete graph or an odd cycle, or*

$$\chi(G) \leq \Delta(G).$$

Theorem 0.2.14 (Johansson-Molloy theorem [65, 90]). *For every triangle-free graph G,*

$$\chi(G) \leq (1 + o(1)) \frac{\Delta(G)}{\ln \Delta(G)}.$$

The strength of all these theorems lies in the fact that they are all sharp, since for every one of them there exist infinite families of graphs satisfying its constraints and asymptotically reaching the upper bound. The sharpness is exact for all of them, except the Johansson-Molloy theorem, for which there remains an asymptotic multiplicative gap of 2 between the upper bound and the value reached by the known infinite family of graphs, constructed through a pseudo-random process which we describe in Section 0.3.2.3.

There remains of course numerous open problems concerning the chromatic number of graphs. A first one concerns a generalisation of Theorem 0.2.14 to any H-free graph.

Conjecture 0.2.1 (Alon, Krivelevich, Sudakov, 1999 [7]). *Let H be some fixed graph. There exists a constant $C_H > 0$ such that, for every H-free graph G,*

$$\chi(G) \leq C_H \frac{\Delta(G)}{\ln \Delta(G)}.$$

Note that in order to prove Conjecture 0.2.1, it is enough to prove it when H is a complete graph. Indeed, since any fixed H is a subgraph of $K_{n(H)}$, every H-free graph is also in particular $K_{n(H)}$-free. This conjecture deals with the asymptotic value of the chromatic number of graphs of fixed clique number, when their maximum degree grows to infinity. A related conjecture, known as the Reed conjecture, deals with the case when the maximum degree might be arbitrarily close to the clique number.

Conjecture 0.2.2 (Reed, 1998 [104])**.** *For every graph G,*

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.$$

One of the best partial results to date in favour of Reed's conjecture is the following.

Theorem 0.2.15 (Delcourt, Postle, 2017 [35])**.** *There exists a constant $\Delta_0 > 0$ such that, for every graph G of maximum degree $\Delta(G) \geq \Delta_0$,*

$$\chi(G) \leq \left\lceil \frac{\omega(G) + 12(\Delta(G) + 1)}{13} \right\rceil.$$

0.2.7 Variants of the chromatic number

One can think of many variants of the colouring problem, depending on the context needed for its application. Some are strengthened versions of the problem, some are relaxed. Let us present the most notorious ones in this section.

0.2.7.1 The list chromatic number

In a k-colouring, every vertex receives a colour from a common set of colours $[k]$. In order to be even more expressive, we consider a generalisation of k-colourings where every vertex picks its colour from a private list of allowed ones. These are *list colourings*.

Definition 0.2.3. Let G be a graph.

1. Given a *list assignment* $L\colon V(G) \to \mathbb{N}$, we say that G is *L-colourable* if there exists a proper colouring c of G such that
$$\forall v \in V(G), \quad c(v) \in L(v).$$
We say that c is a proper *L-colouring* of G.

2. For a given $k \in \mathbb{N}$, G is *k-choosable* if it is L-choosable for every list assignment L satisfying
$$\forall v \in V(G), \quad |L(v)| \geq k.$$

3. The minimum k such that G is k-choosable is the *list chromatic number*, or *choice number* of G. It is denoted $\chi_\ell(G)$, or $\mathrm{ch}(G)$.

Remark 0.2.3. The naive bounds for the chromatic number remain valid.

1. A k-colouring of a given graph G is an L-colouring of G, where

$$\forall v \in V(G), \quad L(v) = [k].$$

Therefore,

$$\forall G, \quad \chi(G) \leq \chi_\ell(G).$$

2. The greedy colouring is as efficient for list colourings as it is for usual colourings;

$$\forall G, \quad \chi_\ell(G) \leq \delta^*(G) + 1 \leq \Delta(G) + 1.$$

The list chromatic number of a given graph might be arbitrarily larger than its chromatic number. This is notably the case for complete bipartite graphs.

Lemma 0.2.16. *Let $n = \binom{2k-1}{k}$. Then $\chi_\ell(K_{n,n}) > k$.*

Proof. Let us call X, Y the two parts of $K_{n,n}$. We let L be the list assignment on $K_{n,n}$ such that every set $S \in \binom{[2k-1]}{k}$ appears as the lists $L(x)$ and $L(y)$ assigned to some vertices $x \in X$ and $y \in Y$. We now show that $K_{n,n}$ is not L-colourable.

Let us assume for the sake of contradiction that there exists a proper L-colouring c of $K_{n,n}$. Assume that at most $k-1$ different colours appear in $c(X)$. Let $S \subseteq [2k-1] \setminus c(X)$ be any set of cardinality k, there exists a vertex $x \in X$ such that $L(x) = S$, and so $L(x) \cap c(X) = \varnothing$, a contradiction. So at least k colours appear in $c(X)$, and the same holds by symmetry for $c(Y)$. Since there are $2k-1$ available different colours, $c(X)$ and $c(Y)$ must intersect in at least one colour; so there exist $x \in X$ and $y \in Y$ such that $c(x) = c(y)$. But since we are in $K_{n,n}$, there is an edge between x and y, which contradicts the fact that c is proper. □

Lemma 0.2.16 has been generalised to every graphs with minimum degree d — and so in particular bipartite ones — by Alon [5], and later improved by Saxton and Thomason [106].

Theorem 0.2.17 (Saxton, Thomason, 2015 [106])**.** *For every graph G with minimum degree d,*

$$\chi_\ell(G) \geq (1 + o(1)) \log_2(d).$$

In 1998, Alon and Krivelevich [6] had shown that the lower bound of Theorem 0.2.17 corresponds asymptotically almost surely to the value of the list chromatic number of a random bipartite graphs. The random model that they used corresponds to the model $G^k(n, p)$ defined in Section 0.3.2.3 when $k = 2$, and the results holds for any value of p as a function of n. In the same paper, they conjectured that this actually holds for every regular graph, up to some multiplicative constant.

Theorem 0.2.18 (Alon, Krivelevich, 1998 [6])**.** *For every graph G drawn from the bipartite Erdős-Rényi model $G(n, n, p)$, it asymptotically almost surely holds that*

$$\chi_\ell(G) = (1 + o(1)) \log_2(np),$$

and so asymptotically almost surely

$$\chi_\ell(G) = (1 + o(1)) \log_2(\Delta(G)).$$

Conjecture 0.2.3 (Alon, Krivelevich, 1998 [6]). *There exists an absolute constant $C > 0$ such that, for every bipartite graph G,*

$$\chi_\ell(G) \leq C \ln \Delta(G).$$

Quite surprisingly, while estimating the largest possible list chromatic number of a bipartite graph remains an open problem, the same problem has been settled sharply for planar graphs.

Theorem 0.2.19 (Thomassen, 1994 [114]). *For every planar graph G,*

$$\chi_\ell(G) \leq 5.$$

Theorem 0.2.20 (Voigt, 1993 [118]). *There exist planar graphs which are not 4-choosable.*

Theorem 0.2.21 (Thomassen, 1995 [115]). *For every planar graph G of girth at least 5,*

$$\chi_\ell(G) \leq 3.$$

The intermediary result between Theorem 0.2.19 and Theorem 0.2.21 concerns the list chromatic number of triangle-free planar graphs. The fact that it is at most 4 is actually a corollary of Theorem 0.1.6, which states that the degeneracy of triangle-free planar graphs is at most 3.

0.2.7.2 The fractional chromatic number

For some applications, the chromatic number is too restrictive in order to formalise the corresponding problem. We need refinements of the chromatic number, and a most elegant and relevant one is the fractional chromatic number.

Definition 0.2.4. Let G be a graph.

1. A proper $(a : b)$-*colouring* of G is a function $c\colon V(G) \to \binom{[a]}{b}$ such that

   $$\forall uv \in E(G), \quad c(u) \cap c(v) = \varnothing.$$

   We say that c is a *fractional colouring* of G, of *weight* a/b.

2. The *fractional chromatic number* of G, denoted $\chi_f(G)$, is defined by

   $$\chi_f(G) = \inf \left\{ \frac{a}{b} \;\middle|\; \text{there exists a proper } (a : b)\text{-colouring of } G \right\}.$$

Remark 0.2.4. Bounds for the fractional chromatic number can be derived from those of the chromatic number.

1. A proper k-colouring is the same as a proper $(k : 1)$-colouring. Therefore,

   $$\forall G, \quad \chi_f(G) \leq \chi(G).$$

2. The Hall ratio is also a lower bound for the fractional chromatic number;

   $$\forall G, \quad \frac{n(G)}{\alpha(G)} \leq \rho(G) \leq \chi_f(G).$$

3. A proper $(a:b)$-colouring of G is the same as a proper a-colouring of $G \boxtimes K_b$, so

$$\chi_f(G) = \inf_{b \in \mathbb{N}^*} \frac{\chi(G \boxtimes K_b)}{b}.$$

Figure 0.2.4: A proper $(5:2)$-colouring of C_5, equivalent to a 5-colouring of $C_5 \boxtimes K_2$

We might wonder, as it has been done for the chromatic number, how far from the Hall ratio the fractional chromatic number can get. The Kneser graphs are of no help here, since their fractional chromatic number equals their Hall ratio — this property holds for every vertex-transitive graph; a proof of this fact lies in Section 0.3.2.4.

Question. Does there exist some function $f\colon \mathbb{Q} \to \mathbb{Q}$ such that, for every graph G,

$$\chi_f(G) \leq f\big(\rho(G)\big)?$$

This question has been answered only very recently by the negative.

Theorem 0.2.22 (Dvořák, Ossona de Mendez, Wu, 2018+ [37])**.** *For every $c \geq 1$, there exists a graph G such that $\chi_f(G) \geq c$ and $\rho(G) \leq 18$.*

The fractional chromatic number can equivalently be defined as the fractional relaxation of the integer linear program (2) computing the chromatic number.

$$\begin{aligned} \chi_f(G) \quad = \quad & \min \sum_{I_{\max} \in \mathcal{I}(G)} w_I, \\ \text{such that} \quad & \begin{cases} \forall v \in V(G), & \displaystyle\sum_{\substack{I \in \mathcal{I}_{\max}(G) \\ v \in I}} w_I \geq 1, \\ \forall I \in \mathcal{I}_{\max}(G), & w_I \in [0,1]. \end{cases} \end{aligned} \tag{4}$$

This equivalent formulation implies that the value of the fractional chromatic number is always attained as the weight of some fractional colouring. The infimum in Definition 0.2.4.2 can therefore be replaced by a minimum. In particular, the fractional chromatic number is always a rational number.

Remark 0.2.5. Fractional colourings can be defined as probability distributions on the independent sets. Given a graph G, there exists a fractional colouring of G of weight w if and only if there exists a probability distribution on the independent sets $\mathcal{I}(G)$ of G — here again we can restrict to the set $\mathcal{I}_{\max}(G)$ of maximal independent sets of G — such that, if $\mathbf{I}$ is a random independent set drawn according to this distribution,

$$\forall v \in V(G), \quad \mathbb{P}\left[v \in \mathbf{I}\right] \geq \frac{1}{w}.$$

Remark 0.2.6. Kneser graphs play the same role for proper $(a : b)$-colourings as complete graphs for proper k-colourings. An $(a : b)$-colouring of some graph G is the same as a homomorphism from G to $KG_{a,b}$.

A practical illustration of the fractional chromatic number. We have just defined two variants of the chromatic number; the list chromatic number and the fractional chromatic number. The usefulness of the first one is quite transparent from its definition; it is simply a more constrained version of the colouring problem where each vertex is selective on the colours he may be assigned. The second one, on the other hand, seems more arbitrary, and could be misleadingly thought of as a purely mathematical construction, whose mere purpose is to yield a neater parameter than the chromatic number, whose behaviour is more convenient in the mathematical theory. While these facts might be subjectively true, the fractional chromatic number is far from being only an abstract parameter; it is actually way more fit to some real life applications than the chromatic number, as we may see right now.

Let us provide an example close to the one presented in Section 0.2.2 where graph colouring is needed. Nowadays, computing power arises mainly from parallel processes. Even a simple smartphone comes with 8 cores, and supercomputer have thousands, if not millions of them. Assume now that you are a scientist, and that you have a massive computation to perform. Luckily for you, this computation consists in some large number of similar tasks, all of which need roughly the same computing time, say one hour. Moreover, you have access to an unlimited number of cores which can execute these tasks in parallel, or at least more than you will ever need. If no further constraint arises, you could simply run all your tasks in parallel at once, thus finishing your computation within one hour.

Unfortunately, more constraints do arise in your case. Your tasks share common resources, and for this reason some of them cannot be run at the same time. You need to use a conflict graph G, where every vertex is a task, and two tasks which cannot be executed simultaneously share an edge. An independent set S in your conflict graph is a set of tasks which can be executed simultaneously.

A proper k-colouring c of your graph yields a conflict-free sequence of executions of your tasks within k hours, where the i-th hour is devoted to completely executing the tasks coloured with the i-th colour. You can finish all your tasks within $\chi(G)$ hours.

However, nothing forces you to fully run a given task in one setting. You could very well run it for half an hour, then pause it some time, and run it until the end later. In order to exploit this strategy, you need an $(a : b)$-colouring of G. Each colour corresponds to the execution of pairwise conflictless tasks during $1/b$ hour, so that each task is run and paused b times, until it is finished. You will therefore be able to complete your computation within $\chi_f(G) \leq \chi(G)$ hours.

Results on the fractional chromatic number. The fractional chromatic number is also interesting as an intuition for the chromatic number. For instance, Reed's conjecture has been shown to hold in the fractional setting.

Theorem 0.2.23 (Molloy, Reed, 2002 [93])**.** *For every graph G,*

$$\chi_f(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}.$$

In general, there is no known classical class of graphs whose fractional chromatic number differs significantly from their chromatic number. Kneser graphs are an example of graphs where

the difference between both parameters can get arbitrarily large, however they are too specific to be consider as a class of graphs in a whole. In general, a class $\mathscr{G}$ of graph should be *hereditary*, which means that for any graph $G \in \mathscr{G}$, all induced subgraphs H should also be in the class $\mathscr{G}$ — the class $\mathscr{G}$ can be defined as the subset of graphs satisfying some hereditary property.

A plausible classical class of graphs where both parameters could diverge is the class of triangle-free d-degenerate graphs. While the chromatic number of such a graph can be as large as $d+1$, the only known example of graphs reaching this bound have a much smaller fractional chromatic number. Thus, Harris has made the following conjecture.

Conjecture 0.2.4 (Harris, 2019 [59])**.** *There exists an absolute constant $C \geq \frac{1}{2}$ such that, for every triangle-free d-degenerate graph G,*

$$\chi_f(G) \leq C\frac{d}{\ln d}.$$

0.2.7.3 Edge colourings

The vertices of a graph are not the only objects which might be properly coloured. The edges of the graph are the next natural ones to consider in the context of proper colourings. In this setting, matchings are the analogue of independent sets in vertex colouring.

Definition 0.2.5. Let G be a graph.

1. A proper *k-edge-colouring* of G is a function $c\colon E(G) \to [k]$ assigning colours to the edges of G such that no two incident edges are assigned the same colour;

   $$\forall uv, vw \in E(G), \quad c(uv) \neq c(vw).$$

   When G admits a k-edge-colouring, we say that G is *k-edge-colourable*.

2. The *chromatic index* $\chi'(G)$ of G is the minimum k such that G is k-edge-colourable.

Remark 0.2.7. An edge-colouring of a given graph G is equivalent to a colouring of its line graph $L(G)$. Therefore

$$\chi'(G) = \chi(L(G)) \geq \omega(L(G)) \geq \Delta(G).$$

You can then easily define list and fractional versions of the chromatic index, respectively $\chi'_\ell(G) = \chi_\ell(L(G))$ and $\chi'_f(G) = \chi_f(L(G))$.

The most famous result on edge colourings is due to Vizing [117], who determined that the chromatic index of a graph of fixed maximum degree can take only two different values.

Theorem 0.2.24 (Vizing's theorem [117])**.** *For every graph G,*

$$\chi'(G) \in \{\Delta(G), \Delta(G)+1\}.$$

There also exists a notorious conjecture related to the list chromatic index, known as the list colouring conjecture.

Conjecture 0.2.5 (The list colouring conjecture)**.** *For every graph G,*

$$\chi'_\ell(G) = \chi'(G).$$

A partial result in favor of Conjecture 0.2.5 states that it holds for bipartite graphs. It is due to Galvin [55].

Theorem 0.2.25 (Galvin, 1995 [55]). *For every bipartite graph G,*

$$\chi'_\ell(G) = \Delta(G) = \chi'(G).$$

The list colouring conjecture has been shown to hold asymptotically by Kahn in 1996.

Theorem 0.2.26 (Kahn, 1996 [67]). *For every graph G of maximum degree Δ,*

$$\chi'_\ell(G) = (1 + o(1))\Delta$$

as $\Delta \to \infty$.

0.2.7.4 Total colourings

We now define total colourings, which are simultaneous colourings of the edges and vertices of a graph.

Definition 0.2.6. Let G be a graph.

1. A proper *k-total-colouring* of G is a function $c\colon (V(G) \cup E(G)) \to [k]$ assigning colours to the vertices and edges of G such that no two incident edges, no two adjacent vertices, and no vertex and incident edge are assigned the same colour;

   $$\forall u \in V(G), \forall v \in N(u), c(u) \neq c(v) \text{ and } c(u) \neq c(uv) \text{ and } \forall w \in N(u) \setminus \{v\}, c(uv) \neq c(uw).$$

   When G admits a k-total-colouring, we say that G is *k-totally-colourable*.

2. The *total chromatic number* $\chi''(G)$ of G is the minimum k such that G is k-totally-colourable.

Remark 0.2.8. If you define the *total graph* $T(G)$ of a given graph G as the graph obtained by adding an edge between every vertex $v \in V(G)$ and every edge $uv \in V(L(G))$ in the disjoint union of G and $L(G)$, then

$$\chi''(G) = \chi(T(G)).$$

You can then easily define list and fractional versions of the total chromatic number, respectively $\chi''_\ell(G) = \chi_\ell(T(G))$ and $\chi''_f(G) = \chi_f(T(G))$.

There is a notorious conjecture on total graph colourings.

Conjecture 0.2.6 (The total colouring conjecture). *For every graph G,*

$$\chi''(G) \leq \Delta(G) + 2.$$

There are two results in favour of Conjecture 0.2.6.

Theorem 0.2.27 (Molloy, Reed, 1998 [92]). *For every graph G,*

$$\chi''(G) \leq \Delta(G) + 10^{26}.$$

Theorem 0.2.28 (Kilakos, Reed, 1993 [77])**.** *For every graph G,*

$$\chi''_f(G) \leq \Delta(G) + 2.$$

Note that, given a graph G together with a proper $(\Delta(G) + 1)$-colouring, each edge of G is incident to at most 2 colours. So

$$\chi''(G) \leq \chi'_\ell(G) + 2,$$

and therefore Conjecture 0.2.5, if true, would almost directly imply Conjecture 0.2.6, namely

$$\chi''(G) \leq \Delta(G) + 3.$$

0.3 Tools and strategies

Let us sum up a bit what has been developed in the two previous sections. We have given a broad overview of what graph theory is about, and why this is such an important domain to explore. Graphs are powerful both as an abstract combinatorial object and as a concrete representation of real-life structures, for they manage to combine a high expressiveness, all the while being generic enough to capture countless possible scenarios. There are many problems which can be raised in the domain of graph theory, most of them arising from a demand for concrete applications. The graph colouring problem is one of the most notorious, and translates the problem of resource allocation under some constraints.

Finding an optimal proper colouring of a graph is a highly desirable goal in many situations, since this is the solution in order to optimise many processes in life. Unfortunately, this problem reaches a strong complexity barrier. It belongs to the family of NP-hard problems, even when relaxing the desired goal to some reasonable approximation of the optimum. Therefore, generic algorithms computing an optimal colouring of a graph are doomed to fail at terminating in a non astronomical time when working on graphs with merely a hundred vertices, barring a really unlikely proof that P = NP.

In this context, research in graph colouring has brought numerous colourability results relying on some specific structure of the considered graphs. Organising graphs into structural classes appears to be one of the most efficient ways to increase the general knowledge in the domain of graph colouring. All the results depicted in Section 0.2.6 fall into this branch of research. The content of this thesis will be devoted to bringing new results to this branch, and to improve on some already established ones.

In this new section, we are going to present some of the most common techniques used in order to obtain the aforementioned results. All of these techniques will be illustrated with classical applications raising interesting facts on graph colouring.

0.3.1 Structural lemmas

The first tools ever described in graph theory, which provoked the emergence of this domain, rely on a structural analysis of the graphs. Their common goal is to extract regular structures from graphs satisfying some conditions as permissive as possible. Most of the results presented in this subsection are the historical foundations of modern graph theory, most of them established in the first half of the XX-th century.

0.3.1.1 Structure on the edges of a graph

The first ever result from graph theory was established by Leonard Euler in 1736 as the solution to a famous problem known as the Seven Bridges of Königsberg. We begin by stating its formal formulation.

Definition 0.3.1. Given a graph G, a *Eulerian trail* (or *Eulerian path*) of G is a walk (a sequence of consecutively incident edges, i.e. a path where vertices and edges can be visited several times) of G visiting all edges exactly once. A *Eulerian circuit* (or *Eulerian cycle*) is a closed Eulerian trail. A graph containing a Eulerian circuit is said to be *Eulerian*.

The Seven Bridges of Königsberg problem consists in finding a circuit of the city of Königsberg which traverses each of its seven bridges exactly once. This can be formulated as finding a Eulerian

circuit of the *multigraph* (a graph with possibly several edges between two vertices) associated to the city of Königsberg, where each edge represents a bridge, as represented in Figure 0.3.1.

Figure 0.3.1: The multigraph representation of the seven bridges of Königsberg

What Euler demonstrated is that there is no solution to the Seven Bridges of Königsberg problem, and established a full characterisation of the Eulerian (multi)graphs.

Theorem 0.3.1 (Euler, 1736)**.** *A connected graph G is Eulerian if, and only if, the degrees of all its vertices are even.*

It is now time to mention Hall's theorem, another essential result concerning edge structures in graph theory. It establishes a sufficient condition for a bipartite graph to admit a perfect matching.

Theorem 0.3.2 (Hall, 1935 [58])**.** *Let $H = (U, V, E)$ be a bipartite graph. Then, a sufficient and necessary condition for H admitting a matching incident to every vertex $u \in U$ is that*

$$\forall X \subseteq U, \quad |N_H(X)| \geq |X|\,.$$

Among the many applications of Hall's theorem, a notable one deals with the chromatic index of regular bipartite graphs.

Corollary 0.3.2.1. *Every d-regular bipartite graph H contains a perfect matching. Therefore,*

$$\chi'(H) = d.$$

Proof. Let $H = (U, V, E)$ be a d-regular graph. Let us apply Hall's theorem to U. Let $X \subseteq U$ be any subset of vertices, and let $Y = N_H(X) \subseteq V$. Since H is d-regular, the number of edges incident to X is exactly $d\,|X|$; this is the number of edges between X and Y, and this is at most the number of edges incident to Y, that is $d\,|Y|$;

$$\begin{aligned} d\,|X| &\leq d\,|Y| \\ |X| &\leq |Y|\,. \end{aligned}$$

So Hall's theorem yields a matching M incident to every vertex in U. Since H is regular and bipartite, it holds that $|U| = |V|$ (since $d\,|U| = e(H) = d\,|V|$). So M is also incident to every vertex in V; it is a perfect matching of H.

Now we prove that $\chi'(H) = d$ by induction. When $d = 1$, the graph is a matching, and therefore only one colour is needed to colour its edges. When $d > 1$, we apply induction on $H \setminus M$, a $(d-1)$-regular bipartite graph, which is therefore $(d-1)$ edge-colourable. By colouring M with one extra colour, we obtain a d-colouring of the edges of H. □

Definition 0.3.2. Given a graph G, a *vertex cover* of G is a subset of vertices $X \subseteq V(G)$ such that every edge $e \in E(G)$ has an extremity contained in X.

Remark 0.3.1. The complement of a vertex cover X in G is an independent set. Indeed, if there is an edge $e \in E(G)$ between two vertices in $V(G) \setminus X$, this edge is not covered by X, a contradiction.

Theorem 0.3.3 (Kőnig, 1931 [81])**.** *If G is a bipartite graph, then the size of a maximum matching in G equals the size of a minimum vertex cover of G.*

Since the complement of a minimum vertex cover is a maximum independent set, König's theorem implies that computing the independence number of a bipartite graph is equivalent to computing its matching number. In particular, this can be done in polynomial time.

Both Hall's and Kőnig's theorems are actually special cases of a much stronger statement, known as the max-flow min-cut theorem.

Definition 0.3.3. Let G be a directed graph, of source vertex $s \in V(G)$ (the in-degree of s is 0), of sink vertex $t \in V(G)$ (the out-degree of t is 0), given with some capacity $c_e \in \mathbb{R}^+$ for every $e \in E(G)$.

1. A *s,t-flow* of G is a mapping $f\colon E(G) \to \mathbb{R}^+$ satisfying

   (i) $\forall e \in E(G), \quad f(e) \leq c_e$, and
   (ii) $\forall v \in V(G) \setminus \{s,t\}, \quad \sum_{u \in N^-(v)} f(uv) = \sum_{w \in N^+(v)} f(vw)$.

   The *value* of a given flow f is defined by

   $$|f| = \sum_{v \in N^+(s)} f(sv) = \sum_{u \in N^-(t)} f(ut).$$

2. A *s,t-cut* of G is the set of edges $E(S,T)$ between the two parts of a bipartition (S,T) of $V(G)$ satisfying $s \in S$ and $t \in T$. The *capacity* of a given cut C is defined by

   $$|C| = \sum_{e \in C} c_e.$$

Theorem 0.3.4 (Max-flow min-cut)**.** *Given a directed graph G of source s and sink t, the maximum value of a s,t-flow of G equals the minimum capacity of a s,t-cut of G.*

Remark 0.3.2. The max-flow problem can be formulated as the fractional solution of some linear program, and the min-cut problem can be formulated as the fractional solution of the dual of this linear program. The max-flow min-cut theorem is therefore a consequence of the strong duality theorem for linear programs, which states that the optimal solution (if it exists) of a primal linear program equals the optimal solution of its dual.

0.3.1.2 Regularisation of graphs

Most of the content in this thesis focuses on the class of graphs of maximum degree bounded by some constant d. It is often more convenient to work within the class of regular graphs, so a useful and classical trick, given a graph G of maximum degree $\Delta(G) = d$, consists in injecting G into a d-regular supergraph $\varphi(G)$. This supergraph should share the same properties as G, so that both can be freely interchanged.

Theorem 0.3.5 (Exoo, Jajcay, 2008 [50]). *For every integers d and g both at least 3, there exists a d-edge-coloured d-regular graph with girth at least g which is vertex transitive as a coloured graph.*

Note that Theorem 0.3.5 is stated in a weaker version in [50]. However, the graph constructed in the proof of the theorem is a Cayley graph $C(\Gamma, X)$ where the generating set X is composed of involutions. So the labelling of the edges induced by X yields a d-edge-colouring which is stable by the automorphisms induced by the left multiplication action of Γ on itself.

Lemma 0.3.6. *From any graph G of maximum degree d and girth g, we can construct a d-regular graph $\varphi(G)$ of girth g whose vertex set can be partitioned into induced copies of G, and such that any vertex $v \in G$ can be sent to any of its copies through an automorphism.*

Proof. Set $k \coloneqq \sum_{v \in G} (d - \deg(v))$. Let G' be the supergraph of G obtained by adding k vertices $(v'_i)_{i \in [k]}$, each of degree 1, such that all other vertices have degree d. We let e'_i be the edge of G' incident to v'_i, for each $i \in [k]$. By Theorem 0.3.5, there exists a vertex-transitive k-regular graph H of girth at least g together with a proper edge-colouring c using k colours. Let $n(H)$ be the number of vertices of H and write $V(H) = [n(H)]$.

We construct $\varphi(G)$ by starting from the disjoint union of $n(H)$ copies $(G_i)_{i \in [n(H)]}$ of G. For each edge $e = \{i, j\} \in E(H)$, letting u_e be the vertex of G incident to the edge $e'_{c(e)}$ in G', we add an edge between the copy of u_e in G_i and that in G_j.

Any cycle in $\varphi(G)$ either is a cycle in G, and hence has length at least g, or contains all the edges of a cycle in H, and hence has length at least g. It follows that $\varphi(G)$ has girth g.

The last statement follows readily from the fact that H is vertex transitive as a coloured graph. □

0.3.1.3 Decomposition

Instead of finding one well structured object within a graph, one might want to fully decompose it into structured objects. This is notably what graph colouring is about regarding its independent set; a graph colouring consists in a decomposition of its vertex set into independent sets.

We here state a result of Petersen concerning the decomposition of the edge set of a graph into 2-factors.

Theorem 0.3.7 (Petersen, 1891 [99]). *For any $k > 0$, every $2k$-regular graph can be decomposed into k 2-factors.*

Proof. We know by Euler's theorem that a graph with only even degrees has a Eulerian circuit. A traversal of this Eulerian circuit yields an orientation $\overrightarrow{G}$ of the edges of G, such that every vertex has k in-going arcs and k out-going arcs. Let $H = (V^-, V^+, E_H)$ be the k-regular bipartite graph obtained by duplicating each vertex $v \in V(G)$ into $v^- \in V^-$ and $v^+ \in V^+$, and putting an edge between u^- and v^+ whenever there is an arc going from u to v in $\overrightarrow{G}$.

Since H is bipartite and regular, Corollary 0.3.2.1 implies that H can be decomposed into $2k$ perfect matchings. It is possible to organise them into k pairs, so that each pair of matchings yields a 2-factor of G after collapsing each pair of vertices $(v^-, v^+) \in V^- \times V^+$ back into $v \in V(G)$. □

Petersen's theorem is useful to provide an upper bound on the number of colours needed in an edge-colouring of a *multigraph*, that is a graph where there might be several edges linking the same pair of vertices.

Theorem 0.3.8 (Shannon, 1949 [107]). *For every multigraph G,*

$$\chi'(G) \leq \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor,$$

and this is attained by Shannon's triangles.

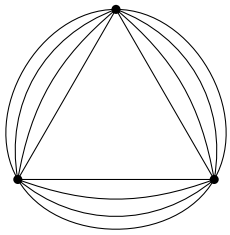


Figure 0.3.2: Shannon's theorem is sharp for Shannon's triangles.

Proof. We give a proof of Shannon's theorem in the case when $\Delta(G) = 2k$. Up to adding edges and vertices to G, which would not decrease its chromatic index, we might assume that G is $2k$-regular. It is therefore possible to decompose G into k 2-factors by Petersen's theorem, all of which are the disjoint union of some cycles. Since every cycle is 3-edge-colourable, it is possible to colour the edges of G with at most $3k = \frac{3\Delta(G)}{2}$ colours. □

0.3.1.4 Cubic graphs

Theorem 0.3.9 (Petersen, 1891 [99]). *If G is* cubic *(3-regular) and* bridgeless *(2-edge-connected), then G can be decomposed into a perfect matching and a 2-factor.*

Lemma 0.3.10 (The parity condition [60]). *Let G be a cubic graph, and c be a proper 3-edge-colouring of G. Then every cut X of G satisfies*

$$\left|c^{-1}(1) \cap X\right| \equiv \left|c^{-1}(2) \cap X\right| \equiv \left|c^{-1}(3) \cap X\right| \pmod 2.$$

Proof. Let $(i,j) \in \binom{[3]}{2}$, and let E_{ij} denote the edges of G coloured either with i or j. E_{ij} is a 2-factor of G, so the disjoint union of cycles. Every cycle intersects X in an even number of edges, therefore $|E_{ij}| \bmod 2 = 0$. So

$$\forall i \neq j, \quad \left|c^{-1}(i) \cap X\right| + \left|c^{-1}(j) \cap X\right| \equiv 0 \pmod 2.$$

This implies the desired result. □

Lemma 0.3.10 is used notably in the proof that the 3-edge colouring problem is NP-complete [60] in the class of cubic graph, through a reduction to 3-SAT.

0.3.2 Probabilistic method

The probabilistic method has shown to be really effective in order to prove the existence of special objects which we have no peculiar idea how to construct. In the context of graph colouring, the typical problem consists in finding — or at least giving a good estimation of — the maximum $\chi(\mathscr{G})$ of the chromatic number over all graphs of a given class $\mathscr{G}$. In order to demonstrate that $\chi(\mathscr{G}) \leq k$ for some $k \in \mathbb{N}$, it suffices to find a proper k-colouring of every graph from the class. The colouring can typically be shown to exist through the probabilistic method. On the other hand, in order to demonstrate that $\chi(\mathscr{G}) > k$ for some $k \in \mathbb{N}$, it suffices to find a graph $G_0 \in \mathscr{G}$ which is not k-colourable. Again, such a graph can be shown to exist through the probabilistic method.

The probabilistic method is particularly efficient for proving the existence of objects which have a positive density within their probability space. When the objects looked for are too rare, the probabilistic method often needs to be combined with carefully chosen deterministic structural arguments in order to be effective.

0.3.2.1 Lovász Local lemma

The main probabilistic tool for proving the existence of a special instance of a random object which satisfies a given set of random events sharing dependencies is the Lovász Local Lemma, generally referred to as LLL for brevity.

Theorem 0.3.11 (LLL)**.** *Let $\mathscr{E} = \{E_1, \ldots, E_n\}$ be a finite set of random events in a probability space. For every $E_i \in \mathscr{E}$, let $\Gamma_i \subseteq \mathscr{E} \backslash \{E_i\}$ be a subset of events such that E_i is mutually independent of $\mathscr{E} \setminus (\Gamma_i \cup \{E_i\})$. If there exist $x_1, \ldots, x_n \in (0,1)$ such that*

$$\forall i, \quad \mathbb{P}[E_i] \leq x_i \prod_{E_j \in \Gamma_i} (1 - x_j),$$

then the probability that no event E_i occurs is lower bounded by

$$\prod_{i=1}^{n} (1 - x_i) > 0.$$

Remark 0.3.3. It is possible to have a version of LLL with no independence requirement, called the *lopsided LLL*, which has the same proof as LLL. In this version, we bound $\mathbb{P}\left[E_i \,\middle|\, \bigcup_{E_j \in \overline{\Gamma_i \cup \{E_i\}}} \overline{E_j}\right]$ rather than $\mathbb{P}[E_i]$.

It is often the case that LLL needs to be applied in a symmetric setting. The symmetric formulation of LLL is therefore the one which is mostly used in the literature.

Theorem 0.3.12 (Symmetric LLL)**.** *Let $E_1, \ldots, E_n$ be a finite set of random events, all of which occur with probability p. Assume that every E_i is mutually independent from a set of all but at most D of the other random events. Then, provided that*

$$ep(D+1) \leq 1,$$

the probability that no event E_i occurs is positive.

Remark 0.3.4. Shearer [109] later improved the condition of the symmetric version of LLL up to optimality, by showing that

$$\begin{cases} p < 1 & \text{if } D = 1 \\ p < \frac{(D-1)^{D-1}}{D^D} & \text{if } D \geq 2 \end{cases}$$

is a sufficient condition on p. This implies in particular that $epD \leq 1$ is enough as well, since it always holds that $1/e < (\frac{D-1}{D})^{D-1}$.

Let us illustrate some use of LLL in the context of list colouring. Given a graph G, and some k-list assignment $L\colon V(G) \to \binom{[n]}{k}$, we say that L has *separation* c if it holds that

$$\forall uv \in E(G), \quad |L(u) \cap L(v)| \leq c.$$

Theorem 0.3.13 (Kratochvil, Tuza, Voigt, 1998 [82])**.** *Given a graph G, let L be a k-list assignment of G with separation c. Then, provided that $k \geq \sqrt{2ec(\Delta(G)-1)}$, G is L-colourable.*

Proof. Let c be a (most likely improper) random L-colouring of G obtained by drawing uniformly at random a colour from $L(v)$ for every vertex $v \in V(G)$. For every edge $uv \in E(G)$, we let E_{uv} be the event that $c(u) = c(v)$. The event that c is a proper L-colouring of G is equivalent to the negation of all the events E_{uv}, for $uv \in E(G)$.

If E_{uv} occurs, it means that $c(u)$ is one of the at most c common colours between $L(u)$ and $L(v)$, and provided that, $c(v)$ is that exact same colour. So

$$\begin{aligned} \mathbb{P}[E_{uv}] &= \mathbb{P}[c(u) \in L(u) \cap L(v)] \cdot \mathbb{P}[c(v) = c(u) \mid c(u) \in L(u) \cap L(v)] \\ &\quad + \mathbb{P}[c(u) \in L(u) \setminus L(v)] \cdot \mathbb{P}[c(v) = c(u) \mid c(u) \in L(u) \setminus L(v)] \\ &\leq \frac{c}{k} \cdot \frac{1}{k} + \frac{k-c}{k} \cdot 0 = \frac{c}{k^2}. \end{aligned}$$

If two edges uv and $u'v'$ are not incident, the events E_{uv} and $E_{u'v'}$ are independent. So the number of events not independent from E_{uv} is the number of edges incident to uv, which is at most $D = 2(\Delta(G) - 1)$.

We apply the symmetric version of LLL with $p = \frac{c}{k^2}$ and $D = 2\Delta(G) - 2$.

$$epD = \frac{2ec(\Delta(G)-1)}{k^2} \leq 1,$$

so we conclude that, with positive probability, the L-colouring c is proper.

□

0.3.2.2 Probability bounds

When analysing a random process, computing the exact probability of a given event can quickly become a tedious and technical task to perform, if not entirely out of reach. Fortunately, there exist quite a few probability bounds which can be of great help here. There is such a variety of them that it is very unlikely that none can be efficiently applied in a given probabilistic setting. Let us here present some of the inescapable ones for applications in graph colouring.

Theorem 0.3.14 (Union bound: Boole's inequality)**.** *Given a countable set of random events $(E_i)_i$, it holds that*

$$\mathbb{P}\left[\bigcup_i E_i\right] \leq \sum_i \mathbb{P}[E_i].$$

Theorem 0.3.15 (Markov's inequality)**.** *Let X be a non negative random variable, and $a > 0$ be some constant. Then*

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

Theorem 0.3.16 (Chebyshev's inequality)**.** *Let X be a non negative random variable, and $a > 0$ be some constant. Then*

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq a] \leq \frac{\mathrm{Var}(X)}{a^2},$$

where $\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right]$ *is the* variance *of X.*

Theorem 0.3.17 (Chernoff bounds)**.** *Let $X_1, \ldots, X_n$ be independent $\{0,1\}$-valued random variables. Let $X = \sum_{i=1}^{n} X_n$, and $\delta \geq 0$ be some constant. Then*

$$\mathbb{P}[X \leq (1-\delta)\mathbb{E}[X]] \leq e^{-\frac{\delta^2}{2}\mathbb{E}[X]}, \text{ and}$$
$$\mathbb{P}[X \geq (1+\delta)\mathbb{E}[X]] \leq e^{-\frac{\delta^2}{2+\delta}\mathbb{E}[X]}.$$

These bounds also hold when each inequality is strict.

Remark 0.3.5. Chernoff bounds also hold when $X_1, \ldots, X_n$ are *negatively correlated* rather than independent, that is when

$$\mathbb{E}\left[\prod_{i=1}^{n} X_i\right] \leq \prod_{i=1}^{n} \mathbb{E}[X_i].$$

The inequalities of Markov and Chebyshev are usually useful for the first and second moment method, which is used in order to analyse phase transitions in random graphs.

Chernoff bounds have a much wider range of applications, mainly because of their strength due to the exponential decrease of its bounds. Its statement implies that the distribution function of any random variable counting independent events is highly concentrated around its mean. As it turns out, the number of applications of such a result is huge, and we shall illustrate an application to fractional colourings. To this end, let us first define an analogue of list colouring in the fractional setting.

Definition 0.3.4. Given a graph G, we say that G is $(a:b)$*-choosable* if for any list assignment $L\colon V(G) \to \binom{\mathbb{N}}{a}$ there exists a colouring $c_L\colon V(G) \to \binom{\mathbb{N}}{b}$ satisfying

$$\forall v \in V(G), \quad c_L(v) \in \binom{L(v)}{b}, \text{ and}$$
$$\forall uv \in E(G), \quad c_L(u) \cap c_L(v) = \varnothing.$$

The *fractional choosability* $\mathrm{ch}_f(G)$ is the infimum of a/b such that G is $(a:b)$-choosable.

Theorem 0.3.18 (Alon, Tuza, Voigt, 1997 [9])**.** *For every graph G,*

$$\mathrm{ch}_f(G) = \chi_f(G).$$

Proof. Note that, in the definition of $(a : b)$-choosability, if the list assignment L is such that $L(v) = [a]$ for every vertex $v \in V(G)$, the corresponding colouring c_L is a $(a : b)$-colouring of G. In particular, $\mathrm{ch}_f(G) \geq \chi_f(G)$.

We will now prove the other side of the inequality. Let $1 \geq \varepsilon > 0$ be an arbitrarily small rational positive constant, and let c be a proper $(p : q)$-colouring of G such that $\chi_f(G) = p/q$. We let $a = (1 + \varepsilon)pm$ and $b = qm$ for some integer m such that a is an integer. We shall later precise the value of m, which will be large enough for our needs, that is for G to be $(a : b)$-choosable. Let L be a list assignment of G such that $|L(v)| = a$ for every vertex $v \in V(G)$, and let $X = \bigcup_{v \in V(G)} L(v)$ be the (finite) set of colours used by L.

For any colour $x \in X$, we let $\mathbf{Y}_x$ be drawn uniformly at random from $[p]$, and for every $i \in [p]$ we let $\mathbf{X}_i$ be the (random) subset of X containing all colours $x \in X$ satisfying $\mathbf{Y}_x = i$. We now let $\mathbf{L}_i(v) = L(v) \cap \mathbf{X}_i$ for every vertex $v \in V(G)$, and apply Chernoff bound on the random variable $|\mathbf{L}^i(v)|$ of expectancy $\mu = a/p = (1 + \varepsilon)m$;

$$\begin{aligned}
\mathbb{P}\left[\left|\mathbf{L}^i(v)\right| < m\right] &\leq \mathbb{P}\left[\left|\mathbf{L}^i(v)\right| < \left(1 - \frac{\varepsilon}{2}\right)\mu\right] \\
&< e^{-\frac{\left(\frac{\varepsilon}{2}\right)^2}{2}\mu} \\
&< e^{-\frac{\varepsilon^2}{8}m}.
\end{aligned}$$

Let B be the bad random event that there exist $v \in V(G)$ and $i \in [p]$ such that $|\mathbf{L}_i(v)| < m$. We can bound from above the probability of B through a union bound, which yields

$$\begin{aligned}
\mathbb{P}[B] &< np \cdot e^{-\frac{\varepsilon^2}{8}m} \\
&< 1,
\end{aligned}$$

provided that $m \geq 8\ln(np)/\varepsilon^2$. It means that with positive probability, every $\mathbf{L}_i(v)$ is of size at least m. Let $C_i(v)$ be the corresponding value of each $\mathbf{L}_i(v)$ in such a random draw. Note that $C_i(u)$ and $C_j(v)$ are disjoint for any $u, v \in V(G)$ provided that $i \neq j$. We define the colouring C_L by

$$\forall v \in V(G), \quad C_L(v) \coloneqq \bigcup_{i \in c(v)} C_i(v).$$

For every L, the colouring C_L satisfies that $C_L(v) \subseteq L(v)$ and $|C_L(v)| \geq qm$ for every vertex $v \in V(G)$, and $C_L(u) \cap C_L(v) = \varnothing$ for every edge $uv \in E(G)$. We conclude that indeed G is $(a : b)$-choosable. So

$$\mathrm{ch}_f(G) \leq \inf_{\varepsilon > 0} \frac{(1 + \varepsilon)pm}{qm} = \frac{p}{q} = \chi_f(G).$$

□

0.3.2.3 Random graphs

The most widely analysed random graph model is the Erdős-Rényi one, as well for its simplicity than for its high potential in finding special graphs. Note in particular that a graph drawn from

$G(n, 1/2)$ corresponds to a graph chosen uniformly at random among the labelled graphs on n vertices.

A classical analysis performed on a graph G drawn from $G(n, p)$ is, given a graph property P, to find a threshold on the value of p for the property P to appear asymptotically almost surely in G. Formally, a *threshold* for a given property P is a function f on integers such that if $p = o(f(n))$ the property P asymptotically almost surely does not hold, while if $p = \omega(f(n))$ the property P asymptotically almost surely holds. It can be the case that the conditions can be replaced with $p \leq (1-\varepsilon)f(n)$ and $p \geq (1+\varepsilon)f(n)$ respectively, for any $\varepsilon > 0$, in order to have the same behaviour. We say in this case that f is a *sharp threshold*. In the paper [48] where they introduced their model, Erdős and Rényi proved that there exists a sharp threshold for connectivity.

Theorem 0.3.19 (Erdős, Rényi, 1960 [48])**.** *The function $n \mapsto \ln n/n$ is a sharp threshold for the property that G drawn from $G(n, p)$ is connected.*

A sharp threshold for unique colourability. We describe a model of random balanced k-partite graph directly inspired from the Erdős-Rényi one, for which we exhibit a sharp threshold for the property that a graph G is *uniquely k-colourable*, that is there exists exactly one proper k-colouring of G.

Definition 0.3.5. We define the random k-partite graph model $G^k(n, p)$ as the set of random graphs $G = \left(\bigcup_{i\in[k]} V_i, E\right)$ consisting of k independent parts V_i of size n, and such that the events that $xy \in E$, for $x \in V_i, y \in V_j, i \neq j$, are independent and identically distributed, of probability p.

Remark 0.3.6.

1. A graph drawn from $G^k(n, 1/2)$ corresponds to a labelled subgraph of the complete k-partite graph K_{k*n} drawn uniformly at random.

2. Drawing a graph G from $G^k(n, p)$ is equivalent to drawing a graph G' from $G(kn, p)$, and removing from G' the edges with both extremities in a same part V_i, for any $i \in [k]$.

Theorem 0.3.20. *Let $1 > \varepsilon > 0$ and $G \in G^k(n, p)$, for some $p \leq (1-\varepsilon)\frac{\ln n}{n}$. Then asymptotically almost surely, there exist several proper k-colourings of G.*

Proof. Let $G = (V_1 \cup \ldots \cup V_k, E) \in G^k(n, p)$, such that $p \leq (1-\varepsilon)\frac{\ln n}{n}$. For every $x \in V_i$ and $j \neq i$, we define the event U_x^j that x is uncovered by V_j;

$$U_x^j \equiv N_G(x) \cap V_j = \varnothing.$$

The probability that the event U_x^j occurs is then

$$\mathbb{P}\left[U_x^j\right] = (1-p)^n \geq \left(1 - \frac{(1-\varepsilon)\ln n}{n}\right)^n \underset{n\to\infty}{\sim} e^{-(1-\varepsilon)\ln n} = n^{\varepsilon-1}.$$

The events $(U_x^j)x, j$ are all pairwise independent, so we can use Chernoff's inequality to bound from above the probability that none of them occurs. Let $\mathbf{X}_x^j$ be 1 if U_x^j occurs, and 0 otherwise.

