# Extremal properties of the subchromatic number

Internship proposal

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**General context:** A proper *k-colouring* of a graph $G$ is a partition of its vertex set $V(G)$ into $k$ independent sets (i.e. set of pairwise non-adjacent vertices). This combinatorial object has been (and still is!) extensively studied because of its many applications (in rooting, in scheduling, and more generally in resource allocation). Many variants of graph colourings have been proposed, and subcolourings are one of them. A *k-subcolouring* of $G$ is a partition of $V(G)$ into $k$ colour classes, each of which induces a disjoint union of cliques. The *subchromatic number* $\chi_{\text{sub}}(G)$ of $G$ is the minimum $k$ such that a $k$-subcolouring of $G$ exists. This graph parameter has been introduced in 1989 by Albertson *et. al.* [1], who proved a few extremal properties on $\chi_{\text{sub}}$: for every graph $G$ of maximum degree $\Delta$,

$$\chi_{\text{sub}}(G) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil, \tag{1}$$

and if $G$ has $n$ vertices then

$$\chi_{\text{sub}}(G) \leq (1+o(1))\frac{n}{\log_4 n} \tag{2}$$

as $n \to \infty$ (this is a consequence of the upper bound $R(k,k) \leq 4^k$ for the *diagonal Ramsey numbers* [2]). We don't known whether any of the bounds (1) or (2) is tight, and this is an interesting line of prospect. Natural lower bounds can be obtained by considering graphs of large chromatic number and small clique number, since it is straightforward that $\chi(G) \leq \omega(G)\chi_{\text{sub}}(G)$ for every graph $G$ (one can trivially properly colour a disjoint union of cliques in $G$ with $\omega(G)$ colours). So, using the extremal properties of the chromatic number of triangle-free graphs, we infer that there exists $G$ such that $\chi_{\text{sub}}(G) \geq \frac{\Delta}{4\ln\Delta}$, or $\chi_{\text{sub}}(G) = \Omega\left(\sqrt{n/\ln n}\right)$. Actually, it was showed in [1] that $\chi_{\text{sub}}(K_{m,\ldots,m}) = m$ for the complete $m$-partite graph, which shows that we may have $\chi_{\text{sub}}(G) \geq \sqrt{n}$.

A new line a research was open more recently by Nešetřil *et. al.* [3], who studied $\chi_{\text{sub}}$ in the context of powers of graphs of bounded extension. They showed in particular that for every planar graph $G$, $\chi_{\text{sub}}(G^2) \leq 135$ (given an integer $t$, $G^t$ is the $t$-th power of $G$, obtained by linking every pair of vertices at distance at most $t$ in $G$). The best lower bound that they could provide is 5, so there is much room for tightening this result.

**Objectives:** The aim of the internship is to study the extremal values of the subchromatic number in natural classes of graphs (graphs of bounded degree, powers of graphs, ...). This could be the occasion to learn many techniques from graph theory, such as the probabilistic method for graph colouring. For instance, a natural goal would be to try and improve either (1) or (2). One could also study $\chi_{\text{sub}}(G)$ when $G \leftarrow G(n,1/2)$ is a uniformly random graph on $n$ vertices. Another interesting goal would be to find better bounds for $\chi(G^2)$ when $G$ is planar, or more generally when $G$ is $K_t$-minor-free.

## References

[1] M. O. Albertson, R. E. Jamison, S. T. Hedetniemi, and S. C. Locke. The subchromatic number of a graph. In *Annals of Discrete Mathematics*, volume 39, pages 33–49. Elsevier, 1989.

[2] P. Erdös and G. Szekeres. A combinatorial problem in geometry. *Compositio mathematica*, 2:463–470, 1935.

[3] J. Nešetřil, P. O. de Mendez, M. Pilipczuk, and X. Zhu. Clustering powers of sparse graphs. *The Electronic Journal of Combinatorics*, pages P4–17, 2020.