Graph Algorithms

DM

Rules Every result from the course and the exercise sessions may be referred to by quoting it and stating the precise reference (e.g. "Every graph G is $\Delta(G)+1$-colourable [course, Chapter 2]"). Brainstorming between students is allowed, but the redaction in each assignment must be distinct. In case of an identical redaction in two assignments, the concerned section will be graded 0 for both.

1 The chromatic number when a tree is forbidden as a subgraph

The goal of this exercise is to study the chromatic number of graphs that do not contain a given tree as a subgraph. For a fixed graph H, we say that a graph G is H-free if H does not appear as a subgraph in G.

Let us fix some tree T on t vertices.

1. Let G be a graph of minimum degree $t-1$. Show that G contains T as a subgraph. 3 pts
2. Let G be a T-free graph. Deduce from the previous result that there exists a function f such that $\chi(G) \leq f(t)$ (provide an explicit value for $f(t)$). What is the complexity of constructing a proper $f(t)$-colouring of G? 3 pts
3. Show that your value $f(t)$ is tight (exhibit a specific tree T on t vertices and a T-free graph G such that $\chi(G) = f(t)$). 2 pts

2 t-improper colourings

Given a graph G, a t-improper k-colouring of G is a partition of $V(G)$ into k colour classes that induce subgraphs of maximum degree at most t. Said otherwise, it assigns a colour $c(v) \in [k]$ to each vertex $v \in V(G)$ such that at most t neighbours of v share its colour. In particular, a 0-improper k-colouring of G is a proper k-colouring of G.

The t-improper chromatic number $\chi^t(G)$ of G is the minimum k such that there exists a t-improper k-colouring of G.

1. Show that $\chi^t(G) \leq \left\lfloor \frac{\Delta(G)}{t+1} \right\rfloor + 1$ for every graph G (you may analyse the performance of a greedy algorithm). 3 pts
2. Show that $\chi(G) \leq (t+1)\chi^t(G)$ for every graph G. 2 pts
3. Show that $\chi(G) \leq t\,\chi^t(G)$ for every $t \geq 3$ and every graph G of clique number $\omega(G) \leq t$. 2 pts

3 Intersection graph of subtrees

Let T be a tree, and let $\mathcal{T} = \{T_1, \ldots, T_r\}$ be a set of subtrees (connected subgraphs) of T. Let $G_{\mathcal{T}}$ be the intersection graph of $\mathcal{T}$, that is $V(G_{\mathcal{T}}) = \mathcal{T}$ and $T_iT_j \in E(G_{\mathcal{T}})$ iff $V(T_i) \cap V(T_j) \neq \emptyset$.

1. If T is a path, show that $G_{\mathcal{T}}$ is an interval graph. 1 pt
2. Show that $G_{\mathcal{T}}$ contains a simplicial vertex, that is a vertex $v_0 \in V(G_{\mathcal{T}})$ such that $N(v_0)$ is a clique in $G_{\mathcal{T}}$. 2 pts
3. Describe a polytime algorithm that returns a proper $\chi(G_{\mathcal{T}})$-colouring of $G_{\mathcal{T}}$. 2 pts
4. Describe a polytime algorithm that returns an independent set of $G_{\mathcal{T}}$ of size $\alpha(G_{\mathcal{T}})$. 3 pts