We let $\mathbf{S}$ be the sum of all the random variables $\mathbf{X}_x^j$;

$$\mathbf{S} \coloneqq \sum_{i=1}^{k} \sum_{x \in V_i} \sum_{j \neq i} \mathbf{X}_x^j.$$

Note that $\mathbb{E}[\mathbf{S}] \underset{n\to\infty}{\sim} k(k-1)n^{\varepsilon}$, and the negation of all events U_x^j is equivalent to the event that $\mathbf{S} = 0$. Chernoff bound applied with $\delta = 1$ yields

$$\mathbb{P}\left[\bigwedge_{x,j} \overline{U_x^j}\right] = \mathbb{P}[\mathbf{S} \leq 0] \leq e^{-\frac{1}{2}\mathbb{E}[\mathbf{S}]} = e^{-\frac{1+o(1)}{2}k(k-1)n^{\varepsilon}} \underset{n\to\infty}{\to} 0.$$

This means that, asymptotically almost surely, there exists a vertex $x_0 \in V_{i_0}$ with no neighbours in V_{j_0}, for some $(i_0, j_0) \in \binom{[k]}{2}$. Therefore, there exist at least two proper k-colourings of G, the canonical one c_0 satisfying $c_0(x) = i$ for every i and $x \in V_i$, and the one obtained from c_0 by changing the colour of x_0 from i_0 to j_0. □

Remark 0.3.7. Let $k \geq 2$. In a properly k-coloured graph G, if the subgraph induced by two different colours is disconnected, then G contains several proper colourings. If G is drawn from $G^k(n,p)$, then the subgraph H of G induced by V_0 and V_1 is a subgraph of a graph drawn from $G(2n,p)$, as noted in Remark 2. A weaker version of Theorem 0.3.20 can therefore be obtained by applying Theorem 0.3.19 on H, with $p \leq (1-\varepsilon)\frac{\ln(2n)}{2n}$.

Lemma 0.3.21. *Let $\frac{1}{2} > \varepsilon > 0$ and $G \in G^k(n,p)$, for some $p \geq (1+\varepsilon)\frac{\ln n}{n}$. Then asymptotically almost surely, G admits a unique proper k-colouring.*

Proof. Let c be a proper k-colouring of G. For every $i \in [k]$, we denote $a_i = |c^{-1}(\{i\})|$ and $a_{i,j} = |c^{-1}(\{i\}) \cap V_j|$, and we define $a_i^* \coloneqq \max_{j\in[k]} a_{i,j}$ and $a_i^+ \coloneqq a_i - a_i^*$.

If c is not the canonical colouring of G, there exists some i such that $a_i^+ \geq 1$. We are going to roughly estimate the probability that there exists a non canonical colouring of G, through the union bound

$$\sum_{a_{1,1}+\ldots+a_{k,1}=n} \cdots \sum_{a_{1,k}+\ldots+a_{k,k}=n} \binom{n}{a_{1,1},\ldots,a_{k,1}} \cdots \binom{n}{a_{1,k},\ldots,a_{k,k}} (1-p)^{\sum_{i=1}^{k} \sum_{j\neq j'} a_{i,j}a_{i,j'}}.$$

- Let us first upper bound the contribution of the colourings satisfying $a_{i_0}^+ \geq \frac{\varepsilon}{2k}n$ for some $i_0 \in [k]$. Note by pigeonhole principle that $a_{i_0}^* \geq \frac{a_{i_0}^+}{k-1} \geq \frac{\varepsilon}{2k(k-1)}n$, and that the sum of all multinomial coefficients is $\sum_{a_1+\ldots+a_k=n} \binom{n}{a_1,\ldots,a_k} = k^n$. The upper bound is therefore

$$\begin{aligned}
&\sum_{a_{1,1}+\ldots+a_{k,1}=n} \cdots \sum_{a_{1,k}+\ldots+a_{k,k}=n} \binom{n}{a_{1,1},\ldots,a_{k,1}} \cdots \binom{n}{a_{1,k},\ldots,a_{k,k}} (1-p)^{\sum_{i=1}^{k} \sum_{j\neq j'} a_{i,j}a_{i,j'}} \\
&\leq \sum_{a_{1,1}+\ldots+a_{k,1}=n} \cdots \sum_{a_{1,k}+\ldots+a_{k,k}=n} \binom{n}{a_{1,1},\ldots,a_{k,1}} \cdots \binom{n}{a_{1,k},\ldots,a_{k,k}} e^{-pa^*_{i_0}a^+_{i_0}} \\
&\leq (k^n)^k \, n^{-\frac{\varepsilon^2}{4k^2(k-1)}n} \underset{n\to\infty}{\to} 0.
\end{aligned}$$

- We bound from above the contribution of the colourings satisfying $a_i^+ \leq \frac{\varepsilon}{2k}n$ for every $i \in [k]$. Note that, for all $i \in [k]$, $a_i^* \leq n$, and so $a_i = a_i^* + a_i^+ \leq \left(1 + \frac{\varepsilon}{2k}\right) n$. On the other hand, since $\sum_{i=1}^{k} a_i = kn$, it means that $a_i \geq \left(1 - \frac{(k-1)\varepsilon}{2k}\right) n$ and $a_i^* = a_i - a_i^+ \geq \left(1 - \frac{\varepsilon}{2}\right) n$, for every $i \in [k]$.

  Therefore, for every $j \in [k]$, there exists at most one value of i such that $a_i^* = a_{i,j}$. Without loss of generality, we may assume that $a_i^* = a_{i,i}$ for every $i \in [k]$. So the choice of the values a_i^+ and the distribution of the corresponding colours among the sets V_j fully determines the colouring c. For some fixed m, the contribution of the colourings satisfying $\sum_i a_i^+ = m$ is bounded from above by

$$\begin{aligned}
&\underbrace{\binom{kn}{m}}_{\text{choice of the vertices}} \sum_{a_1^+ + \ldots + a_k^+ = m} \underbrace{\binom{m}{a_1^+,\ldots,a_k^+}}_{\text{attribution of the colours}} (1-p)^{\sum_{i=1}^{k} a_i^* a_i^+} \\
&\leq \left(\frac{kne}{m}\right)^m \sum_{a_1^+ + \ldots + a_k^+ = m} \binom{m}{a_1^+,\ldots,a_k^+} e^{-p\left(1-\frac{\varepsilon}{2}\right)nm} \\
&\leq \left(\frac{k^2e}{m}\right)^m n^m n^{-(1+\varepsilon)\left(1-\frac{\varepsilon}{2}\right)m} \\
&\leq c_k n^{-\frac{\varepsilon}{4}m} \text{ where } c_k = \max_{m\in\mathbb{N}^*} \left(\frac{k^2e}{m}\right)^m \leq e^{k^2}.
\end{aligned}$$

  When we sum these contributions for every $m \geq 1$, we obtain an upper bound of

$$c_k \frac{n^{-\frac{\varepsilon}{4}}}{1 - n^{-\frac{\varepsilon}{4}}} \underset{n\to\infty}{\to} 0.$$

□

A lower bound on the fractional chromatic number. The study of random graphs lets us prove in a non constructive way the existence of graphs with interesting properties. The core of this kind of study lies in the relevant choice of the random model of graphs. By analysing a random model closely related to d-regular graphs, Bollobás could prove the following surprising lower bound on the (fractional) chromatic number of graphs of large girth.

Theorem 0.3.22 (Bollobás, 1981 [18]). *For every $d, g \geq 3$, there exist graphs of maximum degree at most d, girth at least g, and of fractional chromatic number at least $\frac{d}{2\ln d}$.*

The proof of Bollobás relies on two important facts on random d-regular graphs on n vertices: asymptotically almost surely as $n \to \infty$, their independence number is of order $2n \ln d/d$, and they contain no more than $\ln n$ cycles of length less than g for any fixed g. By removing one vertex from each cycle of length less than g, the independence number remains of the same order, while the girth of the graph becomes at least g. The fact that $\chi_f(G) \geq n(G)/\alpha(G)$ for every graph G concludes his proof.

The random model studied in order to establish those two facts on random d-regular graphs is obtained by considering a configuration drawn uniformly at random from the set of all configurations of dn unordered pairs among $2dn$ labelled vertices, partitioned into $2n$ sets V_i. By contracting each set V_i of a configuration into a single vertex v_i, one obtains a d-regular graph on $2n$ vertices, which might contain loops and multiple edges. Yet, the probability that the obtained graph is simple and loopless is bounded away from 0, namely it tends to $e^{-(d^2-1)/4}$ as n grows to infinity. As a conclusion, any event which occurs asymptotically almost surely on a random configuration also occurs asymptotically almost surely on a (simple loopless) d-regular graph drawn uniformly at random.

0.3.2.4 Fractional chromatic number

The fractional chromatic number combines well with probabilistic arguments, notably because of its probabilistic formulation.

Definition 0.3.6. A fractional colouring of weight w of some graph G is a probability distribution on $\mathcal{I}(G)$ such that, if $\mathbf{I}$ denotes a random independent set drawn according to this distribution,

$$\forall v \in V(G), \quad \mathbb{P}[v \in \mathbf{I}] \geq \frac{1}{w}.$$

Theorem 0.3.23. *If G is a vertex-transitive graph, then*

$$\chi_f(G) = \frac{n(G)}{\alpha(G)}.$$

Proof. Let $\mathbf{I}$ be an independent set drawn uniformly at random among the collection of maximum independent sets $\mathcal{I}_\alpha(G)$ of G.

Let $u \neq v \in V(G)$, and f be an automorphism of G such that $f(u) = v$. The automorphism f yields a bijection between the maximum independent sets of G containing u and the maximum independent sets of G containing v. Therefore,

$$\mathbb{P}[u \in \mathbf{I}] = \frac{\#\{I \in \mathcal{I}_\alpha(G) \mid u \in I\}}{|\mathcal{I}_\alpha(G)|} = \frac{\#\{I \in \mathcal{I}_\alpha(G) \mid v \in I\}}{|\mathcal{I}_\alpha(G)|} = \mathbb{P}[v \in \mathbf{I}].$$

We have shown that $\mathbb{P}[v \in \mathbf{I}]$ is equal to some constant p for every $v \in V(G)$. We know that

$$\alpha(G) = \mathbb{E}[|\mathbf{I}|] = \sum_{v \in V(G)} \mathbb{P}[v \in \mathbf{I}] = pn(G),$$

and so for every $v \in V(G), \mathbb{P}[v \in \mathbf{I}] = \frac{\alpha(G)}{n(G)}$. □

0.3.3 Local recolouring

In some classes of well enough structured graphs, it is possible to obtain colouring results through methods of local recolouring. The idea of a recolouring process is, from a proper colouring of all but one vertex v of a graph obtained by an inductive algorithm, to remove one colour from the neighbourhood of v in order to colour v with this colour. This removal can be obtained by recolouring a (relatively) small part of the graph.

In the context of graph recolouring, one of the most powerful tools are Kempe chains, which are the key for the proof of Vizing's theorem.

Definition 0.3.7. Let G be a graph, and c a proper colouring of G. For two given colours a, b of c, and a vertex $v \in V(G)$ such that $c(v) \in \{a, b\}$, we let $H_{a,b}$ be the subgraph of G induced by the vertices coloured a or b by c;

$$H_{a,b} \coloneqq G\left[c^{-1}\big(\{a, b\}\big)\right].$$

The (a, b)-*Kempe chain* of v is the connected component of $H_{a,b}$ containing v.

Given a proper partial colouring c of a graph G, the colour of a vertex v can be changed from a to b by inverting the colours in the (a, b)-Kempe chain of v. The new colouring obtained after this process remains a proper partial colouring of G. Such a strategy can be used to provide a short proof of the 5-colour theorem for planar graphs, a weakening of the 4-colour theorem.

Theorem 0.3.24. *For every planar graph G,*

$$\chi(G) \leq 5.$$

Proof. As observed in Section 0.1.5.3, every planar graph is 5-degenerate. For the sake of contradiction, let G be a minimum counterexample of the theorem, so G is a planar graph with $\chi(G) > 5$ and $\chi(H) \leq 5$ for every subgraph $H \subsetneq G$. Let v be a vertex of degree 5 in G, and let c be a proper 5-colouring of $G - x$. Since G is not 5-colourable, every colour appears in $N_G(v)$. Let us fix an embedding of G in the plane, and denote $v_1, \ldots, v_5$ the neighbours of v, such that they respect the trigonometric order in the embedding of G. Without loss of generality, we assume that $c(v_i) = i$. If v_3 is not contained in the $(1, 3)$-Kempe chain $C_{1,3}$ of v_1, we can invert colours 1 and 3 in $C_{1,3}$, and thus recolour v_1 with 3 without changing the colours of the other v_i's. We can then properly colour v with 1, which is a contradiction, so v_3 must be contained in $C_{1,3}$.

The subgraph $G[C_{1,3} \cup \{v\}]$ separates the plane into two regions, one containing v_2, and another one containing v_4 and v_5. The $(2, 4)$-Kempe chain $C_{2,4}$ of v_2 cannot contain v_4, for otherwise it would cross $C_{1,3}$. This cannot be done through a vertex since they are of different colours, and this cannot be done through an edge since G is planar. Therefore, inverting the colours 2 and 4 in $C_{2,4}$ recolours v_2 with 4 without changing the colours of the other v_i's. We can then properly colour v with 2, a contradiction. □

0.3.4 The discharging method

The discharging method is a really efficient tool for proving results on planar graphs, when it is associated to Euler's formula. In a classical use, one may want to prove that a certain subclass $\mathscr{G}$ of the planar graphs has some desired property P. The discharging method is then two-folds.

First, one would assume that there is a counterexample to the property P within the class $\mathscr{G}$, and would chose a minimum such counterexample G, in terms of the number of vertices, the

number of edges, or any other minimised parameter. It is then usually possible to derive a set of properties that G must have as a minimum counterexample of P, which most often translate into some local configurations which cannot appear within G; we call these *reducible configurations* for the property P.

Second, the discharging method enters into action in order to demonstrate that no graph in $\mathscr{G}$ can avoid all the reducible configurations for P. This is done by assigning a charge to the vertices, and/or faces, and/or edges of G such that the sum of the charges is negative — the charges have to be chosen so that Euler's formula can be applied to them in order to ensure this. The discharging step then averages the value of each individual charge by transferring the excess of charge of some objects to neighbouring objects which would have a negative charge, in such a way that, given the set of reducible configurations of P, every object has a non negative charge in the end of the discharging process. This raises a contradiction, and demonstrates that the property P holds in the class $\mathscr{G}$.

Hereafter is an illustration of this method of proof for a classical problem of graph colouring in the class of planar graphs avoiding some cycles.

Theorem 0.3.25. *For every planar graph G containing no cycle of length in $\{4, \ldots, 10\}$,*

$$\chi(G) \leq 3.$$

Lemma 0.3.26 (Reducible configurations)**.** *Let G be a counterexample of Theorem 0.3.25 with minimum number of vertices. Then*

1. *$\delta(G) \geq 3$,*
2. *G contains no two adjacent triangles,*
3. *for any non-triangular face f of G, we let $n_t(f)$ be the number of triangles incident to f, and $n_4(f)$ be the number of vertices of degree at least 4 incident to f. Then*

$$2n_t(f) \leq \deg_G(f) + n_4(f).$$

Proof. Let G be a minimum counterexample of Theorem 0.3.25.

1. If G contains a vertex v of degree at most 2, then $G - v$ is 3-colourable by minimality of G, and any 3-colouring of $G - v$ can be extended into a 3-colouring of G by assigning to v the colour which does not appear among its neighbours. This contradicts the fact that G is not 3-colourable.

2. Two adjacent triangles would form a C_4, which G does not contain.

3. Let f be a non-triangular face of G. Let f_0 be a triangular face adjacent to f, and f_1 be the face adjacent to f which comes next in the trigonometric order. If f_1 is a triangle, then f_0 and f_1 cannot be adjacent, so they share exactly one common vertex v_0, which is incident to f and of degree at least 4. Let $\varphi(f_0) = v_0$, or $\varphi(f_0) = f_1$ if f_1 is a non-triangular face. The mapping φ is an injection between the triangular faces adjacent to f, and the union of the non-triangular faces and the vertices of degree at least 4 incident to f. By counting twice each triangle adjacent to f, we obtain a value which is at most the total number of faces adjacent to f plus the number of vertices of degree at least 4 incident to f.

□

Proof of Theorem 0.3.25 (Discharging method). Let G be a minimum counterexample to the theorem. Let $\mathrm{ch}_0 \colon (V(G) \cup F(G)) \to \mathbb{R}$ be an initial charge function on the vertices and faces of G, such that the charge is $\mathrm{ch}_0(v) = 2\deg_G(v) - 6$ for every vertex $v \in V(G)$ and $\mathrm{ch}_0(f) = \deg_G(f) - 6$ for every face $f \in F(G)$.

By Euler's formula and the handshaking's lemma, we know that the total charge on G is

$$\begin{aligned}\sum_{v \in V(G)} \mathrm{ch}_0(v) + \sum_{f \in F(G)} \mathrm{ch}_0(f) &= 2 \sum_{v \in V(G)} \deg_G(v) - 6\,|V(G)| + \sum_{f \in F(G)} \deg_G(f) - 6\,|F(G)| \\ &= 6\Big(|E(G)| - |V(G)| - |F(G)| \Big) = -12.\end{aligned}$$

In order to yield a contradiction, we apply two discharging rule, which will ensure that the charge $\mathrm{ch}(x)$ obtained after application of these rules is non negative for every $x \in V(G) \cup F(G)$.

1. Every triangular face receives charge 1 from each of its adjacent faces.
2. Every non-triangular face receives charge 1 from each of its incident vertices of degree at least 4.

The charge of any degree-3 vertex v is unchanged by the discharging rules, therefore $\mathrm{ch}(v) = 0$. Let v be a vertex of degree $d_v \geq 4$; it is incident to at most $\lfloor d_v/2 \rfloor$ triangles since no two triangles can be adjacent. After application of the discharging rule 2, the charge of v is

$$\mathrm{ch}(v) \geq 2d_v - 6 - \lfloor d_v/2 \rfloor \geq \frac{3}{2} d_v - 6 \geq 0.$$

The charge of any triangular face f, after application of the discharging rule 1, is 0. Let f be a non-triangular face of G, so of degree $d_f \geq 11$. After application of the discharging rules 1 and 2, Lemma 0.3.26 ensures that the charge of g is

$$\mathrm{ch}(f) = d_f - 6 + n_4(f) - n_t(f) \geq \left\lceil \frac{d_f + n_4(f)}{2} \right\rceil - 6 \geq 0.$$

We have shown that the charge $\mathrm{ch}(x)$ is non negative on every vertex or face $x \in V(G) \cup F(G)$, while the total charge on G is $-12 < 0$. This is the desired contradiction, so there is no counterexample to Theorem 0.3.25. □

This is not the only context in which the discharging method can be useful. In a more general setting, it can be used in order to estimate the number of objects with some property in a graph, or in any other combinatorial structure. If the total charge of the charge function corresponds to the number of such objects, the discharging step provides an averaging of the number of desired objects within the neighbourhoods of the vertices/edges/faces/any other configurations of the graph, which eases the estimation of the total charge as a linear function in the number of vertices/edges/faces/other configurations.

0.4 Ramsey theory

After this broad general introduction of graph theory, graph colouring, and the usual general methods to prove results in this domain, let me focus more precisely on one specific branch of research which has been inspirational for most of the content in this thesis. This is Ramsey theory, which informally seeks for regularities among disorder.

0.4.1 Ramsey's theorem

One of the most fundamental results in Ramsey theory, which later gave its name to the theory, is Ramsey's theorem, which can be formulated as follows in the context of finite graphs.

Theorem 0.4.1 (Ramsey, 1930 [103])**.** *Given any two integers $s, t \geq 1$, there exists some integer $R(s,t)$ such that any graph G on $n \geq R(s,t)$ vertices contains either a clique of size s, or an independent set of size t.*

The integer $R(s,t)$ is the minimum value for which Theorem 0.4.1 holds, and is called the *Ramsey number* of parameters s and t. Note that if G is a graph on n vertices with either a clique of size s or an independent set of size t, then $\overline{G}$ is a graph on n vertices with either a clique of size t or an independent set of size s. Therefore, $R(s,t) = R(t,s)$ for any integers $s, t \geq 1$. It is also straightforward that $R(2,t) = t$ for any integer $t \geq 1$, since a graph which does not contain any clique of size 2, that is with no edge, is itself an independent set. This remark together with a simple inductive argument proves Ramsey's theorem in a quantitative way, as was first proved by Erdős and Szekeres in 1935.

Theorem 0.4.2 (Erdős, Szekeres, 1935 [40])**.** *For all integers $s, t \geq 2$,*

$$R(s,t) \leq R(s-1,t) + R(s,t-1),$$

and so by induction

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

Proof. Let G be a graph on $R(s-1,t) + R(s,t-1)$ vertices. Let $v \in V(G)$ be any vertex, then either $|N(v)| \geq R(s-1,t)$ or $|\overline{N(v)}| \geq R(s,t-1)$. In the former case, $G[N(v)]$ either contains an independent set of size t, and so does G, or it contains a clique W of size $s-1$ which may be extended into the clique $W \cup \{v\}$ of size s in G. The latter case is treated in a symmetric way. □

The upper bound on $R(s,t)$ in Theorem 0.4.2 is still the best known one in the general case. Determining the exact value of $R(s,t)$ for specific values of s and t is a really hard problem, and has been solved only for small values of s and t through computer search.

There are two specific settings of the Ramsey numbers which have raised a particular interest. Those are the *diagonal Ramsey numbers* $R(s,s)$ for any $s \geq 3$, and the *off-diagonal Ramsey numbers* $R(3,t)$ for any $t \geq 3$.

In the case of the diagonal Ramsey numbers, Theorem 0.4.2 yields the upper bound

$$R(s,s) \leq \binom{2s-2}{s-1} \underset{s\to\infty}{\lesssim} \frac{4^{s-1}}{\sqrt{\pi s}}.$$

Erdős [39] was the first to provide an exponential lower bound on the diagonal Ramsey number $R(s,s)$. This bound has then been improved by a factor 2 by Spencer in 1975 [111], and there has been no further improvement since then.

Theorem 0.4.3 (Erdős, 1947 [39]). *For every $s \geq 3$,*

$$R(s,s) \geq \frac{s}{e\sqrt{2}} 2^{s/2}.$$

Proof. Let $s \geq 3$ and $N < \frac{s}{e\sqrt{2}} 2^{s/2}$ be fixed integers. The number of graphs on N labelled vertices which contain a fixed clique of size s is

$$2^{\binom{N}{2} - \binom{s}{2}},$$

and therefore the number of graph on N labelled vertices which contain at least one clique of size s is less than

$$\binom{N}{s} \frac{2^{\binom{N}{2}}}{2^{\binom{s}{2}}} < \frac{2^{\binom{N}{2}}}{2}. \tag{5}$$

Indeed, using the lower bound $s! \geq (s/e)^s \sqrt{2\pi s}$ given by Stirling's formula for any integer s, we know that

$$\begin{aligned} \frac{\binom{N}{s}}{2^{\binom{s}{2}}} &< \frac{N^s}{s! \cdot 2^{\binom{s}{2}}} \\ &< \frac{N^s}{(s/e)^s \sqrt{2\pi s} \cdot 2^{\binom{s}{2}}} \\ &< \left(\frac{N}{s/e \cdot 2^{(s-1)/2}} \right)^s \frac{1}{\sqrt{2\pi s}} < \frac{1}{\sqrt{2\pi s}} < \frac{1}{2}. \end{aligned}$$

We know by (5) that less than half of all the graphs on N labelled vertices contain at least one clique of size s. This ensures that there exists a graph G on N vertices such that neither G nor $\overline{G}$ contains a clique of size s, which concludes the proof. □

This proof was fundamental as a demonstration of the importance of the probabilistic method, which was introduced by Erdős and is central — even in its most basic form — in this thesis. It illustrates how it is possible to ensure the existence of a graph with a specific property in a non-constructive way, by showing that a random graph would have this property with non-zero probability. This argument is impressive both for its surprising simplicity, and its exceptional strength. Indeed, there is still no known non-probabilistic construction of a graph which would certify an exponential lower bound on $R(s,s)$.

More than 70 years later, the multiplicative gap of 4 that lies between the lower and upper bounds on $\log_2 R(s,s)$ given by Theorems 0.4.2 and 0.4.3 has not been improved by any non-negligible value;

$$s/2 < \log_2 R(s,s) < 2s. \tag{6}$$

It is an open question of Erdős whether $\lim_{s\to\infty} \frac{\log_2 R(s,s)}{s}$ exists [44]. There are \$100 to earn for the proof of its existence, or \$10000 for the proof of its non-existence. Furthermore, in case of its existence, there are extra \$250 to earn for the determination of its exact value.

0.4.2 Off-diagonal Ramsey numbers

By definition, the off-diagonal Ramsey number $R(3,t)$ provides a correlation between the number of vertices and the independence number in a triangle-free graph. Indeed, there is an equivalence between $R(3,t) \leq f(t)$ and $\alpha(G) \geq f^{-1}(n(G))$ for every triangle-free graph G, given any bijective function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$. Triangle-free graphs have the special property that every neighbourhood is an independent set. Therefore, in a triangle-free graph G, it holds that

$$\alpha(G) \geq \Delta(G).$$

Let me make a small digression before continuing. Because of this property, triangle-free graphs with large minimum degree have a lot of large independent sets, to the point that there exists a threshold such that they can be coloured with a constant number of colours. This was a problem of Erdős and Simonovits [47]. If G is a triangle-free graph of minimum degree δ, then $\delta > \frac{2}{5}n(G)$ ensures that G is bipartite [10]; $\delta > \frac{10}{29}n(G)$ ensures that $\chi(G) \leq 3$ [64]; $\delta > \frac{1}{3}n(G)$ ensures that $\chi(G) \leq 4$ [23]; and the Kneser graph $K_{k,3k-1}$ demonstrates that there exist triangle-free graphs with arbitrarily large chromatic number, and minimum degree arbitrarily close to $\frac{1}{3}$ of their number of vertices.

Going back to the general case, we have already seen that the maximum degree of G is a fundamental parameter when it comes to properly colouring G, which lets us find large independent sets in G. Notably, the largest colour class of a proper colouring of G produced by any greedy algorithm is of size at least

$$\frac{n(G)}{\Delta(G)+1},$$

and this immediately yields that

$$\alpha(G) \geq \max\left(\Delta(G),\ \frac{n(G)}{\Delta(G)+1}\right) \geq \sqrt{n(G)} - 1.$$

From this, we can derive the upper bound $R(3,t) \leq (t+1)^2$, which is worse than the upper bound $R(3,t) \leq \binom{t+1}{2}$ yielded by Theorem 0.4.2 roughly by a factor of 2. Although this is close to the right asymptotic order, this can be improved. Indeed, the bound $\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$ is not sharp for triangle-free graphs in general, especially as Δ grows, and the right order of magnitude was found in 1980 by Ajtai, Komlós, and Szemerédi [2].

Theorem 0.4.4 (Ajtai, Komlós, Szemerédi, 1980 [2])**.** *For every triangle-free graph G,*

$$\alpha(G) \geq 0.01\frac{\ln \operatorname{ad}(G)}{\operatorname{ad}(G)}n(G).$$

In particular, this implies that

$$R(3,t) = \underset{t\to\infty}{O}\left(\frac{t^2}{\ln t}\right).$$

This was later improved by Shearer in 1983 [108] through an elegant inductive argument. This provides the best known asymptotic upper bound on the independence number of triangle-free graphs of fixed density.

Theorem 0.4.5 (Shearer, 1983 [108]). *For every triangle-free graph G of average degree d,*

$$\frac{\alpha(G)}{n(G)} \geq \frac{d\ln d - d + 1}{(d-1)^2} \underset{d\to\infty}{\sim} \frac{\ln d}{d}.$$

In particular, this implies that

$$R(3,t) \underset{t\to\infty}{\lesssim} \frac{t^2}{\ln t}.$$

One of the main motivations of this thesis is to generalise Theorem 0.4.5 in several qualitative and quantitative ways. Namely, instead of excluding triangles, we may exclude other cycles, or allow only a bounded proportion of them to appear. We also focus on refined versions of the independence number, like the average size of an independent set, or the fractional chromatic number.

0.4.3 Random graphs are mostly extremal in Ramsey theory

Random graphs provide the most extremal constructions that we know in Ramsey theory. This is true to the point that in most scenarios, there is no known deterministic construction which can reach the same order of magnitude as random graphs with respect to the relevant parameters.

The most direct application of random graphs in Ramsey theory arises in diagonal Ramsey numbers, since a random graph certifies the left hand-side of (6) asymptotically almost surely.

Theorem 0.4.6 (cf. Erdős [39]). *Let G be drawn from $G(n,1/2)$. Then asymptotically almost surely,*

$$\omega(G) \leq 2\log_2 n, \text{ and}$$
$$\alpha(G) \leq 2\log_2 n.$$

Proof. We prove that asymptotically almost surely, there is no clique of order larger than $2\log_2 n$ in G. Let $k \geq 2\log_2 n + 1$ be a fixed integer. Given any subset of k vertices in G, the probability that they form a clique is $2^{-\binom{k}{2}}$. Let $\mathbf{X}$ be the random variable counting the number of cliques of size k in G. By linearity of expectation,

$$\mathbb{E}[\mathbf{X}] = \binom{n}{k} 2^{-\binom{k}{2}}.$$

We apply Markov's inequality and deduce that the probability that there exists a clique of size k in G is

$$\begin{aligned}\mathbb{P}[\mathbf{X} \geq 1] \leq \mathbb{E}[\mathbf{X}] &\leq \frac{n^k}{k!} 2^{-k(k-1)/2} \\ &\leq \frac{1}{k!}\Big(\frac{n}{2^{(k-1)/2}}\Big)^k \leq \frac{1}{k!} \underset{n\to\infty}{\to} 0.\end{aligned}$$

This concludes that indeed asymptotically almost surely, a graph drawn from $G(n,1/2)$ contains no clique of size k.

Since $\overline{G}$ is also drawn from $G(n,1/2)$, there is asymptotically almost surely no clique of size k in $\overline{G}$, and therefore no independent set of size k in G. The conclusion follows from a union bound. □

The application of random graphs for off-diagonal Ramsey numbers gets a bit more complicated. Recall from Section 0.3.2.3 that random d-regular graphs almost surely have independence ratio at most $2\ln d/d$, and since they can be made triangle-free by removing a sub-linear number of vertices, they certify the sharpness of the first half of Theorem 0.4.5 up to an asymptotic factor 2. This however is true only when d is fixed, by letting $n \to \infty$, and so cannot directly provide a lower bound on $R(3,t)$.

In 1961, Erdős [43] proved a lower bound of the form $R(3,t) = \Omega((t/\ln t)^2)$, again with the help of the probabilistic method. This result was reiterated later by Spencer [112] with an explicit constant and a shorter proof relying again on random graphs, which we sketch hereafter.

Theorem 0.4.7 (Spencer, 1977 [112])**.**

$$R(3,t) \geq \left(\frac{1}{27} - o(1)\right)\left(\frac{t}{\ln t}\right)^2$$

Sketch of the proof. In order to prove this result, Spencer considers a graph G drawn from $G(n,p)$, by fixing $p = 1/\sqrt{3n}$. He shows through an application of LLL that, with non-zero probability, the independence number of G is at most $t = \left(3\sqrt{3}/2 + o(1)\right)\sqrt{n}\ln n$ and G is triangle-free. This yields the result by expressing n in terms of t. □

This lower bound was however still not of the right order. In order to improve it using a graph G drawn from $G(n,p)$, one would need to use a value of p with a larger order of magnitude, namely $p = \Theta\left(\sqrt{\ln n/n}\right)$ in order to obtain an independence number $\alpha(G) = \Theta\left(\sqrt{n\ln n}\right)$ with non-zero probability. Unfortunately, it seems no longer possible to ensure that G is moreover triangle-free with non-zero probability in this setting.

In 1995, Kim [79] obtained the right order of magnitude of $R(3,t)$ in a breakthrough paper, using some random model of triangle-free graphs relying on the Rödl nibble. This method consists in repetitively selecting a small random fraction of possible edges to add in a random triangle-free graph.

Theorem 0.4.8 (Kim, 1995 [79])**.**

$$R(3,t) = \Theta\left(\frac{t^2}{\ln t}\right).$$

Kim's lower bound was then improved substantially through a thorough analysis of the *triangle-free process* [15, 16, 52]. This process selects a random maximal triangle-free graph on a fixed number of vertices as follows.

(i) Start with G_0 being an independent set of size n, and fix a random ordering of all $\binom{n}{2}$ pairs of vertices.

(ii) At step $i \geq 1$, let e_i be the edge corresponding to the i-th pair of vertices. Then

$$G_i \coloneqq \begin{cases} G_{i-1} + e_i & \text{if } G_{i-1} + e_i \text{ is triangle-free,} \\ G_{i-1} & \text{otherwise.} \end{cases}$$

The random triangle-free graph obtained at the end of the triangle-free process is denoted $G_{n,\Delta}$.

Theorem 0.4.9 (Bohman, Keevash, 2013+ [16]; Fiz Pontiveros, Griffiths, Morris, 2013+ [52]; cf. also [15])**.** *Asymptotically almost surely,*

$$\Delta(G_{n,\Delta}) = (1+o(1))\sqrt{\frac{n\ln n}{2}},$$
$$\alpha(G_{n,\Delta}) = (1+o(1))\sqrt{2n\ln n}.$$

In particular, this implies that

$$\frac{t^2}{4\ln t} \underset{t\to\infty}{\lesssim} R(3,t) \underset{t\to\infty}{\lesssim} \frac{t^2}{\ln t}. \qquad (*)$$

To this date, $(*)$ is the best known asymptotic estimation of $R(3,t)$. This is a reference for the sharpness of the results that we will present throughout this thesis. In particular, most of them will imply one side of $(*)$, which means that they are best possible barring a breakthrough in quantitative Ramsey theory.

0.4.4 Hard-core distributions

Let S be a given set of elements, and $\mathscr{F}$ be a family of subsets of S. For every real $\lambda > 0$, a random subset $\mathbf{X}$ drawn from the *hard-core distribution* on $\mathscr{F}$ at fugacity λ satisfies

$$\mathbb{P}[\mathbf{X} = X] = \frac{\lambda^{|X|}}{Z_{\mathscr{F}}(\lambda)}$$

for every $X \in \mathscr{F}$, where

$$Z_{\mathscr{F}}(\lambda) = \sum_{X\in\mathscr{F}} \lambda^{|X|}$$

is the partition function of $\mathscr{F}$. Given an element $x \in S$, we say that x is *occupied* if $x \in \mathbf{X}$. The *occupancy fraction* is then defined to be the average probability of an element of S being occupied, that is

$$\frac{\mathbb{E}[|\mathbf{X}|]}{|S|}.$$

When $\lambda \leq 1$, the occupancy fraction is therefore a lower bound on the average size of an element in $\mathscr{F}$.

Note that the hard-core distribution on $\mathscr{F}$ at fugacity λ corresponds to the uniform distribution on $\mathscr{F}$ when $\lambda = 1$, and to the uniform distribution on the elements of $\mathscr{F}$ of maximal cardinality when $\lambda = \infty$.

In general, we may allow λ to be a function from S to the positive reals, in which case $\lambda^{|X|}$ is replaced by $\prod_{x\in X} \lambda(x)$. However, all the applications of the hard-core distributions throughout this thesis will use a constant fugacity λ.

These distributions are used in statistical physics, and more specifically in condensed matter physics. They happen to be of particular interest when used with the probabilistic method in combinatorics, notably because of their *spatial Markov property*.

Proposition 0.4.10 (Spatial Markov Property)**.** *Let $\mathscr{F} \subseteq 2^S$ be a family of subsets of a given set of elements S, and fix some $T \subseteq S$.*

Let $Y \subseteq S \setminus T$ be fixed, and set $\mathscr{F}_Y = \{X \subseteq T \mid X \cup Y \in \mathscr{F}\}$ to be the family of subsets of T which extend Y into an element of $\mathscr{F}$.

For every real $\lambda > 0$, if $\mathbf{X}$ follows the hard-core distribution on $\mathscr{F}$ at fugacity λ, then conditioned on the fact that $\mathbf{X} \setminus T = Y$, $\mathbf{X} \cap T$ follows the hard-core distribution on $\mathscr{F}_Y$ at fugacity λ.

Proof. Conditioned on the fact that $\mathbf{X} \setminus T = Y$, the set of possible realisations of $\mathbf{X} \cap T$ is precisely $\mathscr{F}_Y$ by definition. Now, for every $X \in \mathscr{F}_Y$,

$$\mathbb{P}\left[\mathbf{X} \cap T = X \mid \mathbf{X} \setminus T = Y\right] = \mathbb{P}\left[\mathbf{X} = X \cup Y\right] = \frac{\lambda^{|X \cup Y|}}{\sum\limits_{X' \in \mathscr{F}_Y} \lambda^{|X' \cup Y|}} = \frac{\lambda^{|X|}}{\sum\limits_{X' \in \mathscr{F}_Y} \lambda^{|X'|}} \cdot \frac{\lambda^{|Y|}}{\lambda^{|Y|}} = \frac{\lambda^{|X|}}{Z_{\mathscr{F}_Y}(\lambda)}.$$

□

The strength of the spatial Markov property lies in the possibility to perform a local analysis of a global random model. For this reason, hard-core distributions are helpful in order to analyse the behaviour of matchings, independent sets, and different kinds of colourings of various classes of graphs with local constraints.

In 1996, Kahn [68] used the hard-core distributions in the framework of graph theory in order to prove an asymptotic result on the chromatic index of multigraphs, namely that it asymptotically matches its fractional counterpart. In this setting, given a multigraph G, the set of elements S is the set of edges $E(G)$, and the family $\mathscr{F}$ is the family of all matchings $\mathcal{M}(G)$. More recently, hard-core distributions have been proven useful when applied on the independent sets, notably in order to prove a version of Theorem 0.4.5 for occupancy fraction. In this setting, S is the set of vertices of a triangle-free graph G, and $\mathscr{F}$ is its whole family of independent sets $\mathcal{I}(G)$.

Theorem 0.4.11 (Davies, Jenssen, Perkins, Roberts, 2018 [34])**.** *Given a triangle-free graph G of maximum degree $\Delta \geq 3$, let $\mathbf{I}$ be a random independent set drawn according to the hard-core distribution on $\mathcal{I}(G)$ at fugacity $\lambda = 1/\ln \Delta$. Then the occupancy fraction of $\mathbf{I}$ is*

$$\frac{\mathbb{E}\left[|\mathbf{I}|\right]}{n(G)} \underset{\Delta \to \infty}{\gtrsim} \frac{\ln \Delta}{\Delta}.$$

In particular, since $\lambda \leq 1$, this is a lower bound on the average size of an independent set in G.

Throughout Chapter 1, we will use hard-core distributions in the setting of independent sets with various regimes in order to prove several results on fractional colourings of sparse graphs, which generalise Theorem 0.4.11.

0.4.5 Random colourings of triangle-free graphs

We have seen so far that random graphs are mostly extremal in off-diagonal Ramsey theory. More precisely, the independence number of a triangle-free graph of fixed density is at least half that of a random triangle-free graph of the same density, as is stated in Theorem 0.4.5. It is important to mention here that the methods determining the value of the independence number of random graphs drawn from $G(n,p)$, or random regular graphs, are all non constructive; they rely on the probabilistic method. To this date, no polytime algorithm is known to return an independent set of size more than half the independence ratio of a given random graph, and it has even been conjectured by Karp in 1976 [74] that no such algorithm exists. This is called the *algorithmic gap*, and it has been justified to some extent by an analysis of the geometry of the space formed by the independent sets of fixed size k in random graphs [28]. It is shown that there is a threshold at $k \sim n \ln d/d$, where n is the number of vertices and d the average degree of the random graph,

above which independent sets of size k form highly disconnected small clusters. This explains why local exploration algorithms fail at finding independent sets of size above that threshold, while it is known that there exist some of size $k \sim 2n \ln d/d$. So, if we restrict to independent sets which we know how to find in an efficient way, random graphs are asymptotically extremal among triangle-free graphs.

In this subsection, we discuss how this fact extends to the case of proper colourings. Like for independence number, there is an algorithmic gap for the chromatic number of random graphs of fixed density d. The greedy colouring algorithm will return a proper k-colouring where $k \sim d/\ln d$ asymptotically almost surely [89], while the probabilistic method ensures in a non constructive way that there exists a proper k-colouring where $k \sim d/(2 \ln d)$ asymptotically almost surely [54].

In a proper random colouring of a triangle-free graph G of maximum degree Δ, the coupon collector problem gives insight in the needed number of colours. Indeed, if you assume that the colours in the neighbourhood of a given vertex $v \in V(G)$ are drawn independently uniformly at random from $[k]$, the number of neighbours that v needs to have for all the k colours to appear in $N(v)$ is then $k \ln k$. So if $\Delta < k \ln k$, there is a good chance that there remains a colour available for v after its neighbours have been coloured randomly.

Of course, we cannot assume independence between the colours which are assigned to the neighbours of v in a random proper colouring. However, Kim [78] managed to prove that with graphs of girth 5, the insight given by the coupon collector problem was correct.

Theorem 0.4.12 (Kim, 1995 [78]). *Let $\varepsilon > 0$ be fixed. For every graph G of girth at least 5, and maximum degree Δ large enough,*

$$\chi_\ell(G) \leq (1+\varepsilon)\frac{\Delta}{\ln \Delta}.$$

The proof relies again on the Rödl nibble. Instead of colouring the whole graph in one round, Kim colours it with a succession of random rounds where only a small random fraction of the vertices select a uniformly random colour from their (updated) list, and every pair of conflicting vertices are uncoloured. By carefully keeping track of the size of the lists, and of the number of uncoloured neighbours at each round, this random colouring process is shown to achieve at some point a partial proper colouring where every uncoloured vertex v has a list of available colours of size $|L(v)| \geq \ell$, and for every colour $x \in L(v)$ no more than $\ell/8$ neighbours u such that $x \in L(u)$, for some value of ℓ. At this point, it is possible to extend the partial proper colouring into a proper colouring of the whole graph with non-zero probability through a last random round, the *finishing blow*, where every uncoloured vertex picks a uniformly random colour from their list.

The case of triangle-free graphs was more complicated because there were more dependencies between the bad events when trying to apply the Rödl nibble. Eventually though, Johansson [65] was able to adapt Kim's proof up to a worse multiplicative constant through an involved analysis of the Rödl nibble, which was never published.

Theorem 0.4.13 (Johansson, 1996 [65]). *There exists an absolute constant $C > 0$ such that, for every triangle-free graph of maximum degree Δ,*

$$\chi_\ell(G) \leq C \cdot \frac{\Delta}{\ln \Delta}.$$

More than 20 years later, a breakthrough was achieved by Molloy [90], who used a new probabilistic method, *entropy compression*, in order to randomly colour triangle-free graphs. He obtained a bound matching Kim's one for graphs of girth 5. A few months later, Bernshteyn [13] simplified

the proof of Molloy, notably by replacing the use of entropy compression by a direct application of LLL. The strength of his proof relied in the fact that he was able to analyse directly a uniformly random partial proper colouring of any given triangle-free graph G, with the help of the lopsided LLL which works even when the random events have no mutual independence.

Theorem 0.4.14 (Molloy, 2019 [90]; Bernshteyn, 2018 [13])**.** *Let $\varepsilon > 0$ be fixed. For every triangle-free graph G of maximum degree Δ large enough,*

$$\chi_\ell(G) \leq (1+\varepsilon)\frac{\Delta}{\ln \Delta}.$$

Overview of the proof. We overview here the ideas behind the proof of Bernshteyn, which is simpler and on which we rely in order to prove a generalised version of Theorem 0.4.14 in Chapter 1.

Let G be a triangle-free graph of maximum degree Δ large enough in terms of ε, and L be a list-assignment of G satisfying $|L(v)| = (1+\varepsilon)\Delta/\ln\Delta$ for every vertex $v \in V(G)$. Fix also $\ell \coloneqq \Delta^{\varepsilon/2}$.

The proof consists in finding a random proper L-colouring of G in two steps. The first step draws a partial proper L-colouring $\mathbf{c}$ uniformly at random (by adding an artificial colour `Blank` to every list, which stands for the vertex being left uncoloured if it is drawn). After this first step, we let $L'(v) \coloneqq L(v) \setminus \mathbf{c}(N(v))$ be the list of available colours left for every uncoloured vertex v. With non-zero probability, for every uncoloured vertex v of G, it holds that

(i) $|L'(v)| \geq \ell$, and

(ii) for every colour $x \in L'(v)$, the number of neighbours u of v such that $x \in L(u)$ is at most $\ell/2$.

This is shown first by showing that the corresponding random bad events happen with small probability, with an application of Chernoff's inequality.

Claim. Let $v \in V(G)$ be a fixed vertex, and c_0 be a fixed partial proper colouring of $G\left[\overline{N[v]}\right]$. We let L_0 be the list assignment of $N(v)$ induced by L after removal of the colours in conflict with c_0. For a uniformly random partial proper L_0-colouring $\mathbf{c}$ of $N(v)$, it holds that

(i) $\mathbb{P}\left[|L(v) \setminus \mathbf{c}(N(v))| < \ell\right] \leq (2\Delta)^{-3}$, and

(ii) $\mathbb{P}\left[\exists x \in L(v) \setminus \mathbf{c}(N(v)), \#\left\{u \in N(v) \mid \mathbf{c}(u) = \texttt{Blank} \text{ and } x \in L_0(u)\right\} > \ell/2\right] \leq (2\Delta)^{-3}$.

The conclusion holds by noting that a uniformly random partial proper colouring $\mathbf{c}$ of G has the spatial Markov property, and so by conditioning on $\mathbf{c} \setminus N[v] = c_0$ for any $v \in V(G)$, the outcome of $\mathbf{c} \cap N(v)$ is a uniform random partial proper L_0-colouring of $N(v)$, which lets us apply the claim together with the lopsided LLL.

The final step, referred to as the *finishing blow*, consists in drawing one colour independently uniformly at random from $L'(v)$ for every uncoloured vertex v of G. This extends $\mathbf{c}$ into a proper L-colouring of G with non-zero probability, as can be shown using LLL. □

Chapter 1

Independent sets and local colourings of sparse graphs

Given a hereditary class of graphs $\mathscr{G}$ (closed by induced subgraphs), we say that $\mathscr{G}$ is a class of *dense* graphs if

$$\forall G \in \mathscr{G}, \quad e(G) = \Theta\big(n(G)^2\big)\,,$$

and on the other hand we say that $\mathscr{G}$ is a class of *sparse* graphs if

$$\forall G \in \mathscr{G}, \quad e(G) = o\big(n(G)^2\big)\,.$$

Equivalently, in a hereditary class of sparse graphs, the maximum average degree of a graph on n vertices is $o(n)$. We have seen in the introduction that the maximum average degree is an upper bound on the degeneracy of a graph, which itself is an upper bound of the chromatic number minus one. So sparse graphs are easier to colour.

In general, the notion of sparse graphs is extended to *locally sparse* graphs, which are graphs where each neighbourhood induces a sparse graph. With this extended definition of sparseness, the first class of (locally) sparse graphs which comes to mind is the class of triangle-free graphs, that is the class of graphs where each neighbourhood induces an independent set.

In triangle-free graphs, the maximum average degree is not sublinear in the number of vertices in general; this can be illustrated with complete bipartite graphs. However, those graphs are the densest ones among the class of triangle-free graphs, as was demonstrated by Mantel in 1907 [88].

Theorem 1.1 (Mantel, 1907 [88])**.** *In any triangle-free graph G,*

$$e(G) \leq \left\lfloor \frac{n(G)^2}{4} \right\rfloor,$$

with equality if, and only if, $G \cong K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ for some $n \in \mathbb{N}$.

Noting that the densest possible triangle-free graphs — so intuitively the hardest to colour — are actually bipartite, one could wonder whether the chromatic number of any triangle-free graph is bounded. We have already seen in the introduction of this thesis that it is far from being the case. The first ever construction of a family of triangle-free graphs of unbounded chromatic number is the one of Zykov in 1949 [121] — interestingly enough, the chromatic number of any graph of this family is exactly one plus its degeneracy, demonstrating that this natural upper bound of the chromatic number is sharp in the class of triangle-free graphs. The construction of Mycielski in

1955 [95] provides another example of a family of triangle-free graphs with unbounded chromatic number, with much fewer vertices — in fact, the Mycielski graphs M_0, M_1, M_2, and M_3 reach the minimum possible number of vertices in a triangle-free graph of chromatic number respectively $1, 2, 3$, and 4.

It was then quickly established that the chromatic number is unbounded also in sparser classes of graphs, namely of higher girth. Kelly and Kelly constructed a family of graphs of girth 6 with unbounded chromatic number in 1954 [75]. In 1966, an explicit family of girth 8 was constructed by Nešetřil [96], and two years later Lovász [84] achieved the last step of constructing such a family of arbitrary large girth.

