Graph Algorithms

TD : Introduction

1 To begin

1. *Show that a graph always has an even number of odd degree vertices.*

   Let G be a graph. We have
   $$\sum_{v\in V(G)} \deg(v) = 2|E(G)|.$$
   Therefore $\sum_{v\in V(G)} \deg(v)$ is an even number. The parity of that sum is given by the number of odd operands, that is the number of odd degree vertices in G. We infer that the number of odd degree vertices in G is even.

2. *Show that a graph with at least* 2 *vertices contains* 2 *vertices of equal degree.*

   Let $n := |V(G)|$. First assume that G contains no isolated vertices. Since G has no loop, one has $1 \leq \deg(v) \leq n-1$ for every vertex $v \in V(G)$. Since G contains n vertices, by the pigeonhole principle, there exists two vertices in G with the same degree.

   If G contains two isolated vertices, those vertices have the same degree.

   We can now assume that G contains exactly one isolated vertex v_0. Let $G' := G - v_0$, and observe that $\deg_{G'}(v) = \deg_G(v)$ for every vertex $v \in V(G')$. Since G' has no isolated vertex, we have already showed that it must contain two vertices with the same degree, which also have the same degree in G.

3. *Let G be a graph of minimum degree $\delta(G) \geq 2$. Show that G contains a cycle.*

   Let $P = v_0, \ldots, v_\ell$ be a path of maximal length in G (or any maximal path, i.e. a path that cannot be further extended). Let $v \in N(v_\ell) \setminus \{v_{\ell-1}\}$ (this is non empty since $\deg(v_\ell) \geq 2$). Since P is maximal, we have $v \in V(P)$, so there exists $0 \leq i \leq \ell - 2$ such that $v = v_i$. We conclude that $v_i, \ldots, v_\ell$ is a cycle in G.

4. *Let G be a graph of minimum degree d, and of girth $2t+1$. Given any vertex $v \in V(G)$, show that there are at least $d(d-1)^{i-1}$ vertices at distance exactly i from v in G, for every $1 \leq i \leq t$. Deduce a lower bound on the number of vertices of G.*

   Let X_i be the set of vertices at distance i from v. Let us first prove the following claim.

   Claim For every $0 \leq i \leq t-1$, every set X_i is an independent set, and every vertex in X_{i+1} has at most 1 neighbour in X_i.

   Proof. Let $y, z \in X_i$, and let P_y and P_z be paths of length i from y to v and from z to v, respectively (they exist by definition of X_{i-1}. These path are not disjoint since they both contain v; let v_0 be the first vertex in which they intersect, and so there is a path P_{yz} of length at most $2i \leq 2t-2$ from y to z. If yz is an edge, then together with P_{yz} this forms a cycle of length at most $2t-1$, a contradiction. If y and z have a common neighbour $v \in X_{i+1}$, then the union of the path $y - v - z$ together with P_{yz} forms a cycle of length at most $2t$, again a contradiction. □

   Note that from the Claim, we can deduce that $G[\bigcup_{i\leq t} X_i]$ is a tree. The result follows from the well-known lower bound on the size of a layer in a tree of given minimum degree. Let us repeat the proof of that lower bound, that is done by induction on i.

For $i = 1$, we have $|X_1| = |N(v)| = \deg(v) \geq d$, as desired. Let us assume that induction hypothesis holds from some $1 \leq i < t$, i.e. we have $|X_{i-1}| \geq d(d-1)^{i-2}$. By the Claim, every vertex $x \in X_i$ has all but one of his neighbours in X_{i+1}, so at least $d-1$. Moreover, again by the Claim, the neighbourhoods of the vertices in X_i are disjoint, so we have $|X_{i+1}| \geq (d-1)|X_i| \geq d(d-1)^{i-1}$, as desired.

2 Dense subgraphs

1. *Show that every graph of average degree d contains a subgraph of minimum degree at least $\frac{d}{2}$.*

   Let H be a subgraph of G such that $\mathrm{ad}(H) = \mathrm{mad}(G) \geq d$. Assume for the sake of contradiction that there exists a vertex v of degree less than $d/2$ in H. Then

$$\begin{aligned}\mathrm{ad}(H \setminus v) &= \frac{2|E(H)| - 2\deg_H(v)}{|V(H)|-1} > \frac{2|E(H)| - d}{|V(H)|-1} \geq \frac{2|E(H)| - 2|E(H)|/|V(H)|}{|V(H)|-1} \\ &> \frac{2|E(H)| \times |V(H)| - 2|E(H)|}{(|V(H)|-1) \times |V(H)|} = \frac{2|E(H)|}{|V(H)|} = d\end{aligned}$$

   The average degree of $H \setminus v$ is more than $\mathrm{mad}(G)$, a contradiciton.

2. *Can you find a similar relation between the maximum degree and the minimum degree? And between the maximum degree and the average degree?*

   No, for instance the star $K_{1,n}$ has maximum degree n (unbounded as $n \to \infty$), minimum degree 1, and average degree less than 2.

3. *Show that every graph of average degree d contains a bipartite subgraph of average degree at least $\frac{d}{2}$.*

   Let $H = (X, Y, E)$ be a bipartite subgraph of G given by a maximal cut. Assume for the sake of contradiction and without loss of generality that there exists a vertex $v \in X$ such that $\deg_H(v) = \deg_Y(v) < \deg_G(v)/2$. Observe that $\deg_G(v) = \deg_X(v) + \deg_Y(v)$, so $\deg_X(v) > \deg_G(v)/2 > \deg_Y(v)$. Let H' be given by the cut $(X \setminus \{v\}, Y \cup \{v\})$. Then $|E(H')| = |E(H)| - \deg_Y(v) + \deg_X(v) > E(H)$; this contradicts the maximality of the cut (X, Y).

3 Cuts and trees

1. *If G is connected, and $e = uv$ is a bridge in G, how many connected components does $G \setminus e$ contain? Show that u and v are cut-vertices, unless they have degree 1.*

   By assumption, $G \setminus e$ contains at least two connected components. