Graph Algorithms

DM

Consignes Tous les résultats du cours et des TD peuvent être utilisés, en répétant leur énoncé avec une référence explicite (e.g. "Tout graphe G est $(\Delta(G)+1)$-colorable [cours, Chapitre 2]."). Les échanges d'idées entre étudiants sont autorisés, mais il doit y avoir un rendu distinct par étudiant. Une rédaction identique entre deux copies sera considérée comme tricherie, et vaudra 0 sur la partie concernée pour les deux.

1 A consequence of Brooks' Theorem for triangle-free graphs

The goal of this exercice is to prove the following theorem.

Theorem 1 *Let G be a triangle-free graph (so $\omega(G) = 2$) of maximum degree Δ. Then*

$$\chi(G) \leq 3 \left\lceil \frac{\Delta(G)+1}{4} \right\rceil .$$

Let G be a triangle-free graph. Set $k := \left\lceil \frac{\Delta(G)+1}{4} \right\rceil$. Let $(V_1, \ldots, V_k)$ be a partition of $V(G)$ that minimises the number of internal edges (i.e. the number of edges uv such that $u, v \in V_i$ for some $1 \leq i \leq k$).

1. Show that $\Delta(G[V_i]) \leq 3$, for every $1 \leq i \leq k$. 2 pts
2. Show that $\chi(G) \leq 3k$, with the help of Brooks' theorem. 1 pt
3. Using the above, write an algorithm that computes a proper $3k$-colouring of G. What is its complexity? 3 pts

2 List colouring

Given a graph G, a *list assignment of* G is a function $L\colon V(G) \to 2^{\mathbb{N}}$. If $|L(v)| = k$ for all $v \in V(G)$, we say that L is a *k-list assignment* of G. The elements in $\bigcup_{v \in V(G)} L(v)$ are the *colours* of L, and $L(v)$ is the *list of colours* allowed for each vertex $v \in V(G)$. A proper L-colouring of G is a proper colouring c of G such that $c(v) \in L(v)$ for every vertex $v \in V(G)$ (every vertex gets a colour from its list). In particular, a proper k-colouring of G is a proper L-colouring with $L(v) = \{1, \ldots, k\}$ for all $v \in V(G)$.

The minimum k such that G is L-colourable for every k-list assignment L of G is the *list-chromatic number* of G, denoted $\chi_\ell(G)$. The goal of this exercise is to study some properties of list colourings.

1. Given a cycle C, and a 2-list-assignment L of C, show that C is not L-colourable if and only if C is odd and all lists are the same. 1.5 pt
2. Let $n = \binom{2k-1}{k}$ for some $k \geq 1$, and let $G = (U, V, E)$ be the complete bipartite graph $K_{n,n}$ (i.e. $E = \{uv : u \in U, v \in V\}$). Let L be a list assignment of G such that, for every subset X of $\{1, \ldots, 2k-1\}$ of cardinality k, there exists $u \in U$ and $v \in V$ such that $L(u) = L(v) = X$. Show that G is not L-colourable. 1.5 pt
   Advice: Consider the cases $k = 1$ and $k = 2$ first, then try to generalise the result.
3. The problem $(2,3)$-LIST-COLOUR consists in determining whether a graph G, given with a list-assignment L where all lists have size 2 or 3, is L-colourable. Prove that $(2,3)$-LIST-COLOUR is NP-complete on the class of bipartite graphs. The reduction is from 3-SAT. 3 pts
   Hint: Consider the incidence graph of the literals and the clauses of a 3-SAT instance.
4. Let G be a complete graph, and L a k-list-assignment of G such that each colour appears in at most k lists. Show that G is L-colourable. 2 pts
   Hint: Reduce the problem of finding a proper L-colouring of G to that of finding a specific matching in a bipartite graph, and use Hall's Theorem.
5. Prove that $\chi_\ell(G) \leq \delta^*(G) + 1$ for every graph G. 2 pts
   Hint: Adapt the Greedy Colouring algorithm.

3 Degree choosability

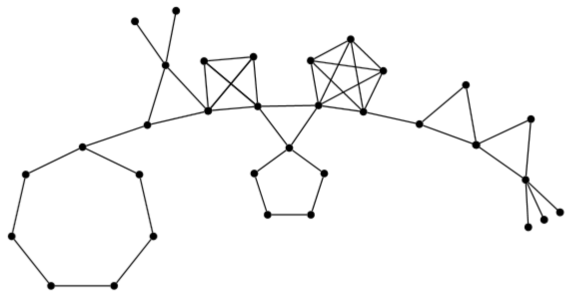

Figure 1: A (connected) Gallai forest.

A graph G is a Gallai forest if every block of G (i.e. every maximal 2-connected subgraph of G) is either a clique or an odd cycle. A graph G is degree-choosable if it is L-colourable for every list assignment $L\colon V(G) \to 2^{\mathbb{N}}$ with $|L(v)| = \deg(v)$ for every vertex $v \in V(G)$. The goal of this exercise is to prove the following generalisation of Brooks' theorem.

Theorem 2 *For every graph G, either G is a Gallai forest, or G is degree-choosable.*

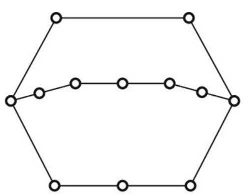

Figure 2: An example of a Theta-graph.

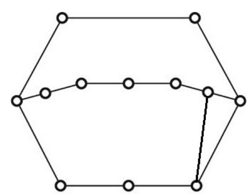

Figure 3: This is not a Theta-graph.

A Theta-graph is the union of three internally disjoint paths with the same extremities. Equivalently, it is a connected graph with degree sequence $3, 3, 2, \ldots, 2$.

1. Show that every Theta-graph is degree-choosable. 1.5 pt

2. Let G be a 2-connected graph that is neither complete nor a cycle. Let us show that G contains a Theta-graph as an **induced** subgraph. Let u, v be two vertices at distance 2 in G, and let $x \in N(u) \cap N(v)$. Let P be a shortest path from u to v in $G \setminus x$, and let C be the cycle formed by $P + x$.

   a) If C contains at least one chord, show that there exists a subset $X \subseteq V(C)$ such that $G[X]$ is a Theta-graph. 1 pt

   b) If C contains no chord, then C does not cover all vertices in G. Let $y \notin V(C)$ be a neighbour of some vertex $a \in V(C)$, and let Q be a shortest path from y to $V(C) \setminus \{a\}$ in $G \setminus a$. Show that there exists a subset $X \subseteq V(C) \cup V(Q)$ such that $G[X]$ is a Theta-graph. 1.5 pt

3. Let G be a 2-connected graph that is neither complete nor a cycle, and $L\colon V(G) \to 2^{\mathbb{N}}$ a list assignment of G with $|L(v)| = \deg(v)$ for every $v \in V(G)$. Show that there exists an ordering on $V(G)$ such that one can compute a proper L-colouring of G greedily by colouring the vertices in that order. 2 pts
   Hint: Colour some Theta-subgraph last.

4. If G is connected, show that G is degree-choosable if one of its blocks is. 2 pts

5. Write an algorithm that computes a proper L-colouring of a graph G whenever G is not a Gallai forest, and $|L(v)| = \deg(v)$ for every vertex $v \in V(G)$. What is its complexity? 3 pts