The existence of such a family had actually been known for almost a decade, since using the probabilistic method, Erdős [42] had proved in 1959 that the class of graphs of any fixed girth g has unbounded chromatic number. He did so through the so-called deletion method, which was a milestone of probabilistic methods applied to graph theory. The same deletion method was used two decades later by Bollobas in 1981 [18] to exhibit his more explicit lower bound of $d/(2\ln d)$ for the maximum of the chromatic number among d-regular graphs of arbitrary large girth, which we discussed in Section 0.3.2.3.

Since the chromatic number of the class of graphs of arbitrary large girth is unbounded, one needs another parameter in order to restrain it, and thus establish interesting bounds on it. A possible one would be the density, which is captured in particular by the degeneracy. But since Zykov's construction demonstrates that $d+1$ is a sharp bound for the chromatic number of d-degenerate triangle-free graphs, it appears that the degeneracy is not a relevant choice.

The first ever result in the other direction, which provides an upper bound on the chromatic number in a given class of sparse graphs, was obtained by Borodin and Kostochka in 1977 [22] by considering graphs of bounded maximum degree and clique number. What they proved is the following.

Theorem 1.2 (Borodin, Kostochka, 1977 [22]). *For every graph G of maximum degree $\Delta(G) \geq 3$,*

$$\chi(G) \leq \Delta(G) - \left\lfloor \frac{\Delta(G) - \max(3, \omega(G))}{\max(3, \omega(G)) + 1} \right\rfloor.$$

So in particular for every K_4-free graph G of maximum degree at least 3,

$$\chi(G) \leq \left\lceil \frac{3(\Delta(G)+1)}{4} \right\rceil.$$

This first chapter revolves around the celebrated result of Johansson on the chromatic number of triangle-free graphs of bounded maximum degree.

Theorem 1.3 (Johansson, 1996 [65]). *There exists an absolute constant $C > 0$ such that, for every triangle-free graph G,*

$$\chi(G) \leq C \frac{\Delta(G)}{\ln \Delta(G)}.$$

This result was a milestone in the domain of sparse graphs colouring, since it provides an upper bound asymptotically optimal, up to some multiplicative constant. This opened a new branch of research in the area, so that the chromatic number of triangle-free graphs is now a classic topic, and has been deeply studied from many perspectives, including algebraic, probabilistic, and algorithmic. It is attractive because of its elegance and its close connection to quantitative Ramsey theory [3, 108].

In this chapter, we consider sparse graphs on different levels. While triangle-free graphs are the flagships of sparse graphs, we also strengthen the sparsity by considering graphs of higher girth, or loosen it by considering graphs which have the property that all their neighbourhoods contain only a fraction of the maximum possible number of edges, or equivalently that all their vertices belong to a bounded number of triangles.

Throughout this chapter, we will alternate between the consideration of independent sets, fractional colourings, and DP-colourings — a generalisation of list colourings — in sparse graphs. We also introduce a notion of locality in all of our colouring results. The content of this chapter is mainly covered by the three submitted articles [31, 32, 101].

1.1 Context and presentation of the results

1.1.1 Triangle-free graphs

Recently, Molloy [90] obtained a breakthrough in the domain of triangle-free graph colourings. He used entropy compression, a new powerful probabilistic tool, in order to show that, given $\varepsilon > 0$, every triangle-free graph of maximum degree Δ has chromatic number at most $\lceil (1+\varepsilon)\Delta/\ln\Delta \rceil$, provided Δ is sufficiently large. This achievement improved on the seminal work of Johansson [65] in two ways, one by lowering the leading asymptotic constant (perhaps even to optimality) and the other by giving a much simpler proof. Molloy's result actually holds for the list chromatic number of triangle-free graphs, an even stronger statement. This result provoked a resurgence of interest in the domain, and several related works emerged soon afterwards.

Theorem 1.1.1 (Molloy, 2019 [90])**.** *Let $\varepsilon > 0$ be fixed. There exists Δ_ε such that every triangle-free graph G of maximum degree $\Delta \geq \Delta_\varepsilon$ satisfies*

$$\chi(G) \leq (1+\varepsilon)\frac{\Delta}{\ln\Delta}.$$

Among those, a notable one was proposed by Bernshteyn [13]. He proved again Molloy's result, by using LLL instead of entropy compression, thus demonstrating that the breakthrough from Molloy was not due to a supposed superiority of the entropy compression method compared to the LLL. He also extended the result to hold in the context of *DP-colourings*, a generalisation of list colourings proposed by Dvořák and Postle [36]. The idea behind DP-colouring is, given any list assignment L of the vertices of a graph G, and given a permutation of the colours σ_{uv} for every ordered pair $uv \in E(G)$ (such that $\sigma_{vu} = \sigma_{uv}^{-1}$), to find a colour $c(v) \in L(v)$ for every vertex v, such that for every edge $uv \in E(G)$, it holds that $c(u) \neq \sigma_{uv}(c(v))$. Note that a list colouring corresponds to a special instance of a DP-colouring where $\sigma_{uv} = \mathrm{id}$ for every edge $uv \in E(G)$.

In this chapter, we aim to extend results on colourings of triangle-free graphs, and more generally of sparse graphs, in a local setting. This has a natural interest in many applications. For instance, if we consider the use of fractional colourings in parallel computing as presented in Section 0.2.7.2, a local fractional colouring would ensure that some tasks are finished before the end of the whole process, namely the ones corresponding to vertices *easier* to colour in the conflict graph.

A local list colouring relies on a list assignment with variable list sizes, depending on each vertex. Since the maximum degree is a key parameter for bounding from above the (list) chromatic number of a graph, it is natural to consider a list assignment L where the size of $L(v)$ is a (non-decreasing) function of $\deg(v)$, for every vertex v. Indeed one might expect the low degree vertices to be easier

to colour in a quantifiable way. The general idea of having "local" list sizes is far from new; it can be traced at least back to degree-choosability as introduced in one of the originating papers for list colouring [41]. Recently Bonamy, Kelly, Nelson, and Postle [19] initiated a modern and rather general treatment of this idea, including with respect to triangle-free graphs (a conjecture of King [80] and related work are in the same vein). We show the following result.

Theorem 1.1.2. *Fix $\varepsilon > 0$, let Δ be sufficiently large, and $\delta = (72 \ln \Delta)^{2/\varepsilon}$. Let G be a triangle-free graph of maximum degree Δ and $L\colon V(G) \to 2^{\mathbb{N}}$ be a list assignment of G such that*

$$|L(v)| \geq (1+\varepsilon) \max\left\{ \frac{\deg(v)}{\ln \deg(v)}, \frac{\delta}{\ln \delta} \right\}$$

for every vertex $v \in V(G)$. Then there exists a proper L-colouring of G.

This can be considered a local strengthening of Molloy's theorem. When the graph G in Theorem 1.1.2 is of minimum degree δ, the list size condition is local in the sense that the lower bound on $|L(v)|$ reduces to a function of $\deg(v)$ and no other parameter of G. Theorem 1.1.2 (or rather the stronger Theorem 1.2.2 below) improves the asymptotic leading constant $4 \ln 2$ of [19, Thm. 1.12] to 1, at the expense of requiring a larger minimum list size. Our proof relies heavily on the work of Bernshteyn [13]. For Theorem 1.1.2, it has sufficed to prove a local version of the so-called "finishing blow" (see Lemma 1.2.1 below) and to notice that there is more than enough slack in Bernshteyn's (and indeed Molloy's) argument to satisfy the new blow's hypothesis.

We also provide a local version of Molloy's theorem for fractional colourings. To this end, we use a definition of fractional colourings compatible with local parameters. Writing $\mathcal{I}(G)$ for the set of independent sets of G, and μ for the standard Lebesgue measure on $\mathbb{R}$, a *fractional colouring* of a graph G is an assignment $w\colon \mathcal{I}(G) \to \mathcal{P}(\mathbb{R})$ of pairwise disjoint measurable subsets of $\mathbb{R}$ to the independent sets of G such that, for every vertex $v \in V(G)$, the induced assignment

$$w(v) \coloneqq \bigcup_{\substack{I \in \mathcal{I}(G) \\ v \in I}} w(I)$$

is of measure $\hat{w}(v) \coloneqq \mu\big(w(v)\big) \geq 1$. In actuality, we may and will only use subset which each consist in a finite disjoint union of intervals, so that there Lebesgue measure is simply the (finite) sum of the lengths of those intervals. It naturally holds that $w(u)$ and $w(v)$ are disjoint whenever $uv \in E(G)$. The total weight of the fractional colouring is $\hat{w}(G) \coloneqq \sum_{I \in \mathcal{I}(G)} \mu(w(I))$.

Theorem 1.1.3. *Let $\varepsilon > 0$ be fixed. There exists $\delta_\varepsilon > 0$ such that every triangle-free graph G admits a fractional colouring w where, for every vertex $v \in V(G)$,*

$$w(v) \subseteq \left[0, (1+\varepsilon) \max\left\{ \frac{\deg(v)}{\ln \deg(v)}, \frac{\delta_\varepsilon}{\ln \delta_\varepsilon} \right\}\right].$$

Again, when G is of minimum degree δ_ε, our condition on $w(v)$ reduces to a function of $\deg(v)$ alone, yielding a local condition. Clearly Theorem 1.1.3 is not implied by Molloy's theorem nor is the converse true, but both results imply that the fractional chromatic number of a triangle-free graph of maximum degree Δ is at most $(1 + o(1))\Delta / \ln \Delta$. We believe that the main interest in Theorem 1.1.3 will be in its derivation. We give a short and completely self-contained proof by performing a local analysis of the hard-core model in triangle-free graphs (Lemma 1.4.4), and

demonstrate that to obtain the desired result it suffices to feed the hard-core model as input to a greedy fractional colouring algorithm (Lemma 1.3.1). Since it makes no use of the Lovász Local Lemma, the proof is unlike any other derivation of a Johansson-type colouring result (regardless of local list sizes). This may be of independent interest.

The asymptotic leading constant of 1 in the conditions of both Theorems 1.1.2 and 1.1.3 cannot be improved below 1/2 due to random regular graphs [54]. In fact, as a corollary of either result we match asymptotically the upper bound of Shearer [108] for off-diagonal Ramsey numbers. So any improvement below 1, or even to 1 precisely (i.e. removal of the ε term), would be a significant advance. The relative sharpness of the colouring results can be contextualised by the analysis of the independence ratio of sparse graphs.

The result of Molloy, and our extensions in Theorem 1.2.2 and Theorem 1.1.2 all suffer from the same lack of consideration of vertices of small degree. Namely, for any degree d smaller than Δ_ε or δ_ε, none of these theorems provides a satisfactory explicit value of the amount of colour needed to colour the vertices of degree at most d in a sparse graph. To this date, the best known general upper bound in terms of clique number and maximum degree for the *fractional* chromatic number is due to Molloy and Reed [93, Theorem 21.7, p. 244].

Theorem 1.1.4 (Molloy and Reed, 2002 [93])**.** *For every graph G,*

$$\chi_f(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}.$$

If one considers a convex combination of the clique number and the maximum degree plus one for an upper bound on the (fractional) chromatic number of a graph, then because the chromatic number of a graph never exceeds its maximum degree plus one, the aim is to maximise the coefficient in front of the clique number. The convex combination provided by Theorem 1.1.4 (which is conjectured to hold also for the chromatic number once rounded up), is best possible. Indeed, for every positive integer k the graph $G_k \coloneqq C_5 \boxtimes K_k$ is such that

$$\begin{aligned} \omega(G_k) &= 2k, \\ \Delta(G_k) &= 3k - 1, \\ \chi_f(G_k) &= \frac{5k}{2} = \frac{\omega(G_k) + \Delta(G_k) + 1}{2}. \end{aligned}$$

By increasing the sparsity condition on the considered graph from being triangle-free to having girth at least 7, we can provide local fractional colourings which do not require any special consideration of the vertices of small degree.

Theorem 1.1.5. *Set $\gamma(x) \coloneqq \min_{k \in \mathbb{N}_{\geq 3}} \dfrac{2x + 2^{k-3} + k}{k}$.*
Every graph G of girth at least 7 admits a fractional colouring w where, for every vertex $v \in V(G)$,

$$w(v) \subseteq \big[0, \gamma(\deg(v))\big].$$

In particular, $\chi_f(G) \leq \gamma(\Delta(G))$.

Remark 1.1.1. In Theorem 1.1.5, when $x \geq 3$, the minimum of the function γ is attained in $k = \lfloor 4 + \log_2 x - \log_2 \log_2 x \rceil$. More generally, when $x \geq 0$, the minimum is attained for a value of k greater or equal to 4. The asymptotic behaviour of γ is therefore $\gamma(x) = (2 \ln 2 + o(1))x / \ln x$,

and this is off by a multiplicative factor $2\ln 2$ from Molloy's result for large degrees. However, up to x of the order of 10^7, the value $\gamma(x)$ is smaller than the bound for a triangle-free graph G of maximum degree x which can be derived from the proof of Theorem 1.1.2, namely

$$\chi_f(G) \leq \min_{\lambda>0} \quad \frac{\lambda+1}{\lambda} e^{W(x\ln(1+\lambda))},$$

where W is the Lambert function, defined as the reciprocal of the function $z \mapsto ze^z$.

1.1.2 Independence ratio and girth

Independent sets in graphs are fundamental objects, at the heart of several problems and notions such as graph colouring. Of particular interest is the order $\alpha(G)$ of a largest independent set in a graph G, which often is divided by the number of vertices of G: this is the *independence ratio* of G,

$$\mathrm{ir}(G) \coloneqq \frac{\alpha(G)}{n(G)}.$$

Since a k-colouring of a graph is a partition of the vertex set into k independent sets, it follows that the independence ratio of a graph is a lower bound on its chromatic number. For instance, the 4-colour theorem thus implies that every planar graph has independent ratio at least $\frac{1}{4}$. Interestingly enough, no one seems to know how to prove this last statement, sometimes called the "Erdős-Vizing conjecture", without using the 4-colour theorem — or a proof of a similar nature and length.

The independence ratio of a graph has often been studied in relation with the *girth*, which is the length of a smallest cycle in the graph. A first result in this direction is the celebrated introduction of the so-called "deletion method" in graph theory by Erdős, who used it to demonstrate the existence of graphs with arbitrarily large girth and chromatic number. The latter is actually established by proving that the independence ratio of the graph is arbitrarily small. As a large girth is not strong enough a requirement to imply a constant upper bound on the chromatic number, a way to pursue this line of research is to express the upper bound in terms of the maximum degree $\Delta(G)$ of the graph G considered. This also applies to the independence ratio. As seen in the introduction of this thesis, the Hall ratio of a graph G is a hereditary version of the inverse of its independence ratio;

$$\rho(G) \coloneqq \max_{H\subseteq G} \frac{n(H)}{\alpha(H)}.$$

Letting $\mathrm{girth}(G)$ stand for the girth of the graph G, that is, the length of a shortest cycle in G if G is not a forest and $+\infty$ otherwise, we define $\rho(d,g)$ to be the supremum of the Hall ratios among all graphs of maximum degree at most d and girth at least g. We also denote $\rho(d,\infty)$ the limit as $g \to \infty$ of $\rho(d,g)$ — note by definition that, for fixed d, $\rho(d,g)$ is a non-increasing function of g.

$$\rho(d,g) \coloneqq \sup\left\{ \frac{n(G)}{\alpha(G)} \;\middle|\; G \text{ graph with } \Delta(G) \leq d \text{ and } \mathrm{girth}(G) \geq g \right\},$$
$$\rho(d,\infty) \coloneqq \lim_{g\to\infty} \rho(d,g).$$

In 1979, Staton [113] established that $\rho(d,4) \leq \frac{5d-1}{5}$, in particular implying that $\rho(3,4) \leq \frac{14}{5}$. The two graphs depicted in Figure 1.1.1, called the graphs of Fajtlowicz and of Locke, have fourteen vertices each, girth 5, and no independent set of order 6. It follows that $\rho(3,4) = \frac{14}{5} = \rho(3,5)$. It is known that the graphs of Fajtlowicz and of Locke are the only two cubic triangle-free and connected graphs with Hall ratio $\frac{14}{5}$. This follows from a result of Fraughnaugh and Locke [53] for

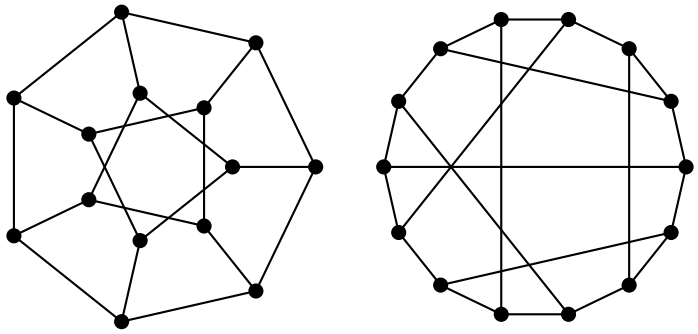


Figure 1.1.1: The two cubic triangle-free connected graphs with Hall ratio $\frac{14}{5}$.

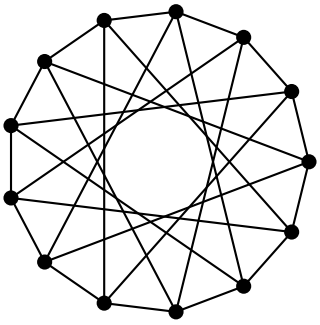


Figure 1.1.2: The only known 4-regular triangle-free connected graph of Hall ratio $\frac{13}{4}$

graphs with more than 14 vertices completed by an exhaustive computer check on graphs with at most 14 vertices performed by Bajnok and Brinkmann [11].

In 1983, Jones [66] reached the next step by establishing that $\rho(4,4) = \frac{13}{4}$. Only one connected graph is known to attain this value: it has 13 vertices and is represented in Figure 1.1.2. The value of $\rho(d,4)$ when $d \geq 5$ is still unknown; the best general upper bound is due to Shearer [110]. He also provides an upper bound for $\rho(d,6)$ as a consequence of a stronger result on graphs with no cycle of length 3 or 5.

Theorem 1.1.6 (Shearer, 1991 [110])**.** *For every non-negative integer d, set*

$$f(d) \coloneqq \begin{cases} 1 & \text{if } d = 0, \\ \frac{1+(d^2-d)f(d-1)}{d^2+1} & \text{if } d \geq 1. \end{cases}$$

If G is a triangle-free graph on n vertices with degree sequence $d_1, \ldots, d_n$, then

$$\alpha(G) \geq \sum_{i=1}^{n} f(d_i).$$

Theorem 1.1.7 (Shearer, 1991 [110])**.** *For every non-negative integer d, set*

$$f(d) \coloneqq \begin{cases} 0 & \text{if } d = 0, \\ \frac{4}{7} & \text{if } d = 1, \\ \frac{1+(d^2-d)f(d-1)}{d^2+1} & \text{if } d \geq 2. \end{cases}$$

If G is a $\{C_3, C_5\}$-free graph on n vertices with degree sequence $d_1, \ldots, d_n$, then

$$\alpha(G) \geq \sum_{i=1}^{n} f(d_i) - \frac{n_{11}}{7},$$

where n_{11} is the number of pairs of adjacent vertices of degree 1 in G.

Theorems 1.1.6 and 1.1.7 allow us to compute upper bounds on $\rho(d, 4)$ and on $\rho(d, 6)$ for small values of d, as indicated in Table 1.1. When $d \geq 5$, these bounds are the best known ones.

d	upper bound of $\rho(d,4)$		upper bound on $\rho(d,6)$	
2	$\frac{5}{2}$	$= 2.5$	$\frac{7}{3}$	≈ 2.33333
3	$\frac{50}{17}$	≈ 2.94118	$\frac{14}{5}$	$= 2.8$
4	$\frac{425}{127}$	≈ 3.34646	$\frac{119}{37}$	≈ 3.21622
5	$\frac{2210}{593}$	≈ 3.72681	$\frac{3094}{859}$	≈ 3.60186
6	$\frac{8177}{2000}$	≈ 4.0885	$\frac{57239}{14432}$	≈ 3.96612
7	$\frac{408850}{92177}$	≈ 4.43549	$\frac{408850}{94769}$	≈ 4.31417
8	$\frac{13287625}{2785381}$	≈ 4.77049	$\frac{13287625}{2857957}$	≈ 4.64934
9	$\frac{1089585250}{213835057}$	≈ 5.09545	$\frac{1089585250}{219060529}$	≈ 4.9739
10	$\frac{11004811025}{2033474038}$	≈ 5.41183	$\frac{11004811025}{2080503286}$	≈ 5.28949

Table 1.1: Upper bounds on $\rho(d, 4)$ and $\rho(d, 6)$ for $d \leq 10$ yielded by Theorems 1.1.6 and 1.1.7.

We are not aware of any non trivial lower bounds on $\rho(5, 4)$ and $\rho(6, 4)$. Figures 1.1.3 and Figure 1.1.4 show graphs illustrating that $\rho(5, 4) \geq \frac{10}{3} \approx 3.33333$ and $\rho(6, 4) \geq \frac{29}{8} = 3.625$.

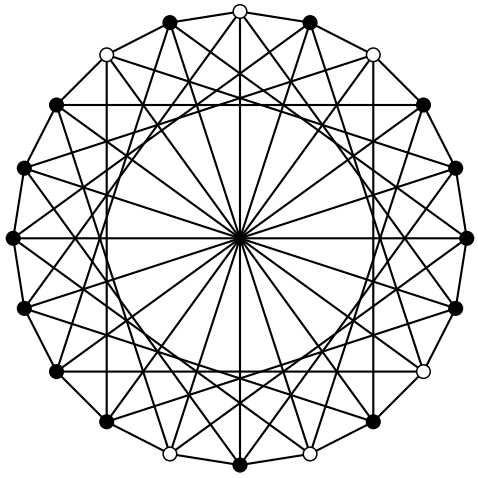


Figure 1.1.3: A 5-regular triangle-free (vertex-transitive) graph with Hall ratio $\frac{10}{3}$. Its vertex set is $[20]$, and its edge set is $\{ij \mid (i-j) \bmod 20 \in \{1, 6, 10\}\}$. There is no independent set of order 7, and the white vertices form an independent set of order 6.

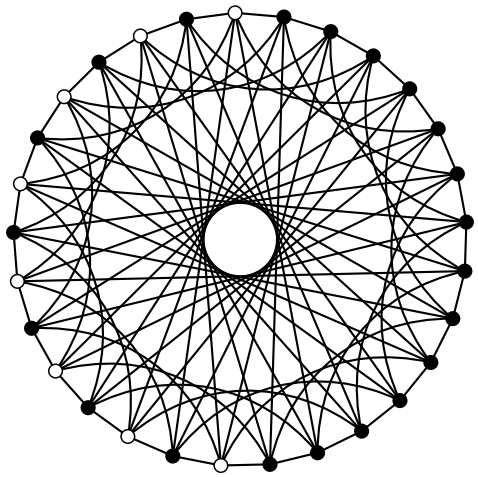


Figure 1.1.4: A 6-regular triangle-free (vertex-transitive) graph with Hall ratio $\frac{29}{8}$. Its vertex set is $[29]$, and its edge set is $\{ij \mid (i-j) \bmod 29 \in \{1,5,13\}\}$. There is no independent set of order 9, and the white vertices form an independent set of order 8.

The value of $\rho(3,g)$ has also been studied when g goes to infinity. Kardoš, Král' and Volec [73] proved the existence of an integer g_0 such that $\rho(3,g_0) \leq 2.2978$. More strongly, their upper bound holds for the fractional chromatic number of every (sub)cubic graph of girth at least g_0. In the other direction, Bollobás [18] proved a general lower bound on $\rho(d,g)$.

Theorem 1.1.8 (Bollobás, 1981 [18])**.** *Let $d \geq 3$. Let α be a real number in $(0,1)$ such that*

$$\alpha(d\ln 2 - \ln(\alpha)) + (2-\alpha)(d-1)\ln(2-\alpha) + (\alpha-1)d\ln(1-\alpha) < 2(d-1)\ln 2.$$

For every $g \geq 3$, there exists a d-regular graph with girth at least g and Hall ratio more than $2/\alpha$.

Theorem 1.1.8 allows us to compute lower bounds on $\rho(d,\infty)$ for any value of d, the smaller ones being represented in Table 1.2. All these values can be generalised into a looser but asymptotically equivalent general lower bound of $d/(2\ln d)$ [18, Corollary 3].

d	lower bound on $\rho(d,\infty)$
2	2
3	2.17835
4	2.3775
5	2.57278
6	2.76222
7	2.94606
8	3.1249
9	3.29931
10	3.46981
d	$d/(2\ln d)$

Table 1.2: Lower bounds on $\rho(d,\infty)$ implied by Theorem 1.1.8.

Our contribution is to provide improved upper bounds on the Hall ratio of graphs of maximum degree in $\{3, 4, 5\}$ and girth in $\{6, \ldots, 12\}$. In particular, these are upper bounds on the fractional chromatic number of vertex-transitive graphs in these classes. These upper bounds are obtained *via* a systematic computer-assisted method.

d \ g	6	7	8	9	10	11	12
3	$30/11 \approx 2.727272$	$30/11$	2.625224	2.604167	2.557176	2.539132	2.510378
4	$41/13 \approx 3.153846$	$41/13$	3.038497	3.017382	3		
5	$69/19 \approx 3.631579$	3.6	3.5				

Table 1.3: Upper bounds on $\rho(d, g)$ for $d \in \{3, 4, 5\}$ and $g \in \{6, \ldots, 12\}$.

Theorem 1.1.9. *The values presented in Table 1.3 are upper bounds on $\rho(d, g)$ for $d \in \{3, 4, 5\}$ and $g \in \{6, \ldots, 12\}$.*

The bounds provided by Theorem 1.1.9 when $d \in \{3, 4\}$ and $g = 7$ are the same as those for $g = 6$. It seems that this could be a general phenomenon. A computation is currently running to determine an upper bound on $\rho(3, 13)$, which we expect to be 2.5. We therefore propose the following conjecture.

Conjecture 1.1.1. *The values presented in Table 1.4 are upper bounds on $\rho(d, g)$ for $d \in \{3, 4, 5\}$ and $g \in \{6, 8, 10, 12\}$.*

d \ g	6	8	10	12
3		2.604167	2.539132	2.5
4		3.017382	3	
5	3.6	3.5		

Table 1.4: Conjectured upper bounds on $\rho(d, g)$ for $d \in \{3, 4, 5\}$ and $g \in \{6, \ldots, 12\}$.

1.1.3 Loosening of the sparsity

By a deletion argument, Ajtai, Komlós and Szemerédi [3] noted a more general statement as corollary to their seminal bound on the independence number of triangle-free graphs. There is some $C > 0$ and some Δ_0 such that, for every graph G containing at most $Tn(G)$ triangles, and of maximum degree $\Delta \geq \Delta_0\sqrt{T}$, the Hall ratio is at most $C \cdot \Delta / \ln\left(\Delta/\sqrt{T}\right)$. In other words, an upper bound on the number of triangles in the graph yields a corresponding lower bound on independence number. Somewhat later, Alon, Krivelevich and Sudakov [7] proved a stronger version of this in terms of an upper bound on the chromatic number. Recently, using a sophisticated "stochastic local search" framework, Achlioptas, Iliopoulos and Sinclair [1] tightened the result of [7], corresponding

to a constant C above of around $1/4$ in general[1]. In fact, shortly after the work in [3], using a sharper bootstrapping from the triangle-free case, Shearer [108] had improved the above statement on the independence number as follows.

Theorem 1.1.10 (Shearer [108])**.** *Given a rational $T > 0$, for every graph G of average degree $d \geq e^{7/4}\sqrt{T}$ where each vertex is contained in an average of T triangles (so G contains a total of $n(G)T/3$ triangles), it holds that*

$$\frac{n(G)}{\alpha(G)} \leq \begin{cases} \dfrac{d}{\ln \frac{d}{\sqrt{T}} - \frac{1}{2}\Big(\ln\Big(\ln \frac{d}{\sqrt{T}} - 1\Big) + \ln \frac{4}{3} + 3\Big)} & \text{if } T \geq \dfrac{7}{4\Big(\ln \frac{d}{\sqrt{T}} - 1\Big)}, \\ \dfrac{d}{\big(1 - \frac{2}{3}T\big)(\ln d - 1)} & \text{otherwise.} \end{cases}$$

So as a general upper bound,

$$\frac{n(G)}{\alpha(G)} \leq \Big(1 + \underset{d\to\infty}{o(1)}\Big) \frac{d}{\ln \dfrac{d}{\sqrt{T}}}.$$

The case $T = o(d)$ includes the triangle-free case and yields the best-to-date asymptotic lower bound on the off-diagonal Ramsey numbers. The asymptotic factor 1 cannot be improved below $1/2$, due to random regular graphs; see Section 1.6.3 for more details on sharpness.

Our main contribution is to give two stronger forms of Theorem 1.1.10, one on occupancy fraction, that is the expectancy of the size of a random independent set (see Theorem 1.6.1 below), the other on fractional chromatic number. We show how either easily implies Theorem 1.1.10 in Section 1.6.

Theorem 1.1.11. *Given $\varepsilon, T > 0$, there exists Δ_ε such that, for every graph G of maximum degree $\Delta \geq \Delta_\varepsilon\sqrt{T}$ where each vertex is contained in at most $T > 0$ triangles,*

(i) *the average size of the independent sets of G is at least*

$$(1 - \varepsilon)\frac{n(G)\ln \frac{\Delta}{\sqrt{T}}}{\Delta}, \text{ and}$$

(ii) *there exists a fractional colouring w of G such that, for every vertex $v \in V(G)$ of degree at least $\Delta_\varepsilon\sqrt{T}$,*

$$w(v) \subseteq \left[0, (1 + \varepsilon)\frac{\deg(v)}{\ln \frac{\deg(v)}{\sqrt{T}}}\right],$$

and so in particular

$$\chi_f(G) \leq (1 + \varepsilon)\frac{\Delta}{\ln \frac{\Delta}{\sqrt{T}}}.$$

We prove Theorem 1.1.11 by an analysis of the hard-core model. In Section 1.6.3, we give some indication that our application of this analysis is essentially tight.

Theorem 1.1.11(ii) and the results in [1] hint at their common strengthening.

[1]They also obtained an asymptotic estimate of $1/(2 + \varepsilon)$ for small T, namely $\sqrt{T} \leq \Delta^\varepsilon / \ln \Delta$.

Conjecture 1.1.2. *Given $\varepsilon, T > 0$, there exists Δ_ε such that, for every graph G of maximum degree $\Delta \geq \Delta_\varepsilon \sqrt{T}$ where each vertex is contained in at most T triangles,*

$$\chi_\ell(G) \leq (1+\varepsilon)\frac{\Delta}{\ln \frac{\Delta}{\sqrt{T}}}.$$

1.1.4 Preliminaries and structure of the chapter

In Section 1.2, we prove the result on local list colourings of triangle-free graphs. We do so by first reviewing the definition of correspondence colouring and proving for it a local version of the "finishing blow" (Lemma 1.2.1) used by Bernshteyn in his proof of Molloy's theorem for correspondence colouring [13]. Next, we sketch how Bernshteyn's argument can then be adapted to prove Theorem 1.1.2, and we present a simple construction (Proposition 1.2.5) to show that even some bipartite graphs cannot satisfy the conclusions of Theorem 1.1.2 without a suitable lower bound on the minimum list size.

In Section 1.3, we present tools which take advantage of the local occupancy of a given probability distribution on the independent sets in order to derive results concerning local colourings and the size of the independent sets. Among these tools is a greedy fractional colouring algorithm, which we use in order to give various bounds on the local fractional colourings of sparse graphs in Section 1.4. We prove a fractional local version of Reed's bound using the uniform distribution on maximum independent sets, a fractional local version of Molloy's theorem using the hard-core distribution on all independent sets, and a fractional local bound for graphs of girth 7 (which beats Molloy's for small degree vertices, but is worse in asymptotics) using the hard-core distribution on maximal independent sets.

In Section 1.5, we improve on the known bounds for the independence ratio of d-regular graphs of some prescribed (not too large) girth, for $d \in \{3, 4, 5\}$.

Finally, in Section 1.6, we generalise all our results on triangle-free graphs to graphs with a bounded number of triangles containing each vertex. We give a sharp bound for the average size of the independent sets in such graphs, and a bound sharp up to a multiplicative constant 4 for the fractional chromatic number of such graphs, in a local setting.

Notation. Given a graph G, if v is a vertex of G and r a non-negative integer, then $N^r_G(v)$ is the set of all vertices of G at distance exactly r from v in G, while $N^r_G[v] \coloneqq \bigcup_{j=0}^{r} N^j_G(v)$. Note in particular that $N^0_G(v) = \{v\}$. We write $\mathcal{I}(G)$ for the set of independent sets (including the empty set) of G. We write $\mathcal{I}_{\max}(G)$ and $\mathcal{I}_\alpha(G)$ for the restriction to maximal and maximum independent sets of G, respectively. If I in an independent set of G, a vertex v is *covered* by I if v belongs to I or has a neighbour in I. A vertex that is not covered by I is *uncovered* (by I). If w is a mapping from $\mathcal{I}(G)$ to measurable subsets of $\mathbb{R}$ then for every vertex $v \in V(G)$ we set

$$w(v) \coloneqq \bigcup_{\substack{I \in \mathcal{I}(G) \\ v \in I}} w(I),$$

and $w(X) \coloneqq \bigcup_{v \in X} w(v)$ for every subset of vertices $X \subseteq V(G)$. We write $\hat{w}(x) \coloneqq \mu(w(x))$ for the Lebesgue measure of $w(x)$, for every x in the domain of w. By extension, $\hat{w}(G)$ denotes $\hat{w}(V(G)) = \sum_{I \in \mathcal{I}(G)} \mu(w(I))$, the total weight of w on the graph G.

Given $\lambda > 0$, the *hard-core distribution* on G at *fugacity* λ is a probability distribution on

$\mathcal{I}(G)$, where each $I \in \mathcal{I}(G)$ occurs with probability proportional to $\lambda^{|I|}$. Writing $\mathbf{I}$ for the random independent set drawn according to this probability distribution, we have

$$\mathbb{P}[\mathbf{I} = I] = \frac{\lambda^{|I|}}{Z_G(\lambda)},$$

where the normalising term in the denominator is the *partition function* (or *independence polynomial*) $Z_G(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$. The *occupancy fraction* is $\mathbb{E}[|\mathbf{I}|/n(G)]$. Note that this is a lower bound on the independence ratio of G.

The function $W : [-1/e, \infty) \to [-1, \infty)$ is the inverse of $z \mapsto ze^z$, also known as the *Lambert W-function*. It is monotonic and satisfies $W(x) = \ln x - \ln\ln x + o(1)$ and $W((1+o(1))x) = W(x) + o(1)$ as $x \to \infty$, and $e^{W(y)} = y/W(y)$ for all y.

1.2 Local list colouring

1.2.1 The finishing blow for DP-colouring

Just as in [13], we will establish Theorem 1.1.2 for a generalised form of list colouring called *correspondence colouring* (or *DP-colouring*). We here state the definition given in [13].

Definition 1.2.1. Given a graph G, a *cover* of G is a pair $\mathscr{H} = (L, H)$, consisting of a graph H and a function $L\colon V(G) \to 2^{V(H)}$, satisfying the following requirements:

1. the sets $\{L(u) : u \in V(G)\}$ form a partition of $V(H)$;
2. for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
3. if $E_H(L(u), L(v)) \neq \varnothing$, then either $u = v$ or $uv \in E(G)$;
4. if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (possibly empty).

An *$\mathscr{H}$-colouring* of G is an independent set in H of size $n(G)$.

A reader who prefers not to concern herself with this generalised notion may merely read L as an ordinary list assignment and $V(H)$ as the disjoint union of all lists. For usual list colouring, there is an edge in H between equal colours of two lists if and only if there is an edge between their corresponding vertices in G.

To state and prove our local version of the finishing blow, we will need some further notation. Define H^* to be the spanning subgraph of H such that an edge $c_1c_2 \in E(H)$ belongs to $E(H^*)$ if and only if c_1 and c_2 are in different parts of the partition $\{L(u) : u \in V(G)\}$. We write $\deg^*_{\mathscr{H}}(c)$ instead of $\deg_{H^*}(c)$.

Lemma 1.2.1. *Let $\mathscr{H} = (L, H)$ be a cover of a graph G. Suppose there is a function $\ell\colon V(G) \to \mathbb{N}_{\geq 4}$, such that, for all $u \in V(G)$, $|L(u)| \geq \ell(u)$ and $\deg^*_{\mathscr{H}}(c) \leq \frac{1}{6} \min_{v \in N_G(u)} \ell(v)$ for all $c \in L(u)$. Then G is $\mathscr{H}$-colourable.*

For clarity, we separately state the corollary this lemma has for conventional list colouring.

Corollary 1.2.1.1. *Let $L\colon V(G) \to 2^{\mathbb{N}}$ be a list assignment of a graph G. Suppose there is a function $\ell\colon V(G) \to \mathbb{N}_{\geq 4}$ such that, for all $u \in V(G)$, $|L(u)| \geq \ell(u)$ and the number of neighbours $v \in N_G(u)$ for which $c \in L(v)$ is at most $\frac{1}{6} \min_{v \in N_G(u)} \ell(v)$ for all $c \in L(u)$. Then there exists a proper L-colouring of G.*

Proof of Lemma 1.2.1. Remove, if needed, some vertices from H to ensure that $|L(u)| = \ell(u)$ for all $u \in V(G)$. Let $\mathbf{I}$ be a random subset of $V(H)$ obtained by choosing, independently and uniformly, one vertex from each list $L(u)$. For $c_1c_2 \in E(H^*)$, let $B_{c_1c_2}$ denote the event that both c_1 and c_2 are chosen in $\mathbf{I}$. So, if none of the events $B_{c_1c_2}$ occurs, then $\mathbf{I}$ is an independent set and hence an $\mathscr{H}$-colouring. Let u_i be the vertex of G such that $c_i \in L(u_i)$, for $i \in \{1,2\}$. By definition, $\mathbb{P}[B_{c_1c_2}] = \frac{1}{\ell(u_1)\ell(u_2)}$. Define

$$\Gamma(c_1c_2) \coloneqq \{c_1'c_2' \in E(H^*) \mid c_1' \in L(u_1) \text{ or } c_2' \in L(u_2)\}.$$

Note that $B_{c_1c_2}$ is mutually independent of the events $B_{c_1'c_2'}$ with $c_1'c_2' \notin \Gamma(c_1c_2)$. All that remains is to define weights $x_{c_1c_2} \in [0,1)$ to satisfy the hypothesis of the General Lovász Local Lemma. Specifically, we need that

$$\frac{1}{\ell(u_1)\ell(u_2)} = \mathbb{P}[B_{c_1c_2}] \leq x_{c_1c_2} \prod_{c_1'c_2' \in \Gamma(c_1c_2)} \left(1 - x_{c_1'c_2'}\right).$$

Since $\exp(-1.1x) \leq 1 - x$ if $0 \leq x \leq 0.17$, it suffices to find weights $x_{c_1c_2} \in [0, 0.17]$ satisfying

$$\frac{1}{\ell(u_1)\ell(u_2)} \leq x_{c_1c_2} \exp\left(-1.1 \sum_{c_1'c_2' \in \Gamma(c_1c_2)} x_{c_1'c_2'}\right). \tag{1.1}$$

If we choose weights of the form $x_{c_1c_2} = \dfrac{k}{\ell(u_1)\ell(u_2)}$ for some constant $k > 0$, then (1.1) becomes

$$\ln k \geq 1.1k \sum_{c_1'c_2' \in \Gamma(c_1c_2)} \frac{1}{\ell(u_1')\ell(u_2')}$$

(where u_i' is such that $c_i' \in L(u_i')$, for $i \in \{1,2\}$).

Now note that

$$\sum_{c_1'c_2' \in \Gamma(c_1c_2)} \frac{1}{\ell(u_1')\ell(u_2')} \leq \sum_{c_1' \in L(u_1)} \frac{\deg^*_{\mathscr{H}}(c_1')}{\ell(u_1) \min_{v \in N_G(u_1)} \ell(v)} + \sum_{c_2' \in L(u_2)} \frac{\deg^*_{\mathscr{H}}(c_2')}{\ell(u_2) \min_{v \in N_G(u_2)} \ell(v)} \leq \frac{1}{3},$$

by the assumption on $\deg^*_{\mathscr{H}}$. So (1.1) is fulfilled if there is $k > 0$ such that

$$\ln k \geq \frac{1.1k}{3} \text{ and } \frac{k}{\ell(u_1)\ell(u_2)} \leq 0.17 \text{ for all } u_1, u_2 \in V(G).$$

Noting the lower bound condition on ℓ, the choice $k = e$ works. □

1.2.2 The proof of the list colouring theorem

In this section, we prove Theorem 1.1.2. Let us remark that an alternative to the following derivation would be to similarly follow Molloy's original proof and apply Corollary 1.2.1.1. We will sketch a proof of the following stronger form of Theorem 1.1.2.

Theorem 1.2.2. *Fix $\varepsilon \in (0,1)$. There exists some Δ_ε such that, for every graph G of maximum degree $\Delta \geq \Delta_\varepsilon$ and minimum degree at least $\delta \coloneqq (72 \ln \Delta)^{2/\varepsilon}$, and for every cover $\mathscr{H} = (L, H)$ of G satisfying*

$$|L(u)| \geq (1+\varepsilon) \max\left\{ \frac{\deg(u)}{\ln \deg(u)}, \frac{\delta}{\ln \delta} \right\},$$

it holds that G is $\mathscr{H}$-colourable.

We will need further notation. Given a cover $\mathscr{H} = (L, H)$, the *domain* of an independent set I in H is $\mathrm{dom}(I) = \{u \in V(G) : I \cap L(u) \neq \varnothing\}$. Let $G_I = G - \mathrm{dom}(I)$ and let $\mathscr{H}_I = (L_I, H_I)$ denote the cover of G_I defined by

$$H_I = H - N_H[I] \qquad \text{and} \qquad L_I(u) = L(u) \setminus N_H(I) \text{ for all } u \in V(G_I).$$

Note that, if I' is an $\mathscr{H}_I$-colouring of G_I, then $I \cup I'$ is an $\mathscr{H}$-colouring of G.

For the rest of this section, fix some reals ε, Δ, δ, G, and $\mathscr{H}$ to satisfy the conditions of Theorem 1.2.2. Write

$$k(u) = (1+\varepsilon) \max\left\{ \frac{\deg_G(u)}{\ln \deg_G(u)}, \frac{\delta}{\ln \delta} \right\},$$

so that $|L(u)| \geq k(u)$, and set $\ell(u) = \max\{\deg_G(u)^{\varepsilon/2}, \delta^{\varepsilon/2}\}$ so that $\ell(u) \geq 72 \ln \Delta$ for all u.

With this notation, and in view of Lemma 1.2.1, it suffices to establish the following analogue of Lemma 3.5 in [13].

Lemma 1.2.3. *The graph H contains an independent set I such that*

(i) $|L_I(u)| \geq \ell(u)$ *for all $u \in V(G_I)$, and*

(ii) $\deg^*_{\mathscr{H}_I}(x) \leq \ell(u)/6 \leq 12 \ln \Delta$ *for all $x \in V(H_I)$.*

Through an application of a lopsided version of the LLL, Lemma 1.2.3 reduces to the following result. The proof of this reduction appears in [13] with a fixed value of ℓ instead of a varying function; yet the proof is still valid within this local setting.

Lemma 1.2.4. *Fix a vertex $u \in V(G)$ and an independent set $J \subseteq L(\overline{N_G[u]})$. Let $\mathbf{I}'$ be a uniformly random independent subset of $L_J(N_G(u))$ and let $\mathbf{I} = J \cup \mathbf{I}'$. Then*

(i) $\mathbb{P}\left[|L_I(u)| < \ell(u)\right] \leq \Delta^{-3}/8$, *and*

(ii) $\mathbb{P}\left[\exists c \in L_{\mathbf{I}}(u), \deg^*_{\mathscr{H}_{\mathbf{I}}}(c) > 12 \ln \Delta\right] \leq \Delta^{-3}/8$.

Proof. This proof is adapted from the proof of Lemma 3.6 in [13].

Since G is triangle-free, it holds that $\mathbf{I}' \cap L_J(v)$ contains an element from $L_J(v) \cup \{\bullet\}$ selected uniformly at random, where $\mathbf{I}' \cap L_J(v)$ is empty whenever $\bullet$ is selected.

Given a vertex $u \in V(G)$, for every $x \in L(u)$ we denote $\widetilde{N}(x)$ the set of neighbours v of u in G such that $x \in L(v)$. Then

$$\mathbb{P}\left[x \in L_{\mathbf{I}}(u)\right] = \mathbb{P}\left[\mathbf{I}' \cap N_H(x) = \varnothing\right] = \prod_{v \in \widetilde{N}(x)} \left(1 - \frac{1}{|L_J(v)| + 1}\right),$$

and hence

$$\exp\left(-\sum_{v \in \widetilde{N}(x)} \frac{1}{|L_J(v)|}\right) \leq \mathbb{P}\left[x \in L_{\mathbf{I}}(u)\right] \leq \exp\left(-\sum_{v \in \widetilde{N}(x)} \frac{1}{1 + |L_J(v)|}\right).$$

(i) We notice that

$$\sum_{x \in L(u)} \sum_{v \in \widetilde{N}(x)} \frac{1}{|L_J(v)|} \leq \sum_{\substack{v \in N_G(u) \\ L_J(v) \neq \varnothing}} \sum_{y \in L_J(v)} \frac{1}{|L_J(v)|} \leq \deg_G(u).$$

By using the convexity of the exponential function, we obtain that

$$\begin{aligned}
\mathbb{E}\left[|L_{\mathbf{I}}(u)|\right] &= \sum_{x \in L(u)} \mathbb{P}\left[x \in L_{\mathbf{I}}(u)\right] \\
&\geq \sum_{x \in L(u)} \exp\left(-\sum_{v \in \widetilde{N}(x)} \frac{1}{|L_J(v)|}\right) \\
&\geq k(u) \exp\left(-\frac{\deg_G(u)}{k(u)}\right) \\
&= (1+\varepsilon) \max\left\{\frac{\deg_G(u)^{1-\frac{1}{1+\varepsilon}}}{\ln \deg_G(u)}, \frac{\delta^{1-\frac{1}{1+\varepsilon}}}{\ln \delta}\right\} \\
&> 2 \max\left\{\deg_G(u)^{\frac{\varepsilon}{2}}, \delta^{\frac{\varepsilon}{2}}\right\} = 2\ell(u),
\end{aligned}$$

where the final inequality holds for Δ (and hence δ) large enough in terms of ε, because by convexity $1 - 1/(1+\varepsilon) > \varepsilon/2$ for $0 < \varepsilon < 1$. We now apply a Chernoff bound for negatively correlated random variables, which yields that

$$\mathbb{P}\left[|L_I(u)| < \ell(u)\right] \leq e^{-\ell(u)/4} \leq \Delta^{-18},$$

which is at most $\Delta^{-3}/8$ for $\Delta \geq 2$.

(ii) For every $x \in L(u)$ we define

$$p_x \coloneqq \mathbb{P}\left[x \in L_{\mathbf{I}}(u) \text{ and } \deg^*_{\mathscr{H}_{\mathbf{I}}}(x) > 12 \ln \Delta\right].$$

By a union bound, $\mathbb{P}\left[\exists x \in L_{\mathbf{I}}(u), \deg^*_{\mathscr{H}_{\mathbf{I}}}(x) > 12 \ln \Delta\right] \leq \sum_{x \in L(u)} p_x$, and since $|L(u)| \leq \frac{\deg(u)}{8}$ assuming that Δ is large enough, it suffices to prove that $p_x \leq \Delta^{-4}$ for every $x \in L(u)$ in order to obtain the desired bound.

