Graph Algorithms — Home Assignment

Rules Every result from the course and the exercise sessions may be used by making an explicit referrence to it (e.g. "Every graph G is $(\Delta(G)+1)$-colourable [course, Chapter 2]."). Brainstorming between students is allowed, but the redaction in each assignment must be distinct.

1 A consequence of Brooks' Theorem

The goal of this exercice is to prove the following theorem.

Theorem 1 *Let G be a graph of clique number $\omega(G) \leq 3$ and maximum degree Δ. Then*

$$\chi(G) \leq 3\left\lceil \frac{\Delta+1}{4} \right\rceil.$$

Let G be a graph of maximum degree Δ. Set $k := \left\lceil \frac{\Delta+1}{4} \right\rceil$. Let $(V_1, \ldots, V_k)$ be a partition of $V(G)$ that minimises the number of internal edges (i.e. the number of edges uv such that $u, v \in V_i$ for some $1 \leq i \leq k$).

1. Show that $\Delta(G[V_i]) \leq 3$, for every $1 \leq i \leq k$.
2. Assume now that $\omega(G) \leq 3$, and show that this implies $\chi(G) \leq 3k$.

   Hint. Use Brooks' theorem.
3. Using the above, write a polytime algorithm that computes a proper $3k$-colouring of G.

2 List colouring

Given a graph G, a *list assignment of* G is a function $L\colon V(G) \to 2^{\mathbb{N}}$. If $|L(v)| = k$ for all $v \in V(G)$, we say that L is a k*-list assignment* of G. The elements in $\bigcup_{v \in V(G)} L(v)$ are the *colours* of L, and $L(v)$ is the *list of colours* allowed for each vertex $v \in V(G)$. A proper L-colouring of G is a proper colouring c of G such that $c(v) \in L(v)$ for every vertex $v \in V(G)$ (every vertex gets a colour from its list). In particular, a proper k-colouring of G is a proper L-colouring with $L(v) = \{1, \ldots, k\}$ for all $v \in V(G)$.

The minimum k such that G is L-colourable for every k-list assignment L of G is the *list-chromatic number* of G, denoted $\chi_\ell(G)$. The goal of this exercise is to study some properties of list colourings.

1. Given a cycle C, and a 2-list-assignment L of C, show that C is not L-colourable if and only if C is odd and all lists are the same.
2. Let $n = \binom{2k-1}{k}$ for some $k \geq 1$, and let $G = (U, V, E)$ be the complete bipartite graph $K_{n,n}$ (i.e. $E = \{uv : u \in U, v \in V\}$). Let L be a list assignment of G such that, for every subset X of $\{1, \ldots, 2k-1\}$ of cardinality k, there exists $u \in U$ and $v \in V$ such that $L(u) = L(v) = X$. Show that G is not L-colourable.

   Hint. Consider the cases $k = 1$ and $k = 2$ first, then try to generalise the result.
3. The problem $(2,3)$-LIST-COLOUR consists in determining whether a graph G, given with a list-assignment L where all lists have size 2 or 3, is L-colourable. Prove that $(2,3)$-LIST-COLOUR is NP-complete on the class of bipartite graphs. The reduction is from 3-SAT.

   Hint. Consider the incidence graph of the literals and the clauses of a 3-SAT instance.
4. The problem 2-LIST-COLOUR consists in determining whether a graph G, given with a list-assignment L where all lists have size 2, is L-colourable. Prove that 2-LIST-COLOUR is in P.

   Hint: You may show that 2-LIST-COLOUR $\leq_{\mathrm{P}}$ 2-SAT.
5. Prove that $\chi_\ell(G) \leq \delta^*(G) + 1$ for every graph G.

   Hint. Adapt the Greedy Colouring algorithm.

3 A polytime algorithm for solving 3-COLOUR in dense graphs

In this exercise, G is a graph on n vertices. We say that G is *dense* if $\delta(G) \geq n/2$.

1. A *dominating set* of G is a set of vertices $D \subseteq V(G)$ such that $N_G[D] = V(G)$ (every vertex $v \in V(G) \setminus D$ has a neighbour in D). We denote $\gamma(G)$ the minimum size of a dominating set of G. We will show that the following greedy algorithm returns a dominating set of G of size at most $\log_2 n + 1$ when G is dense.

Algorithm 1: Greedy algorithm

Data: G: graph
Result: D: dominating set of G
$V_0 \leftarrow V(G)$, $i \leftarrow 0$
while $V_i \neq \emptyset$ **do**
  $v_i \leftarrow$ vertex of $V(G)$ with maximum degree in V_i
  $V_{i+1} \leftarrow V_i \setminus N[v_i]$
  $i \leftarrow i + 1$
end
return $\{v_0, \ldots, v_i\}$

   (a) Let $H = (X, Y, E)$ be a bipartite graph. Show that $|X| \operatorname{ad}(X) = |Y| \operatorname{ad}(Y)$.

   (b) Assume that G is dense. Show that, at each iteration of the loop, the degree of v_i in V_i is at least $|V_i|/2$.
   Hint. Use the previous result on the bipartite subgraph of G induced by the cut $(V_i, V(G) \setminus V_i)$.

   (c) Show that the algorithm returns a dominating set of G of size at most $\log_2 n + 1$ when G is dense.

2. Let D be a dominating set of G, and $\phi\colon D \to [3]$ a proper 3-coloring of $G[D]$. Show that it is possible to test in polynomial time whether ϕ extends to a proper 3-colouring c of G (we must have $c(x) = \phi(u)$ for every $u \in D$).

   Hint: Show that this reduces to solving an instance of 2-LIST-COLOUR.

3. Using the above, describe a polytime algorithm to solve 3-COLOUR on G when G is dense.

4. What can you say when $\delta(G) \geq c \cdot n$ for some absolute constant $c > 0$?