Since $p_x \leq \mathbb{P}[x \in L_{\mathbf{I}}(u)]$, we may assume that

$$\exp\left(-\sum_{v\in\widetilde{N}(x)}\frac{1}{1+|L_J(v)|}\right) \geq \mathbb{P}[x \in L_{\mathbf{I}}(u)] \geq \Delta^{-4},$$

hence

$$\mathbb{E}\left[\deg^*_{\mathscr{H}_{\mathbf{I}}}(x)\right] = \sum_{v\in\widetilde{N}(x)}\mathbb{P}[\mathbf{I}'\cap L_J(v)=\varnothing] = \sum_{v\in\widetilde{N}(x)}\frac{1}{1+|L_J(v)|} \leq 4\ln\Delta.$$

There remains to apply a Chernoff bound in order to obtain

$$\begin{aligned} p_x &\leq \mathbb{P}\left[\deg^*_{\mathscr{H}_{\mathbf{I}}}(x) > 12\ln\Delta\right] \\ &\leq \mathbb{P}\left[\deg^*_{\mathscr{H}_{\mathbf{I}}}(x) > 3\mathbb{E}\left[\deg^*_{\mathscr{H}_{\mathbf{I}}(c)}\right]\right] \\ &\leq \Delta^{-4}, \end{aligned}$$

as required.

□

1.2.3 A construction to prove the need for a minimum degree condition

In Theorem 1.1.2 the condition is only truly local when the graph is of minimum degree $\delta = (72\ln\Delta)^{2/\varepsilon}$, which grows with the maximum degree Δ. The result is made strictly stronger by reducing δ. In this section we show that even for bipartite graphs the conclusion of Theorem 1.1.2 requires some $\omega(1)$ bound on δ as $\Delta\to\infty$. We state and prove the result specifically with $\deg(u)/\ln\deg(u)$ as the target local list size per vertex u. The reader can check that any sublinear and superlogarithmic function will do, but with a different tower of exponentials.

Proposition 1.2.5. *For any $\delta \geq 3$, there is a bipartite graph of minimum degree δ and maximum degree $\exp^{\delta-1}(\delta)$ (so a tower of exponentials of height $\delta-1$) that is not L-colourable for some list assignment $L\colon V(G)\to 2^{\mathbb{N}}$ satisfying*

$$|L(u)| \geq \frac{\deg(u)}{\ln\deg(u)}$$

for all $u\in V(G)$.

Proof. The construction is a recursion, iterated $\delta-1$ times.

For the basis of the recursion, let G_0 be the star $K_{1,\delta}$ of degree $\delta\geq 3$. We write A_0 as the set containing the centre v_0 of the star and B_0 as the set of all non-central vertices. Note that, with the assignment L_0 that assigns the list $\{1_0,\dots,\delta_0\}$ to the centre and lists $\{i_0\}$, $i\in[\delta]$, to the non-central vertices, G_0 is not L_0-colourable.

We recursively establish the following properties for G_i, A_i, B_i, L_i, where $0\leq i\leq\delta-1$:

1. G_i is bipartite with partite sets A_i and B_i;

2. A_i has all vertices of degree at least δ and at most $\exp^i(\delta)$, with some vertex v_i attaining the maximum $\exp^i(\delta)$;

3. B_i has $\exp^i(\delta)$ vertices of degree $i+1$;
4. $|L_i(a)| \geq \deg(a)/\ln\deg(a)$ for all $a \in A_i$ and $|L_i(b)| \geq \deg(b)$ for all $b \in B_i$;
5. G_i is not L_i-colourable.

These properties are clearly satisfied for $i = 0$.

From step i to step $i+1$, we form G_{i+1} by taking $\exp(\exp^i(\delta))/\exp^i(\delta)$ copies of G_i and adding a vertex v_{i+1} universal to all of the B_i-vertices. Let A_{i+1} be v_{i+1} together with all A_i-vertices, and B_{i+1} be all of the B_i-vertices. Label each copy of G_i with j from 1 to $\exp(\exp^i(\delta))/\exp^i(\delta)$. We set $L_{i+1}(a) = L_i(a)$ for every $a \in A_i$, $L_{i+1}(v_{i+1}) = \{1_{i+1}, \ldots, \exp(\exp^i(\delta))/\exp^i(\delta)_{i+1}\}$ and add colour j_{i+1} to $L_i(b)$ to form $L_{i+1}(b)$ for every B_i-vertex b in the j-th copy of G_i. It is routine to check then that G_{i+1}, A_{i+1}, B_{i+1}, L_{i+1} satisfy the promised properties.

The proposition follows by taking $G_{\delta-1}$. □

As a final remark on minimum degree or minimum list size conditions, we note that our proof of Theorem 1.1.2 can be adapted to reduce $\delta = (72\ln\Delta)^{2/\varepsilon}$ as a function of Δ by increasing the leading constant '1' in the list size condition. Indeed, this removes the dependence on ε and brings the result much closer to the triangle-free case of the significantly more general local colouring result of Bonamy *et al.* [19], which has a minimum degree condition of $(\ln\Delta)^2$. Here, as we focus on triangle-free graphs we prefer to aim for the best possible constant at the expense of the cutoff value δ.

1.3 Using probability distributions on the independent sets

In this section we present some lemmas needed for the proofs of the main theorems on fractional colourings and independence ratio. All of these rely on a given probability distribution on the independent sets.

1.3.1 Greedy fractional colouring algorithm

Our results on fractional colouring use a greedy algorithm. This algorithm is a generalisation of an algorithm first described in the book of Molloy and Reed [93, p. 245] for the uniform distribution over maximum independent sets. It relies on a given probability distribution over the independent sets of any induced subgraph of the input graph that we wish to fractionally colour.

Lemma 1.3.1. *Fix a positive integer r. Let G be a graph and suppose that every vertex $v \in V(G)$ is assigned a list $(\alpha_j(v))_{j=0}^r$ of $r+1$ real numbers. Suppose that for each induced subgraphs H of G, there is a probability distribution on $\mathcal{I}(H)$ such that, writing $\mathbf{I}_H$ for the random independent set from this distribution,*

$$\forall v \in V(H), \qquad \sum_{j=0}^{r} \alpha_j(v)\mathbb{E}\left[\left|N_H^j(v) \cap \mathbf{I}_H\right|\right] \geq 1.$$

The greedy fractional algorithm defined by Algorithm 3 produces a fractional colouring w of G such that $w(v) \subseteq [0, \gamma_\alpha(v)]$ for every vertex $v \in V(G)$, where

$$\gamma_\alpha(v) \coloneqq \sum_{j=0}^{r} \alpha_j(v)\left|N_G^j(v)\right|.$$

In particular,

$$\chi_f(G) \leq \max_{v \in V(G)} \gamma_\alpha(v).$$

Algorithm 3 The greedy fractional algorithm

for $I \in \mathcal{I}(G)$ **do**
  $\hat{w}(I) \leftarrow 0$
end for
$H \leftarrow G$
while $|V(H)| > 0$ **do**
  $\tau \leftarrow \min \left\{ \min_{v \in V(H)} \frac{1 - \hat{w}(v)}{\mathbb{P}[v \in \mathbf{I}_H]}, \min_{v \in V(H)} \gamma_\alpha(v) - \hat{w}(G) \right\}$
  for $I \in \mathcal{I}(H)$ **do**
    $\hat{w}(I) \leftarrow \hat{w}(I) + \mathbb{P}[\mathbf{I}_H = I]\,\tau$
  end for
  $H \leftarrow H - \{v \in V(H) \mid \hat{w}(v) = 1\}$
end while

Proof. We present a refinement of an algorithm given in the book of Molloy and Reed [93], and show that under the assumptions of the lemma, it returns the desired fractional colouring. The idea of the algorithm is to greedily add weight to independent sets according to the probability distribution induced on all not yet fully coloured vertices.

We build a fractional colouring w in several iterations, and we write $\hat{w}(I)$ for $\mu(w(I))$ so that $\hat{w}(I)$ is a non-negative rational representing the measure w assigns to I. Through the iterations, w is a partial fractional colouring in the sense of not yet having satisfied the condition that the measure $\hat{w}(v)$ induced on each vertex $v \in V(G)$ is at least 1.

We next show that this algorithm certifies the desired fractional colouring. For the analysis, it is convenient to index the iterations: let H_i, $\hat{w}_i(I)$, $\hat{w}_i(v)$, $\hat{w}_i(G)$, τ_i denote the corresponding H, $\hat{w}(I)$, $\hat{w}(v)$, $\hat{w}(G)$, τ in the i-th iteration of the while loop, prior to updating the sequence. Note then that $H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots$. We also have $\hat{w}_{i+1}(v) = \sum_{k=0}^{i} \mathbb{P}[v \in \mathbf{I}_{H_k}]\,\tau_k$ for any $v \in V(H_i)$ and $\hat{w}_{i+1}(G) = \sum_{k=0}^{i} \tau_k$.

Let us first describe the precise fractional colouring (rather than its sequence of measures) that is constructed during the algorithm. During the update from $\hat{w}_i$ to $\hat{w}_{i+1}$, in actuality we do the following. Divide the interval $[\hat{w}_i(G), \hat{w}_i(G) + \tau_i)$ into a sequence $(B_I)_{I \in \mathcal{I}(G)}$ of consecutive right half-open intervals such that B_I has length $\mathbb{P}[\mathbf{I}_{H_i} = I]\,\tau_i$. We then let $w_{i+1}(I) = w_i(I) \cup B_I$ for each $I \in \mathcal{I}(G)$. Note that $\mu(w_i(I)) = \hat{w}_i(I)$ for all $I \in \mathcal{I}(G)$ and i. Moreover, by induction, $w_i(G) \subseteq [0, \hat{w}_i(G))$ for all i.

By the choice of τ_i, if there is some $v \in V(H_i)$ (i.e. with $\hat{w}_i(v) < 1$), then $\hat{w}_{i+1}(G) \leq \gamma_\alpha(v)$ and so $w_{i+1}(G) \subseteq [0, \gamma_\alpha(v)]$. So we only need to show that the algorithm terminates. To do so, it suffices to show that $|V(H_{i+1})| < |V(H_i)|$ for all i.

If

$$\tau_i = \min_{v \in V(H_i)} \frac{1 - \hat{w}_i(v)}{\mathbb{P}[v \in \mathbf{I}_{H_i}]},$$

then there must be some vertex $v \in V(H_i)$ such that $\hat{w}_i(v) < 1$ and $\hat{w}_{i+1}(v) = 1$, so $|V(H_{i+1})| < |V(H_i)|$ and we are done. We may therefore assume that there is some vertex $v \in V(H_i)$ such that $\tau_i = \gamma_\alpha(v) - \hat{w}_i(G)$, and so $\hat{w}_{i+1}(G) = \gamma_\alpha(v)$.

For any $k \in [i+1]$, we know that

$$\sum_{j=0}^{r} \alpha_j(v) \mathbb{E}\left[\left|N^j_{H_k}(v) \cap \mathbf{I}_{H_k}\right|\right] \geq 1,$$

and so

$$\sum_{j=0}^{r} \alpha_j(v) \sum_{u \in N^j_{H_k}(v)} \mathbb{P}\left[u \in \mathbf{I}_{H_k}\right] \tau_k \geq \tau_k.$$

By summing this last inequality over all such k, we obtain

$$\sum_{j=0}^{r} \alpha_j(v) \left|N^j_G(v)\right| \geq \sum_{j=0}^{r} \alpha_j(v) \sum_{u \in N^j_G(v)} \hat{w}_{i+1}(u) \geq \hat{w}_{i+1}(G) = \gamma_\alpha(v).$$

We then have that

$$\alpha_0(v)\hat{w}_{i+1}(v) \geq \gamma_\alpha(v) - \sum_{j=1}^{r} \alpha_j(v) \left|N^j_G(v)\right| = \alpha_0(v).$$

So $\hat{w}_{i+1}(v) = 1$, hence $|V(H_{i+1})| < |V(H_i)|$. □

1.3.2 Independence ratio

We state two lemmas which can be proved in similar ways. We only present the proof of the second one, the argument for the first lemma being very close but a little simpler.

Lemma 1.3.2. *Let r be a positive integer and G be a d-regular graph on n vertices. Assume that there exists a probability distribution p on $\mathcal{I}(G)$ such that*

$$\forall v \in V(G), \quad \sum_{i=0}^{r} \alpha_i \mathbb{E}\left[\mathbf{X}_i(v)\right] \geq 1, \tag{1.2}$$

where $\mathbf{X}_i(v)$ is the random variable counting the number of paths of length i between v and a vertex belonging to a random independent set $\mathbf{I}$ chosen following p. Then

$$\frac{n}{\alpha(G)} \leq \alpha_0 + \sum_{i=1}^{r} \alpha_i d(d-1)^{i-1}. \tag{1.3}$$

Lemma 1.3.3. *Let r be a positive integer and G be a d-regular graph on n vertices. Assume that there exists a probability distribution p on $\mathcal{I}(G)$ such that*

$$\forall e \in E(G), \quad \sum_{i=0}^{r} \alpha_i \mathbb{E}\left[\mathbf{X}_i(e)\right] \geq 1, \tag{1.4}$$

where $\mathbf{X}_i(e)$ is the random variable counting the number of paths of length $i+1$ starting with e

and ending at a vertex belonging to a random independent set $\mathbf{I}$ *chosen following* p*. Then*

$$\frac{n}{\alpha(G)} \leq \sum_{i=0}^{r} 2\alpha_i (d-1)^i. \tag{1.5}$$

Proof. Given an edge e of G, the contribution of an arbitrary vertex $v \in \mathbf{I}$ to $\mathbf{X}_i(e)$ is the number of paths of length $i+1$ starting at v and ending with e. It follows that the total contribution of any vertex $v \in \mathbf{I}$ to $\sum_{e \in E(G)} \mathbf{X}_i(e)$ is the number of paths of G with length $i+1$ that start at v, which is $d(d-1)^i$ since G is a d-regular graph. Consequently,

$$\sum_{e \in E(G)} \mathbf{X}_i(e) = \sum_{v \in V(G)} \mathbb{P}\left[v \in \mathbf{I}\right] d(d-1)^i.$$

We now sum (1.4) over all edges of G.

$$\begin{aligned}
\sum_{e \in E(G)} \sum_{i=0}^{r} \alpha_i \mathbb{E}\left[\mathbf{X}_i(e)\right] &\geq |E(G)| = \frac{nd}{2} \\
\sum_{i=0}^{r} \alpha_i \sum_{e \in E(G)} \mathbb{E}\left[\mathbf{X}_i(e)\right] &\geq \frac{nd}{2} \\
\sum_{i=0}^{r} \alpha_i \sum_{v \in V(G)} \mathbb{P}\left[v \in \mathbf{I}\right] d(d-1)^i &\geq \frac{nd}{2} \\
\sum_{i=0}^{r} 2\alpha_i \mathbb{E}\left[|\mathbf{I}|\right] (d-1)^i &\geq n \\
\sum_{i=0}^{r} 2\alpha_i (d-1)^i &\geq \frac{n}{\alpha(G)}
\end{aligned}$$

□

We can proceed to a generalisation of Lemmas 1.3.2 and 1.3.3 to non-regular graphs through a regularisation of the considered graph, and so by heredity of the class of graphs of fixed maximum degree and girth this generalisation holds for the Hall ratio. This can be done by an application of Lemma 0.3.6.

Lemma 1.3.4. *Let* d *and* g *be integers greater than two. If there exists a constant* $B = B(d,g)$ *such that every* d*-regular graph* H *with girth* g *has independence ratio at least* B*, then every graph* G *with maximum degree* d *and girth* g *also has independence ratio at least* B*. In particular, if Lemma 1.3.2 or Lemma 1.3.3 can be applied to the class of* d*-regular graphs of girth* g*, then the conclusion also holds for the class of graphs with maximum degree* d *and girth* g*, that is, for* $i(d,g)$*.*

Proof. Let G be a graph with maximum degree d and girth g on n vertices. Let $\varphi(G)$ be the graph provided by Lemma 0.3.6. In particular, $n(\varphi(G)) = kn$ where k is the number of induced copies of G partitioning $V(\varphi(G))$. By assumptions, $\varphi(G)$ contains an independent set I of order at least $B \cdot kn$. Letting I_i be the set of vertices of the i-th copy of G contained in I, by the pigeon-hole principle there exists $i \in [k]$ such that $|I_i| \geq B \cdot n$, and hence G has independence ratio at least B. □

1.4 Local fractional colourings

1.4.1 A local Reed bound

For the sake of illustration, we begin by showing how Lemma 1.3.1 can be used to prove Theorem 1.1.4. We actually establish a local form of Theorem 1.1.4, which strengthens a result first devised by McDiarmid (unpublished) and appearing as an exercise in Molloy and Reed's book [93].

Theorem 1.4.1 (McDiarmid, unpublished)**.** *Let G be a graph, and let us denote $\omega(v)$ for the order of a largest clique in G containing v, for every vertex $v \in V(G)$. Then*

$$\chi_f(G) \leq \max_{v \in V(G)} \frac{\omega(v) + \deg(v) + 1}{2}.$$

A published version of Theorem 1.4.1 can be found in the thesis of Andrew King [80, Theorem 2.10, p. 12]. The argument relies on the relation (1.6) below [80, Lemma 2.11], which is a local version of the relation (21.10) appearing in Molloy and Reed's book [93]. The short argument, however, stays the same and we provide it here only for explanatory purposes, since it is the inspiration for the argument used in the proof of Theorem 1.1.5.

Theorem 1.4.2. *Let G be a graph, and let us denote $\omega(v)$ for the order of a largest clique in G containing v, for every vertex $v \in V(G)$. Setting*

$$\gamma(v) \coloneqq \frac{\omega(v) + \deg(v) + 1}{2},$$

there exists a local fractional colouring w of G such that, for every vertex $v \in V(G)$,

$$w(v) \subseteq [0, \gamma(v)].$$

Proof. We demonstrate the statement by applying Lemma 1.3.1. To this end, we use the uniform distribution on maximum independent sets. Specifically, for every induced subgraph H of G we let $\mathbf{I}_H$ be a maximum independent set of H, drawn uniformly at random. Let $v \in V(H)$ be any vertex. We shall prove that

$$\frac{\omega(v) + 1}{2} \mathbb{P}\left[v \in \mathbf{I}_H\right] + \frac{1}{2} \mathbb{E}\left[|N(v) \cap \mathbf{I}_H|\right] \geq 1. \tag{1.6}$$

The conclusion then follows by applying Lemma 1.3.1, with $r = 1$, $\alpha_0(v) = \frac{1}{2} \cdot (\omega(v)+1)$ and $\alpha_1(v) = \frac{1}{2}$ for every vertex $v \in V(G)$.

It remains to establish (1.6). We let $\mathbf{J}_v \coloneqq \mathbf{I}_H \setminus N[v]$, and we condition on the following random events.

(i) Let X_k be the random event that $\mathbf{W} \coloneqq N[v] \setminus N(\mathbf{J}_v)$ is a clique of size $k \leq \omega(v)$. It follows that exactly one vertex from $\mathbf{W}$ belongs to $\mathbf{I}_H$, and every vertex in $\mathbf{W}$ has equal probability $1/k$ to be in $\mathbf{I}_H$. It follows that

$$\frac{\omega(v) + 1}{2} \mathbb{P}\left[v \in \mathbf{I}_H \mid X_k\right] + \frac{1}{2} \mathbb{E}\left[|N(v) \cap \mathbf{I}_H| \mid X_k\right] = \frac{\omega(v) + 1}{2k} + \frac{k - 1}{2k} \geq 1.$$

(ii) Let Y be the random event that $\mathbf{W}$ is not a clique. Note that Y is the complementary event to the union of the events X_k. In this case, $|(\mathbf{W} \setminus \{v\}) \cap \mathbf{I}_H| \geq 2$, and $v \notin \mathbf{I}_H$, since $\mathbf{I}_H$ is a maximum independent set. It follows that

$$\frac{\omega(v)+1}{2}\mathbb{P}\left[v \in \mathbf{I}_H \mid Y\right] + \frac{1}{2}\mathbb{E}\left[|N(v) \cap \mathbf{I}_H| \mid Y\right] \geq \frac{1}{2} \times 2 = 1$$

The validity of (1.6) follows by summing over all possible sets J for which there exists a maximum independent set I of H such that $J = I \setminus N[v]$. □

We finish by noting that the bound provided by Theorem 1.4.1 is best possible over the class of unicyclic triangle-free graphs if one uses the fractional greedy colouring of Lemma 1.3.1 together with any probability distribution on the *maximum* independent sets of the graph.

Lemma 1.4.3. *If the probability distribution used in Lemma 1.3.1 gives positive probability only to maximum independent sets, then the greedy fractional colouring algorithm can return a fractional colouring of weight up to $\frac{d+3}{2}$ in general for graphs of degree d, should they be acyclic when d is odd, or have a unique cycle (of length 5) when d is even.*

Proof. We prove the statement by induction on the positive integer d.

- If $d = 1$, then let G_1 consist only of an edge. The algorithm returns a fractional colouring of G_1 of weight 2.

- If $d = 2$, then let G_2 be the cycle of length 5. The algorithm returns a fractional colouring of G_2 of weight $\frac{5}{2}$.

- If $d > 2$, then let G_d be obtained from G_{d-2} by adding two neighbours of degree 1 to every vertex. This creates no new cycles, so G_d is acyclic when d is odd, and contains a unique cycle, which is of length 5, when d is even.

  For every $d \geq 3$, the graph G_d contains a unique maximum independent set, namely $I_0 \coloneqq V(G_d) \setminus V(G_{d-2})$. After the first step of the algorithm applied to G_d, all the vertices in I_0 have weight 1, and we are left with the graph G_{d-2} where every vertex has weight 0. By the induction hypothesis, the total weight of the fractional colouring returned by the algorithm is therefore $1 + \frac{(d-2)+3}{2} = \frac{d+3}{2}$.

□

1.4.2 Local fractional colourings of triangle-free graphs

Lemma 1.4.3 implies that if we are to prove a better bound than that given by Theorem 1.4.1, we need to use a probability distribution that gives a non-zero probability to non-maximum independent sets. Moreover, we need to be able to make a local analysis of the possible outcomes for the random independent set, independently from its exterior shape. Only few probability distributions have this property, which is referred to as the *spatial Markov property*. One of them is the hard-core distribution, which we use together with Lemma 1.3.1 in order to prove Theorem 1.1.3 and Theorem 1.1.5.

1.4.2.1 The hard-core model

Given a graph G, and a parameter $\lambda > 0$, the *hard-core model on G at fugacity λ* is a probability distribution on the independent sets $\mathcal{I}(G)$ (including the empty set) of G, where each $I \in \mathcal{I}(G)$ occurs with probability proportional to $\lambda^{|I|}$. Writing $\mathbf{I}$ for the random independent set, we have

$$\mathbb{P}[\mathbf{I} = I] = \frac{\lambda^{|I|}}{Z_G(\lambda)},$$

where the normalising term in the denominator is the *partition function* (or independence polynomial) $Z_G(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$.

Given a choice of $I \in \mathcal{I}(G)$, we say that a vertex $u \in V(G)$ is *uncovered* if $N(u) \cap I = \varnothing$, and that u is *occupied* if $u \in I$. Note that u can be occupied only if it is uncovered. We note the following useful facts (which appear verbatim in [33, 34]).

Fact 1 For every vertex $v \in V(G)$,

$$\mathbb{P}[v \in \mathbf{I} \mid v \text{ uncovered}] = \frac{\lambda}{1+\lambda}.$$

Fact 2 If G is triangle-free, then for every vertex $v \in V(G)$,

$$\mathbb{P}[v \text{ uncovered} \mid v \text{ has } j \text{ uncovered neighbours}] = (1+\lambda)^{-j}.$$

Fact 1 holds because, for each realisation J of $\mathbf{I} \setminus \{v\}$ such that $J \cap N(v) = \varnothing$ (i.e. v is uncovered), there are two possible realisations of $\mathbf{I}$, namely J and $J \cup \{v\}$. Now, $\mathbf{I}$ takes these values with probabilities proportional to $\lambda^{|J|}$ and $\lambda^{1+|J|}$ respectively, so for such J we have

$$\mathbb{P}[v \in \mathbf{I} \mid \mathbf{I} \setminus \{v\} = J] = \frac{\lambda^{1+|J|}}{\lambda^{|J|} + \lambda^{1+|J|}},$$

and the fact follows.

Fact 2 holds because, for each realisation J of $\mathbf{I} \setminus N(v)$ such that $|N(v) \setminus N(J)| = j$, every possible subset of $N(v) \setminus N(J)$ (the uncovered neighbours of v) extends J into a valid realisation of $\mathbf{I}$. Only the empty set extends J into a realisation of $\mathbf{I}$ where v is uncovered, so we have

$$\mathbb{P}[v \text{ uncovered} \mid \mathbf{I} \setminus N(v) = J] = \frac{\lambda^{|J|}}{\sum_{X \subseteq N(v) \setminus N(J)} \lambda^{|X|+|J|}} = (1+\lambda)^{-j}.$$

We apply these facts to give a lower bound on a linear combination of the probability that v is occupied and the expected number of occupied neighbours of v. This is a slight modification of the arguments of [33, 34], but here we focus on individual vertices, rather than averaging over a uniformly random choice of vertex.

Lemma 1.4.4. *Let G be a triangle-free graph and let $(\alpha_v)_{v \in V(G)}$ and $(\beta_v)_{v \in V(G)}$ be sequences of positive real numbers. Write $\mathbf{I}$ for a random independent set drawn from the hard-core model on*

G at fugacity $\lambda > 0$. Then for every $v \in V(G)$, we have

$$\alpha_v \mathbb{P}[v \in \mathbf{I}] + \beta_v \mathbb{E}[|N(v) \cap \mathbf{I}|] \geq \frac{\beta_v \lambda (\ln(\alpha_v/\beta_v) + \ln\ln(1+\lambda) + 1)}{(1+\lambda)\ln(1+\lambda)}.$$

Proof. Fix a vertex $v \in V(G)$ and let $\mathbf{Z}$ be the number of uncovered neighbours of v given the random independent set $\mathbf{I}$. By Facts 1 and 2, and Jensen's inequality we have

$$\begin{aligned}\mathbb{P}[v \in \mathbf{I}] = \frac{\lambda}{1+\lambda}\mathbb{P}[v \text{ uncovered}] &= \frac{\lambda}{1+\lambda}\mathbb{E}\left[(1+\lambda)^{-\mathbf{Z}}\right] \\ &\geq \frac{\lambda}{1+\lambda}(1+\lambda)^{-\mathbb{E}[\mathbf{Z}]}.\end{aligned}$$

Similarly, each of the $\mathbf{Z}$ uncovered neighbours of v is occupied with probability $\lambda/(1+\lambda)$ independently of the others (since G is triangle-free). Hence

$$\mathbb{E}[|N(v) \cap \mathbf{I}|] = \frac{\lambda}{1+\lambda}\mathbb{E}[\mathbf{Z}].$$

Minimising over the value of $\mathbb{E}[\mathbf{Z}] \in \mathbb{R}$, we have

$$\begin{aligned}\alpha_v \mathbb{P}[v \in \mathbf{I}] + \beta_v \mathbb{E}[|N(v) \cap \mathbf{I}|] &\geq \frac{\lambda}{1+\lambda}\Big(\alpha_v(1+\lambda)^{-\mathbb{E}[\mathbf{Z}]} + \beta_v \mathbb{E}[\mathbf{Z}]\Big) \\ &\geq \frac{\lambda}{1+\lambda}\min_{z\in\mathbb{R}}\Big\{\alpha_v(1+\lambda)^{-z} + \beta_v z\Big\}.\end{aligned}$$

When $\alpha_v, \lambda > 0$ this is a strictly convex function of z, with a minimum at

$$z = \frac{\ln(\alpha_v/\beta_v) + \ln\ln(1+\lambda)}{\ln(1+\lambda)},$$

from which the result follows. □

1.4.2.2 Proof of the theorem

We are now ready to prove Theorem 1.1.3.

Proof of Theorem 1.1.3. The method is to combine Lemmas 1.3.1 and 1.4.4 by carefully choosing $(\alpha_v)_{v\in V(G)}$ and $(\beta_v)_{v\in V(G)}$. For every $v \in V(G)$, we want to minimise $\alpha_v + \beta_v \deg(v)$ subject to the condition

$$\frac{\beta_v \lambda (\ln(\alpha_v/\beta_v) + \ln\ln(1+\lambda) + 1)}{(1+\lambda)\ln(1+\lambda)} = 1. \tag{1.7}$$

For then the hypothesis of Lemma 1.3.1 (with $\alpha_0(v) = \alpha_v$ and $\alpha_1(v) = \beta_v$ for all $v \in V(G)$) follows from the conclusion of Lemma 1.4.4. Given the assumptions on G, we can apply Lemma 1.4.4 to all induced subgraphs of G since they are also triangle-free, and the local parameters α_v and β_v are invariant under taking induced subgraphs.

Note that (1.7) is equivalent to

$$\alpha_v = \frac{\beta_v(1+\lambda)^{\frac{1+\lambda}{\beta_v\lambda}}}{e\ln(1+\lambda)},$$

so that $\alpha_v + \beta_v \deg(v)$ is a convex function of β_v with a minimum at

$$\beta_v = \frac{1+\lambda}{\lambda} \cdot \frac{\ln(1+\lambda)}{1+W\big(\deg(v)\ln(1+\lambda)\big)},$$

giving

$$\alpha_v + \beta_v \deg(v) = \frac{1+\lambda}{\lambda} \cdot e^{W(\deg(v)\ln(1+\lambda))}.$$

For any fixed λ this is an increasing function of $\deg(v)$. We take $\lambda = \varepsilon/2$, and we are done by Lemma 1.3.1 if we can show that there exists $\delta > 0$ such that for all $\deg(v) \geq \delta$ we have

$$(2/\varepsilon + 1) \cdot e^{W(\deg(v)\ln(1+\varepsilon/2))} \leq (1+\varepsilon)\frac{\deg(v)}{\ln\deg(v)}. \tag{1.8}$$

Let us first assume that $\deg(v)$ is at least some large enough multiple of $1/\varepsilon$ so that

$$e^{W(\deg(v)\ln(1+\varepsilon/2))} \leq \frac{(1+\varepsilon/2)\deg(v)\ln(1+\varepsilon/2)}{\ln(\deg(v)\ln(1+\varepsilon/2))},$$

where we used the fact that $W(x) = \ln x - \ln\ln x + o(1)$ as $x \to \infty$. Then by (1.8), it suffices to have

$$(2/\varepsilon+1)(1+\varepsilon/2)\ln(1+\varepsilon/2) \cdot \ln\deg(v) \leq (1+\varepsilon)\ln(\deg(v)\ln(1+\varepsilon/2)).$$

This last inequality holds for $\deg(v)$ large enough (as a function of ε) provided

$$(2/\varepsilon+1)(1+\varepsilon/2)\ln(1+\varepsilon/2) < 1+\varepsilon.$$

This is easily checked to hold true for small enough ε, namely $\varepsilon \leq 4$. □

1.4.3 A stronger bound for graphs of girth 7

In the proof of Theorem 1.1.3, we have used the hard-core distribution on the independent sets, including the non maximal ones. Moreover, the fugacity which let us derive the proof was $\lambda = \varepsilon/2 < 1$, which means that more weight was given to smaller independent sets in the obtained fractional colouring. This is quite counter-intuitive, since one would expect to be able to colour a graph more easily using larger independent sets, in particular only maximal ones.

By considering graphs of girth 7, we have been able to analyse the hard-core distribution restricted to the maximal independent sets. This restriction yields a certain amount of technicalities, since the local analysis of the independent sets is no longer entirely independent from the outside, because of the maximality condition. We now proceed with the analysis of the hard-core distribution on the maximal independent sets in order to prove Theorem 1.1.5.

Proof of Theorem 1.1.5. For any induced graph H of a graph G, we let $\mathbf{I}_H$ be a random independent set of H, drawn from $\mathcal{I}_{\max}(H)$ according to the hard-core distribution at fugacity $\lambda > 0$. This

means that

$$\forall I_0 \in \mathcal{I}_{\max}(H), \quad \mathbb{P}\left[\mathbf{I}_H = I_0\right] = \frac{\lambda^{|I_0|}}{\sum_{I \in \mathcal{I}_{\max}(H)} \lambda^{|I|}}.$$

From now on, let G be a graph of girth (at least) 7 and H an induced subgraph of G. If $v \in V(H)$ then

$$\mathcal{J}_{H,v} \coloneqq \left\{ I \setminus N_H^2[v] \;\middle|\; I \in \mathcal{I}_{\max}(H) \right\}.$$

Set $\mathbf{J}_v \coloneqq \mathbf{I}_H \setminus N_H^2[v] \in \mathcal{J}_{H,v}$, and $\mathbf{X}_i(v) \coloneqq \mathbf{I}_H \cap N_H^i(v)$. We establish the following claim.

Claim 1.4.5. *Using the hard-core distribution on $\mathcal{I}_{\max}(H)$ at fugacity $\lambda = 4$, it holds that for every vertex $v \in V(H)$, every set $J_0 \in \mathcal{J}_{H,v}$ and every integer $k \geq 4$,*

$$\frac{2^{k-3}+k}{k}\mathbb{E}\left[\mathbf{X}_0(v) \mid \mathbf{J}_v = J_0\right] + \frac{2}{k}\mathbb{E}\left[\mathbf{X}_1(v) \mid \mathbf{J}_v = J_0\right] \geq 1.$$

Proof. The subset $\mathbf{I}_H \setminus \mathbf{J}_v$ consists of an independent set of G contained in $\mathbf{W}_0 \coloneqq N_H^2[v] \setminus N(\mathbf{J}_v)$. It could hold that some vertices in $\mathbf{W}_0$ are forced to belong to this independent set, namely when one of their neighbours in $V(H) \setminus \mathbf{W}_0$ is not covered by $\mathbf{J}_v$. Let $\mathbf{W}_f$ be the set of those vertices, and W_0 be obtained by removing those vertices and their neighbours;

$$\begin{aligned} \mathbf{W}_f &\coloneqq \left\{ v \in \mathbf{W}_0 \mid (N(v) \setminus \mathbf{W}_0) \nsubseteq N(\mathbf{J}_v) \right\}, \\ \mathbf{W} &\coloneqq \mathbf{W}_0 \setminus N[\mathbf{W}_f]. \end{aligned}$$

Note that the subgraph of H induced by $\mathbf{W}$ is a forest composed of trees of maximum degree d and depth at most 2. In the rest of the proof, we may assume without loss of generality that $H[\mathbf{W}]$ is connected, and therefore a tree of maximum degree at most d and depth at most 2, since adding any disjoint component to $H[\mathbf{W}]$ would only increase the value of one or several of the $\mathbf{X}_i(v)$'s.

Let $J_0 \in \mathcal{J}_{H,v}$ be any fixed realisation of $\mathbf{J}_v$, and let us condition on the random event that $\mathbf{J}_v = J_0$. Let W, W_f and W_0 be the respective (deterministic) values of $\mathbf{W}$, $\mathbf{W}_f$ and $\mathbf{W}_0$ in this setting. It turns out that $\mathbf{I}_H \cap W$ is an independent set drawn according to the hard-core distribution at fugacity λ from $\mathcal{I}_{\max}(H[W])$.

To see this, let $I \in \mathcal{I}_{\max}(H)$ be any realisation of $\mathbf{I}_H$ such that $I \setminus N_H^2[v] = J_0$. Let $I_W \coloneqq I \cap W$, we show that $I_W \in \mathcal{I}_{\max}(H[W])$. First, we show that $W_f \subseteq I$. Indeed, if $u \in W_f$, then u has at least one neighbour $u' \in V(H) \setminus W_0$ that is uncovered by J_0. Because H is of girth 7, the vertex u is the only neighbour of u' in W_0. The maximality of I implies that u' must be covered by W_0, hence $u \in I$. Second, if there is a vertex $u \in W$ that is uncovered by I_W, then the maximality of I implies that u must be covered by $I \setminus W$, and hence either by J_0 or by W_f. None is possible since $N(J_0)$ and $N(W_f)$ are both disjoint from W by construction, so we have a contradiction.

On the other hand, given any set $I_W \in \mathcal{I}_{\max}(H[W])$, the set $J_0 \cup W_f \cup I_W$ is a valid realisation of $\mathbf{I}_H$. Indeed, any vertex in W is covered by I_W, and any vertex in $V(H) \setminus W$ is covered either by J_0 or by W_f, so $J_0 \cup W_f \cup I_W$ is a maximal independent set of H.

In conclusion, the set of realisations of $\mathbf{I}_H \cap W$ is exactly $\mathcal{I}_{\max}(H[W])$, and each such realisation I_W has a probability proportional to $\lambda^{|I_W|+|W_f|+|J_0|}$, and hence proportional to $\lambda^{|I_W|}$ since J_0 and W_f are fixed. This finishes to establish that $\mathbf{I}_H \cap W$ follows the hard-core distribution at fugacity λ on $\mathcal{I}_{\max}(H[W])$.

We let W_i be the set of vertices of W at distance i from v in W, for $i \in \{0,1,2\}$, and $W_{1,j}$ be

the subset of vertices of W_1 with j neighbours in W_2. We set $x_j \coloneqq |W_{1,j}|$. Thus

$$|W_1| = \sum_{j=0}^{d-1} x_j, \text{ and}$$
$$|W_2| = \sum_{j=1}^{d-1} j x_j.$$

Note that $W_1 \in \mathcal{I}_{\max}(H[W])$ and that $\mathbb{P}[\mathbf{I} \cap W = W_1]$ is proportional to $\lambda^{\sum_{j=0}^{d-1} x_j}$. In order to ease the following computations and verifications, we compute a weight $w(I)$ for each independent set $I \in \mathcal{I}_{\max}(H[W])$ that is proportional to $\mathbb{P}[\mathbf{I} \cap W = I]$, such that $w(W_1) = 1$.

There is exactly one maximal independent set I_0 that contains v, namely $I_0 \coloneqq \{v\} \cup W_2$, of normalised weight $w_0 \coloneqq \lambda^{1+\sum_{j\geq 0}(j-1)x_j}$. Every other maximal independent set $I \in \mathcal{I}_{\max}(W) \backslash \{I_0, W_1\}$ contains $W_{1,0}$. In addition, for every vertex $u \in W_1 \setminus W_{1,0}$, the set I either contains u or it contains all the neighbours of u in W_2. Therefore, it follows that if $x_0 > 0$, then the sum of the weights of these other independent sets is

$$T \coloneqq \sum_{i_1 \leq x_1, \ldots, i_{d-1} \leq x_{d-1}} \prod_{j=1}^{d-1} \binom{x_j}{i_j} \left(\lambda^{j-1}\right)^{i_j} = \prod_{j=1}^{d-1} \left(1 + \lambda^{j-1}\right)^{x_j}.$$

If $x_0 = 0$, then the sum of their weights is $T - \frac{w_0}{\lambda}$, since there is no independent set containing W_2 in whole and not v in this case.
We let $D \coloneqq T + w_0$ if $x_0 > 0$, and $D \coloneqq T + w_0\left(1 - \frac{1}{\lambda}\right)$ otherwise. It follows that

$$\mathbb{E}[\mathbf{X}_0] = \frac{w_0}{D}, \text{ and}$$
$$\mathbb{E}[\mathbf{X}_1] = \frac{T}{D}\left(x_0 + \sum_{j=1}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right).$$

There remains to check that, up to a good choice of λ, it holds that

$$\frac{2^{k-3}+k}{k}\mathbb{E}[X_0] + \frac{2}{k}\mathbb{E}[X_1] \geq 1.$$

This translates to

$$2^{k-3}w_0 + \frac{kw_0}{\lambda} \geq T\left(k - 2\sum_{j=1}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \qquad \text{if } x_0 = 0, \text{ and to}$$
$$2^{k-3}w_0 \geq T\left(k - 2x_0 - 2\sum_{j=1}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \qquad \text{if } x_0 \neq 0.$$

We use the two following facts.

Fact 1 For every positive integer j, the function $\lambda \mapsto \left(1 + \frac{1}{\lambda^{j-1}}\right)^{1+\lambda^{j-1}}$ is non increasing on $(0, +\infty)$, and in particular always bounded from above by $\frac{3125}{1024}$ when $\lambda \geq 4$ and $j \geq 2$,

and by $\left(1+\frac{1}{\lambda^{j_0-1}}\right)^{1+\lambda^{j_0-1}}$ when $\lambda \geq 1$ and $j \geq j_0$.

Fact 2 For all real numbers y_0, A and B with $A > 1$ and $B \geq 0$, the maximum of the function $f\colon y \mapsto A^y(B-2y)$ on the domain $[y_0, +\infty)$ is $f(y_0)$ when $B/2 - 1/\ln A \leq y_0$, and $\frac{2A^{B/2}}{e \ln A}$ otherwise.

Let us discriminate on the possible values for x_0, noting that $w_0 \geq \lambda^{1-x_0}$. When $x_0 = 0$, it suffices to show that

$$2^{k-3}\lambda + k \geq \prod_{j=1}^{d-1}\left(1+\frac{1}{\lambda^{j-1}}\right)^{x_j}\left(k - 2\sum_{j=1}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right), \tag{1.9}$$

and when $1 \leq x_0 \leq k/2$, that

$$2^{k-3}\lambda^{1-x_0} \geq \prod_{j=1}^{d-1}\left(1+\frac{1}{\lambda^{j-1}}\right)^{x_j}\left(k - 2x_0 - 2\sum_{j=1}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right). \tag{1.10}$$

Recall by definition that each value x_j is an integer. Note that the right side of inequality (1.9) and that of inequality (1.10) are both at most 0 if $x_1 \geq k - 2x_0$; so we may assume that $x_1 \in \{0, \ldots, k-2x_0-1\}$. Let us fix $\lambda = 4$, and prove the stronger statement that the right side of inequality (1.10), which we call R, is always at most 2^{k-2x_0-1}. This implies both (1.9) and (1.10). We define $y_j \coloneqq \dfrac{x_j}{1+\lambda^{j-1}}$, for every $j \in \{1, \ldots, d-1\}$.

- If $x_1 = k - 2x_0 - 1$, then

$$\begin{aligned}
\mathrm{R} &= 2^{k-2x_0-1}\cdot\prod_{j=2}^{d-1}\left(1+\frac{1}{\lambda^{j-1}}\right)^{x_j}\left(1-2\sum_{j=2}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right)\\
&= 2^{k-2x_0-1}\cdot\prod_{j=2}^{d-1}\left(1+\frac{1}{\lambda^{j-1}}\right)^{(1+\lambda^{j-1})y_j}\left(1-2\sum_{j=2}^{d-1}y_j\right)\\
&\leq 2^{k-2x_0-1}\cdot\prod_{j=2}^{d-1}\left(\frac{3125}{1024}\right)^{y_j}\left(1-2\sum_{j=2}^{d-1}y_j\right) && \text{by Fact 1}\\
&= 2^{k-2x_0-1}\cdot\left(\frac{3125}{1024}\right)^{y}(1-2y) && \text{where } y \coloneqq \sum_{j=2}^{d-1}y_j\\
&\leq 2^{k-2x_0-1}\cdot\max_{y\in\mathbb{R}^+}\left(\frac{3125}{1024}\right)^{y}(1-2y)\\
&= 2^{k-2x_0-1} && \text{by Fact 2.}
\end{aligned}$$

- If $x_1 = k - 2x_0 - 2$, then

$$\mathrm{R} = 2^{k-2x_0-2}\cdot\prod_{j=2}^{d-1}\left(1+\frac{1}{\lambda^{j-1}}\right)^{x_j}\left(2-2\sum_{j=2}^{d-1}\frac{x_j}{1+\lambda^{j-1}}\right).$$

If $x_j = 0$ for every $2 \le j \le d-1$, then $\mathrm{R} = 2^{k-2x_0-1}$. Let us now assume otherwise, and let $j_0 = \min\{j \mid x_j > 0\}$. So in particular $x_{j_0} \ge 1$ and $y_{j_0} \ge \frac{1}{1+\lambda^{j_0-1}}$. Then

$$\begin{aligned}
\mathrm{R} &= 2^{k-2x_0-2} \cdot \prod_{j=j_0}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{x_j} \left(2 - 2\sum_{j=j_0}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \\
&= 2^{k-2x_0-2} \cdot \prod_{j=j_0}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{\left(1+\lambda^{j-1}\right)y_j} \left(2 - 2\sum_{j=j_0}^{d-1} y_j\right) \\
&\le 2^{k-2x_0-2} \cdot \prod_{j=j_0}^{d-1} \left(1 + \frac{1}{\lambda^{j_0-1}}\right)^{\left(1+\lambda^{j_0-1}\right)y_j} \left(2 - 2\sum_{j=j_0}^{d-1} y_j\right) && \text{by Fact 1} \\
&= 2^{k-2x_0-2} \cdot \left(1 + \frac{1}{\lambda^{j_0-1}}\right)^{\left(1+\lambda^{j_0-1}\right)y} (2-2y) && \text{where } y \coloneqq \sum_{j=j_0}^{d-1} y_j \ge \frac{1}{1+\lambda^{j_0-1}} \\
&\le 2^{k-2x_0-2} \cdot \max_{y \ge \frac{1}{1+\lambda^{j_0-1}}} \left(1 + \frac{1}{\lambda^{j_0-1}}\right)^{\left(1+\lambda^{j_0-1}\right)y} (2-2y) \\
&= 2^{k-2x_0-1} && \text{by Fact 2.}
\end{aligned}$$

- If $x_1 \le k - 2x_0 - 3$, then

$$\begin{aligned}
\mathrm{R} &= 2^{x_1} \cdot \prod_{j=2}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{x_j} \left(k - 2x_0 - x_1 - 2\sum_{j=2}^{d-1} \frac{x_j}{1+\lambda^{j-1}}\right) \\
&= 2^{x_1} \cdot \prod_{j=2}^{d-1} \left(1 + \frac{1}{\lambda^{j-1}}\right)^{\left(1+\lambda^{j-1}\right)y_j} \left(k - 2x_0 - x_1 - 2\sum_{j=2}^{d-1} y_j\right) \\
&\le 2^{x_1} \cdot \prod_{j=2}^{d-1} \left(\frac{3125}{1024}\right)^{y_j} \left(k - 2x_0 - x_1 - 2\sum_{j=2}^{d-1} y_j\right) && \text{by Fact 1} \\
&= 2^{x_1} \cdot \left(\frac{3125}{1024}\right)^{y} (k - 2x_0 - x_1 - 2y) && \text{where } y \coloneqq \sum_{j=2}^{d-1} y_j \\
&\le 2^{x_1} \cdot \max_{y \in \mathbb{R}} \left(\frac{3125}{1024}\right)^{y} (k - 2x_0 - x_1 - 2y) \\
&= 2^{x_1} \cdot \frac{2\left(\frac{3125}{1024}\right)^{\frac{k-2x_0-x_1}{2}}}{e \ln\left(\frac{3125}{1024}\right)} && \text{by Fact 2} \\
&< 2^{k-2x_0-1}.
\end{aligned}$$

We have shown that when $\lambda = 4$,

$$\frac{2^{k-3}+k}{k}\mathbb{E}\left[\mathbf{X}_0\right] + \frac{2}{k}\mathbb{E}\left[\mathbf{X}_1\right] \geq 1.$$

□

We let $\lambda = 4$, and apply Lemma 1.3.1 with $\Big(\alpha_0(v), \alpha_1(v), \alpha_2(v)\Big) = \left(\frac{2^{k(v)-3}+k(v)}{k(v)}, \frac{2}{k(v)}, 0\right)$ for every vertex $v \in V(G)$, where $k(v)$ is chosen such that $\frac{2\deg(v)+2^{k-3}+k}{k}$ is minimised in $k = k(v)$, and is always at least 4 since $\deg(v)$ is a non-negative integer. The explicit value of $k(v)$ is

$$k(v) = \lfloor 4 + \log_2 \deg(v) - \log_2 \log_2 \deg(v) \rceil .$$

□

1.5 Bounds on the Hall ratio

We focus on establishing upper bounds on the Hall ratios of graphs with bounded maximum degree and girth. These bounds are obtained by using the uniform distribution on $\mathcal{I}_\alpha(G)$, for G in the considered class of graphs, into Lemma 1.3.2 or Lemma 1.3.3.

1.5.1 Structural analysis of a neighbourhood

We start by introducing some terminology.

Definition 1.5.1.

1. A *pattern of depth* r is any graph P given with a root vertex v such that

$$\forall u \in V(G), \quad \mathrm{dist}_G(u, v) \leq r.$$

   The *layer at depth* i of P is the set of vertices at distance i from its root vertex v.

2. A pattern P of depth r and root v is *d-regular* if all its vertices have degree exactly d, except maybe in the two deepest layers where the vertices have degree at most d.

Definition 1.5.2. Let P be a pattern of depth r and root v. Let $\mathbf{I}$ be a maximum independent set chosen uniformly at random from $\mathcal{I}_\alpha(P)$. We define $e_i(P) \coloneqq \mathbb{E}\left[|\mathbf{I} \cap N_P^i(v)|\right]$ for each $i \in [r+1]$.

1. The *constraint* associated to P is the pair $c(P) = (\mathbf{e}(P), n(P))$, where $\mathbf{e}(P) = (e_i(P))_{i=0}^r \in (\mathbb{Q}^+)^{r+1}$, and $n(P) \in \mathbb{N}$ is the *cardinality* of the constraint, which is the number of maximum independent sets of P. Most of the time, we only need to know the value of $\mathbf{e}(P)$, in which case we characterise the constraint $c(P) = (\mathbf{e}(P), n(P))$ only by $\mathbf{e}(P)$. The value of $n(P)$ is only needed for a technical reason, in order to be able to compute constraints inductively.

2. Given two constraints $\mathbf{e}, \mathbf{e}' \in (\mathbb{Q}^+)^{r+1}$, we say that $\mathbf{e}$ is *weaker* than $\mathbf{e}'$ if, for any vector $\alpha \in (\mathbb{Q}^+)^{r+1}$ it holds that

$$\alpha^\top \mathbf{e}' \geq 1 \quad \Longrightarrow \quad \alpha^\top \mathbf{e} \geq 1.$$

   If the above condition holds only for all vectors $\alpha \in (\mathbb{Q}^+)^{r+1}$ with non-increasing coordinates, then we say that $\mathbf{e}$ is *relatively weaker* than $\mathbf{e}'$.

Note that $\mathbf{e}$ is weaker than $\mathbf{e}'$ if and only if

$$\forall i \in [r+1], \quad e_i \geq e'_i,$$

and $\mathbf{e}$ is relatively weaker than $\mathbf{e}'$ if and only if

$$\forall i \in [r+1], \quad \sum_{j=0}^{i} e_j \geq \sum_{j=0}^{i} e'_j.$$

Remark 1.5.1. Let P be a pattern such that one of its vertices u is adjacent with some vertices $u_1, \ldots, u_k$ of degree 1 in the next layer, where $k \geq 2$. Then every maximum independent set of P contains $\{u_1, \ldots u_k\}$ and not u. Consequently, $\mathbf{e}(P)$ is weaker than $\mathbf{e}(P \setminus \{u_3, \ldots, u_k\})$ since, letting i be the distance of u_1 to the root of P, one has

$$e_j(P) = \begin{cases} e_j(P \setminus \{u_3, \ldots, u_k\}) & \text{if } j \neq i, \text{ and} \\ e_i(P \setminus \{u_3, \ldots, u_k\}) + (k-2) & \text{if j=i.} \end{cases}$$

1.5.2 Tree-like patterns

1.5.2.1 Rooting at a vertex

Fix a depth $r \geq 2$. Let G be a d-regular graph of girth at least $2r+2$, and let $\mathbf{I} \in \mathcal{I}_\alpha(G)$ be a maximum independent set drawn uniformly at random. For any fixed vertex v, we set $\mathbf{J} \coloneqq \mathbf{I} \setminus N^r[v]$, and $\mathbf{X_i}(v) \coloneqq \mathbf{I} \cap N^i(v)$, for each $i \in [r+1]$. Finally, we set $\mathbf{W} \coloneqq N^r[v] \setminus N(\mathbf{J})$. So $\mathbf{J}$ is the set of vertices in $\mathbf{I}$ at distance at least $r+1$ from v, and $\mathbf{W}$ is the set of vertices at distance at most r from v uncovered by $\mathbf{J}$. In particular, we know that $\mathbf{I} \cap N^r[v] \subseteq \mathbf{W}$.

Because $\mathbf{I}$ is a maximum independent set of G, it holds that $\mathbf{I} \cap N^r[v]$ is a maximum independent set of $G[\mathbf{W}]$. Conversely, if I_W is a maximum independent set of $G[\mathbf{W}]$, then $\mathbf{J} \cup I_W$ is a maximum independent set of G. Thus, for any independent set J of $G \setminus N^r[v]$, if one conditions on the fact that $\mathbf{J} = J$, then $\mathbf{I} \cap N^r[v]$ is a uniform random maximum independent set of $G[\mathbf{W}]$. The subgraph $G[\mathbf{W}]$ is a d-regular pattern of depth r with root vertex v, and since G has girth at least $2r+2$, it follows that $G[\mathbf{W}]$ is a tree. Let $\mathcal{T}_r(d)$ be the set of acyclic d-regular patterns of depth r.

We seek parameters $(\alpha_i)_{i \leq r}$ such that the inequality $\sum_{i=0}^{r} \alpha_i \mathbb{E}\left[|\mathbf{X}_i(v)|\right] \geq 1$ is satisfied regardless of the choice of v. To this end, it is enough to pick the rational numbers α_is in such a way that the inequality is satisfied in any tree $T \in \mathcal{T}_r(d)$, when v is the root vertex. In a more formal way, given any $T \in \mathcal{T}_r(d)$, the vector $\alpha = (\alpha_0, \ldots, \alpha_r)$ must be compatible with the constraint $\mathbf{e}(T)$, that is, $\alpha^\top \mathbf{e}(T) \geq 1$ for each $T \in \mathcal{T}_r(d)$.

An application of Lemma 1.3.2 then lets us conclude that the desired bound is the solution to the following linear program.

$$\frac{|G|}{\alpha(G)} \leq \min \quad \alpha_0 + \sum_{i=1}^{r} \alpha_i d(d-1)^{i-1} \tag{1.11}$$
$$\text{such that} \begin{cases} \forall T \in \mathcal{T}_r(d), & \sum_{i=0}^{r} \alpha_i e_i(T) \geq 1 \\ \forall i \leq r, & \alpha_i \geq 0. \end{cases}$$

The end of the proof is made by computer generation of $\mathcal{T}_r(d)$, in order to generate the desired linear program, which is then solved again by computer computation. For the sake of illustration, we give a complete human proof of the case where $r = 2$ and $d = 3$. There are 10 trees in $\mathcal{T}_2(3)$. One can easily compute the constraint $(e_0(T), e_1(T), e_2(T))$ for each $T \in \mathcal{T}_2(3)$; they are depicted in Figure 1.5.1. Note that constraints $\mathbf{e}_8$, $\mathbf{e}_9$ and $\mathbf{e}_{10}$ are weaker than constraint $\mathbf{e}_7$, so we may disregard these constraints in the linear program to solve. Note also that constraint $\mathbf{e}_0$ is relatively weaker than constraint $\mathbf{e}_1$, and so may be disregarded as well, provided that the solution of the linear program is attained by a vector α with non-increasing coordinates, which will have to be checked. The linear program to solve is therefore the following.

$$\begin{array}{ll} \text{minimise} & \alpha_0 + 3\alpha_1 + 6\alpha_2 \\ \text{such that} & \begin{cases} 5/2 \cdot \alpha_1 + 1/2 \cdot \alpha_2 & \geq 1 \\ 2\alpha_1 + 2\alpha_2 & \geq 1 \\ 1/5 \cdot \alpha_0 + 8/5 \cdot \alpha_1 + 6/5 \cdot \alpha_2 & \geq 1 \\ 1/3 \cdot \alpha_0 + \alpha_1 + 8/3 \cdot \alpha_2 & \geq 1 \\ 1/2 \cdot \alpha_0 + 1/2 \cdot \alpha_1 + 4\alpha_2 & \geq 1 \\ \alpha_0 + 3\alpha_2 & \geq 1 \\ \alpha_0, \alpha_1, \alpha_2 \geq 0. \end{cases} \end{array}$$

The solution of this linear program is $\frac{85}{31} \approx 2.741935$, attained by $\alpha = \left(\frac{19}{31}, \frac{14}{31}, \frac{4}{31}\right)$, which indeed has non-increasing coordinates. This is an upper bound on $\rho(3, 6)$, though we prove a stronger one through a more involved computation in Section 1.5.3.2.

1.5.2.2 Inductive computation of the vectors $\mathbf{e}(T)$

To compute $\mathbf{e}(T)$ for each $T \in \mathcal{T}_r(d)$, one can enumerate all the maximum independent sets of T and average the size of their intersection with each layer of T. For general graphs, there might be no better way of doing so, however the case of $\mathcal{T}_r(d)$ can be treated inductively by a standard approach: we distinguish between the maximum independent sets that contain the root and those that do not. We introduce the following notation.

Definition 1.5.3. Let $c = (\mathbf{e}, n)$ and $c' = (\mathbf{e}', n')$ be two constraints.

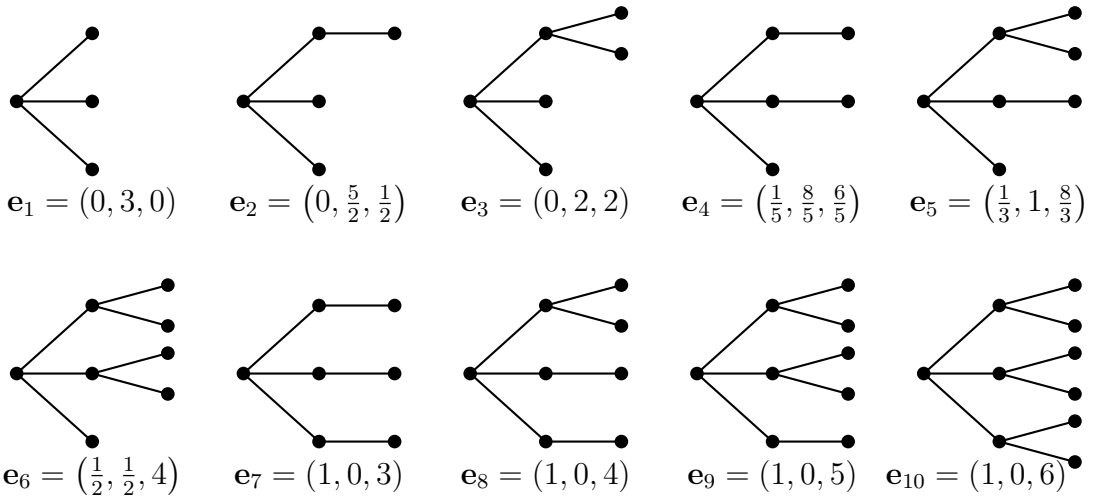


Figure 1.5.1: An enumeration of $\mathbf{e}(T)$ for all trees $T \in \mathcal{T}_2(3)$.

1. The operation $\vee$ on c and c' returns the constraint

$$c \vee c' \coloneqq \begin{cases} \left(\dfrac{n}{n+n'}\mathbf{e} + \dfrac{n'}{n+n'}\mathbf{e}', n+n'\right) & \text{if } \|\mathbf{e}\|_1 = \|\mathbf{e}'\|_1, \\ c & \text{if } \|\mathbf{e}\|_1 > \|\mathbf{e}'\|_1, \\ c' & \text{if } \|\mathbf{e}\|_1 < \|\mathbf{e}'\|_1. \end{cases}$$

2. The operation $\oplus$ on c and c' returns the constraint $c \oplus c' \coloneqq \big(\mathbf{e} + \mathbf{e}', n \cdot n'\big)$.

For a given tree $T \in \mathcal{T}_r(d)$ with root v, let $c_0(T)$ be the constraint associated to T where v is forced (we restrict to the maximum independent sets which contain v when computing the constraint $c_0(T)$), and $c_1(T)$ be the constraint associated to T where v is forbidden. It readily follows from Definition 1.5.3 that

$$c(T) = c_0(T) \vee c_1(T).$$

If $(T_i)_{i\in[d]}$ are the subtrees of T rooted at the children of the root v (some of which might be empty). It holds that

$$c_0(T) = \big((0,\mathbf{e}), n\big) \qquad \text{where } (\mathbf{e}, n) = \bigoplus_{i\in[d]} c(T_i), \text{ and}$$

$$c_1(T) = \big((1,\mathbf{e}), n\big) \qquad \text{where } (\mathbf{e}, n) = \bigoplus_{i\in[d]} c_0(T_i).$$

We thus obtain an inductive way of computing $\mathbf{e}(T)$ by using the following initial values.

$$c_0(\varnothing) \coloneqq \big((0),1\big) \qquad c_1(\varnothing) \coloneqq \big((0),0\big)$$
$$c_0(\{v\}) \coloneqq \big((0),1\big) \qquad c_1(\{v\}) \coloneqq \big((1),1\big)$$

Following the enumeration of the vectors $\mathbf{e}(T)$ for $T \in \mathcal{T}_r(d)$ described in Section 1.5.2.2, one obtained the following statement by computer calculus.

Lemma 1.5.1. *The solution to the linear program* (1.11) *is*

$$\mathcal{T}_3(3)\colon \quad \frac{5849}{2228} \approx 2.625224 \qquad \text{with } \alpha = \left(\frac{953}{2228}, \frac{162}{557}, \frac{81}{557}, \frac{21}{557}\right),$$

$$\mathcal{T}_4(3)\colon \quad \frac{2098873192}{820777797} \approx 2.557176 \qquad \text{with } \alpha = \left(\frac{225822361}{820777797}, \frac{18575757}{91197533}, \frac{10597368}{91197533}, \frac{5054976}{91197533}, \frac{1172732}{91197533}\right),$$

$$\mathcal{T}_5(3)\colon \quad \frac{29727802051155412}{11841961450578397} \approx 2.510378 \quad \text{with } \alpha = \left(\frac{3027359065168972}{11841961450578397}, \frac{2216425114872980}{11841961450578397}, \frac{2224040336719575}{23683922901156794}, \frac{2026654050681425}{47367845802313588}, \frac{403660478424775}{23683922901156794}, \frac{51149140376400}{11841961450578397}\right),$$

$$\mathcal{T}_3(4)\colon \quad \frac{7083927}{2331392} \approx 3.038497 \qquad \text{with } \alpha = \left(\frac{123345}{333056}, \frac{68295}{291424}, \frac{12283}{145712}, \frac{2911}{145712}\right),$$

$$\mathcal{T}_4(4)\colon \quad 3 \qquad \text{with } \alpha = \left(\frac{7}{43}, \frac{6}{43}, \frac{19}{258}, \frac{7}{258}, \frac{1}{258}\right),$$

$$\mathcal{T}_2(5)\colon \quad \frac{69}{19} \approx 3.631579 \qquad \text{with } \alpha = \left(\frac{37}{57}, \frac{6}{19}, \frac{4}{57}\right),$$

$$\mathcal{T}_3(5)\colon \quad \frac{7}{2} = 3.5 \qquad \text{with } \alpha = \left(\frac{77}{282}, \frac{25}{141}, \frac{17}{282}, \frac{2}{141}\right).$$

1.5.2.3 Rooting in an edge

Definition 1.5.1 can be extended to a pattern with a root-edge instead of a root-vertex. The distance in a pattern P between a vertex w and an edge uv is defined to be $\min\{\mathrm{dist}_P(w,u), \mathrm{dist}_P(w,v)\}$. The depth of a pattern P rooted in an edge e is then the largest distance between e and a vertex in P. It is then possible to follow the same analysis as in Section 1.5.2.1 with edge-rooted patterns: in order for the edge-rooted pattern of depth r to always be a tree, the graph G must have girth at least $2r+3$. Let $\mathcal{T}'_r(d)$ be the set of acyclic edge-rooted d-regular patterns of depth r. By Lemma 1.3.3, the linear program to solve is now the following.

$$\frac{|G|}{\alpha(G)} \leq \min \quad 2\sum_{i=0}^{r} \alpha_i (d-1)^i \tag{1.12}$$

$$\text{such that} \begin{cases} \forall T \in \mathcal{T}'_r(d), & \displaystyle\sum_{i=0}^{r} \alpha_i e_i(T) \geq 1 \\ \forall i \leq r, & \alpha_i \geq 0. \end{cases}$$

For a given tree $T \in \mathcal{T}'_r(d)$ rooted in $e = uv$, it is possible to compute $\mathbf{e}(T)$ using the constraints associated to vertex-rooted trees. If T_u and T_v are the subtrees of T respectively rooted at u and

at v, then it readily follows from Definition 1.5.3 that

$$c(T) = \Big(c_0(T_u) \oplus c_0(T_v)\Big) \vee \Big(c_0(T_u) \oplus c_1(T_v)\Big) \vee \Big(c_1(T_u) \oplus c_0(T_v)\Big). \tag{1.13}$$

Following the enumeration of the vectors $\mathbf{e}(T)$ for $T \in \mathcal{T}'_r(d)$ described earlier, the next statement is obtained by computer calculus.

Lemma 1.5.2. *The solution to the linear program* (1.12) *is*

$$\begin{aligned}
&\mathcal{T}'_2(3)\colon && \frac{30}{11} \approx 2.72727 && \text{with } \alpha = \left(\frac{1}{2}, \frac{13}{44}, \frac{3}{44}\right),\\
&\mathcal{T}'_3(3)\colon && \frac{125}{48} \approx 2.604167 && \text{with } \alpha = \left(\frac{11}{32}, \frac{5}{24}, \frac{3}{32}, \frac{1}{48}\right),\\
&\mathcal{T}'_4(3)\colon && \frac{14147193}{5571665} \approx 2.539132 && \text{with } \alpha = \left(\frac{98057}{506515}, \frac{159348}{1114333}, \frac{3688469}{44573320}, \frac{1752117}{44573320}, \frac{402569}{44573320}\right),\\
&\mathcal{T}'_2(4)\colon && \frac{41}{13} \approx 3.153846 && \text{with } \alpha = \left(\frac{11}{26}, \frac{3}{13}, \frac{2}{39}\right),\\
&\mathcal{T}'_3(4)\colon && \frac{127937}{42400} \approx 3.017382 && \text{with } \alpha = \left(\frac{5539}{16960}, \frac{1737}{10600}, \frac{257}{5300}, \frac{399}{42400}\right),\\
&\mathcal{T}'_2(5)\colon && \frac{18}{5} = 3.6 && \text{with } \alpha = \left(\frac{17}{45}, \frac{8}{45}, \frac{2}{45}\right).
\end{aligned}$$

The bounds obtained in Lemma 1.5.2 are valid for graphs of girth at least $2r+3$. It turns out that the same bounds, with the same α, remain valid for graphs of girth $2r+2 = 6$, when $r = 2$ and $d \in \{3, 4\}$. We were not able to check this for higher values of r or d, but we propose the following conjecture which would explain and generalise this phenomenon.

Conjecture 1.5.1. *Let P be a d-regular edge-rooted pattern of depth r and of girth $2r+2$. Then the constraint $\mathbf{e}(P)$ is weaker than some convex combination of constraints $\mathbf{e}(T)$ with $T \in \mathcal{T}'_r(d)$. More formally, there exist $T_1, \dots, T_m \in \mathcal{T}'_r(d)$ and $\lambda_1, \dots, \lambda_m \in [0,1]$ with $\sum_{i=1}^m \lambda_i = 1$ such that for any $\alpha \in (\mathbb{Q}^+)^{r+1}$,*

$$\alpha^\top \left(\sum_{i=1}^m \lambda_i \mathbf{e}(T_i)\right) \geq 1 \quad \Longrightarrow \quad \alpha^\top \mathbf{e}(P) \geq 1.$$

1.5.3 More complicated patterns

1.5.3.1 Rooting at a vertex

Let us fix a depth $r \geq 2$. Let G be a d-regular graph of girth $g \leq 2r+1$. We repeat the same analysis as in Section 1.5.2.1: we end up having to find a vector $\alpha \in \mathbb{Q}^{r+1}$ compatible with all the constraints generated by vertex-rooted d-regular patterns of depth r and girth g. Letting $\mathcal{P}_r(d, g)$ be the set of such patterns, we thus want that

$$\forall P \in \mathcal{P}_r(d, g), \quad \alpha^\top \mathbf{e}(P) \geq 1.$$

In this setting, we could do no better than performing an exhaustive enumeration of every possible pattern $P \in \mathcal{P}_r(d, g)$, and computing the associated constraint $\mathbf{e}(P)$ through an exhaustive

enumeration of $\mathcal{I}_\alpha(P)$. The complexity of such a process grows fast, and we considered only depth $r \leq 2$ and degree $d \leq 4$. Since the largest value of the Hall ratio over the class of 3-regular graphs of girth 4 or 5 is known to be $\frac{14}{5} = 2.8$, and the one of 4-regular graphs of girth 4 is known to be $\frac{13}{4} = 3.25$, the only open value in these settings is for the class of 4-regular graphs of girth 5. Unfortunately, this method is not powerful enough to prove an upper bound lower than $\frac{13}{4}$, the obtained bound for $\mathcal{P}_2(4,5)$ being $\frac{82}{25} = 3.28$. It is more interesting to root the patterns in an edge.

1.5.3.2 Rooting in an edge

Similarly, we define $\mathcal{P}'_r(d,g)$ to be the set of edge-rooted d-regular patterns of girth g. For fixed r and g, we seek for the solution of the following linear program.

$$\frac{|G|}{\alpha(G)} \leq \min \quad 2\sum_{i=0}^{r} \alpha_i (d-1)^i \tag{1.14}$$
$$\text{such that} \begin{cases} \forall P \in \mathcal{P}'_r(d,g), & \sum_{i=0}^{r} \alpha_i e_i(P) \geq 1 \\ \forall i \leq r, & \alpha_i \geq 0. \end{cases}$$

Again, our computations were limited to the cases where $r \leq 2$ and $d \leq 4$. However, we managed to prove improved bounds for girth 6 when $d \in \{3,4\}$, which seems to support Conjecture 1.5.1.

Lemma 1.5.3. *The solution to the linear program* (1.14) *is*

$$\mathcal{P}'_2(3,6): \quad \frac{30}{11} \approx 2.72727 \quad \text{with } \alpha = \left(\frac{1}{2}, \frac{13}{44}, \frac{3}{44}\right),$$
$$\mathcal{P}'_2(4,6): \quad \frac{41}{13} \approx 3.153846 \quad \text{with } \alpha = \left(\frac{11}{26}, \frac{3}{13}, \frac{2}{39}\right).$$

1.6 Graphs with few triangles

We next discuss our main result, Theorem 1.1.11, in slightly deeper context. We in fact show a sharp, general lower bound on occupancy fraction for graphs of bounded local triangle fraction, to which Theorem 1.1.11(i) is corollary.

Theorem 1.6.1. *Suppose T, Δ, λ satisfy, as $\Delta \to \infty$, that*

$$\Delta \ln(1+\lambda) = \omega(1) \quad \text{and} \quad \frac{2T \ln(1+\lambda)^2}{W(\Delta \ln(1+\lambda))} = o(1).$$

In any graph G of maximum degree Δ in which every vertex is contained in at most T triangles, writing $\mathbf{I}$ for the random independent set from the hard-core model on G at fugacity λ, the occupancy fraction satisfies

$$\frac{\mathbb{E}[|\mathbf{I}|]}{n(G)} \geq (1+o(1)) \frac{\lambda}{1+\lambda} \frac{W(\Delta \ln(1+\lambda))}{\Delta \ln(1+\lambda)}.$$

This may be viewed as generalising [34, Thm. 3]. By monotonicity of the occupancy fraction in λ (see e.g. [34, Prop. 1]), and the fact that a uniform choice from $\mathcal{I}(G)$ is a hard-core distribution with $\lambda = 1$, Theorem 1.1.11(i) follows from Theorem 1.6.1 with $\lambda = \min\left\{\varepsilon, 1/\sqrt{T}\right\}$. Theorem 1.6.1 is asymptotically optimal. More specifically, in [34] it was shown how the analysis of [14] yields that, for any fugacity $\lambda = o(1)$ in the range allowed in Theorem 1.6.1, the random Δ-regular graph (conditioned to be triangle-free) with high probability has occupancy fraction asymptotically equal to the bound in Theorem 1.6.1. In Section 1.6.3, we show our methods break down for λ outside this range, so that new ideas are needed for any improvement in the bound for larger λ.

Moreover, the asymptotic bounds of Theorems 1.1.10 and 1.1.11 cannot be improved, for any valid choice of f as a function of Δ, by more than a factor of between 2 and 4. This limits the hypothetical range of λ in Theorem 1.6.1. This follows by considering largest independent sets in a random regular construction or in a suitable blow-up of that construction [108]; see Section 1.6.3.

Observe that Theorem 1.1.11(ii) trivially fails if we average T on all the vertices of the graph, since the presence of a $(\Delta+1)$-vertex clique is then no longer excluded. It is however still possible to have a version of the theorem with a maximum average value of T taken over all subgraphs. In any case, Theorems 1.1.10 and 1.1.11 may appear incompatible, since the former has a global condition, while the latter has a local one. Nevertheless, either assertion in Theorem 1.1.11 is indeed (strictly) stronger.

Proof of Theorem 1.1.10. Without loss of generality we may assume that $\varepsilon > 0$ is small enough so that $(1-\varepsilon^2)^2(1-\varepsilon/2) \geq 1-\varepsilon$. Let G be a graph on n vertices of maximum degree Δ with at most $Tn/3$ triangles. Call $v \in V(G)$ *bad* if the number of triangles of G that contain v is greater than $\varepsilon^{-2}T$. Let B be the set of all bad vertices. Note that $Tn > |B|\,\varepsilon^{-2}T$ and so $|B| < \varepsilon^2 n$. Let H be the subgraph of G induced by the subset $V(G) \setminus B$. Then H is a graph of maximum degree Δ on at least $(1-\varepsilon^2)\,n$ vertices such that each vertex is contained in at most $\varepsilon^{-2}T$ triangles. Provided $\Delta/\sqrt{T}$ is large enough, either of (i) and (ii) in Theorem 1.1.11 implies that H, and thus G, contains an independent set of size

$$\begin{aligned}\left(1-\varepsilon^2\right)\frac{(1-\varepsilon^2)\,n\ln\frac{\Delta}{\sqrt{\varepsilon^{-2}T}}}{\Delta} &\geq \left(1-\varepsilon^2\right)^2\frac{n\left(\ln\frac{\Delta}{\sqrt{T}}+\ln\varepsilon\right)}{\Delta}\\ &\geq \left(1-\varepsilon^2\right)^2(1-\varepsilon/2)\,\frac{n\ln\frac{\Delta}{\sqrt{T}}}{\Delta}\\ &\geq (1-\varepsilon)\frac{n\ln\frac{\Delta}{\sqrt{T}}}{\Delta},\end{aligned}$$

where on the second line we used that $\ln\varepsilon \geq -\frac{\varepsilon}{2}\ln\frac{\Delta}{\sqrt{T}}$ when $\frac{\Delta}{\sqrt{T}}$ is large enough. □

1.6.1 Another analysis of the hard-core model

A crucial ingredient in the proofs is an occupancy guarantee from the hard-core model, which we establish in Lemma 1.6.2 below. This refines an analysis given in [34]. Given G, $I \in \mathcal{I}(G)$, and $v \in V(G)$, let us call a neighbour $u \in N(v)$ of v *externally uncovered* by I if $u \notin N(I \setminus N(v))$.

Lemma 1.6.2. *Let G be a graph and $\lambda > 0$. Let $\mathbf{I}$ be an independent set drawn from the hard-core model at fugacity λ on G.*

(i) *For every $v \in V(G)$, writing $\mathbf{F}_v$ for the subgraph of G induced by the neighbours of v externally uncovered by $\mathbf{I}$,*

$$\mathbb{P}[v \in \mathbf{I}] \geq \frac{\lambda}{1+\lambda}(1+\lambda)^{-\mathbb{E}[|V(\mathbf{F}_v)|]}.$$

(ii) *Moreover,*

$$\mathbb{E}[|\mathbf{I}|] \geq \frac{\lambda}{1+\lambda} n(G)(1+\lambda)^{-\operatorname{ad}(G)}.$$

Proof. The first part follows from two applications of the spatial Markov property of the hard-core model. First, we have

$$\mathbb{P}[v \in \mathbf{I}] = \frac{\lambda}{1+\lambda}\mathbb{P}[\mathbf{I} \cap N(v) = \varnothing],$$

because conditioned on a value $\mathbf{I} \setminus \{v\} = J$ such that $J \cap N(v) = \varnothing$ there are two realisations of $\mathbf{I}$, namely J and $J \cup \{v\}$, giving

$$\mathbb{P}[v \in \mathbf{I} \mid J] = \frac{\lambda^{|J|+1}}{\lambda^{|J|} + \lambda^{|J|+1}} = \frac{\lambda}{1+\lambda},$$

and conditioned on $\mathbf{I} \setminus \{v\} = J$ such that $J \cap N(v) \neq \varnothing$, v cannot be in $\mathbf{I}$.

Second, the spatial Markov property gives that $\mathbf{I} \cap N(v)$ is a random independent set drawn from the hard-core model on $\mathbf{F}_v$. Then $\mathbf{I} \cap N(v) = \varnothing$ if and only if this random independent set in $\mathbf{F}_v$ is empty. It follows that

$$\mathbb{P}[\mathbf{I} \cap N(v) = \varnothing] = \mathbb{E}\left[\frac{1}{Z_{\mathbf{F}_v}(\lambda)}\right] \geq \mathbb{E}\left[(1+\lambda)^{-n(\mathbf{F}_v)}\right] \geq (1+\lambda)^{-\mathbb{E}[n(\mathbf{F}_v)]},$$

since the graph on $n(\mathbf{F}_v)$ vertices with largest partition function is the graph with no edges, and by convexity. This completes the proof of (i).

By the fact that $|V(\mathbf{F}_v)| \leq \deg(v)$ we also have for all $v \in V(G)$ that

$$\mathbb{P}[v \in \mathbf{I}] \geq \frac{\lambda}{1+\lambda}(1+\lambda)^{-\deg(v)}.$$

Then (ii) follows by convexity:

$$\begin{aligned}\mathbb{E}[|\mathbf{I}|] = \sum_{v \in V(G)} \mathbb{P}[v \in \mathbf{I}] &\geq \frac{\lambda}{1+\lambda} \sum_{v \in V(G)} (1+\lambda)^{-\deg(v)} \\ &\geq \frac{\lambda}{1+\lambda} n(G)(1+\lambda)^{-\operatorname{ad}(G)}.\end{aligned}$$

□

Lemma 1.6.3. *Let G be a graph of maximum degree Δ where every vertex is contained in at most T triangles, and $\lambda > 0$ be a given real. Let $\mathbf{I}$ be an independent set drawn from the hard-core model at fugacity λ on G. Then, for every $v \in V(G)$,*

$$\frac{\mathbb{E}[|\mathbf{I}|]}{n(G)} \geq \min_{z \geq 0} \quad \max\left\{\frac{\lambda}{1+\lambda}(1+\lambda)^{-z}, \frac{\lambda}{1+\lambda}\frac{z}{\Delta}(1+\lambda)^{-\frac{2T}{z}}\right\}.$$

Proof. Note that

$$\sum_{v\in V(G)} \mathbb{E}\left[|N(v)\cap \mathbf{I}|\right] = \sum_{v\in V(G)}\sum_{u\in N(v)} \mathbb{P}\left[u\in\mathbf{I}\right] = \sum_{u\in V(G)} \deg(u)\mathbb{P}\left[u\in\mathbf{I}\right]$$
$$\leq \Delta \sum_{u\in V(G)} \mathbb{P}\left[u\in\mathbf{I}\right] = \Delta\mathbb{E}\left[|\mathbf{I}|\right].$$

Therefore, we may bound $\mathbb{E}\left[|\mathbf{I}|\right]$ by above in two distinct ways:

$$\mathbb{E}\left[|\mathbf{I}|\right] \geq \frac{n(G)}{\Delta}\frac{\lambda}{1+\lambda} z(1+\lambda)^{-\frac{2T}{z}} \quad \text{and}$$
$$\mathbb{E}\left[|\mathbf{I}|\right] \geq n(G)\frac{\lambda}{1+\lambda}(1+\lambda)^{-z},$$

where z is the expected number of externally uncovered neighbours of a uniformly random vertex. The result follows. □

Lemma 1.6.4. *Let G be a graph. We denote by $t(v)$ the number of triangles in G containing v, for every $v\in V(G)$. We let $\lambda>0$, and $\alpha(v),\beta(v)>0$ be associated to every $v\in V(G)$. Let H be any induced subgraph of G, and $\mathbf{I}_H$ be an independent set drawn from the hard-core model at fugacity λ on H. Then, for any $v\in V(H)$, it holds that*

$$\alpha(v)\mathbb{P}\left[v\in\mathbf{I}_H\right] + \beta(v)\mathbb{E}\left[|N_H(v)\cap\mathbf{I}_H|\right] \geq \frac{\lambda}{1+\lambda}\left(\min_{z\geq 0} \quad \alpha(v)(1+\lambda)^{-z} + \beta(v) z(1+\lambda)^{-\frac{2t(v)}{z}}\right).$$

Proof. Write $\mathbf{F}_v$ for the graph induced by the neighbours of v externally uncovered by $\mathbf{I}$ in H, and $z_v = \mathbb{E}\left[n(\mathbf{F}_v)\right]$. By Lemma 1.6.2(i) we have

$$\mathbb{P}\left[v\in\mathbf{I}\right] \geq \frac{\lambda}{1+\lambda}(1+\lambda)^{-z_v}.$$

For the other term, we apply Lemma 1.6.2(ii) to the graph $\mathbf{F}_v$, for which by assumption $\mathrm{ad}(\mathbf{F}_v)\leq \frac{2t(v)}{n(\mathbf{F}_v)}$. If $\mathbf{J}$ is an independent set drawn from the hard-core model at fugacity λ on $\mathbf{F}_v$, then by convexity

$$\mathbb{E}\left[|N(v)\cap\mathbf{I}|\right] = \mathbb{E}\left[|\mathbf{J}|\right] \geq \frac{\lambda}{1+\lambda}\mathbb{E}\left[n(\mathbf{F}_v)(1+\lambda)^{-\frac{2t(v)}{n(\mathbf{F}_v)}}\right]$$
$$\geq \frac{\lambda}{1+\lambda} z_v (1+\lambda)^{-\frac{2t(v)}{z_v}},$$

so the result follows by minimising on all the possible values of z_v. □

1.6.2 The proofs

Proof of Theorem 1.6.1. We optimise the lower bound in Lemma 1.6.3. As the first argument of the maximisation in the lower bound of the lemma is increasing in z while the second is decreasing, the minimum occurs where these two arguments are equal: at $z_0\in\mathbb{R}^+$ satisfying

$$(1+\lambda)^{-z_0} = \frac{z_0}{\Delta}(1+\lambda)^{-\frac{2T}{z_0}}. \tag{1.15}$$

Let us now give an estimate of z_0.

$$\begin{aligned} z_0 e^{z_0 \ln(1+\lambda)} &= \Delta e^{\frac{2T\ln(1+\lambda)}{z_0}} \\ z_0 \ln(1+\lambda) &= W\Big(\Delta \ln(1+\lambda) e^{\frac{2T\ln(1+\lambda)}{z_0}}\Big) \\ &\geq W(\Delta \ln(1+\lambda)), \qquad \text{since } e^{\frac{2T\ln(1+\lambda)}{z_0}} \geq 1 \text{ and } W \text{ is increasing.} \end{aligned}$$

We conclude that

$$W(\Delta \ln(1+\lambda)) \leq z_0 \ln(1+\lambda) \leq W\left(\Delta \ln(1+\lambda) e^{\frac{2T\ln(1+\lambda)^2}{W(\Delta\ln(1+\lambda))}}\right).$$

So by using the assumptions of the theorem,

$$z_0 \ln(1+\lambda) \leq W((1+o(1))\Delta \ln(1+\lambda)) = W(\Delta \ln(1+\lambda)) + o(1).$$

We now use Lemma 1.6.3 to conclude that

$$\begin{aligned} \frac{\mathbb{E}\left[|\mathbf{I}|\right]}{n(G)} &\geq \frac{\lambda}{\lambda+1}(1+\lambda)^{-z_0} \\ &\geq \frac{\lambda}{1+\lambda} e^{-W(\Delta\ln(1+\lambda))+o(1)} \\ &= (1+o(1))\frac{\lambda}{1+\lambda}\frac{W(\Delta\ln(1+\lambda))}{\Delta\ln(1+\lambda)}. \end{aligned}$$

□

Proof of Theorem 1.1.11(ii). Supposing that $\lambda > 0$ is fixed, write for every vertex $v \in V(G)$

$$g_v(x) = \frac{\lambda}{1+\lambda}\Big(\alpha(v)(1+\lambda)^{-x} + \beta(v)x(1+\lambda)^{-\frac{2t(v)}{x}}\Big).$$

By Lemma 1.6.4 and Lemma 1.3.1, we can find the desired local fractional colouring, of total weight at most $\max_{v\in V} \alpha(v) + \beta(v)\deg(v)$, provided $g_v(x) \geq 1$ for all $x \geq 0$ and for every vertex $v \in V(G)$.

It is easy to verify that with $\alpha(v), \beta(v), t(v), \lambda > 0$ the function g_v is strictly convex, so the minimum of $g_v(x)$ occurs when $g_v'(x) = 0$, or

$$(1+\lambda)^{-x} = \frac{\beta(v)}{\alpha(v)}\left(\frac{1}{\ln(1+\lambda)} + \frac{2t(v)}{x}\right)(1+\lambda)^{-\frac{2t(v)}{x}}.$$

Let z_0 satisfy

$$(1+\lambda)^{-z_0} = \frac{z_0}{\deg(v)}(1+\lambda)^{-\frac{2t(v)}{z_0}}.$$

Then by choosing

$$\frac{\alpha(v)}{\beta(v)} = \frac{\deg(v)}{z_0}\left(\frac{1}{\ln(1+\lambda)} + \frac{2t(v)}{z_0}\right), \tag{1.16}$$

the minimum of g occurs at z_0. Now the equations $g(z_0) = 1$ and (1.16) give us values of $\alpha(v)$ and $\beta(v)$ in terms of λ, $d(v)$, and $t(v)$. This means

$$g(z_0) = \frac{\lambda}{1+\lambda}(1+\lambda)^{-z_0}\Big(\alpha(v) + \beta(v)\deg(v)\Big) = 1,$$

and hence for every vertex $v \in V(G)$ it holds that

$$\alpha(v) + \beta(v)\deg(v) = \frac{1+\lambda}{\lambda}(1+\lambda)^{z_0}.$$

We pose $\lambda = \min\left\{\varepsilon, 1/\sqrt{T}\right\}$, and note that the analysis of (1.15) ensures that

$$z_0 \ln(1+\lambda) = W(\deg(v)\ln(1+\lambda)) + o(1).$$

We also use the fact that

$$\frac{(1+x)\ln(1+x)}{x} \leq 1 + \frac{x}{2},$$

for every $x > 0$. An application of Lemma 1.3.1 yields a local fractional colouring w of G such that $w(v) \subseteq [0, \gamma_{\alpha,\beta}(v)]$ for every vertex $v \in V(G)$, where

$$\begin{aligned}
\gamma_{\alpha,\beta}(v) &\leq \frac{1+\lambda}{\lambda} e^{z_0 \ln(1+\lambda)} \\
&= \frac{1+\lambda}{\lambda} e^{W(\deg(v)\ln(1+\lambda)) + o(1)} \\
&= (1+o(1))\frac{(1+\lambda)\ln(1+\lambda)}{\lambda}\frac{\deg(v)}{\ln\big(\deg(v)\ln(1+\lambda)\big)} \\
&\leq (1+o(1))\left(1+\frac{\lambda}{2}\right)\frac{\deg(v)}{\ln(\deg(v)\lambda)} \\
&\leq (1+\varepsilon)\frac{\deg(v)}{\ln\frac{\deg(v)}{\sqrt{T}}},
\end{aligned}$$

provided $\dfrac{\deg(v)}{\sqrt{T}}$ is large enough. □

1.6.3 Sharpness

As stated in Theorem 1.1.8, for every value of Δ there are triangle-free Δ-regular graphs G_Δ in which the independence ratio is at most

$$\frac{2\ln\Delta}{\Delta}.$$

So G_Δ certifies Theorems 1.1.10 and 1.1.11 to be sharp up to an asymptotic factor 2 provided $T = o(\Delta)$. For larger T, let us for completeness reiterate an observation from [108]. For a given triangle-free graph G of maximum degree d, and an integer k, let kG be the graph $G \boxtimes K_k$. It is of maximum degree $\Delta := (d+1)k - 1$, and such that each vertex is contained in at most

$T \coloneqq (d+1)\binom{k}{2} - k + 1$ triangles. Moreover,

$$\frac{\alpha(kG)}{n(kG)} = \frac{1}{k} \cdot \frac{\alpha(G)}{n(G)}.$$

For $T \geq \Delta$, let $d = \left\lfloor \frac{\Delta^2}{2T} \right\rfloor - 1$ and let bG_d be the graph obtained from G_d by substituting each vertex with a clique of size $b = \left\lfloor \frac{2T}{\Delta} \right\rfloor$. Then bG_d is regular of degree $b(d+1) - 1 \leq \Delta$ such that each neighbourhood contains at most $\frac{b^2(d+1)}{2} \leq T$ edges, and so bG_d has at most $\frac{|V(bG_d)|T}{3}$ triangles. In G_d the largest independent set is of size

$$(2 + o(1))\frac{|V(G_d)|}{d} \ln d = (4 + o(1))\frac{|V(bG_d)|}{\Delta} \ln \frac{\Delta}{\sqrt{T}}. \tag{1.17}$$

The same is clearly true in bG_d, and this is an asymptotic factor 4 greater than the lower bound in Theorem 1.1.10. Last, observe that for $T \leq \Delta$, G_Δ certifies that Theorems 1.1.10 and 1.1.11 are asymptotically off by at most a multiplicative factor 4 from extremal, and so this holds throughout the range of T.

Chapter 2

Distance colouring and cycles

The constraint for the colouring c of a graph G to be proper can be formulated in the following way: If two different vertices u and v are a distance at most 1 in G, then $c(u)$ and $c(v)$ must be different. For an edge colouring c', the same kind of reformulation is possible: If two different edges e and e' are at distance at most 1 in the line graph $L(G)$, then $c'(e)$ and $c'(e')$ must be different.

Given such a formulation, most mathematicians have the impulse to wonder about its generalisation to larger distances. What if we extend the constraint up to a given fixed distance $t \geq 1$ in the graph? This is what we call a distance-t (edge-)colouring.

The main content of this chapter is covered in the journal articles [70] and [71].

2.1 Formulation of the problem and motivation

For a positive integer t, let us recall that the *t-th power* G^t of a (simple) graph $G = (V, E)$ is a graph with vertex set V in which two distinct elements of V are joined by an edge if they are at distance at most t in G. We define the *distance-t-chromatic number* $\chi_t(G)$ and *distance-t chromatic index* $\chi'_t(G)$ as, respectively, the chromatic number of the t-th power of G, and the chromatic number of the t-th power of the line graph of G;

$$\begin{aligned}\chi_t(G) &\coloneqq \chi\big(G^t\big)\,, \text{ and}\\ \chi'_t(G) &\coloneqq \chi\big(L(G)^t\big)\,.\end{aligned}$$

With this formulation, the upper bounds yielded by the greedy algorithm for these two parameters are

$$\begin{aligned}\chi_t(G) &\leq \Delta\big(G^t\big) + 1 \leq \Delta(G)^t + 1, \text{ and}\\ \chi'_t(G) &\leq \Delta\big(L(G)^t\big) + 1 \leq 2\Delta(G)^t.\end{aligned}$$

For distance $t = 1$, these are the classical chromatic number and chromatic index. Theorem 0.2.13 (Brooks) states that the upper bound $\Delta(G) + 1$ is reached for the chromatic number, and characterises the graphs reaching the bound as exactly the cliques and odd cycles. Theorem 0.2.24 (Vizing) states that the upper bound for the chromatic index is actually also $\Delta(G) + 1$, thus way smaller than the greedy one of $2\Delta(G) - 1$. Moreover, since $\Delta(G)$ is a lower bound for the chromatic index of G, it can take only two possible values, and it is NP-hard to decide whether a given graph G has chromatic index $\Delta(G)$ or $\Delta(G) + 1$ [60].

For sparser graphs, we have seen throughout Chapter 1 through Johansson's theorem that the chromatic number of triangle-free graphs of degree at most d is $\Theta(d/\ln d)$ as $d \to \infty$. It was observed in [7] that this last statement with C_ℓ-free, $\ell > 3$, rather than triangle-free also holds. In conclusion, excluding any cycle length has a sensible effect on the extremal behaviour of the chromatic number among graphs of fixed maximum degree; its extremal asymptotic value is decreased by a logarithmic factor. This behaviour does not hold for the chromatic index, as it is always lower bounded by $\Delta(G)$ — any copy of $K_{1,\Delta(G)}$ in G yields a clique $K_{\Delta(G)}$ in $L(G)$.

This consideration for distance 1 leads to the following question for larger distances.

Problem 2.1.1. *What is the largest possible value of the distance-t chromatic number and index among all graphs G with maximum degree at most d, when some cycle C_ℓ might be forbidden?*

For both parameters, we are interested in finding those choices of ℓ (depending on t) for which there is an upper bound that is $o(d^t)$ as $d \to \infty$. Moreover, by probabilistic constructions [8, 69], these upper bounds must be $\Omega(d^t/\ln d)$ as $d \to \infty$ regardless of the choice of ℓ. On the other hand, while finding the exact maximum value of the distance-t chromatic number or index with no cycle restrictions seems to be a fundamentally hard problem, there are several constructions — such as the Hamming graph $H(t, d/t) = {K_{d/t}}^{\square t}$ or the t-dimensional De Bruijn graph on $d/2$ symbols — which demonstrate that this value is $\Theta(d^t)$ as $d \to \infty$ for fixed t. We are therefore also interested in the choices of ℓ which do not affect this asymptotic behaviour. We first discuss some previous work.

The distance-2 chromatic index appears in the literature as the *strong chromatic index*. With no cycle restriction, the above question is directly related to a conjecture from Erdős and Nešetřil.

Conjecture 2.1.1 (Erdős, Nešetřil, 1985)**.** *For every graph G of maximum degree d,*

$$\chi'_2(G) \leq d^2 + \left\lfloor \frac{d}{2} \right\rfloor^2 \leq \frac{5}{4}d^2.$$

This conjecture, if true, would be sharp. Indeed, the graph $G_k = C_5 \boxtimes I_k$, where I_k stands for an independent set of size k, is $2k$-regular, contains $5k^2$ edges, and is such that $L(G_k)^2$ is a clique. It took more than a decade in order to obtain a first step in the direction of this conjecture; in 1997 Molloy and Reed [91] manage to provide a slightly better bound than the trivial one for the strong chromatic index of graphs of maximum degree $d \geq d_0$ for some fixed d_0, namely $1.998d^2$. Their method has since been tightened in order to provide what is the best-to-date result in favor of the conjecture, which still remains widely open.

Theorem 2.1.2 (Bonamy, Perret, Postle, 2018 [20])**.** *There exists some integer $d_0 > 0$ such that, for every graph G of maximum degree $d \geq d_0$,*

$$\chi'_2(G) \leq 1.835d^2.$$

The behaviour of the strong chromatic index χ'_2 with some cycle restrictions was considered by Mahdian [87] who showed that the largest strong chromatic chromatic index over all C_4-free graphs of maximum degree at most d is $C \cdot d^2/\ln d$, with $1/2 \leq C \leq 2 + o(1)$ as $d \to \infty$. Vu [119] extended this to hold — up to a worse constant — for any fixed bipartite graph instead of C_4, which in particular implies the statement for any C_ℓ, ℓ even. Since the complete bipartite graph $K_{d,d}$ satisfies $\chi'_2(K_{d,d}) = d^2$, the statement does not hold for C_ℓ, ℓ odd. This completely settles the second question asymptotically for $\chi'_2 = \chi'_s$.

The case of the distance-2 chromatic number was considered and settled asymptotically by Alon and Mohar [8] with restriction on the girth rather than forbidding one cycle. They showed that the largest possible value of the distance-2 chromatic number taken over all graphs of girth 7 and of maximum degree at most d is $\Theta(d^2/\ln d)$ as $d \to \infty$. Moreover, there exist graphs of girth 6 and of arbitrarily large maximum degree d such that their distance-2 chromatic number is $(1+o(1))d^2$ as $d \to \infty$.

It is worth pointing out that the basic question unrestricted, i.e. asking for the extremal value of the distance-t chromatic number or index over graphs of maximum degree d as $d \to \infty$, is likely to be very difficult if we ask for the precise (asymptotic) multiplicative constant. We have seen that the question for χ'_t includes the notorious strong edge-colouring conjecture of Erdős and Nešetřil [44] as a special case. On the other hand, the question for χ_t amounts to a slightly weaker version of a well-known conjecture of Bollobás on the degree–diameter problem [17].

Let us advance a systematic treatment of our basic question. Our main results are as follows, which may be considered as extensions of the results of Johansson [65] and Mahdian [87] to distance-t vertex- and edge-colouring, respectively, for all t.

Theorem 2.1.3. *Let t be a positive integer and $\varepsilon > 0$ be an arbitrary small positive constant.*

(i) *Let $\ell \geq 2t+2$ be a fixed even integer. There exists $\Delta_{\ell,\varepsilon}$ such that, for every C_ℓ-free graph G of maximum degree $\Delta \geq \Delta_{\ell,\varepsilon}$,*

$$\chi_t(G) \leq (4+\varepsilon)\frac{\Delta^t}{\ln \Delta}.$$

(ii) *Let $\ell \geq 2t$ be a fixed even integer. There exists $\Delta_{\ell,\varepsilon}$ such that, for every C_ℓ-free graph G of maximum degree $\Delta \geq \Delta_{\ell,\varepsilon}$,*

$$\chi'_t(G) \leq (8+\varepsilon)\frac{\Delta^t}{\ln \Delta}.$$

Theorem 2.1.4. *Let t and ℓ be odd positive integers such that $\ell \geq 3t$, and $\varepsilon > 0$ be an arbitrary small positive constant. There exists $\Delta_{\ell,\varepsilon}$ such that, for every C_ℓ-free graph G of maximum degree $\Delta \geq \Delta_{\ell,\varepsilon}$,*

$$\chi_t(G) \leq (4+\varepsilon)\frac{\Delta^t}{\ln \Delta}.$$

Note that a lower bound on the supremum of the distance-t chromatic index of graphs of arbitrary large girth and fixed maximum degree can be derived from random graphs.

Theorem 2.1.5 (Kaiser, Kang, 2014 [69])**.** *For every $d, t \geq 1$ and $g \geq 3$, there exists a graph G of maximum degree at most d, girth at least g, and distance-t chromatic index*

$$\chi'_t(G) \geq (1+o(1))\frac{d^t}{t \ln d}.$$

In Section 2.3, we exhibit constructions to certify that Theorems 2.1.3 and 2.1.4 are sharp in terms of odd cycle exclusion. So excluding all odd cycles outside of those mentioned by the theorems does not affect the asymptotics of the upper bounds.

Proposition 2.1.6. *Let t and ℓ be positive integers.*

(i) *For t even, the supremum of the distance-t chromatic number over bipartite graphs of maximum degree at most d is $C_t \cdot d^t$ for some $2^{-t} \leq C_t \leq 1$.*

(ii) *For $t \geq 2$, the supremum of the distance-t chromatic index over bipartite C_ℓ-free graphs of maximum degree at most d is $C'_t \cdot d^t$ for some $2^{1-t} \leq C'_t \leq 2$.*

Proposition 2.1.7. *Let t be an (odd) integer. The supremum of the distance-t chromatic number over graphs of odd girth $3t$ and of maximum degree at most d is $C_t \cdot d^t$ for some $4^{-t} \leq C_t \leq 1$.*

Our proofs of Theorems 2.1.3 and 2.1.4 rely on direct applications of the following result of Achlioptas, Iliopoulos, and Sinclair [1] which improves a previous result from Alon, Krivelevich and Sudakov [7]. Its statement is that the chromatic number of a graph G can be bounded away from $\Delta(G)$ when every vertex of G belongs to a bounded fraction of the maximum possible number of triangles.

Lemma 2.1.8 (Achlioptas, Iliopoulos, Sinclair, 2018 [1])**.** *For every $\varepsilon > 0$, there exist Δ_ε and f_ε such that, for every graph G of maximum degree $\Delta \geq \Delta_\varepsilon$, if the neighbourhood of every vertex $v \in V(G)$ spans at most Δ^2/f edges, with $f \in [f_\varepsilon, \Delta^2 + 1]$, then*

$$\chi(G) \leq (4 + \varepsilon)\frac{\Delta}{\ln f}.$$

Note that this results holds for the list chromatic number, at the cost of a worse constant, as was settled by Vu [119] through a direct use of nibble methods. So Theorems 2.1.3 and 2.1.4 also hold with list versions of χ_t and χ'_t, with a worse (non explicit) constant. On the other hand, Theorem 1.1.11(ii) presented in Chapter 1 provides a fractional version of this result with a leading constant of $2 + \varepsilon$ instead of $4 + \varepsilon$; it can therefore be applied in order to obtain fractional versions of Theorems 2.1.3 and 2.1.4 with halved leading constants. If Conjecture 1.1.2 holds, then the leading constants of these theorems can directly be halved.

Section 2.2 is devoted to showing the requisite density properties for Lemma 2.1.8. In order to do so with respect to Theorem 2.1.3, we in part use some intermediary results that were employed in a recent improvement [100] upon the classic result of Bondy and Simonovits [21] that the Turán number $\mathrm{ex}(n, C_{2k})$ of the even cycle C_{2k}, that is, the maximum number of edges in a graph on n vertices not containing C_{2k} as a subgraph, satisfies $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$ as $n \to \infty$. It is natural that techniques used to show sparsity of C_{2k}-free graphs are helpful for Theorem 2.1.3, since the application of Lemma 2.1.8 demands the verification of a *local* sparsity condition.

A motivating conjecture for us is one of Alon and Mohar, asserting that for every t there is a critical girth g_t such that $\chi^t_{g_t}(d) = \Theta(d^t)$ and $\chi^t_{g_t+1}(d) = \Theta(d^t/\ln d)$, just as for $t = 1$ ($g_1 = 3$) and $t = 2$ ($g_2 = 6$). In light of the strong duality that there is between the distance-t chromatic number and the distance-$(t+1)$ chromatic index, a similar conjecture can be done on the existence of a critical girth g'_t for the distance-t chromatic index. If such critical girths exist, our best guess would be that $g_t = 2t + 2$ and $g'_t = 2t$ when $t \geq 2$. Though we have reasons for suspecting that this is true, we do not get so far as to conjecturing it yet. We are not aware of any previous work, for any $t \geq 3$, showing that g_t or g'_t, should they exist, are greater than 3 or 4 respectively, so Section 2.4 is devoted to proving this, thus making a first step in the direction of the conjecture.

Before continuing, we introduce some abbreviating notation:

$$\chi_t(d, g) \coloneqq \max\{\chi_t(G) \mid \Delta(G) \leq d \text{ and } \mathrm{girth}(G) \geq g\}, \text{ and}$$
$$\chi'_t(d, g) \coloneqq \max\{\chi'_t(G) \mid \Delta(G) \leq d \text{ and } \mathrm{girth}(G) \geq g\}.$$

In this language, we have

$$\chi_1(d,3) = d+1 \quad \text{and} \quad \frac{d}{2\ln d} \le \chi_1(d,4) \le (1+o(1))\frac{d}{\ln d},$$
$$d(d-1)+1 \le \chi_2(d,6) \le d^2+1 \quad \text{and} \quad \chi_2(d,7) = \Theta\left(\frac{d^2}{\ln d}\right),$$
$$d^2 + \left\lfloor \frac{d}{2} \right\rfloor^2 \le \chi_2'(d,4) \le 1.835d^2 \quad \text{and} \quad \frac{d^2}{2\ln d} \le \chi_2'(d,5) \le (2+o(1))\frac{d^2}{\ln d}.$$

Moreover, Theorem 2.1.3 implies that $\chi_t(d, 2t+3) = \Theta(d^t/\ln d)$ and $\chi'(d, 2t+1) = \Theta(d^t/\ln d)$ as $d \to \infty$.

Our last contribution in this chapter is to give $\Omega(d^t)$ lower bounds on $\chi_t(d,g)$ for various choices of t and $g(\le 2t+2)$. We show, in particular, that g_t, if it exists, is at least 5 for $t=3$, at least 6 for all $t \ge 4$, and at least 8 for all $t \ge 11$.

Theorem 2.1.9. *There are constructions to certify the following statements hold.*

(i) $\chi_3(d,5) \sim d^3$ *as* $d \to \infty$ *for infinitely many* d*;*

(ii) $\chi_4(d,6) \gtrsim d^4/2^4$ *as* $d \to \infty$ *and* $\chi_4(d,4) \gtrsim 2d^4/2^4$ *as* $d \to \infty$*;*

(iii) $\chi_5(d,7) \sim d^5$ *as* $d \to \infty$ *for infinitely many* d*;*

(iv) $\chi_6(d,6) \gtrsim d^6/2^6$ *as* $d \to \infty$ *and* $\chi_6(d,6) \gtrsim 3d^6/2^6$ *for infinitely many* d*;*

(v) $\chi_7(d,6) \gtrsim 2d^7/2^7$ *as* $d \to \infty$*;*

(vi) $\chi_8(d,6) \gtrsim d^8/2^8$ *as* $d \to \infty$ *and* $\chi_8(d,6) \gtrsim 3d^8/2^8$ *for infinitely many* d*;*

(vii) $\chi_{10}(d,6) \gtrsim d^{10}/2^{10}$ *as* $d \to \infty$ *and* $\chi_{10}(d,6) \gtrsim 5d^{10}/2^{10}$ *for infinitely many* d*;*

(viii) *for* $t = 9$ *or* $t \ge 11$, $\chi_t(d,8) \gtrsim d^t/2^t$ *as* $d \to \infty$, $\chi_t(d,8) \gtrsim 3d^t/2^t$ *for infinitely many* d*, and, if* $5 \mid t$*, then* $\chi_t(d,8) \gtrsim 5d^t/2^t$ *for infinitely many* d*.*

Moreover, these constructions are bipartite if t *is even.*

These lower bounds are obtained by a few different direct methods, including a circular construction (Section 2.3) and two other somewhat ad hoc methods (Section 2.5).

A summary of current known bounds for Alon and Mohar's problem is given in Table 2.1. When reflecting upon the gaps between entries in the upper and lower rows, one should keep in mind that among graphs G of maximum degree at most d and of girth lying strictly within these gaps, the current best upper and lower bounds on the extremal value of $\chi(G^t)$ are off by only a $\ln d$ factor from one another. We would be intrigued to learn of any constructions that certify $\liminf_{t\to\infty} g_t = \infty$, or of any upper bound on $\limsup_{t\to\infty} g_t/t$ strictly less than 2.

2.2 Proofs of Theorems 2.1.3 and 2.1.4

In this section we prove the main theorems. Before proceeding, let us set notation and make some preliminary remarks.

t	1	2	3	4	5	6	7	8	9	10	11	≥ 12
$g_t \geq$	3	6	5	6	7	6	6	6	8	6	8	8
$g'_t \geq$		4	6	8	6	12	6	6	6	8	6	8

Table 2.1: Bounds on the conjectured critical girths g_t and g'_t (if they exist).

Let $G = (V, E)$ be a graph of maximum degree at most d. We will often need to specify the vertices at some fixed distance from a vertex or an edge of G. Let i be a non-negative integer. If $x \in V$, we write $A_i = A_i(x)$ for the set of vertices at distance exactly i from x. If $e \in E$, we write $A_i = A_i(e)$ for the set of vertices at distance exactly i from an endpoint of e. We shall often abuse this notation by writing $A_{\leq j}$ for $\bigcup_{i \leq j} A_i$ and so forth. We will write $G_i = G[A_i, A_{i+1}]$ to be the bipartite subgraph induced by the sets A_i and A_{i+1}

In proving the distance-t chromatic number upper bounds in Theorems 2.1.3 and 2.1.4 using Lemma 2.1.8, given $x \in V$, we need to consider the number of pairs of distinct vertices in $A_{\leq t}$ that are connected by a path of length at most t. It will suffice to prove that this number is $O\left(d^{2t-1}\right)$ as $d \to \infty$, so that Lemma 2.1.8 may be applied with $f = O\left(d\right)$. In fact, in our enumeration we may restrict our attention to paths of length *exactly* t whose endpoints are in A_t and whose vertices do not intersect $A_{<t}$. This is because $|A_{\leq i}| \leq d^i$ for all i and the number of paths of length exactly j containing some fixed vertex is at most $(j+1)d(d-1)^{j-1}$ for all $j \geq 1$. So the number of pairs of vertices in $A_{\leq t}$ linked by a path of length at most $t-1$ is at most $d^t \cdot d \sum_{i=1}^{t-1}(d-1)^i \leq d^{2t-1}$, and the number of paths of length t containing a vertex in $A_{<t}$ is at most $d^{t-1} \cdot (t+1)d(d-1)^{t-1} \leq (t+1)d^{2t-1}$.

Similarly, in proving the distance-t chromatic index upper bound in Theorem 2.1.3 using Lemma 2.1.8, given $e \in E$, we need to consider the number of pairs of distinct edges that each have at least one endpoint in $A_{<t}$ and that are connected by a path of length at most $t-1$. It will suffice to prove that this number is $O(d^{2t-1})$ as $d \to \infty$. Similarly as above, in our enumeration we may restrict our attention to paths of length exactly $t-1$ whose endpoint edges both intersect A_{t-1} and whose vertices do not intersect $A_{<t-1}$. The number of paths which do not satisfy this restriction is upper bounded by $(2t+6)d^{2t-1}$.

As mentioned in the introduction, for Theorem 2.1.3 we are going to use two intermediate results of [100] concerning the presence of a Θ-*subgraph*, defined to be any subgraph that is a cycle of length at least $2k$ with a chord.

Lemma 2.2.1 ([100])**.** *Let $k \geq 3$. Any bipartite graph of minimum degree at least k contains a Θ-subgraph.*

Lemma 2.2.2 ([100])**.** *If $G = (V, E)$ is C_{2k}-free, then for $i \in [k]$ and $x \in V$, neither $G[A_i, A_{i+1}]$ nor $G[A_i]$ contains a bipartite Θ-subgraph, where A_i is defined based on G as above.*

Proof of Theorem 2.1.3(i). Let $\ell = 2k$ for some $k \geq t+1$, let $G = (V, E)$ be a graph of maximum degree at most d such that G contains no C_ℓ as a subgraph, and let $x \in V$. Let T denote the number of pairs of distinct vertices in A_t that are connected by a path of length exactly t that does not intersect $A_{<t}$. As discussed at the beginning of the section, it suffices for the proof to show that $T \leq Cd^{2t-1}$ where C is a constant independent of d, by Lemma 2.1.8.

We define A' to be A_{t+1} if $|A_{t+1}| \geq |A_t|$, or A_t otherwise, and E_H to be the set of edges in $A_t \times A_{t+1}$ whose endpoint in A' is of degree at least ℓ in $G_t = G[A_t, A_{t+1}]$. If E_H is non-empty, then it induces some bipartite graph $H = (X_H \cup Y_H, E_H)$ of average degree $d(H)$, such that $X_H \subseteq A'$ and $Y_H \subseteq (A_t \cup A_{t+1}) \setminus A'$. It must hold that $d(H) < \ell$, or else from H it would be possible

to extract a bipartite graph H' of minimum degree $d(H)/2 \geq \ell/2 = k$, which by Lemma 2.2.1 would contain a Θ-subgraph. This contradicts Lemma 2.2.2 which says G_t contains no bipartite Θ-subgraph. Therefore,

$$\ell > \frac{2\,|E_H|}{|X_H| + |Y_H|} \geq \frac{2\,|E_H|}{\frac{|E_H|}{\ell} + |Y_H|}$$

and so $|E_H| < \ell\,|Y_H| \leq \ell d^t$, where the last inequality follows from the definition of A'.

Moreover, the graph $G[A_t]$ is of average degree $d(G[A_t]) < 2\ell$, for otherwise it would be possible to extract from $G[A_t]$ a bipartite graph H' of average degree at least ℓ. From H' it would then be possible to extract a bipartite graph of minimum degree at least $\ell/2 = k$, which contains a Θ-subgraph by Lemma 2.2.1. This contradicts Lemma 2.2.2 which says $G[A_t]$ contains no bipartite Θ-subgraph. If we denote by $E[A_t]$ the set of edges of $G[A_t]$, it means that $|E[A_t]| < \frac{2\ell|A_t|}{2} \leq \ell d^t$.

Let us count the possibilities for a path $x_0 \dots x_t$ of length t between two distinct vertices $x_0, x_t \in A_t$ that does not intersect $A_{<t}$. We discriminate based on the first edge $e_0 = x_0x_1$ of this path, which can fall into three different cases.

1. $e_0 \in E_H$. We count the paths by first drawing e_0 from the at most ℓd^t possible choices in E_H, then drawing the remaining $t-1$ vertices of the path one at a time, for which there are at most d choices each. So the number of paths in this case is at most ℓd^{2t-1}.

2. $e_0 \in (A_t \times A_{t+1}) \setminus E_H$. It means that x_0 (resp. x_1) is of degree less than ℓ in A_{t+1} (resp. A_t) if $|A_{t+1}| < |A_t|$ (resp. if $|A_{t+1}| \geq |A_t|$). We count the paths by first drawing x_0 (resp. x_t) from the at most d^t possible choices in A_t, then drawing the other t vertices one at a time with d choices each, except for x_1 (resp. x_0) for which there are fewer than ℓ possible choices. The number of paths in this case is therefore at most ℓd^{2t-1}.

3. $e_0 \in E[A_t]$. We count the paths by first drawing e_0 from the at most ℓd^t possible choices in $E[A_t]$, then drawing the remaining $t-1$ vertices of the path one at a time, for which there are at most d choices each. So the number of paths in this case is at most ℓd^{2t-1}.

Summing over the above cases, the overall number of choices for the path $x_0 \dots x_t$ is at most $3\ell d^{2t-1}$, giving the required bound on T. □

Proof of Theorem 2.1.3(ii). Let $\ell \geq 2t$ be even, let $G = (V, E)$ be a graph of maximum degree at most d such that G contains no C_ℓ as a subgraph, and let $e = xy \in E$. It is straightforward to check that Lemma 2.2.2 is still valid with the sets A_i defined according to a root at the edge e rather than a root vertex, by combining the corresponding statements when the A_i are rooted instead at x or at y. Let T denote the number of pairs of distinct edges in $G[A_{t-1}]$ or $G_{t-1} = G[A_{t-1}, A_t]$ that are connected by a path of length $t-1$ that does not intersect $A_{<t-1}$. As discussed at the beginning of the section, it suffices to show that $T \leq Cd^{2t-1}$ where C is a constant independent of d, by Lemma 2.1.8.

We define A' to be A_t if $|A_t| \geq |A_{t-1}|$, or A_{t-1} otherwise, and E_H to be the set of edges in $A_{t-1} \times A_t$ whose endpoint in A' is of degree at least ℓ in G_{t-1}. Exactly as in the proof of Theorem 2.1.3(i), it follows from Lemmas 2.2.1 and 2.2.2 that $|E_H| < 2\ell d^{t-1}$ and $|E[A_{t-1}]| < 2\ell d^{t-1}$, where $E[A_{t-1}]$ denotes the set of edges of $G[A_{t-1}]$ (the additional factor 2 comes from the fact that the size of $|A_i|$ is upper bounded by $2d^i$ rather than d^i, for every i, in the context of a root edge).

Let us count the possibilities for a path $x_0 \dots x_{t+1}$, where $x_1 \dots x_t$ is a path of length $t-1$ between two distinct edges x_0x_1 and x_tx_{t+1} of $G[A_{t-1}]$ or G_{t-1} that does not intersect $A_{<t-1}$. We discriminate based on the first edge $e_0 = x_0x_1$ of this path, which can fall into three different cases.

1. $e_0 \in E_H$. We count the paths by first drawing e_0 from the at most $2\ell d^{t-1}$ possible choices in E_H, then drawing the remaining t edges of the path one at a time, for which there are at most d choices each. So the number of paths in this case is at most $2\ell d^{2t-1}$.

2. $e_0 = ab$ where $a \in A_{t-1}, b \in A_t$, and $e_0 \notin E_H$. It means that a (resp. b) is of degree less than ℓ in A_t (resp. A_{t-1}) if $|A_t| < |A_{t-1}|$ (resp. if $|A_t| \geq |A_{t-1}|$). There are now three different possible subcases.

   (a) $b = x_1$. We count the paths by first drawing x_0 (resp. x_t if it is in A_{t-1} or $x_{t-1} \in A_{t-1}$ otherwise) from the at most $2d^{t-1}$ possible choices in A_{t-1}, then drawing the other $t+1$ vertices one at a time with d choices each, except for x_1 (resp. x_0) for which there are fewer than ℓ possible choices. The number of paths in this subcase is therefore at most $2\ell d^{2t-1}$ (resp. $4\ell d^{2t-1}$).

   (b) $a = x_1$ and $x_2 \in A_{t-1}$. We count the paths by first drawing $e_1 = x_1x_2$ from the at most $2\ell d^{t-1}$ possible choices in $E[A_{t-1}]$, then drawing the other t edges one at a time with d choices each. The number of paths in this subcase is therefore at most $2\ell d^{2t-1}$.

   (c) $a = x_1$ and $x_2 \in A_t$. We count the paths by first drawing x_t if it is in A_{t-1} or $x_{t-1} \in A_{t-1}$ otherwise (resp. x_0) from the at most $2d^{t-1}$ possible choices in A_{t-1}, then drawing the other $t+1$ vertices one at a time with d choices each, except for x_0 (resp. x_1) for which there are fewer than 2ℓ possible choices. The number of paths in this subcase is therefore at most $4\ell d^{2t-1}$ (resp. $2\ell d^{2t-1}$).

3. $e_0 \in E[A_{t-1}]$. We count the paths by first drawing e_0 from the at most $2\ell d^{t-1}$ possible choices in $E[A_{t-1}]$, then drawing the remaining t edges of the path one at a time, for which there are at most d choices each. So the number of paths in this case is at most $2\ell d^{2t-1}$.

Summing over the above cases, the overall number of choices for the path $x_0 \dots x_t$ is at most $12\ell d^{2t-1}$, giving the required bound on T. □

In the proof of Theorem 2.1.4, we use the following lemma, which bounds the number of vertices at distance at most t from some fixed vertex when we impose intersection conditions on certain paths. The proof of this lemma illustrates the two main methods we use to bound the local density as needed for Lemma 2.1.8.

Lemma 2.2.3. *Let $G = (V, E)$ be a graph of maximum degree at most d and let $x_0 \in V$.*

(i) *Let S be a set of vertices at distance exactly t from x_0 such that any two paths of length t from x_0 to distinct elements of S must intersect in at least one vertex other than x_0. Then $|S| \leq d^{t-1}$.*

(ii) *Let P be a path of length $k > 0$ starting at x_0. Let S be a set of vertices at distance at most t from x_0 such that for every $s \in S$ there is a path of length at most t from x_0 to s that intersects with P in at least one vertex other than x_0. Then $|S| \leq kd^{t-1}$.*

Proof of Lemma 2.2.3(i). Suppose V is given with some ordering. As before, for each $i > 0$ let $A_i = A_i(x_0)$ denote the set of vertices at distance exactly i from x_0 in G. We inductively construct a breadth-first search tree $T = T_t$ as follows.

- T_0 consists only of the root x_0.

- If $i > 0$, then for every $y \in A_i$ let a_y be the vertex in $N(y) \cap A_{i-1}$ whose path from x_0 in T_{i-1} is least in lexicographical order. Then T_i is obtained from T_{i-1} by adding each edge ya_y, $y \in A_i$.

By assumption $S \subseteq A_t$. Let x_t be the vertex in S whose path in T from x_0 is least in lexicographical order, and let $P_x = x_0 \dots x_t$ be that path.

Let $y_t \in S$ be distinct from x_t and moreover suppose for a contradiction that the lowest common ancestor of x_t and y_t in T is x_0. Then y_t is at distance at least t from x_1, or else it would have had x_1 as an ancestor by the definition of T and the choice of P_x. Letting $P_y = y_0 \dots y_t$ (where $y_0 = x_0$) be the path from x_0 to y_t in T, by assumption P_x and P_y must have a common vertex other than x_0. So there are $i, j > 0$ such that $x_i = y_j$. It must be that $j < i$, for otherwise $x_1 \dots x_i y_{j+1} \dots y_t$ would be a path of length $i - 1 + t - j \leq t - 1$ between x_1 and y_t, a contradiction. This means though that $x_i \in A_i$ is at distance at most $j < i$ from x_0, also a contradiction. We have shown that S is contained in the subtree of T rooted at x_1, which then implies that $|S| \leq d^{t-1}$. □

Proof of Lemma 2.2.3(ii). To each vertex in S, there is a path of length at most $t - 1$ from some vertex of P other than x_0. There are at most d^{t-1} vertices within distance $t - 1$ of a fixed vertex of P, so summing over all possible choices of such a vertex, this gives $|S| \leq kd^{t-1}$. □

Proof of Theorem 2.1.4. Let $\ell \geq 3t$ be odd, let $G = (V, E)$ be a graph of maximum degree at most d such that G contains no C_ℓ as a subgraph, and let $x \in V$. For convenience, let us call any path contained in $A_{\geq t}$ *peripheral*. Let T denote the number of pairs of distinct vertices in A_t that are connected by a peripheral path of length t and are not connected by any path of length less than t. As discussed at the beginning of the section, it suffices for the proof to show that $T \leq Cd^{2t-1}$ where C is a constant independent of d, by Lemma 2.1.8.

We specify a unique breadth-first search tree BFS = BFS(x) of G, rooted at x. Having fixed an ordering of V, BFS is a graph on V whose edges are defined as follows. For every $v \in A_i$, $i > 0$, we include the edge to the neighbour of v in A_{i-1} that is least in the vertex ordering.

Since ℓ and t are odd, we know that $\ell = 3t + 2k$ for some non-negative integer k. For $j \in \{0, 1, \dots, 2k\}$, let us call a vertex $v \in A_t$ *j-implantable* if it is the endpoint of some peripheral path of length j, the other endpoint of which is in A_t. In particular, any vertex of A_t is 0-implantable.

We first show that the number of pairs of vertices connected by a peripheral path of length t which has a $2k$-implantable endpoint is $O(d^{2t-1})$. Fix v to be a $2k$-implantable vertex and $P = v_0 v_1 \dots v_{2k}$ a path certifying its implantability, so that $v_0 = v$ and (if $k > 0$) $v_{2k} \in A_t \setminus \{v\}$. By Lemma 2.2.3(ii) applied to $G[A_{\geq t}]$ and P, the number of vertices connected by a peripheral path of length t starting at v which intersects P at another vertex is at most $2kd^{t-1}$. Now consider the set $Y \subseteq A_t \setminus \{v\}$ such that there is a peripheral path of length t between v and y that does not intersect P except at v for all $y \in Y$. If a_Y is the ancestor of v_{2k} in BFS at layer A_1, then Y is contained in the subtree rooted at a_Y. Otherwise, there would be some $y_1 \in Y$ such that its lowest common ancestor with v_{2k} in BFS is x, which gives rise to a cycle of length $3t + 2k$ that contains x, v_{2k}, v, y_1, in that order, a contradiction. Thus $|Y| \leq d^{t-1}$, the number of pairs with v that are counted by T is at most $(1 + 2k)d^{t-1}$, and the number of pairs with a $2k$-implantable vertex that are counted by T is at most $(1 + 2k)d^{2t-1}$.

Observe that we are already done if $k = 0$ since every vertex in A_t is 0-implantable by definition, so assume from here on that $k > 0$. It remains for us to (crudely) count the number of pairs $(z_0, z_t) \in A_t^2$ of non-$2k$-implantable vertices that are connected by a peripheral path $z_0 \dots z_t$ of length t and are not connected by any shorter path.

First suppose $k \leq t$. Trivially the number of choices for z_0 is at most d^t and the number of choices for the sub-path $z_0 \dots z_{t-k}$ is d^{t-k}. Given z_{t-k}, the choice for the remainder sub-path $z_{t-k} \dots z_t$ is restricted by the fact that z_t is not $2k$-implantable; in particular, all such sub-paths must intersect at a vertex other than z_{t-k}. By Lemma 2.2.3(i) applied to $G[A_{\geq t}]$ and z_{t-k}, for a fixed choice of z_{t-k}, the number of possibilities for z_t is at most d^{k-1}, and so the number of pairs (z_0, z_t) in this case is at most $d^t \cdot d^{t-k} \cdot d^{k-1} = d^{2t-1}$.

Next suppose $k > t$. We discriminate based on the smallest possible value $j \equiv 2k \pmod t$ such that z_0, z_t are both not j-implantable. Note that since we are in the case where z_0, z_t are not $2k$-implantable, $j \leq 2k$. More formally, we let $\kappa_0 = t$ if $k \bmod t = 0$, or $\kappa_0 = k \bmod t$ otherwise. Let $j = \min\{2\kappa_0 + it \mid 0 \leq i \leq 2(k-\kappa_0)/t$ and z_0, z_t are not j-implantable$\}$. If $j = 2\kappa_0 \leq 2t$, then we can treat this just like the previous case, which means there are at most d^{2t-1} choices for the pair (z_0, z_t).

So suppose that $2\kappa_0 < j \leq 2k$. By the definition of j, without loss of generality z_0 is $(j-t)$-implantable, and z_0, z_t are not j-implantable. We fix z_0 and let P be a path of length $j-t$ certifying its $(j-t)$-implantability. First note that Lemma 2.2.3(ii) applied to $G[A_{\geq t}]$ and P states that there are at most $(j-t)d^{t-1}$ choices for those z_t such that there is a peripheral path of length t between z_0 and z_t that intersects P in some vertex other than z_0. So consider the set $Y \subseteq A_t \setminus \{z_0\}$ such that y is connected to z_0 by a peripheral path P_y of length t that intersects P only in z_0 for all $y \in Y$. Then every vertex $y \in Y$ is j-implantable as certified by the path P concatenated with P_y. This means that no choice for z_t in Y is possible, and so the number of pairs (z_0, z_t) in this setting is at most $(j-t)d^{2t-1}$.

Summing over all possible j, the number of choices for (z_0, z_t) is at most

$$\left(1 + \sum_{i=1}^{2(k-\kappa_0)/t} (2\kappa_0 + it - t)\right) d^{2t-1} = \frac{k^2 - \kappa_0^2}{t} \cdot 2d^{2t-1} \quad \text{if } k > t.$$

It therefore follows that $T \leq (1 + 2k + 2(k^2 - \kappa_0^2)/t)\, d^{2t-1}$, as required. □

2.3 Circular constructions

In this section, we describe some constructions based on a natural "circular unfolding" of the Hamming graph, or of the De Bruijn graph. All operations on indices in this section are performed modulo t.

All our constructions, of maximum degree d, have the stronger property to have an independent set $U^{(0)}$ of order $\Theta(d^t)$, which is a clique in the t-th power of the construction. Then, the set of edges incident to $U^{(0)}$ forms a clique of cardinality $d \cdot \left|U^{(0)}\right| = \Theta(d^{t+1})$ in the $(t+1)$-th power of the line graph of the construction. So any lower bound which can be derived for the distance-t chromatic number from our construction also derives the same lower bound multiplied by d for the distance-$(t+1)$ chromatic index. In general, there is a strong duality between those two parameters, though our work illustrates that they are not affected in the same manner by odd cycles.

We first give basic versions that have weaker girth properties but provide intuition, and we develop these further later. This first proposition directly implies Proposition 2.1.6(i), and half of Proposition2.1.6(ii) (only the case t odd).

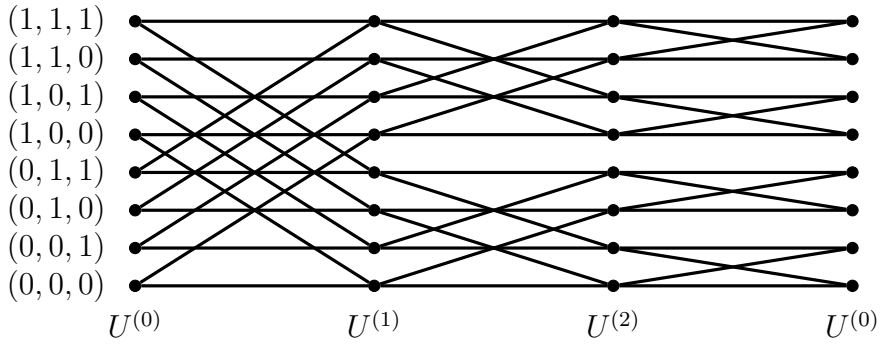


Figure 2.3.1: A schematic for the Hamming-type circular construction G_2 for $t = 3$ and $d = 4$.

Proposition 2.3.1. *For all $t \geq 3$ and all even d, there exists a d-regular graph G such that $\omega(G^t) \geq d^t/2^t$. Moreover, G is bipartite when t is even, and the odd girth of G is t when t is odd.*

Proof. Of course, the t-dimensional De Bruijn graph on $d/2$ symbols already certifies the first part of the statement, but we give two other constructions that satisfy the second part of the statement. For both constructions, the vertex set is $V = \bigcup_{i=0}^{t-1} U^{(i)}$ where each $U^{(i)}$ is a copy of $[d/2]^t$, the set of ordered t-tuples of symbols from $[d/2]$.

A De Bruijn-type construction. We define $G_1 = (V, E_1)$ as follows. For all $i \in [t]$, we join an element $\left(x_0^{(i)}, \ldots, x_{t-1}^{(i)}\right)$ of $U^{(i)}$ and an element $\left(x_0^{(i+1)}, \ldots, x_{t-1}^{(i+1)}\right)$ of $U^{(i+1)}$ by an edge if the latter is a left cyclic shift of the former, i.e. if $x_j^{(i+1)} = x_{j+1}^{(i)}$ for all $j \in [t-1]$ (and $x_0^{(i)}$, $x_{t-1}^{(i+1)}$ are arbitrary from $[d/2]$).

A Hamming-type construction. We define $G_2 = (V, E_2)$ as follows. For all $i \in [t]$, we join an element $\left(x_0^{(i)}, \ldots, x_{t-1}^{(i)}\right)$ of $U^{(i)}$ and an element $\left(x_0^{(i+1)}, \ldots, x_{t-1}^{(i+1)}\right)$ of $U^{(i+1)}$ by an edge if the t-tuples are equal on all coordinates, except maybe the i-th, i.e. if $x_j^{(i+1)} = x_j^{(i)}$ for all $j \neq i$ (and $x_i^{(i)}$, $x_i^{(i+1)}$ are arbitrary from $[d/2]$).

See Figure 2.3.1 for a schematic of G_2.

In both constructions, the maximum degree into $U^{(i+1)}$ from a vertex in $U^{(i)}$ is $d/2$ and the same is true from $U^{(i+1)}$ into $U^{(i)}$, so both constructions have maximum degree d overall. In both constructions, for any pair of elements in $U^{(0)}$ there is a path between them of length at most t, one that passes through every $U^{(i)}$, either by a sequence of t cyclic shifts or a sequence of t one-symbol changes. Therefore, the induced subgraphs ${G_1}^t[U^{(0)}]$ and ${G_2}^t[U^{(0)}]$ are both cliques, implying that $\chi({G_1}^t) \geq \left|U^{(0)}\right| = d^t/2^t$ and similarly for G_2. As these constructions are composed of bipartite graphs connected in sequence around a cycle of length t, G_1 and G_2 are bipartite when t is even, and of odd girth t when t is odd. This ends the proof. □

Our second objective in this section is to introduce a new graph product applicable only to two balanced bipartite graphs. We use it to produce two bipartite constructions for χ'_t, both of which settle the case of t even left open in Proposition 2.1.6(ii), and the second of which also treats what could be interpreted as an edge version of the degree–diameter problem.

Let $H_1 = (V_1 = A_1 \cup B_1, E_1)$ and $H_2 = (V_2 = A_2 \cup B_2, E_2)$ be two balanced bipartite graphs with given vertex orderings, i.e. $A_1 = (a_i^1)_{i \in [n_1]}, B_1 = (b_i^1)_{i \in [n_1]}, A_2 = (a_i^2)_{i \in [n_2]}, B_2 = (b_i^2)_{i \in [n_2]}$ for

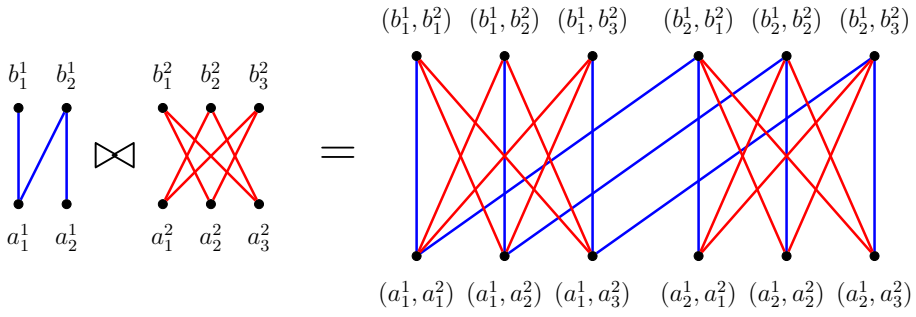


Figure 2.3.2: An illustration of the balanced bipartite product.

some positive integers n_1, n_2. We define the *balanced bipartite product* $H_1 \bowtie H_2$ of H_1 and H_2 as the graph with vertex and edge sets defined as follows:

$$V(H_1 \bowtie H_2) \coloneqq (A_1 \times A_2) \cup (B_1 \times B_2), \text{ and}$$
$$E(H_1 \bowtie H_2) \coloneqq \left\{(a_i^1, a^2)(b_i^1, b^2) \;\middle|\; i \in [n_1], a^2b^2 \in E_2\right\} \cup \left\{(a^1, a_j^2)(b^1, b_j^2) \;\middle|\; j \in [n_2], a^1b^1 \in E_1\right\}.$$

See Figure 2.3.2 for an example of this product.

Usually the given vertex orderings will be of either of the following types. We say that a labelling $A = (a_i)_{i\in[n]}$, $B = (b_i)_{i\in[n]}$ of $H = (V = A \cup B, E)$ is a *matching ordering* of H if $a_ib_i \in E$ for all $i \in [n]$. We say it is a *comatching ordering* if $a_ib_i \notin E$ for all $i \in [n]$. Note by Hall's theorem that every non-empty regular balanced bipartite graph admits a matching ordering, while every non-complete one admits a comatching ordering.

Let us now give some properties of this product relevant to our problem, especially concerning its degree and distance properties. The first of these propositions follow easily from the definition.

Proposition 2.3.2. *Let H_1 and H_2 be two balanced bipartite graphs that have part sizes n_1 and n_2, respectively, and are regular of degrees d_1 and d_2, respectively, for some positive integers n_1, n_2, d_1, d_2. Suppose H_1, H_2 are given in either matching or comatching ordering. Then $H_1 \bowtie H_2$ is a regular balanced bipartite graph with parts $A_{H_1\bowtie H_2} = A_1 \times A_2$ and $B_{H_1\bowtie H_2} = B_1 \times B_2$ each of size n_1n_2. If both are in matching ordering, then $H_1 \bowtie H_2$ has degree $d_1 + d_2 - 1$, otherwise it has degree $d_1 + d_2$.*

Proposition 2.3.3. *Let $H_1 = (V_1 = A_1 \cup B_1, E_1)$ and $H_2 = (V_2 = A_2 \cup B_2, E_2)$ be two regular balanced bipartite graphs.*

(i) *Suppose that for every $a^1, a'^1 \in X_1 \subseteq A_1$ there is a t_1-path between a^1 and a'^1 in H_1 (for some t_1 even). Suppose that for every $a^2, a'^2 \in X_2 \subseteq A_2$ there is a t_2-path between a^2 and a'^2 in H_2 (for some t_2 even). Then for every $(a^1, a^2), (a'^1, a'^2) \in X_1 \times X_2 \subseteq A_{H_1\bowtie H_2}$, there is a $(t_1 + t_2)$-path between (a^1, a^2) and (a'^1, a'^2) in $H_1 \bowtie H_2$.*

(ii) *Suppose that for every $a^1, a'^1 \in X_1 \subseteq A_1$ there is a t_1-path between a^1 and a'^1 in H_1 (for some t_1 even). Suppose that for every $a^2 \in X_2 \subseteq A_2$ and $b^2 \in Y_2 \subseteq B_2$ there is a t_2-path between a^2 and b^2 in H_2 (for some t_2 odd). Then for every $(a^1, a^2) \in X_1 \times X_2 \subseteq A_{H_1\bowtie H_2}$ and $(b^1, b^2) \in Y_1 \times Y_2 \subseteq B_{H_1\bowtie H_2}$ where $Y_1 = \{b_i^1 \mid a_i^1 \in X_1\}$, there is a $(t_1 + t_2)$-path between (a^1, a^2) and (b^1, b^2) in $H_1 \bowtie H_2$.*

Proof. We only show part (ii), as the other part is established in the same manner. Let $(a^1, a^2) \in X_1 \times X_2$ and $(b^1, b^2) \in Y_1 \times Y_2$. Using the distance assumption on H_1, let $a^1_{i_0}, b^1_{i_1}, a^1_{i_2}, \cdots, b^1_{i_{t_1-1}}, a^1_{i_{t_1}}$ be a t_1-path in H_1 between $a^1 = a^1_{i_0}$ and $a^1_{i_{t_1}}$, where i_{t_1} is such that $b^1 = b^1_{i_{t_1}}$. Using the distance assumption on H_2, let $a^2_{j_0} b^2_{j_1} a^2_{j_2} \cdots a^2_{j_{t_2-1}} b^2_{j_{t_2}}$ be a t_2-path in H_2 between $a^2 = a^2_{j_0}$ and $b^2 = b^2_{j_{t_2}}$. The following $(t_1 + t_2)$-path between (a^1, a^2) and (b^1, b^2) in $H_1 \bowtie H_2$ traverses using one of the coordinates, then the other:

$$(a^1, a^2) = (a^1_{i_0}, a^2_{j_0})(b^1_{i_1}, b^2_{j_0})(a^1_{i_2}, a^2_{j_0}) \cdots (b^1_{i_{t_1-1}}, b^2_{j_0})(a^1_{i_{t_1}}, a^2_{j_0})$$
$$(b^1_{i_{t_1}}, b^2_{j_1})(a^1_{i_{t_1}}, a^2_{j_2}) \cdots (a^1_{i_{t_1}}, b^2_{j_{t_2-1}})(b^1_{i_{t_1}}, b^2_{j_{t_2}}) = (b^1, b^2).$$

□

We use the balanced bipartite product to show that no version of Theorem 2.1.4 may hold for χ'_t. In combination with the trivial bound $\chi'_t(G) = O(d^t)$ if $\Delta(G) \leq d$, we deduce Proposition 2.1.6(ii) from Proposition 2.3.1, the following result and the fact that $\chi'_2(K_{d,d}) = d^2$.

Proposition 2.3.4. *Fix $t \geq 4$ even. For every $d \geq 2$ with $d \equiv 0 \pmod{2(t-2)}$, there exists a d-regular bipartite graph G with $\chi'_t(G) \geq d^t/(et2^{t-1})$.*

Proof. Let $t_1 = t - 2$ and $d_1 = (t_1 - 1)d/t_1$. Let $G_1 = (V_1, E_1)$ be the construction promised by Proposition 2.3.1 for d_1 and t_1. Since G_1 is bipartite, we can write $V_1 = A_1 \cup B_1$ where

$$A_1 = \bigcup_{\substack{i \in [t_1] \\ i \text{ even}}} U^{(i)} \quad \text{and} \quad B_1 = \bigcup_{\substack{j \in [t_1] \\ j \text{ odd}}} U^{(j)}.$$

This is a d_1-regular balanced bipartite graph, and for every $a_1, a'_1 \in U^{(0)} \subseteq A_1$ there exists a t_1-path between a_1 and a'_1. Moreover, it is possible to label A_1 and B_1 so that the first $\left|U^{(0)}\right|$ vertices of A_1 are the ones of $U^{(0)}$, and the first $\left|U^{(1)}\right|$ of B_1 are those of $U^{(1)}$. We may also ensure that this labelling is in comatching ordering.

Let $t_2 = 1$ and $d_2 = d - d_1 = d/t_1$. Let $G_2 = (V_2 = A_2 \cup B_2, E_2) = K_{d_2,d_2}$. This is a d_2-regular balanced bipartite graph, and for every $a_2 \in A_2, b_2 \in B_2$, there exists a t_2-path between a_2 and b_2. Trivially any labelling of A_2 and B_2 gives rise to a matching ordering.

Let $G = G_1 \bowtie G_2$, $X = U^{(0)} \times A_2$ and $Y = U^{(1)} \times B_2$. Now G is a d-regular bipartite graph by Proposition 2.3.2, and by Proposition 2.3.3 for every $(a_1, a_2) \in X$ and $(b_1, b_2) \in Y$, there exists a $(t-1)$-path between (a_1, a_2) and (b_1, b_2). Thus the edges of G that span $X \times Y$ induce a clique in $(L(G))^t$. The number of such edges is (since $t > 3$) at least

$$\left(\frac{d_1}{2}\right)^{t_1} d_2 \left(\frac{d_1}{2} + d_2\right) = \left(1 - \frac{1}{t-2}\right)^{t-2} \frac{(t-1)d^t}{(t-3)^2 2^{t-1}} \geq \frac{d^t}{et2^{t-1}}.$$

□

Alternatively, Proposition 2.1.6(ii) follows from the following result, albeit at the expense of a worse dependency on t in the multiplicative factor. For $t \geq 2$, we can take a $(t-1)$-th power of the product operation on the complete bipartite graph to produce a bipartite graph G of maximum degree d with $\Omega(d^t)$ edges such that $(L(G))^t$ is a clique.

Proposition 2.3.5. *Fix $t \geq 2$. For every $d \geq 2$ with $d \equiv 1 \pmod{t-1}$, there exists a d-regular bipartite graph $G = (V, E)$ with $|E| = d \cdot \left(\frac{d-1}{t-1} + 1\right)^{t-1}$ and $\chi'_t(G) = |E|$.*

Proof. Let $d' = (d-1)/(t-1)+1$ and $G = K_{d',d'}^{\bowtie\, t-1}$, the $(t-1)$-th power of $K_{d',d'}$ under the product $\bowtie$, where the factors are always taken in matching ordering. By Proposition 2.3.2, G is a d-regular bipartite graph and has $d \cdot d'^{t-1}$ edges. By Proposition 2.3.3, there is a path of length at most $t-1$ between every pair of vertices in the same part if $t-1$ is even, or in different parts if $t-1$ is odd. It follows that $(L(G))^t$ is a clique. □

We now show how to use the Hamming-type construction of Proposition 2.3.1 in a slightly altered way and obtain a construction of maximum degree d and optimal odd girth, still reaching the $\Theta(d^t)$ bound.

Proposition 2.3.6. *For every odd $t \geq 3$ and d such that $4 \mid d+2$, there exists a graph G of maximum degree d such that $\omega(G^t) \geq (d+2)^t/4^t$, and $g_o(G) = 3t$.*

Proof. Let $t \geq 3$ be an odd integer, and $k \geq 1$ be any integer. We define $G = (V, E)$ as follows. The vertex set is

$$V = \bigcup_{i=1}^{t-1} U^{(i)} \cup \bigcup_{i=0}^{t-2} V^{(i)},$$

where each $V^{(i)}$ is a copy of $[k]^t$, the set of ordered t-tuples of symbols from $[k]$, and each $U^{(i)}$ is the set

$$\left\{ \left(u_0^{(i)}, \ldots, u_{t-1}^{(i)}\right) \;\middle|\; \forall j \in [t], u_j^{(i)} \in [k] \text{ and } u_0^{(i)} \neq u_{t-1}^{(i)} \right\}.$$

Given a fixed $x \in [k]$, the subgraph induced by the vertices in $\bigcup_{i\in[t-1]} V^{(i)}$ whose last coordinate is equal to x is a copy of the Hamming-type construction of Proposition 2.3.1 of order $t-1$. Since $t-1$ is even, this is a $2k$-regular (or $(2k-1)$-regular if $t=3$) bipartite graph. From the properties of the Hamming-type construction, we infer that there is a path of length $t-1$ traversing each $V^{(i)}$ which links any pair of vertices in $V^{(0)}$ with a common last coordinate.

By writing $U^{(0)}$ for $V^{(0)}$, we now define the edge set of the graph induced by the vertices in $\bigcup_{i\in[t]} U^{(i)}$ as follows:

$$\forall i \in [t-1], \quad E \cap \left(U^{(i)} \times U^{(i+1)}\right) \coloneqq \left\{ \left(u_j^{(i)}\right)_{j\in[t]} \left(u_j^{(i+1)}\right)_{j\in[t]} \;\middle|\; \forall j \neq i, u_j^{(i)} = u_j^{(i+1)} \right\}, \text{ and}$$

$$E \cap \left(U^{(t-1)} \times U^{(0)}\right) \coloneqq \left\{ \left(u_j^{(t-1)}\right)_{j\in[t]} \left(u_j^{(0)}\right)_{j\in[t]} \;\middle|\; u_{t-1}^{(0)} = u_0^{(t-1)} \text{ and } \forall 1 \leq j \leq t-2, u_j^{(0)} = u_j^{(t-1)} \right\}.$$

The subgraph induced by the vertices in $\bigcup_{i\in[t]} U^{(i)}$ is of maximum degree $2k$, and there is a path of length t traversing each $U^{(i)}$ which links any pair of vertices in $U^{(0)}$ with a different last coordinate. Namely, a path between $\left(x_0^{(0)}, \ldots, x_{t-1}^{(0)}\right) \in U^{(0)}$ and $\left(y_0^{(0)}, \ldots, y_{t-1}^{(0)}\right) \in U^{(0)}$ where $x_{t-1}^{(0)} \neq y_{t-1}^{(0)}$ can be induced by the vertices $\left(z_0^{(i)}, \ldots, z_{t-1}^{(i)}\right) \in U^{(i)}$ for every $1 \leq i \leq t-1$, where

$$\begin{aligned} \forall 1 \leq j < i \leq t-1, \quad & z_j^{(i)} = y_j^{(0)}, \\ \forall 1 \leq i \leq j \leq t-1, \quad & z_j^{(i)} = x_j^{(0)}, \\ \forall 1 \leq i \leq t-1, \quad & z_0^{(i)} = y_{t-1}^{(0)}. \end{aligned}$$

For every $1 \leq i \leq t-1$, the vertex $\left(z_0^{(i)}, \ldots, z_{t-1}^{(i)}\right) \in U^{(i)}$ satisfies $z_0^{(i)} = y_{t-1}^{(0)} \neq x_{t-1}^{(0)} = z_{t-1}^{(i)}$, which confirms that it belongs to $U^{(i)}$.

It holds that $U^{(0)} = V^{(0)}$ is a clique in G^t, of order k^t. On the other hand, G is a graph of maximum degree $4k-2$ (or $4k-3$ in the case $t=3$), and this degree is attained only by the vertices in $U^{(0)}$ — all the other vertices are of maximum degree $2k$. We now claim that G contains no odd cycle of length less than $3t$. First note that the graph obtained by removing the edges between two consecutive parts $U^{(i)}$ and $U^{(i+1)}$ is bipartite, so an odd cycle must contain at least one edge from $U^{(i)} \times U^{(i+1)}$ for every $i \in [t]$.

Let C be an odd cycle in G, and let P_0 be a maximal path of C which contains no edge from $U^{(t-1)} \times U^{(0)}$. Let us denote $x^{(0)} = \left(x_0^{(0)}, \dots, x_{t-1}^{(0)}\right) \in U^{(0)}$ and $x^{(t-1)} = \left(x_0^{(t-1)}, \dots, x_{t-1}^{(t-1)}\right) \in U^{(t-1)}$ the extremities of P_0, and let $y^{(0)} = \left(y_0^{(0)}, \dots, y_{t-1}^{(0)}\right) \in U^{(0)}$ be the vertex following $x^{(t-1)}$ in C. By construction, we know that $x_{t-1}^{(0)} = x_{t-1}^{(t-1)}$, since the only edges linking two vertices with a different $(t-1)$-th coordinate are the ones from $U^{(t-1)} \times U^{(0)}$. Moreover, since $x^{(t-1)}y^{(0)}$ is an edge in $G\left[U^{(t-1)}, U^{(0)}\right]$, we know by construction that

$$\forall 1 \leq j \leq t-2, \quad y_j^{(0)} = x_j^{(t-1)}, \text{ and}$$
$$y_{t-1}^{(0)} = x_0^{(t-1)} \neq x_{t-1}^{(t-1)}.$$

So we conclude that $x_{t-1}^{(0)} \neq y_{t-1}^{(0)}$. Let $P_1 = C \setminus \left(P_0 + y^{(0)}\right)$; this is a path between $x^{(0)}$ and $y^{(0)}$, two vertices of $U^{(0)}$ which differ in their last coordinate. In particular, P_1 must intersect $G\left[U^{(t-1)}, U^{(0)}\right]$. Let us assume that the winding number of the cycle C_1 obtained by closing P_1 with respect to the cycle $U^{(0)}U^{(1)}\dots U^{(t-1)}U^{(0)}$ is 0. If we orient the path P_1 from $x^{(0)}$ to $y^{(0)}$, each time P_1 traverses the graph $G\left[U^{(t-1)}, U^{(0)}\right]$ (from $U^{(0)}$ to $U^{(t-1)}$), it does not intersect $U^{(0)}$ again before traversing it back (from $U^{(t-1)}$ to $U^{(0)}$). At each traversal from $U^{(0)}$ to $U^{(t-1)}$, the $(t-1)$-th coordinate of the current vertex is copied into the 0-th coordinate of the next one. At the next traversal, from $U^{(t-1)}$ to $U^{(0)}$, the $(t-1)$-th coordinate of the current vertex is changed back to the value of its 0-th coordinate in the next one. It turns out that the 0-th coordinate has remained unchanged since the previous traversal, since the subpath linking those two traversals is disjoint from $U^{(0)}$, and traversing $U^{(0)}$ is a necessary condition in order to change the 0-th coordinate. As a consequence, after each pair of traversals of $G\left[U^{(t-1)}, U^{(0)}\right]$, the last coordinate of the current vertex in P_1 is always $x_{t-1}^{(0)} \neq y_{t-1}^{(0)}$. This is a contradiction, so the winding number of C_1 is at least 1, and hence the winding number of C is at least 2. Finally, since C is an odd cycle, it must have an odd winding number. Therefore, the winding number of C is at least 3, which ensures that $|C| \geq 3t$. □

Remark 2.3.1. In the construction described in the proof of Proposition 2.3.6, the condition for a vertex in $V^{(0)}$ and a vertex in $V^{(1)}$ to share an edge is exactly the same as between $U^{(0)}$ and $U^{(1)}$. It is therefore possible to identify the vertices identically labelled in $V^{(1)}$ and $U^{(1)}$, in such a way that the vertices in $U^{(0)}$ and $U^{(1)}$ are now of degree at most $3k-1$ and $3k$ instead of $4k-2$ and $2k$ respectively. The maximum degree of the construction obtained after this identification is $d = 3k$, and its t-th power contains a clique of size $d^t/3^t$. Moreover, if $t \geq 5$, the odd girth of the construction remains $3t$.

Finally, if $t = 3$ and k is a prime power, we can replace the subgraph of G induced by $V^{(0)}$ and $V^{(1)}$ with k disjoint copies of the incidence graph of a projective plane of order k (described in the next section). By doing so, new vertices are added to $V^{(0)}$ and $V^{(1)}$ — exactly $k(k+1)$ in each — which we do not take into consideration in $U^{(0)}$. This modification yields a graph of odd girth $3t$, maximum degree $d = 3k-1$, and such that $U^{(0)}$ induces a clique of size $(d+1)^3/3^3$ in its third power.

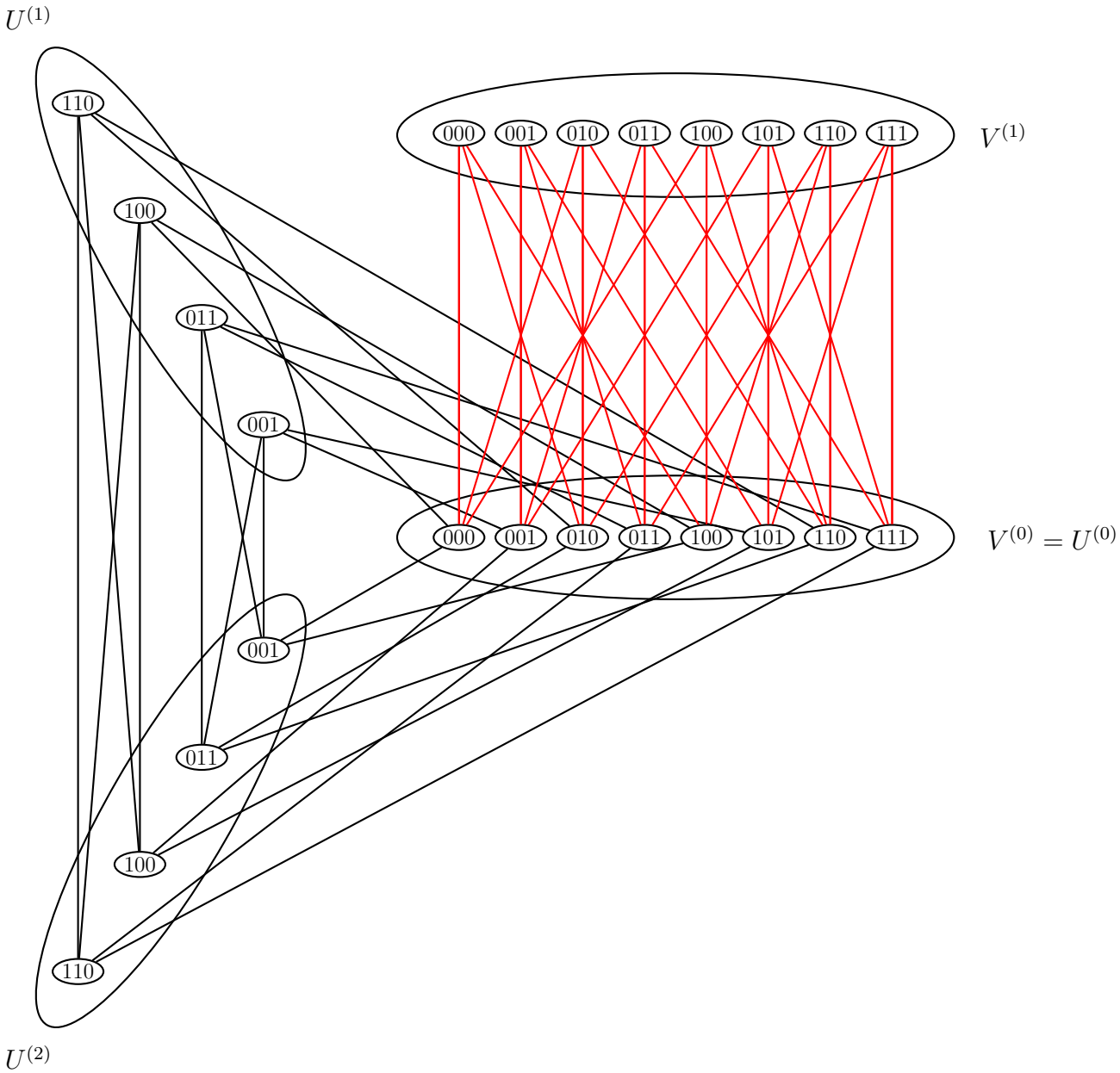


Figure 2.3.3: The construction in the proof of Proposition 2.3.6 when $k = 2$ and $t = 3$.

2.4 Constructions of higher girth

To proceed further with the circular constructions of Proposition 2.3.1, and obtain constructions with higher girth, we certainly have to handle the (many) cycles of length 4 that span only two consecutive parts $U^{(i)}$ and $U^{(i+1)}$. We do this essentially by substituting a subsegment $U^{(i)}, U^{(i+1)}, \ldots, U^{(i+k)}$ with a sparse bipartite structure having good distance properties. Some of the most efficient such sparse structures arise from finite geometries, generalised polygons in particular. We base our substitution operation on these structures, and therefore find it convenient to encapsulate the properties most relevant to us in the following definition.

We say a balanced bipartite graph $H = (V = A \cup B, E)$ with parts A and B, $|A| = |B|$, is a *good conduit* (between A and B) with parameters $(\tau, \Delta, \gamma, c)$ if it has girth γ, it is regular of degree Δ, there is a path of length at most τ between any $a \in A$ and any $b \in B$, and moreover $|A|$ (and so also $|B|$) is of maximum possible order $\Theta(\Delta^\tau)$ such that $|A| \geq c\Delta^\tau$.

The following good conduits are useful in our constructions, because of their relatively high girth. The balanced complete bipartite graph $K_{\Delta,\Delta}$ is a good conduit with parameters $(1, \Delta, 4, 1)$. Let q be a prime power. The point-line incidence graph $\mathcal{Q}_q$ of a symplectic quadrangle with parameters (q, q) is a good conduit with parameters $(3, q+1, 8, 1)$. The point-line incidence graph $\mathcal{H}_q$ of a split Cayley hexagon with parameters (q, q) is a good conduit with parameters $(5, q+1, 12, 1)$. We have intentionally made specific classical choices of generalised polygons here, cf. [98], partly because we know they are defined for all prime powers q and partly for symmetry considerations described later. We remark that no generalised octagon with parameters (q, q) exists and no generalised n-gons for any other even value of n exist [51].

We are now prepared to present the main construction of the section. This is a generalisation of G_2. (It is possible to generalise G_1 in a similar way.)

Theorem 2.4.1. *Let $t = \sum_{i=0}^{\lambda-1} \tau_i$ for some positive odd integers τ_i and $\lambda \geq 2$. Let d be even. Suppose that for every i there is a good conduit H_i with parameters $(\tau_i, d/2, \gamma_i, c_i)$. Then there is a graph G of maximum degree d such that*

$$\omega\big(G^t\big) \geq \frac{d^t}{2^t} \cdot \prod_{i=0}^{\lambda-1} c_i, \text{ and}$$
$$\operatorname{girth}(G) \geq \min\left\{\lambda, 8, \min_i \gamma_i\right\}.$$

Moreover, G is bipartite if and only if t is even.

After the proof, we show how to modify the construction in certain cases to mimic the inclusion of good conduits with τ parameter equal to 2 (note that good conduits with even τ are precluded from the definition), to increase the girth of G, or to improve the bound on $\chi(G^t)$.

Proof of Theorem 2.4.1. For every i, let $H_i = (V_i = A_i \cup B_i, E_i)$ be the assumed good conduit with parameters $(\tau_i, d/2, \gamma_i, c_i)$. Write $A_i = \{a_1^i, \ldots, a_{n_i}^i\}$ and $B_i = \{b_1^i, \ldots, b_{n_i}^i\}$. By the definition of H_i, $n_i \geq c_i d^{\tau_i}/2^{\tau_i}$.

In this proof, the operations on indices are performed modulo λ. We define $G = (V, E)$ as follows. The vertex set is $V = \bigcup_{i\in[\lambda]} U^{(i)}$ where each $U^{(i)}$ is a copy of $\prod_{j=0}^{\lambda-1}[n_j]$, the set of ordered λ-tuples whose j^{th} coordinate is a symbol from $[n_j]$. For all $i \in [\lambda]$, we join an element $\left(x_0^{(i)}, \ldots, x_{\lambda-1}^{(i)}\right)$ of $U^{(i)}$ and an element $\left(x_0^{(i+1)}, \ldots, x_{\lambda-1}^{(i+1)}\right)$ of $U^{(i+1)}$ by an edge only if the λ-tuples are equal on all coordinates except maybe the i-th, in which case we use H_i and its ordering as a template for adjacency. More precisely, join $\left(x_0^{(i)}, \ldots, x_{\lambda-1}^{(i)}\right)$ and $\left(x_0^{(i+1)}, \ldots, x_{\lambda-1}^{(i+1)}\right)$ by an edge in G if $x_j^{(i)} = x_j^{(i+1)}$ for all $j \neq i$ and there is an edge in H_i joining $a^i_{x_i^{(i)}}$ and $b^i_{x_i^{(i+1)}}$. Clearly G is bipartite if and only if λ is even, which holds if and only if t is even.

Since H_i has maximum degree $d/2$, the degree into $U^{(i+1)}$ from a vertex in $U^{(i)}$ is at most $d/2$ (with respect to the edges added between $U^{(i)}$ and $U^{(i+1)}$) and the same is true from $U^{(i+1)}$ into $U^{(i)}$, so overall G has maximum degree d. Between any pair of elements in $U^{(0)}$, there is a path of length at most $t = \sum_{i=0}^{\lambda-1} \tau_i$ passing through every $U^{(i)}$, which changes the symbol at coordinate

i via a path of length at most τ_i in the subgraph induced by $U^{(i)}$ and $U^{(i+1)}$. It follows that the induced subgraph $G^t\left[U^{(0)}\right]$ is a clique, of size

$$\left|U^{(0)}\right| = \prod_{i=0}^{\lambda-1} n_i \geq \prod_{i=0}^{\lambda-1} c_i \cdot \frac{d^{\tau_i}}{2^{\tau_i}} = \frac{d^t}{2^t} \cdot \prod_{i=0}^{\lambda-1} c_i.$$

All that remains is to establish the girth of G. For the statement we essentially only need to consider cycles of length 7 or less, whose winding number with respect to the cycle $U^{(0)} \cdots U^{(\lambda-1)}U^{(0)}$ is 0. Such cycles are of even length, so we only need to consider lengths 4 and 6. We do not need to consider the cycles that only go back and forth between $U^{(i)}$ and $U^{(i+1)}$ (only along the edges added between $U^{(i)}$ and $U^{(i+1)}$), for such cycles are accounted for by the $\min_i \gamma_i$ term. So, for cycles of length 4 of winding number 0, without loss of generality we need only consider one that proceeds in order through $U^{(0)}$, $U^{(1)}$, $U^{(2)}$, and then back through $U^{(1)}$ to $U^{(0)}$, written as $u^{(0)}u^{(1)}u^{(2)}v^{(1)}u^{(0)}$. By construction, the λ-tuples $u^{(0)}$, $u^{(1)}$, $v^{(1)}$ share all but their zeroth coordinate symbols and the tuples $u^{(2)}$, $u^{(1)}$, $v^{(1)}$ share all but their first coordinate; however, this implies that $u^{(1)}$ and $v^{(1)}$ are the same tuple in $U^{(1)}$, a contradiction. For cycles of length 6 that, say, proceed in order through $U^{(0)}$, $U^{(1)}$, $U^{(2)}$, $U^{(3)}$ and back, we argue in a similar fashion as for length 4 to obtain a contradiction. The remaining case (for winding number 0) is a cycle of length 6 that is, without loss of generality, of the form $u^{(0)}u^{(1)}u^{(2)}v^{(1)}v^{(2)}w^{(1)}u^{(0)}$. By construction, the tuples $u^{(0)}$, $u^{(1)}$, $w^{(1)}$ share all but their zeroth coordinate symbols and the tuples $v^{(2)}$, $v^{(1)}$, $w^{(1)}$ share all but their first coordinate as do $u^{(2)}$, $u^{(1)}$, $v^{(1)}$; however, this implies that $u^{(1)}$ and $w^{(1)}$ are the same tuple in $U^{(1)}$, a contradiction. This concludes our determination of the girth of G.

We remark that cycles of length 8 may well occur, for instance when the same good conduit H is used two times consecutively. In particular, supposing H is used from $U^{(0)}$ to $U^{(1)}$ to $U^{(2)}$ and $a_1b_2a_3$ and $b_4a_5b_6$ are two 2-paths in H, then $(4,1,\ldots)^{(1)}$, $(4,2,\ldots)^{(2)}$, $(4,3,\ldots)^{(1)}$, $(5,3,\ldots)^{(0)}$, $(6,3,\ldots)^{(1)}$, $(6,2,\ldots)^{(2)}$, $(6,1,\ldots)^{(1)}$, $(5,1,\ldots)^{(0)}$, $(4,1,\ldots)^{(1)}$ represents an 8-cycle in the construction. □

In two of the small values for t (namely, 4 or 7), we cannot apply the construction of Theorem 2.4.1 without a modification. The intuition is to include another sparse structure with good distance properties, that is, the point-line incidence graph $\mathcal{P}_q$ of the projective plane $PG(2,q)$, for q a prime power. This is a bipartite graph $(V = A \cup B, E)$ of girth 6, that is regular of degree $q+1$, has a 2-path between a and a' for any $a, a' \in A$ (and similarly a 2-path between b and b' for any $b, b' \in B$), and has $|A| = |B| = q^2+q+1$. The graph $\mathcal{P}_{d-1}$ certifies $\chi_2(d,6) \geq d(d-1)+1$ if $d-1$ is a prime power; moreover, since the gap between two successive primes p and p' is $o(p)$ [61], the inequality $\chi_2(d,6) \geq (1-o(1))d^2$ holds for all d as $d \to \infty$.

The graph $\mathcal{P}_q$ has properties similar to what we might require for a good conduit having parameter $\tau = 2$, except that it connects vertices in the same part. One solution to this parity issue is to "unfold a mirror of $\mathcal{P}_q$", that is, add a disjoint copy of one of its parts with the same adjacencies as the original, so that the conduit is between the vertices of two copies of the same part. See Figure 2.4.1 for an illustration of $\mathcal{P}_2$ together with its mirror, denoted by $-\mathcal{P}_2$.

Directly, however, this creates cycles of length 4 (from vertices of degree 2), so we need to segregate the embedding of $\mathcal{P}_q$ and its mirror (i.e. $\mathcal{P}_q$ with A and B switched). More precisely, suppose that we want to use $\mathcal{P}_q$ as a template for the edges between $U^{(0)}$ and $U^{(1)}$ (as in the construction of G in Theorem 2.4.1). For this, we change symbols (chosen from $[q^2+q+1]$) at the 0-th coordinate in one step according to $\mathcal{P}_q$ between $U^{(0)}$ and $U^{(1)}$, and change the 0-th coordinate in a second step according to the mirror of $\mathcal{P}_q$ but later in the cycle, say by adding a new part $U'^{(0)}$ after $U^{(2)}$ in the cycle and adding edges between $U^{(2)}$ and $U'^{(0)}$ according to the mirror of

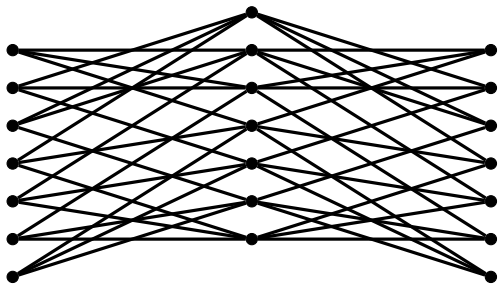


Figure 2.4.1: An illustration of $\mathcal{P}_2$ together with its mirror.

$\mathcal{P}_q$. Although this adds one more part to the cycle of $U^{(j)}$s, it avoids cycles in G of length 4 and appropriately mimics the distance properties of a good conduit with parameter $\tau = 2$. We can also interleave when we want to embed $\mathcal{P}_q$ and its mirror for two or three coordinates.

In all, the girth we obtain for this modification of Theorem 2.4.1 satisfies $\text{girth}(G) \geq \min\{\lambda + \iota, 6, \min_i \gamma_i\}$, where $\iota = \#\{i \mid \tau_i = 2\} \geq 1$, provided that $\lambda \geq 3$ if $\iota = 1$.

For $t \geq 6$, it is possible to improve on the construction in Theorem 2.4.1 either in terms of the girth of G or $\omega(G^t)$ by applying a similar modification as above but instead to good conduits. In particular, we can "unfold" $\mathcal{Q}_q$ or $\mathcal{H}_q$ into three copies (one of which is mirrored), or possibly five copies (two of which are mirrored) in the case of $\mathcal{H}_q$, and distribute the embeddings of these copies around the cycle, doing this for all coordinates. By unfolding into an odd number ($\leq \tau$) of parts, the distance properties of the construction are unhindered. If these embeddings are interleaved so that no two embeddings of the same coordinate are at distance at most 1 in the cycle $U^{(0)}U^{(1)} \cdots U^{(\lambda)}U^{(0)}$, then the same analysis for girth at the end of the proof of Theorem 2.4.1 applies. If they are interleaved so that no two embeddings of the same coordinate are at distance 0, then cycles of length 4 do not occur but cycles of length 6 may well occur.

Furthermore, when all of the coordinates are unfolded into the same number (three or five) of copies and these are distributed evenly so that each segment of length λ contains exactly one embedding for each coordinate, and the good conduits satisfy a symmetry condition (self-duality) that we describe formally in Section 2.5, then $U^{(0)} \cup U^{(\lambda)} \cup U^{(2\lambda)} \cup \cdots$ induces a clique in the t-th power, increasing the bound on $\omega(G^t)$ (by a factor 3 or 5).

The above modifications do not affect the parity of the main cycle, so we still have that the construction is bipartite if and only if t is even. We shall describe a few further special improvements upon Theorem 2.4.1 within the proof of Theorem 2.1.9.

2.5 Distance-t cliques from good conduits

Good conduits of parameters $(\tau, \Delta, \gamma, c)$ are themselves very nearly cliques in the τ^{th} power and indeed there are two simple ways to modify them to create such cliques. This yields better bounds for $\chi_g^t(d)$ in a few situations when $t \in \{3, 5\}$.

The first idea is to contract a perfect matching, thereby merging the parts.

Proposition 2.5.1. *Let H be a good conduit of parameters $(\tau, \Delta, \gamma, c)$ and let $A = (a_i)_{i\in[n]}, B = (b_i)_{i\in[n]}$ be a matching ordering of H. The graph $\mu(H)$, which we call the* matching contraction *of H, formed from H by contracting every edge a_ib_i, $i \in [n]$ and ignoring any duplicate edges satisfies*

the following properties:

$$\begin{aligned} \operatorname{girth}(\mu(H)) &\geq \frac{\gamma}{2}, \\ \Delta(\mu(H)) &\leq 2(\Delta-1), \\ n(\mu(H)) &\geq c\Delta^{\tau}, \\ \mu(H)^{\tau} &\geq \text{ is a clique.} \end{aligned}$$

Proof. The statements about the maximum degree and number of vertices are trivial to check. That every pair of vertices is joined by a τ-path follows from the distance properties of H as a good conduit. Let $\{v_1, \ldots, v_n\}$ be an ordering of the vertices such that v_i corresponds to the contracted edge a_ib_i for every $i \in \{1, \ldots, n\}$. For the girth, suppose $C = v_{i_0}v_{i_1}\cdots v_{i_\ell}v_{i_0}$ is a cycle of length ℓ in $\mu(H)$. Then, for every $j \in [\ell+1]$, either $a_{i_j}b_{i_{j+1}}$ or $a_{i_{j+1}}b_{i_j}$ is an edge of H. Moreover, every $a_{i_j}b_{i_j}$ is an edge of H, so by also including at most ℓ such edges, we obtain a cycle of length at most 2ℓ in H. So $\operatorname{girth}(H) \leq 2\operatorname{girth}(\mu(H))$, as required. □

The second idea is similar to the first, when we have the additional property that the good conduit is symmetric. More precisely, we say a good conduit between A and B is *self-dual* if there is a bijection $\sigma\colon A \to B$ such that the mapping for which every element $a \in A$ is mapped to $\sigma(a)$ and every element $b \in B$ is mapped to $\sigma^{-1}(b)$ is an automorphism of the graph. In other words, a self-dual good conduit has an embedding such that it is isomorphic to its mirror. Note that this corresponds to the notion of self-duality in generalised polygons, so every self-dual generalised polygon gives rise to a self-dual good conduit. It is known that $\mathcal{Q}_q$, resp. $\mathcal{H}_q$, is self-dual when q is a power of 2, resp. of 3, cf. [98].

Proposition 2.5.2. *Let H be a self-dual good conduit of parameters $(\tau, \Delta, \gamma, c)$ for $\tau \geq 3$, and let $A = (a_i)_{i\in[n]}, B = (b_i)_{i\in[n]}$ be an ordering of H such that $\sigma(a_i) = b_i$ for every $i \leq n$, where σ is an automorphism certifying the self-duality of H. The graph $\mu_\sigma(H)$, which we call the* self-dual contraction *of H, obtained by contracting every pair $a_ib_i, i \in [n]$ and ignoring any duplicate edges satisfies the following properties:*

$$\begin{aligned} g_e(\mu_\sigma(G)) &= \gamma, \\ g_o(\mu_\sigma(G)) &= 2\left\lceil \frac{\gamma}{4} \right\rceil + 1, \\ \Delta(\mu_\sigma(G)) &\leq \Delta, \\ n(\mu_\sigma(G)) &\geq c\Delta^{\tau}, \\ \mu_\sigma(G)^{\tau} &\text{ is a clique.} \end{aligned}$$

Proof. The statements about the number of vertices and $\mu(H)^t$ hold for the same reason as in Proposition 2.5.1. The degree of $\mu(H)$ does not increase because the merged neighbourhood after contracting each pair are symmetric. For the girth statement, suppose $C = v_{i_0}v_{i_1}\cdots v_{i_\ell}v_{i_0}$ is a cycle of length ℓ in $\mu(H)$. Then, for every $j \in [\ell+1]$, both $a_{i_j}b_{i_{j+1}}$ and $b_{i_j}a_{i_{j+1}}$ are an edge of H, by self-duality. As a consequence, $C_\ell \times K_2$ is a subgraph of H. If ℓ is even, this is $C_\ell \cup C_\ell$, and so in particular $\ell \geq \gamma$, while if ℓ is odd, this is $C_{2\ell}$, and so in particular $\ell \geq \gamma/2$. The smallest odd integer greater or equal to $\gamma/2$ is $2\lceil\gamma/4\rceil + 1$. □

2.6 Summary

Let us tie things together for Theorem 2.1.9 before proposing further possibilities.

Proof of Theorem 2.1.9. Table 2.2 explicitly indicates which construction from Section 2.3 or 2.5 is used in each lower bound for $\chi_t(d,g)$ listed in Theorem 2.1.9. In the table, we have used the following notation. The largest prime power not exceeding $d/2-1$ is denoted q, so that $2q+2 \sim d$ as $d \to \infty$. The inequalities in rows involving q hold for all d as $d \to \infty$ since the gap between two successive primes p and p' is $o(p)$ [61]. The mirror of a graph G is denoted by $-G$. We have written $({G_0}^{\alpha_0}, \ldots, {G_{\lambda-1}}^{\alpha_{\lambda-1}})$ for the circular construction as described in Section 2.3, with the adjacencies between $U^{(i)}$ and $U^{(i+1)}$ defined according to G_i along the α_i-th coordinate. Bracketed factors 3 and 5 in the lefthand column require self-duality and are not necessarily valid for all values of d as $d \to \infty$, as we describe below.

In the row for $t = 4$ and girth 6 we use a subgraph of the circular construction. It has vertex set $U^{(0)} \cup U^{(1)} \cup U^{(2)}$, with the edges between $U^{(0)}$ and $U^{(1)}$ embedded according to $\mathcal{P}_q$ along the zeroth coordinate (as in the circular construction), and edges between $U^{(1)}$ and $U^{(2)}$ embedded according to $\mathcal{P}_q$ along the first coordinate. There are no edges between $U^{(0)}$ and $U^{(2)}$. By the same arguments used for the circular construction, we conclude that the girth of the graph is 6. Moreover, the distance properties of $\mathcal{P}_q$ ensure that $U^{(1)}$ induces a clique in the fourth power.

In the row for $t = 4$ and girth 4, we have an additional factor 2, which is justified with the observation that $U^{(0)} \cup U^{(2)}$ induces a clique in the fourth power. This holds similarly for $U^{(0)} \cup U^{(1)}$ in the row for $t = 7$.

For the third-to-last row, it is easily checked that, if $t = 9$ or $t \geq 11$, then t is expressible as a sum of at least three terms in $\{3, 5\}$, so that the circular construction as per Theorem 2.4.1 need only be composed using $\mathcal{Q}_q$'s and $\mathcal{H}_q$'s. Then, as described at the end of Section 2.3, we can unfold each coordinate into three copies, distributed evenly around the cycle, to achieve a girth 8 construction. Optionally, if we use $\mathcal{Q}_{q_2}$ and $\mathcal{H}_{q_3}$, where q_2 is a power of 2 and q_3 is a power of 3 , then the use of self-dual embeddings ensures that we can freely change direction around the main cycle so that $U^{(0)} \cup U^{(\lambda)} \cup U^{(2\lambda)}$ induces a clique in the t-th power. By choosing q_2 and q_3 of similar magnitude (say, by using arbitrarily fine rational approximations of $\log_2 3$), we see that the factor 3 improvement in the inequality holds for infinitely many d. This also explains the girth 6 constructions for $t = 6$ and $t = 8$. A similar argument, where we instead unfold each coordinate into five copies, applies for the last two rows. □

Our work is a first systematic attempt at the problem of Alon and Mohar, although their conjecture — which says for every positive t there is a critical girth g_t such that $\chi_t(d, g_t) = \Theta(d^t)$ and $\chi_t(d, g_t + 1) = \Theta(d^t / \ln d)$ — remains wide open. Because of the reliance upon incidence structures, it seems unlikely that our methods or similar ones could produce constructions of girth higher than 12 or 16. We suspect though that g_t exists and is linear in t.

Another interesting problem: what is the smallest possible value of the stability number $\alpha(G^t)$ of G^t, taken over all graphs G of maximum degree at most d and girth at least g? To our knowledge, this natural extremal problem has not been extensively studied thus far.

2.7 Concluding remarks and open problems

Our goal was to address the question, what is the asymptotically largest value of $\chi_t(G)$ or of $\chi'_t(G)$ among graphs G with maximum degree at most d containing no cycle of length ℓ, where $d \to \infty$?

Bound	Construction that certifies the bound
$\chi_2(d,6) \gtrsim d^2$	$\mathcal{P}_{q'}$, with q' the largest prime power at most $d+1$
$\chi_3(d,5) \gtrsim d^3, 2 \mid d-1$	A self-dual contraction of $\mathcal{Q}_{d-1}$
$\chi_4(d,4) \gtrsim 2d^4/2^4$	$\left(\mathcal{P}_q{}^0, -\mathcal{P}_q{}^0, \mathcal{P}_q{}^1, -\mathcal{P}_q{}^1\right)$
$\chi_4(d,6) \gtrsim d^4/2^4$	A "non-circular" $\left(\mathcal{P}_q{}^0, \mathcal{P}_q{}^1\right)$ (see proof)
$\chi_5(d,7) \gtrsim d^5, 3 \mid d-1$	A self-dual contraction of $\mathcal{H}_{d-1}$
$\chi_5(d,6) \gtrsim d^5/2^5$	A matching contraction of $\mathcal{H}_{q'}$
$\chi_6(d,6) \gtrsim (3)d^6/2^6$	$\left(\mathcal{Q}_q{}^0, \mathcal{Q}_q{}^1\right)$, each coordinate unfolded into three copies
$\chi_7(d,6) \gtrsim 2d^7/2^7$	$\left(\mathcal{Q}_q{}^0, \mathcal{P}_q{}^1, -\mathcal{Q}_q{}^0, -\mathcal{P}_q{}^1, \mathcal{P}_q{}^2, \mathcal{Q}_q{}^0, -\mathcal{P}_q{}^2\right)$
$\chi_8(d,6) \gtrsim (3)d^8/2^8$	$\left(\mathcal{Q}_q{}^0, \mathcal{H}_q{}^1\right)$, each coordinate unfolded into three copies
$\chi_t(d,8) \gtrsim (3)d^t/2^t$, $t = 9$ or $t \geq 11$	A circular construction with at least three τ_j's chosen from $\{3,5\}$ such that they sum to t, each coordinate unfolded into three copies
$\chi_{10}(d,6) \gtrsim (5)d^t/2^t$	A circular construction composed of two $\mathcal{H}_q$'s, each coordinate unfolded into five copies
$\chi_t(d,8) \gtrsim (5)d^t/2^t$, $t \geq 15$, $5 \mid t$	A circular construction composed only of $\mathcal{H}_q$'s, each coordinate unfolded into five copies

Table 2.2: A list of constructions used in the proof of Theorem 2.1.9.

The case $t = 1$ for both parameters and the case $t = 2$ for χ'_t followed from earlier work, but we showed more generally that for each fixed t this question for both parameters can be settled apart from a finite number of cases of ℓ. These exceptional cases are a source of mystery. We would be very interested to learn if the cycle length constraints $2t$, $2t+2$ and $3t$ in Theorems 2.1.3 and 2.1.4 could be weakened (or not).

More specifically, writing

$$\chi_t(d, C_\ell) = \sup\left\{\chi_t(G) \mid \Delta(G) \leq d, C_\ell \nsubseteq G\right\}, \text{ and}$$
$$\chi'_t(d, C_\ell) = \sup\left\{\chi'_t(G) \mid \Delta(G) \leq d, C_\ell \nsubseteq G\right\},$$

the following questions are natural, even if there is no manifest monotonicity in ℓ.

1. For each $t \geq 1$, is there a critical even length ℓ_t^{e} such that for any even ℓ, if $\ell < \ell_t^{\mathrm{e}}$ then $\chi_t(d, C_\ell) = \Theta(d^t)$, while if $\ell \geq \ell_t^{\mathrm{e}}$ then $\chi_t(d, C_\ell) = \Theta(d^t/\ln d)$?

2. For each $t \geq 2$, is there a critical even length ℓ'_t such that for any even ℓ, if $\ell < \ell'_t$ then $\chi'_t(d, C_\ell) = \Theta(d^t)$, while if $\ell \geq \ell'_t$ then $\chi'_t(d, C_\ell) = \Theta(d^t/\ln d)$?

3. For each $t \geq 1$ odd, is there a critical odd length ℓ_t^{o} such that for any odd ℓ, if $\ell < \ell_t^{\mathrm{o}}$ then $\chi_t(d, C_\ell) = \Theta(d^t)$, while if $\ell \geq \ell_t^{\mathrm{o}}$ then $\chi_t(d, C_\ell) = \Theta(d^t/\ln d)$?

The combination of Theorem 2.1.4 and Proposition 2.1.7 demonstrates that the third question has a positive answer, and that

$$\forall t \text{ odd}, \quad \ell_t^{\mathrm{o}} = 3t.$$

We can already give a positive answer to the first and second questions for the values of t for which there exist projective geometries, namely

$$\begin{aligned}\ell_1^{\mathrm{e}} &= \ell_2' = 4,\\ \ell_2^{\mathrm{e}} &= \ell_3' = 6,\\ \ell_3^{\mathrm{e}} &= \ell_4' = 8,\\ \ell_5^{\mathrm{e}} &= \ell_6' = 12.\end{aligned}$$

It is tempting to conjecture that the pattern $\ell_t^{\mathrm{e}} = \ell_{t+1}' = 2t+2$ holds for every value of t, yet the high dependency of those results on the existence of projective geometries is a major flaw in this hypothetical conjecture. In this chapter, we showed the desired upper bounds on those critical lengths.

The above three questions are natural analogues to open questions of Alon and Mohar [8] and of Kaiser and Kang that ask for a critical girth g_t (resp. g_t') for which there is an analogous decrease in the asymptotic extremal behaviour of the distance-t chromatic number (resp. index). If these critical values all exist, it would be natural to think that $g_t = \min\{\ell_t^{\mathrm{e}}, \ell_t^{\mathrm{o}}\}$ and $g_t' = \ell_t'$. But there is limited evidence for the existence questions, let alone this stronger set of assertions. We have established a lower bound of 8 for these hypothetical critical values, when $t \geq 12$, but the odd critical length is the only one for which we managed to exhibit a general construction certifying that it should be unbounded as $t \to \infty$.

As mentioned in the introduction, Vu [119] proved that the exclusion of any fixed bipartite graph is sufficient for a $O(d^2/\ln d)$ upper bound on the strong chromatic index of graphs of maximum degree d. One might wonder, similarly, for each $t \geq 2$ is there a natural wider class of graphs than sufficiently large cycles (of appropriate parity) whose exclusion leads to asymptotically non-trivial upper bounds on the distance-t chromatic number or index?

Chapter 3

Bipartite induced density

Chapter 1 was the occasion to illustrate the general gap that lies between the largest Hall ratio of a class of graphs and its largest (fractional) chromatic number. Well, mathematicians — like nature — abhor a vacuum, and where there is a gap, there is an urge to fill it with new notions.

Somewhere between independent sets and fractional colourings of a graph G lies its *bipartite induced density*. Denoted $\mathrm{bid}(G)$, it is defined as the maximum average degree over all the induced bipartite subgraphs of G;

$$\mathrm{bid}(G) \coloneqq \max_{\substack{H \subseteq G \\ \chi(H) \leq 2}} \mathrm{ad}(H).$$

Finding a bipartite subgraph of G which is a certificate of $\mathrm{bid}(G)$ consists in finding two independent sets of G, the union of which spans as many edges as possible given its size. On the other hand, given a proper k-colouring of G, the $\binom{k}{2}$ bipartite graphs induced by any pair of colours partition $E(G)$. Since every vertex is contained in exactly $k-1$ of them, this implies that if G is of average degree d, then $\frac{d}{k-1}$ is a convex combination of the average degrees of those induced bipartite graphs, and thus by the pigeon-hole principle one of them has to be of average degree at least $\frac{d}{k-1}$. We will give a formal proof of a similar result using a fractional colouring.

So we have a parameter which relaxes the notion of fractional colouring with a consideration of a specific structured subset of edges, which could be seen as an edge version of the independence number — indeed the independence number relaxes the notion of fractional colouring with a consideration of a specific structured subset of vertices. What would be of interest would be to prove the analogue of a conjecture concerning fractional colouring in the setting of bipartite induced density. In this chapter, we study the analogue of Conjecture 0.2.4 for bipartite induced density.

The content of this chapter is covered by the submitted articles [26, 31].

3.1 Introduction

Our starting point is a conjecture of Esperet, Kang, and Thomassé, which would be sharp up to the choice of constant if true.

Conjecture 3.1.1 (Esperet, Kang, and Thomassé [49])**.** *There is a constant $C > 0$ such that any triangle-free graph with minimum degree at least d contains a bipartite induced subgraph of minimum degree at least $C \ln d$.*

Although the conjecture is new, it might be difficult. Conjecture 3.1.1 aligns with central challenges in combinatorics, especially about colourings and independent sets in triangle-free graphs. For example, Johansson's theorem [65] for colouring triangle-free graphs (combined with the pigeonhole principle) implies Conjecture 3.1.1 for any triangle-free graph with $O(d)$ maximum degree. Similarly, a result of Ajtai, Komlós and Szemerédi [2] about the off-diagonal Ramsey numbers $R(3,t)$ confirms Conjecture 3.1.1 for any triangle-free graph on n vertices provided $d = \Omega(n^{2/3}\sqrt{\ln n})$ as $n \to \infty$.

Our main result is a stronger, near optimal version of this last statement.

Theorem 3.1.1. *There are constants $C_1, C_2 > 0$ such that, for $0 \le d \le n/2$,*

- *any triangle-free graph on $n \ge 2$ vertices with minimum degree at least d contains a bipartite induced subgraph of minimum degree at least $\max\{C_1 d\sqrt{\ln n/n}, d^2/(2n)\}$; and*
- *provided n/d is large enough, there is a triangle-free graph on between $n/2$ and n vertices with minimum degree at least d such that every bipartite induced subgraph has minimum degree at most $\lceil C_2 d^2/n \rceil \ln n$.*

Thus we observe the following phase transition behaviour: for d there is a critical exponent of n (namely $1/2$) above which we can be assured of bipartite induced minimum degree polynomially large in n (and below which we cannot). Theorem 3.1.1 resolves Problem 4.1 in [49] up to a logarithmic factor. Our constructions for near optimality are blow-ups of an adaptation of Spencer's construction for lower bounds on $R(3,t)$ [112] (see Section 3.3).

Due to a captivating connection between bipartite induced density and fractional colouring [49] (see Section 3.2 for more details), Theorem 3.1.1 is closely related to the following extremal result.

Theorem 3.1.2. *There are constants $C_1, C_2 > 0$ such that, for $0 < d \le n/2$,*

- *any triangle-free graph on n vertices with minimum degree at least d has fractional chromatic number at most $\min\{C_1\sqrt{n/\ln n}, n/d\}$;*
- *provided n/d is large enough, there is a triangle-free graph on between $n/2$ and n vertices with minimum degree at least d and fractional chromatic number at least $C_2 \min\{\sqrt{n/\ln n}, n/d\}$.*

This bound is basic, but has not appeared in the literature as far as we know. There is equality in the n/d bound when $d = n/2$, in which case we must have a complete bipartite graph with two equal-sized parts. As for the $\Theta(\sqrt{n/\ln n})$ bound, it is an interesting problem to sharpen the asymptotic constants. In Section 3.4, we combine Theorem 1.1.1 (Johansson-Molloy) with the proof idea in Theorem 3.1.2 to show the following as a first step.

Theorem 3.1.3. *As $n \to \infty$, any triangle-free graph on n vertices has fractional chromatic number at most $(2 + o(1))\sqrt{n/\ln n}$.*

Note that if one could improve the factor $(2 + o(1))$ to $(\sqrt{2} + o(1))$ (which we formally state as a conjecture below), then it would match the best to date asymptotic upper bound for the Ramsey numbers $R(3,t)$ due to Shearer [108]. By the final outcome of the triangle-free process [16, 52], Theorem 3.1.3 is sharp up to a $(2\sqrt{2} + o(1))$ factor. (The triangle-free process also gives sharpness up to a constant factor in Theorem 3.1.2.)

Theorem 3.1.1 is not far from optimal, but the best constructions we know so far are almost regular. As noted above, we already know Conjecture 3.1.1 in the almost regular case. This motivates the following bound which improves on Theorem 3.1.1 if the graph is irregular. We prove this in Section 3.6.

Theorem 3.1.4. *Any triangle-free graph on $n \geq 2$ vertices with $m \geq 1$ edges and w_3 (directed) three-edge walks contains a bipartite induced subgraph of minimum degree at least $w_3/(4nm)$.*

Observe that $w_3 \geq 2md^2$ if the graph has minimum degree at least d (with equality for d-regular graphs), and so Theorem 3.1.4 directly implies the $d^2/(2n)$ bound. Put another way, Theorem 3.1.4 replaces the squared minimum degree term in the bound of Theorem 3.1.1 by the average over all edges of the product of the two endpoint degrees.

Theorem 3.1.2 inspires the following question, which is left to further study.

Problem 3.1.5. *Given a function $d = d(n)$, letting $\chi(n, d)$ denote the largest chromatic number of a triangle-free graph on n vertices with minimum degree at least d, asymptotically what is $\chi(n, d)/\min\{\sqrt{n/\ln n}, n/d\}$ as $n \to \infty$?*

In a sense, substantial effort has already been devoted to this problem when d is linear in n, in relation to a problem of Erdős and Simonovits [47] discussed in Section 0.4.2, see e.g. [116, 23]. In particular, for $\varepsilon > 0$ fixed, the answer is $\omega(1)$ if $d < (1/3 - \varepsilon)n$ while it is $O(1)$ if $d > (1/3 + \varepsilon)n$ *or* if $d = O(\sqrt{n \ln n})$. A result of Lovász [85] (and Theorem 3.1.2) implies that the ratio is always $O(\ln n)$.

Outline of the chapter. This chapter is organized as follows. In Section 3.2 we prove Theorem 3.1.2, which in turn implies the lower bound of Theorem 3.1.1. In Section 3.3, we use the Local Lemma to construct a random graph that certifies the upper bound of Theorem 3.1.1. In the remaining sections we explore generalizations of Theorems 3.1.1 and 3.1.2. In Section 3.4, we prove Theorem 3.1.3 and we also provide bounds in terms of the number of edges. In Section 3.5, we go a bit further and extend this analysis to graphs with sparse neighbourhoods. Section 3.6 is devoted to the proof of Theorem 3.1.4 and related distance-3 results. Finally, in Section 3.7 we make some concluding remarks concerning the exclusion of an arbitrary subgraph. In particular, we show how our results generalize to excluding any given cycle as a subgraph.

Note added. Shortly after the content of this chapter was posted to a public preprint repository, we learned that Matthew Kwan, Shoham Letzter, Benny Sudakov and Tuan Tran [83] independently obtained a finer version of Theorem 3.1.1 with different methods. Eventually, they moreover proved a marginally weaker form of Conjecture 3.1.1, guaranteeing a bipartite induced subgraph of minimum degree at least $C \ln d / \ln \ln d$.

Probabilistic preliminaries. We use a specific form of the Chernoff bound [62, (2.11)] and an alternative formulation of the Lovász Local Lemma [112, Thm. 1.3].

Theorem 3.1.6. *If $x \geq 7np$, then*

$$\Pr(\mathrm{Bin}(n, p) \geq x) < e^{-x}.$$

Theorem 3.1.7 (General Local Lemma)**.** *Consider a set $\mathscr{E} = \{E_1, \ldots, E_n\}$ of (bad) events, and let $\Gamma_i \subseteq \mathscr{E} \setminus \{E_i\}$ be a subset of events such that E_i is mutually independent of $\mathscr{E} \setminus (\Gamma_i \cup \{E_i\})$, for every i. If there exist real numbers $y_1, \ldots, y_n > 0$ such that for each i,*

$$y_i \mathbb{P}[E_i] < 1 \quad \text{and} \quad \ln y_i > \sum_{E_j \in \Gamma_i} y_j \mathbb{P}[E_j],$$

then the probability that none of the events in $\mathscr{E}$ occur is positive.

3.2 Fractional colouring

Given a graph $G = (V, E)$, we say that a probability distribution $\mathcal{P}$ over the independent sets of G satisfies property Q^*_r if $\mathbb{P}[v \in \mathbf{I}] \geq r$ for every $v \in V$ and $\mathbf{I}$ taken randomly according to $\mathcal{P}$. Recall that the *fractional chromatic number* $\chi_f(G)$ of G is defined as the smallest k such that there is a probability distribution over the independent sets of G satisfying property $\mathrm{Q}^*_{1/k}$. Here is a link between fractional colouring and bipartite induced density.

Theorem 3.2.1 (Esperet, Kang, Thomassé, 2019 [49])**.** *Any graph with fractional chromatic number k and average degree d has a bipartite induced subgraph of average degree at least d/k.*

This result first appeared with a stronger condition that the graph is of minimum degree d rather than of average degree d. Here is the proof of the stronger statement.

Proof of Theorem 3.2.1. Let G be a graph and let $\mathcal{P}$ be a probability distribution over the independent sets of G with property $\mathrm{Q}^*_{1/k}$. Without loss of generality, we may assume that $\mathbb{P}[v \in \mathbf{I}] = 1/k$ for every vertex $v \in V$ and $\mathbf{I}$ a random independent set taken according to $\mathcal{P}$. Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be two independent sets taken independently at random according to $\mathcal{P}$. Note that $\mathbb{E}[|\mathbf{I}_1|] = \mathbb{E}[|\mathbf{I}_2|] = n/k$ by the assumption on $\mathcal{P}$. Moreover, for any edge $uv \in E(G)$, the probability that uv is in the induced subgraph $G[\mathbf{I}_1 \cup \mathbf{I}_2]$ is

$$\mathbb{P}[u \in \mathbf{I}_1]\,\mathbb{P}[v \in \mathbf{I}_2] + \mathbb{P}[u \in \mathbf{I}_2]\,\mathbb{P}[v \in \mathbf{I}_1] = \frac{2}{k^2}.$$

By linearity of expectation, we have that

$$\mathbb{E}\left[e(G[\mathbf{I}_1 \cup \mathbf{I}_2]) - (|\mathbf{I}_1| + |\mathbf{I}_2|)\,\frac{d}{2k}\right] = \frac{2e(G)}{k^2} - \frac{nd}{k^2} = 0.$$

The probabilistic method guarantees the existence of independent sets I_1 and I_2 of G such that

$$e(G[I_1 \cup I_2]) \geq (|I_1| + |I_2|)\frac{d}{2k}.$$

So $G[I_1 \cup I_2]$ is an induced bipartite subgraph of G, of average degree at least d/k. □

There is always a subgraph whose minimum degree is at least half the graph's average degree. Thus to obtain the lower bound in Theorem 3.1.1 we only need the upper bound in Theorem 3.1.2.

Proof of Theorem 3.1.2. Let $G = (V, E)$ be a triangle-free graph on n vertices with minimum degree at least d. Since fractional chromatic number is at most chromatic number, the first term of the upper bound was already observed by Erdős and Hajnal [45] as a consequence of the aforementioned result of Ajtai, Komlós and Szemerédi [2] (see Section 3.4). For the second term of the upper bound, choose $\mathbf{I}$ from full neighbourhood sets uniformly over all n such sets. Since G is triangle-free, $\mathbf{I}$ is an independent set. For all $v \in V$,

$$\mathbb{P}[v \in \mathbf{I}] = \frac{\deg(v)}{n} \geq \frac{d}{n}.$$

We have shown then that this distribution has property $\mathrm{Q}^*_{d/n}$, as required.

For sharpness, fix $\varepsilon > 0$ and let j be the minimum between n and the least value for which $2\sqrt{(2+\varepsilon/4)j/\ln j} \geq n/d$, and consider the final output of the triangle-free process on j vertices.

This is a triangle-free random graph that was shown, independently, by Bohman and Keevash [16] and by Fiz Pontiveros, Griffiths and Morris [52], to have minimum degree $(1+o(1))\sqrt{j\ln j/2}$ and stability number at most $(1+o(1))\sqrt{2j\ln j}$ with high probability as $j\to\infty$. For large enough j (which we can guarantee if n/d is large enough), we may fix a triangle-free graph $\hat{G}$ that has minimum degree at least $\sqrt{j\ln j/(2+\varepsilon/4)}$ and stability number at most $\sqrt{(2+\varepsilon/4)j\ln j}$. Form a new graph G from $\hat{G}$ by replacing each vertex by an independent set of size $\lfloor n/j\rfloor$, and adding a complete bipartite graph between every pair of independent sets that corresponds to an edge in $\hat{G}$. Observe that G is a triangle-free graph on between $n/2$ and n vertices with minimum degree at least $\sqrt{j\ln j/(2+\varepsilon/4)}\lfloor n/j\rfloor\geq d$. Moreover, G has stability number at most $\sqrt{(2+\varepsilon/4)j\ln j}\lfloor n/j\rfloor\leq 2(2+\varepsilon/4)d$ and so has fractional chromatic number at least $(n/2)/(2(2+\varepsilon/4)d)=n/((8+\varepsilon)d)$, as desired.

Note that we essentially lost a factor 2 twice due to rounding, which is only an issue when $d=\Theta(\sqrt{n\ln n})$. Thus when $d=\omega(\sqrt{n\ln n})$ the n/d upper bound is in fact correct up to a factor $(2+o(1))$ as $n\to\infty$, and so in this case we can take the choice $C_2=1/2+o(1)$.

For the other lower bound, recall that the triangle-free process on n vertices yields, with high probability, a random graph with stability number at most $(1+o(1))\sqrt{2n\ln n}$. Thus the fractional chromatic number of that graph is at least $(1+o(1))\sqrt{n/(2\ln n)}$. □

3.3 Near optimality

As mentioned before, the lower bound in Theorem 3.1.1 follows from Theorem 3.1.2, so it just remains to prove near sharpness, which is our next task.

Instead of a dense bipartite *induced subgraph*, we might be satisfied with a dense bipartite *subgraph* where we only require that (at least) one of the two parts induces an independent set. Given $G=(V,E)$, we call an induced subgraph $G'=(V',E')$ of G *semi-bipartite* if it admits a partition $V'=V_1\cup V_2$ such that V_1 is an independent set of G, and we define the degree of a vertex of G' with respect to the semi-bipartition as its degree in the bipartite subgraph $G[V_1,V_2]$ between V_1 and V_2 (and so we ignore any edges in V_2). A version of Conjecture 3.1.1 where 'bipartite' is replaced by 'semi-bipartite' is known [49]. Let us begin by improving on [49, Theorem 3.6] by providing a version where the minimum degree is replaced with the geometric mean of the degrees, all the while increasing the leading constant.

Theorem 3.3.1. *A triangle-free graph G on n vertices contains a semi-bipartite induced subgraph of average degree at least $(2+o(1))\frac{1}{n}\sum_{v\in V(G)}\ln\deg(v)$.*

In the statement of the theorem and in the proof below, the $o(1)$ term tends to zero as the geometric mean of the degree sequence of G tends to infinity. The proof uses a result from Chapter 1, and the method relies on the properties of a random independent set drawn according to the hard-core model, also defined in Chapter 1.

Proof of Theorem 3.3.1. We find a semi-bipartite induced subgraph of G where one of the parts is a random independent $\mathbf{I}$ from the hard-core model, and the other is $V(G)\setminus\mathbf{I}$. The number of edges between the parts is therefore $\mathbf{X}=\sum_{v\in\mathbf{I}}\deg(v)$. We write $\mathbb{E}[\mathbf{X}]$ in two different ways:

$$\mathbb{E}[\mathbf{X}]=\sum_{v\in V(G)}\deg(v)\mathbb{P}[v\in\mathbf{I}]=\sum_{v\in V(G)}\mathbb{E}[|N(v)\cap\mathbf{I}|]. \tag{3.1}$$

The first version is obtained using the linearity of expectation, and the second comes from the fact that $\mathbb{E}\left[|N(v)\cap\mathbf{I}|\right]=\sum_{u\in N(v)}\mathbb{P}\left[u\in\mathbf{I}\right]$ and hence $\mathbb{P}\left[u\in\mathbf{I}\right]$ appears $\deg(u)$ times in the sum as required. For any $\alpha,\beta>0$ we have

$$(\alpha+\beta)\mathbb{E}\left[\mathbf{X}\right]=\sum_{v\in V(G)}\Big(\alpha\deg(v)\mathbb{P}\left[v\in\mathbf{I}\right]+\beta\mathbb{E}\left[|N(v)\cap\mathbf{I}|\right]\Big),$$

hence by Lemma 1.4.4,

$$\mathbb{E}\left[\mathbf{X}\right]\geq\frac{n\lambda\left(\frac{1}{n}\sum_{v\in V(G)}\ln\deg(v)+\ln\frac{\alpha}{\beta}+\ln\ln(1+\lambda)+1\right)}{\left(1+\frac{\alpha}{\beta}\right)(1+\lambda)\ln(1+\lambda)}.$$

Choosing e.g. $\dfrac{\alpha}{\beta}=\lambda=\dfrac{n}{\sum_{v\in V(G)}\ln\deg(v)}$, we observe that

$$\mathbb{E}\left[\mathbf{X}\right]\geq(1+o(1))\sum_{v\in V(G)}\ln\deg(v).$$

To complete the proof, note that the bound on $\mathbb{E}\left[\mathbf{X}\right]$ means that there is at least one independent set I with at least $(1+o(1))\sum_{v\in V(G)}\ln\deg(v)$ edges from I to its complement. This immediately means that the average degree of the semi-bipartite subgraph with parts I and $V(G)\setminus I$ is at least

$$(2+o(1))\frac{1}{n}\sum_{v\in V(G)}\ln\deg(v).$$

□

In what follows we give near optimal constructions for Theorem 3.1.1 not only for bipartite induced density, but also for semi-bipartite induced density. The following result, an adaptation of work of Spencer [112], is central. It might also be possible to adapt an earlier construction due to Erdős [43], but it would produce a construction comparable to Theorem 3.1.1. Although we chose not to pursue it, we suspect that the outcome at the end of the triangle-free process is significantly better, and optimal up to a constant factor. For this reason, we did not optimise either of the constants below.

Theorem 3.3.2. *There exist constants $\delta,\gamma>0$ such that for every large enough n there is a triangle-free graph on n vertices with minimum degree at least $\delta\sqrt{n}$ that contains no semi-bipartite induced subgraph of minimum degree at least $\gamma\ln n$.*

Before proving this, let us see how it implies the second part of Theorem 3.1.1.

Proof of sharpness in Theorem 3.1.1. Let j be the smaller of n and the least value for which $2\sqrt{j}\geq\delta n/d$. Provided j is large enough, we may by Theorem 3.3.2 fix a triangle-free graph $\hat{G}$ that has minimum degree at least $\delta\sqrt{j}$ that contains no semi-bipartite induced subgraph of minimum degree at least $\gamma\ln j$. Consider a new graph G formed from $\hat{G}$ by replacing each vertex by an independent

set of size $\lfloor n/j \rfloor$, and adding a complete bipartite graph between every pair of independent sets that corresponds to an edge in $\hat{G}$. Note that G is a triangle-free graph on between $n/2$ and n vertices with minimum degree at least $\delta\sqrt{j}\lfloor n/j \rfloor \geq d$. Moreover, the largest minimum degree of a semi-bipartite induced subgraph in G is smaller than $\gamma\lfloor n/j \rfloor \ln j \leq \gamma\lceil \delta^2 d^2/(16n) \rceil \ln n$. □

For Theorem 3.3.2 we will need the convenient observation that, if we do not mind constant factors, it suffices to consider only semi-bipartite induced subgraphs with both parts of equal size.

Proposition 3.3.3 ([49]). *Suppose $A, B \subseteq G$ are disjoint with $|A| \geq |B|$ and satisfy that the average degree of $G[A, B]$ is d. Then there exists $A' \subseteq A$ with $|A'| = |B|$ such that the average degree of $G[A', B]$ is at least $d/2$.*

Proof of Theorem 3.3.2. For a sufficiently large positive integer n, let $p = c_1/\sqrt{n}$ and $t = c_2\sqrt{n}\ln n$ for some fixed $c_1, c_2 > 0$. Consider the binomial random graph $G(n, p)$. Fix $0 < \alpha < 1$ and $\beta > 0$. The constants c_1, c_2, α, β will be specified more precisely later in the proof.

With a view to applying the General Local Lemma, let us define four types of (bad) events in $G(n, p)$.

A For a set of three vertices, it induces a triangle.

B For a set of t vertices, it induces an independent set.

C For a single vertex, it has degree at most $(1 - \alpha)np$.

D_i For two disjoint sets of i vertices, the bipartite subgraph induced by the cut between the two sets has average degree at least $\beta \ln n$.

Note that by Proposition 3.3.3 and the choice of p, we obtain the desired graph if there is an element of the probability space $G(n, p)$ for which no event of Types A, B, C, and D_i, $\beta \ln n \leq i \leq t$, occur.

Let us write $P(*)$ for the probability of an event of Type $*$. We have that $P(\mathrm{A}) = p^3$ and $P(\mathrm{B}) = (1-p)^{\binom{t}{2}} < e^{-p\binom{t}{2}}$. Note that $P(\mathrm{C}) = \mathbb{P}\left[\mathrm{Bin}(n, p) \leq (1-\alpha)np\right] \leq e^{-\alpha^2 np/2}$ by a Chernoff Bound. Since $ip \leq tp \leq c_1c_2 \ln n$, $P(\mathrm{D}_i) = \mathbb{P}\left[\mathrm{Bin}(i^2, p) \geq \beta i \ln n\right] \leq e^{-\beta i \ln n}$ by a Chernoff Bound with a choice of β satisfying

$$\beta \geq 7c_1c_2. \tag{3.2}$$

Let us write that each Type $*$ event is mutually independent of all but $N(*, *')$ events of Type $*'$. We have that $N(\mathrm{A}, \mathrm{A}) = 3(n-3) < 3n$, $N(\mathrm{B}, \mathrm{A}) = \binom{t}{2}(n-t) + \binom{t}{3} < t^2 n/2$, $N(\mathrm{C}, \mathrm{A}) = \binom{n-1}{2} < n^2/2$, and $N(\mathrm{D}_i, \mathrm{A}) < i^2 n$. More crudely, we have $N(\mathrm{A}, \mathrm{B})$, $N(\mathrm{B}, \mathrm{B})$, $N(\mathrm{C}, \mathrm{B})$, and $N(\mathrm{D}_i, \mathrm{B})$ are all at most $\binom{n}{t} < (en/t)^t = e^{t\ln(en/t)}$; $N(\mathrm{A}, \mathrm{C})$, $N(\mathrm{B}, \mathrm{C})$, $N(\mathrm{C}, \mathrm{C})$, and $N(\mathrm{D}_i, \mathrm{C})$ are all at most n; and $N(\mathrm{A}, \mathrm{D}_j)$, $N(\mathrm{B}, \mathrm{D}_j)$, $N(\mathrm{C}, \mathrm{D}_j)$, and $N(\mathrm{D}_i, \mathrm{D}_j)$ are all at most $\binom{n}{2j} < (en/2j)^{2j} = e^{2j\ln(en/2j)}$,

Let us write $\mathscr{E} = \{A, B, C\} \cup \{D_i \mid \beta \ln n \leq i \leq t\}$. By the General Local Lemma, we only need to find, for $\beta \ln n \leq i \leq t$, positive reals $Y(\mathrm{A})$, $Y(\mathrm{B})$, $Y(\mathrm{C})$, and $Y(\mathrm{D}_i)$ such that the following inequalities hold:

$$\forall X \in \mathscr{E}, \quad Y(X)P(X) < 1$$
$$\forall X \in \mathscr{E}, \quad \ln Y(X) > \sum_{X' \in \mathscr{E} - X} Y(X')P(X')N(X, X')$$

The estimates that we derived earlier imply that it instead suffices to find positive reals $Y(\mathrm{A})$, $Y(\mathrm{B})$, $Y(\mathrm{C})$, and $Y(\mathrm{D}_i)$ for which the following hold:

$$\begin{aligned}
&Y(\mathrm{A}) \cdot p^3 < 1;\\
&Y(\mathrm{B}) \cdot e^{-p\binom{t}{2}} < 1;\\
&Y(\mathrm{C}) \cdot e^{-\frac{\alpha^2 np}{2}} < 1;\\
\forall i,\quad &Y(\mathrm{D}_i) \cdot e^{-\beta i \ln n} < 1;\\
&Z < \ln Y(\mathrm{A}) - Y(\mathrm{A})p^3 \cdot 3n;\\
&Z < \ln Y(\mathrm{B}) - Y(\mathrm{A})\frac{p^3 t^2 n}{2};\\
&Z < \ln Y(\mathrm{C}) - Y(\mathrm{A})\frac{p^3 n^2}{2};\text{ and}\\
\forall i,\quad &Z < \ln Y(\mathrm{D}_i) - Y(\mathrm{A})\frac{p^3 i^2}{n};
\end{aligned}$$

where

$$Z = Y(\mathrm{B})e^{-p\binom{t}{2}+t\ln\frac{en}{t}} + Y(\mathrm{C}) \cdot e^{-\frac{\alpha^2 np}{2}} \cdot n + \sum_j Y(\mathrm{D}_j)e^{-\beta j \ln n + 2j \ln \frac{en}{2j}}.$$

Let us choose $Y(\mathrm{A}) = 1+\varepsilon$, $Y(\mathrm{B}) = e^{c_3\sqrt{n}(\ln n)^2}$, $Y(\mathrm{C}) = e^{c_4\sqrt{n}}$, and $Y(\mathrm{D}_i) = e^{c_5 i \ln n}$ for some fixed $\varepsilon, c_3, c_4, c_5 > 0$.

We first consider the asymptotic behaviour of the three constituents of Z as $n \to \infty$. For the first two, we have

$$\begin{aligned}
Y(\mathrm{B})e^{-p\binom{t}{2}+t\ln\frac{en}{t}} &= e^{\left(c_3 - \frac{c_1 c_2^2}{2} + c_2 + o(1)\right)\sqrt{n}(\ln n)^2},\text{ and}\\
Y(\mathrm{C}) \cdot 2e^{-\frac{\alpha^2 np}{3}} n &= e^{\left(c_4 - \frac{\alpha^2 c_1}{2}\right)\sqrt{n}}.
\end{aligned}$$

For the third, note since $\beta \ln n \le j \le t$ that

$$\begin{aligned}
\sum_j Y(\mathrm{D}_j)e^{-\beta j \ln n + 2j \ln \frac{en}{2j}} &\le \sum_j e^{(c_5 - \beta + 2 + o(1))j \ln n}\\
&\le e^{(c_5 - \beta + 2 + o(1))\beta (\ln n)^2}.
\end{aligned}$$

We may therefore conclude that Z is superpolynomially small in n provided

$$c_3 - \frac{c_1 c_2^2}{2} + c_2 < 0, \tag{3.3}$$

$$c_4 - \frac{\alpha^2 c_1}{2} < 0,\text{ and} \tag{3.4}$$

$$c_5 - \beta + 2 < 0. \tag{3.5}$$

Of the remaining terms in the inequalities required for the application of the General Local

Lemma, the critical ones can be seen to be polylogarithmic or greater in magnitude (as $n \to \infty$):

$$\ln Y(\mathrm{B}) - Y(\mathrm{A})\frac{p^3t^2n}{2} = \left(c_3 - (1+\varepsilon)\frac{c_1^3c_2^2}{2} + o(1)\right)\sqrt{n}(\ln n)^2;$$
$$\ln Y(\mathrm{C}) - Y(\mathrm{A})\frac{p^3n^2}{2} = \left(c_4 - (1+\varepsilon)\frac{c_1^3}{2} + o(1)\right)\sqrt{n}; \text{ and}$$
$$\ln Y(\mathrm{D}_i) - Y(\mathrm{A})\frac{p^3i^2}{n} \geq \left(c_5 - (1+\varepsilon)c_1^3c_2\right) i \ln n$$

(where we used $i \leq c_2\sqrt{n}\ln n$ in the last line).

We therefore also want that

$$c_3 - (1+\varepsilon)\frac{c_1^3c_2^2}{2} > 0, \tag{3.6}$$
$$c_4 - (1+\varepsilon)\frac{c_1^3}{2} > 0, \text{ and} \tag{3.7}$$
$$c_5 - (1+\varepsilon)c_1^3c_2 > 0. \tag{3.8}$$

It remains only to show that there is some choice of $c_1, \ldots, c_5, \alpha, \beta, \varepsilon$ so that (3.2)–(3.8) are fulfilled. Note that, whatever the other choices, the inequalities (3.2), (3.5), and (3.8) are satisfied with a sufficiently large choice of β or of c_5. By pairing inequalities (3.3) and (3.6) as well as (3.4) and (3.7), we need

$$c_2\left(\frac{c_1c_2}{2} - 1\right) > c_3 > (1+\varepsilon)\frac{c_1^3c_2^2}{2}, \text{ and}$$
$$\frac{\alpha^2c_1}{2} > c_4 > (1+\varepsilon)\frac{c_1^3}{2}.$$

We have that

$$c_2\left(\frac{c_1c_2}{2} - 1\right) > (1+\varepsilon)\frac{c_1^3c_2^2}{2} \iff c_2 > \frac{2}{c_1 - (1+\varepsilon)c_1^3}, \text{ and}$$
$$\frac{\alpha^2c_1}{2} > (1+\varepsilon)\frac{c_1^3}{2} \iff \alpha > \sqrt{1+\varepsilon}\cdot c_1.$$

Let us then fix $c_1 = 1/\sqrt{3}$. Therefore with a small enough choice of $\varepsilon > 0$ it is possible to choose, say, $c_2 = 21/4$ and $\alpha = 3/4$ (< 1 particular) and then take, say, $c_3 = 0.51$ and $c_4 = 0.97$. This completes the proof. □

3.4 Fractional colouring redux

In this section, we make a first step towards optimising the asymptotic constant for the first term in Theorem 3.1.2. It turns out that this is related to two problems of Erdős and Hajnal [45], concerning the asymptotic order of the chromatic number of a triangle-free graph with a given number of vertices or edges. In terms of edges, the correct order upper bound was first shown by Poljak and Tuza [102]. Matching lower bounds to settle both problems were established as byproduct to the determination of the asymptotic order of the Ramsey numbers $R(3,t)$ by Kim [79], cf. [56, 97].

For completeness we reiterate more precisely the observation of Erdős and Hajnal [45] mentioned in the proof of Theorem 3.1.2 (see also [63, pp. 124–5] and [79]). An application of Shearer's lower bound on the stability number [108] in a greedy colouring procedure bounds the chromatic number by at most $(4 + o(1))$ times optimal (as certified by the triangle-free process [16, 52]).

Lemma 3.4.1 (cf. Jensen and Toft [63])**.** *Let $\mathscr{G}$ be a class of graphs that is closed under vertex-deletion. Suppose for some $x_0 \geq 2$ that there is a continuous, non-decreasing function $f_{\mathscr{G}} : [x_0, \infty) \to \mathbb{R}^+$ such that every $G' \in \mathscr{G}$ on $x \geq x_0$ vertices has an independent set of at least $f_{\mathscr{G}}(x)$ vertices. Then every $G \in \mathscr{G}$ on $n \geq x_0$ vertices has chromatic number at most*

$$x_0 + \int_{x_0}^{n} \frac{dx}{f_{\mathscr{G}}(x)}.$$

Corollary 3.4.1.1. *As $n \to \infty$, any triangle-free graph on n vertices has chromatic number at most $(2\sqrt{2} + o(1))\sqrt{n/\ln n}$.*

Proof. Shearer [108] showed that, for any $\varepsilon > 0$, there exists $x_0 \geq 2$ such that the function $f_{\mathscr{G}}(x) = (1/\sqrt{2} - \varepsilon)\sqrt{x \ln x}$ satisfies the hypothesis of Lemma 3.4.1 for the class $\mathscr{G}$ being the triangle-free graphs. Lemma 3.4.1 yields the desired outcome after an exercise in analysis to show that

$$\lim_{n\to\infty} \frac{\int_{x_0}^{n} \frac{dx}{\sqrt{x \ln x}}}{\sqrt{n/\ln n}} = 2.$$

□

Is the factor $(2 + o(1))$ contribution from the above limit truly necessary? We were unable to address this issue, but Theorem 3.1.3 shows it possible to reduce the bound by a factor $(\sqrt{2}+o(1))$ if we only wish to bound the fractional chromatic number. Note that definitive progress on whether it is possible to improve by *strictly* more than a factor $(2+o(1))$ in Corollary 3.4.1.1, for fractional or not, either positively or negatively, would likely constitute a major breakthrough in combinatorics. A factor $(2 + o(1))$ improvement is indeed plausible, especially for fractional.

Conjecture 3.4.1. *As $n \to \infty$, any triangle-free graph on n vertices has fractional chromatic number at most $(\sqrt{2} + o(1))\sqrt{n/\ln n}$.*

Relatedly, in the spirit of [45], we also conjecture the following.

Conjecture 3.4.2. *As $m \to \infty$, any triangle-free graph with m edges has fractional chromatic number at most $(2^{4/3} + o(1))m^{1/3}/(\ln m)^{2/3}$.*

Both of the above conjectures hold for regular triangle-free graphs.

Here it seems natural to use Theorem 1.1.1 (Molloy-Johansson). Since it is a much stronger form of Shearer's bound, one might wonder if it alone is enough to verify Conjecture 3.4.1. This does not seem to be the case. We remark however that Theorems 3.1.2 and 1.1.1 together immediately yield Conjectures 3.4.1 and 3.4.2 for *regular* triangle-free graphs. (Let G be a D-regular triangle-free graph. If $D \geq \sqrt{n \ln n/2}$, then it follows from Theorem 3.1.2; otherwise, it follows from Theorem 1.1.1.) Moreover, as we will shortly see, an iterated application of Theorem 1.1.1 combined with the simple idea in the proof of Theorem 3.1.2 yields Theorem 3.1.3.

Recall that we may equivalently define the fractional chromatic number of a graph as the smallest k such that there is an assignment of measurable subsets of the interval $[0, k]$ (or rather of any subset of $\mathbb{R}$ of measure k) to the vertices such that each vertex is assigned a subset of measure 1 and subsets assigned to adjacent vertices are disjoint.

Proof of Theorem 3.1.3. Fix $\varepsilon > 0$. Without loss of generality, assume $\varepsilon < 1/2$. Let $G = (V, E)$ be a triangle-free graph on n vertices and let $D \leq n$ be some positive integer to be specified later in the proof. We first associate n disjoint intervals of measure $1/D$ to each of the full neighbourhood sets (each of which is an independent set), and assign each such interval to its neighbourhood's vertices. By independently, arbitrarily de-assigning some (parts) of these intervals, we may assume each vertex of degree at least D has an assignment of measure exactly 1. On the other hand, the subgraph of vertices of measure less than 1 has maximum degree less than D. More precisely, let V_i be the set of vertices of degree exactly i in G, for $i < D$: this initial partial fractional colouring gives each vertex of V_i an assignment of measure exactly $i/D < 1$. We have essentially shown how it suffices to restrict our attention to G having maximum degree D (by the simple idea in the proof of Theorem 3.1.2).

For each $D^{1/(1+\varepsilon/5)} \leq i < D$, let us write G_i for the subgraph of G induced by $\cup_{j=0}^{i} V_j$. Since G_i is a triangle-free graph of maximum degree at most i, it follows from Theorem 1.1.1 that G_i admits a proper colouring c_i of its vertices with at most $(1+\varepsilon/5)i/\ln i$ colours, provided i is large enough (since $i \geq D^{1/(1+\varepsilon/5)}$, i is arbitrarily large if D is.) For each colour class $\mathcal{C}$ of c_i, we choose an interval of measure $1/D$ (that is disjoint from all previously used intervals), and assign it to each vertex of $\mathcal{C}$.

This extends the initial partial fractional colouring to nearly all of G. If $D^{1/(1+\varepsilon/5)} \leq i < D$, then each vertex of V_i has been assigned $D - i$ additional intervals of measure $1/D$, resulting in an assignment of measure 1. Note that, for D large enough, the total measure of the subsets we have thus used is

$$\begin{aligned}\frac{n}{D} + \frac{1}{D} \sum_{i=\lceil D^{1/(1+\varepsilon/5)} \rceil}^{D} \frac{(1+\varepsilon/5)i}{\ln i} &\leq \frac{n}{D} + \frac{(1+\varepsilon/5)^2}{D \ln D} \sum_{i=0}^{D} i \\ &= \frac{n}{D} + \frac{(1+\varepsilon/5)^2(D+1)}{2 \ln D} \\ &\leq \frac{n}{D} + \frac{(1+\varepsilon/2)D}{2 \ln D}.\end{aligned}$$

We have extended the initial partial fractional colouring so that every vertex of G has measure 1 apart from those vertices of degree less than $D^{1/(1+\varepsilon/5)}$. Since the above bound on the total measure used is strictly more than $D^{1/(1+\varepsilon/5)}$ if D is large enough, we can greedily extend the partial fractional colouring to all remaining vertices without any additional measure.

It remains to specify D so that we use at most $(\sqrt{2}+\varepsilon)\sqrt{n/\ln n}$ measure in total. Provided n is large enough, the choice $D = \left\lfloor \sqrt{n \ln n} \right\rfloor$ suffices. Note that under this choice D is arbitrarily large if n is. □

It is worth observing from the proof that a hypothetical sharp example for the bound in Theorem 3.1.3 has maximum degree $(1 + o(1))\sqrt{n \ln n}$.

To conclude the section, we comment that a straightforward substitution of Theorem 3.1.3 or Corollary 3.4.1.1 together with Theorem 1.1.1 into the proof by Gimbel and Thomassen [56] (the

proofs in [97, 102] being slightly less efficient) yields the following bounds. The constants are roughly 2.5 and 3 times larger than the constant in Conjecture 3.4.2.

Proposition 3.4.2. *As $m \to \infty$, any triangle-free graph with m edges has fractional chromatic number at most $(3^{5/3} + o(1))m^{1/3}/(\ln m)^{2/3}$ and chromatic number at most $(3^{5/3}2^{1/3} + o(1))m^{1/3}/(\ln m)^{2/3}$.*

The approach for Theorem 3.1.3 could possibly be adapted to more directly improve upon Proposition 3.4.2, but we have not yet managed to do so.

3.5 Ramsey-type for sparse neighbourhoods

Let us now consider the independence number and the chromatic number of a graph with no maximum degree restriction, where each neighbourhood is sparse instead of triangle-free. Namely, let G be a graph on n vertices and of maximum degree Δ, where Δ might be as large as $n-1$. Moreover, assume that each vertex belongs to at most $T = \Delta^2/f$ triangles, for some fixed $f \in [1, \Delta(G)^2]$.

As we have seen in Chapter 1,

$$\alpha(G) \geq (1 - o(1))\frac{n \ln \frac{\Delta}{\sqrt{T}}}{\Delta} = (1 - o(1))\frac{n \ln \sqrt{f}}{\Delta}. \tag{3.9}$$

On the other hand, let v be a vertex of maximum degree in G, and let H be the subgraph induced by $N(v)$. So in particular $n(H) = \Delta$, and since every edge in H yields a (distinct) triangle containing v, it holds that $e(H) \leq \Delta^2/f$. At this point, we can use the Caro-Wei theorem which provides a lower bound on the independence number of graphs with a given degree sequence.

Theorem 3.5.1 (Caro, 1979, Wei, 1981 [27, 120])**.** *For every graph G,*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1 + \deg_G(v)} \geq \frac{n(G)}{1 + \mathrm{ad}(G)}.$$

The average degree of H is at most $2\Delta/f$, so we infer from Theorem 3.5.1 that

$$\alpha(H) \geq \frac{\Delta}{1 + 2\Delta/f}. \tag{3.10}$$

Since $\alpha(G) \geq \alpha(H)$, (3.9) and (3.10) provide two different lower bounds on $\alpha(G)$, respectively decreasing and increasing as functions of Δ. The maximum of these two lower bounds is therefore minimised when they are equal. This happens when $\Delta = \Delta_0$, where

$$\left(\frac{1}{2} - o(1)\right)\frac{n \ln f}{\Delta_0} = \frac{\Delta_0}{1 + 2\Delta_0/f},$$

and so

$$\Delta_0 \sim \frac{n \ln f + \sqrt{n \ln f (2f^2 + n \ln f)}}{2f}.$$

Substituting $\Delta = \Delta_0$ into (3.9), we obtain that

$$\alpha(G) \geq (1 - o(1)) \frac{n \ln f \left(\sqrt{1 + \frac{2f^2}{n \ln f}} - 1 \right)}{2f}. \tag{3.11}$$

The lower bound provided by (3.11) is asymptotic to $f/2$ if $f = o(\sqrt{n \ln n})$ and to $\sqrt{n \ln f/2}$ if $f = \omega(\sqrt{n \ln n})$, so this in particular extends Shearer's bound on off-diagonal Ramsey numbers to cover any $f \geq n^{1-o(1)}$. Over the range of f as a function of n, (3.11) is asymptotically sharp up to some reasonably small constant factor by considering the final output of the triangle-free process [16, 52] or a blow-up of that graph by cliques.

By applying Lemma 3.4.1 with independent sets of the size guaranteed in (3.11), it follows that as $f \to \infty$ the chromatic number of G is at most

$$(4 + o(1)) \frac{f}{\ln f \left(\sqrt{1 + \frac{2f^2}{n \ln f}} - 1 \right)}. \tag{3.12}$$

Is this the correct asymptotic order for the largest list chromatic number? Is the extra factor 2 unnecessary for the chromatic number? Even improving the bound only for the fractional chromatic number by a factor 2 would be very interesting.

3.6 Bounds involving cubes

In this section, we prove Theorem 3.1.4. We also make some additional observations that link our results with the fractional distance-3 chromatic number.

Let us first remark that, given the adjacency matrix A of a graph G, the total number of directed three-edge walks in G is the sum of all entries in the matrix A^3. The proof of Theorem 3.1.4 combines the proofs of Theorems 3.1.2 and of 3.2.1, without needing to bound the fractional chromatic number.

Proof of Theorem 3.1.4. Let $G = (V, E)$ be a triangle-free graph with $|V| = n$ and $|E| = m$ and suppose G has w_3 directed three-edge walks. We write $q = w_3/(2nm)$ and note that

$$q = \frac{\sum_{x \in V} \sum_{v \in N(x)} \sum_{w \in N(v)} \deg(w)}{n \sum_{x \in V} \deg(x)}.$$

Let x_1, x_2 be two vertices chosen uniformly at random and let $\mathbf{I}_1 := N(x_1)$ and $\mathbf{I}_2 := N(x_2)$ denote their neighbourhoods, which are independent sets by triangle-freeness. Note that $\mathbb{E}[|\mathbf{I}_1|] = \mathbb{E}[|\mathbf{I}_2|] = \frac{1}{n} \sum_{x \in V} \deg(x)$, so also $\frac{1}{2}\mathbb{E}[|\mathbf{I}_1| + |\mathbf{I}_2|] = \frac{1}{n} \sum_{x \in V} \deg(x)$.

The number of edges in the subgraph induced by $\mathbf{I}_1 \cup \mathbf{I}_2$ satisfies

$$\begin{aligned}
\mathbb{E}\left[e\Big(G[\mathbf{I}_1 \cup \mathbf{I}_2]\Big)\right] &= \sum_{I_1,I_2} \mathbb{P}\left[(\mathbf{I}_1 = I_1) \cap (\mathbf{I}_2 = I_2)\right] \cdot e\Big(G[I_1 \cup I_2]\Big) \\
&= \sum_{I_2} \mathbb{P}\left[\mathbf{I}_2 = I_2\right] \sum_{I_1} \mathbb{P}\left[\mathbf{I}_1 = I_1\right] \cdot e\Big(G[I_1 \cup I_2]\Big) \\
&= \sum_{I_2} \mathbb{P}\left[\mathbf{I}_2 = I_2\right] \cdot \mathbb{E}\left[e\Big(G[\mathbf{I}_1 \cup I_2]\Big)\right] \\
&= \frac{1}{n} \sum_{x \in V} \mathbb{E}\left[e\Big(G[\mathbf{I}_1 \cup N(x)]\Big)\right] \\
&= \frac{1}{n} \sum_{x \in V} \sum_{v \in N(x)} \mathbb{E}\left[|\mathbf{I}_1 \cap N(v)|\right] \\
&= \frac{1}{n} \sum_{x \in V} \sum_{v \in N(x)} \sum_{w \in N(v)} \frac{\deg(w)}{n} \\
&= \frac{\sum_{x \in V} \sum_{v \in N(x)} \sum_{w \in N(v)} \deg(w)}{n \sum_{x \in V} \deg(x)} \cdot \frac{\mathbb{E}\left[|\mathbf{I}_1| + |\mathbf{I}_2|\right]}{2} \\
&= \frac{q}{2} \mathbb{E}\left[|\mathbf{I}_1| + |\mathbf{I}_2|\right].
\end{aligned}$$

By linearity of expectation,

$$\mathbb{E}\left[|E(G[\mathbf{I}_1 \cup \mathbf{I}_2])| - \frac{q}{2}(|\mathbf{I}_1| + |\mathbf{I}_2|)\right] \geq 0.$$

It follows that there are two independent sets I_1 and I_2 of G with at least $\frac{q}{2}(|I_1| + |I_2|)$ edges in the subgraph induced by $I_1 \cup I_2$. Discarding the vertices of $I_1 \cap I_2$ (if any) yields a bipartite induced subgraph of average degree at least q. Therefore G contains a bipartite induced subgraph of minimum degree at least $q/2$, as desired. □

Next we indicate a mild improvement upon our bounds in terms of fractional distance-3 colouring. Given a graph G, the cube G^3 of G is the simple graph formed from G by including all edges between vertices that are connected by a path in G of length at most 3. The fractional distance-3 chromatic number of G is the fractional chromatic number $\chi_f(G^3)$ of G^3. Observe that, if G is triangle-free and I is an independent set of G^3, then the union $\cup_{v \in I} N(v)$ of neighbourhood sets taken over I is an independent set in G. In the proofs of Theorems 3.1.2 and 3.1.4, if we sample independent sets by taking such neighbourhood unions according to the distribution given by $\chi_f(G^3)$ rather than uniformly taking a neighbourhood set, then we obtain the following.

Theorem 3.6.1. *Letting χ_f^3 denote the fractional distance-3 chromatic number of the host triangle-free graph,*

- *the upper bound in Theorem 3.1.2 holds with χ_f^3/d instead of n/d; and*
- *Theorem 3.1.4 holds with $w_3/\big(4\chi_f^3 m\big)$ or $d^2/\big(2\chi_f^3\big)$ instead of $w_3/(4nm)$.*

Curiously, as the triangle-free process is sharp in Theorem 3.1.2, we obtain the following. (Perhaps this same result with distance-2 also holds.)

Corollary 3.6.1.1. *With high probability, the final output of the triangle-free process has $\Omega(n)$ fractional distance-3 chromatic number as $n \to \infty$.*

3.7 Concluding remarks

Although we were preoccupied with triangle-free graphs, one could naturally investigate graphs not containing H as a subgraph for any fixed graph H. The following is in essence a more general form of Problem 4.1 in [49].

Problem 3.7.1. *Given a graph H, is there $c_H \in (0,1)$ such that, as $n \to \infty$,*

- *if $c > c_H$, then any H-free graph on n vertices with minimum degree n^c has $n^{\Omega(1)}$ bipartite induced minimum degree; and*
- *if $c < c_H$, then there is an H-free graph on n vertices with minimum degree n^c and $O(\ln n)$ bipartite induced minimum degree?*

We have shown that $c_H = 1/2$ if H is a triangle.

Problem 3.7.1 is particularly enticing when H is the complete graph K_r on $r \geq 4$ vertices. It was noted in [49] that the work of Ajtai, Komlós and Szemerédi [2] implies $c_{K_r} \leq 1 - 1/r$ (if it exists). It is possible to adapt Theorem 3.3.2 and [112] to show that $c_{K_r} \geq 1 - (r-2)\big/\left(\binom{r}{2} - 1\right)$ (if it exists). It is conceivable that $c_{K_r} = 1 - 1/(r-1)$. A motivation for this is that, even though *prima facie* there is no extremely close connection between bipartite induced density and large independent sets, we are tempted to speculate that, denoting the H versus K_t Ramsey number by $R(H, K_t)$,

$$c_H = 1 - \lim_{t\to\infty} \frac{\ln t}{\ln R(H, K_t)} \text{ (if they exist).}$$

The right-hand side is conjectured to be $1 - 1/(r-1)$ when H is K_r.

Motivated by Problem 3.7.1, we observe the following partial extensions of the bounds in Theorems 3.1.1 and 3.1.2.

Proposition 3.7.2. *Fix an integer $r \geq 3$. For $r - 2 \leq d \leq n/2$, any $K_{1,1,r-2}$-free graph on n vertices with minimum degree at least d has fractional chromatic number at most $\binom{n}{r-2}\big/\binom{d}{r-2}$, and thus contains a bipartite induced subgraph of minimum degree at least $\frac{d}{2}\binom{d}{r-2}\big/\binom{n}{r-2}$.*

Proof. Let $G = (V, E)$ be a graph on n vertices with minimum degree at least d that contains no copy of $K_{1,1,r-2}$. We note that $K_{1,1,r-2}$-freeness implies that the joint neighbourhood of every vertex subset of size $r-2$ is an independent set. Choose $\mathbf{I}$ from the joint neighbourhood sets of $(r-2)$-vertex subsets uniformly over all $\binom{n}{r-2}$ such sets. Then it holds for all $v \in V$ that

$$\binom{n}{r-2} \cdot \mathbb{P}\left[v \in \mathbf{I}\right] = \#\left\{T \subseteq N(v) \mid |T| = r-2\right\} \geq \binom{d}{r-2}.$$

We have shown then that this distribution has property $\mathrm{Q}^*_{\binom{d}{r-2}/\binom{n}{r-2}}$. □

Given a graph $H = (V_H, E_H)$ and a positive integer x, let us define

$$\chi_f(H, x) = \max_{v \in V_H} \quad \max\left\{\chi_f(J) \mid J \text{ is an } (H-v)\text{-free graph on } x \text{ vertices}\right\}.$$

Proposition 3.7.3. *Given a graph H, any H-free graph on n vertices with minimum degree at least d and maximum degree at most Δ has fractional chromatic number at most $\chi_f(H,\Delta)\cdot n/d$.*

Proof. Let $G=(V,E)$ be a graph on n vertices with minimum degree at least d and maximum degree at most Δ that contains no copy of H. For any $v\in V$, there is by definition of $\chi_f(H,x)$ a distribution $\mathcal{P}_v$ over the independent sets of $N(v)$ such that any given $w\in N(v)$ is in $\mathbf{I}_v$ with probability at least $(\chi_f(H,\deg(v)))^{-1}\geq(\chi_f(H,\Delta))^{-1}$ for a random $\mathbf{I}_v$ chosen according to $\mathcal{P}_v$. For v uniformly chosen from V, let $\mathbf{I}=\mathbf{I}_v$ be the corresponding random independent set. Then any given $u\in V$ is in $\mathbf{I}$ with probability at least $\deg(u)/(n\cdot\chi_f(H,\Delta))$, so this distribution has property $\mathrm{Q}^*_{d/(n\cdot\chi_f(H,\Delta))}$. □

Let P_r (C_r) denote a path (cycle, respectively) on $r\geq 2$ vertices. Every P_r-free graph is $(r-2)$-degenerate and therefore $(r-1)$-colourable. Thus $\chi_f(C_{r+1},x)\leq r-1$ and we have the following corollary.

Corollary 3.7.3.1. *Fix an integer $r\geq 3$. Any C_r-free graph on n vertices with minimum degree d has fractional chromatic number at most $\frac{n}{d}(r-2)$, and thus contains a bipartite induced subgraph of minimum degree at least $\frac{1}{r-2}\cdot\frac{d^2}{2n}$.*

In Section 3.4, we pursued sharper but fractional versions of the original problems of Erdős and Hajnal [45]. In another direction, the natural list colouring versions are open to the best of our knowledge.

Conjecture 3.7.1. *There are constants $C_1,C_2>0$ such that any triangle-free graph on n vertices with m edges has list chromatic number at most $C\sqrt{n/\ln n}$ and at most $C_2 m^{1/3}/(\ln m)^{2/3}$.*

Note that by a result of Alon [4], the two terms in Conjecture 3.7.1 are correct up to $\ln n$ and $\ln m$ factors, respectively.

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Bibliography

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Curriculum Vitae

François Pirot was born on December 19, 1993 in Orléans, France. He remained in Orléans for his whole childhood. In 2010, he passed the Baccalauréat with highest honours, with a major in mathematics. He then spent two years in the so-called *classes préparatoires* at Lycée Pothier in Orléans, with a major in mathematics and physics. This consisted of two years of intensive training for the competitive exams in order to enter *les Grandes Écoles*. During those two years, he was introduced to computer sciences, and found a great interest in this novel field. In 2012, he successfully passed the entrance exam for the ENS de Lyon with a major in theoretical computer sciences. He pursued his superior studies during four years in Lyon, where he obtained his licence degree with honours in 2013, and his master degree with honours in 2015. At the occasion of his master thesis, he spent five months in Nijmegen in the Netherlands, under the supervision of Dr. Ross Kang, during which he could have a first taste of research in graph theory. Following this rewarding experience, he was proposed a joint PhD under the supervision of Dr. Ross Kang at Radboud University, Nijmegen, and Prof. Dr. Jean-Sébastien Sereni at the Université de Lorraine, Nancy. Before beginning this PhD, he finished his last year of studies at the ENS de Lyon, which was the occasion to perform long internships, under the supervision of Mickaël Rao in Lyon, working on the entropy on bidimensional subshifts, then under the supervision of Guillem Perarnau in Birmingham, working on the probabilistic method in graph theory. He began his PhD in Nancy in September 2016, then moved to Strasbourg after one year in order to follow Jean-Sébastien, and spent the two remaining years of his PhD there. Those three years were interspersed with fruitful summer visits in Nijmegen. From September to November 2019, he will have a short term postdoctoral position at the Université Libre de Bruxelles, in the group of Dr. Gwenaël Joret. He will then start as a postdoctoral researcher in Grenoble, in the group of Dr. Louis Esperet and Dr. Nicolas Bousquet.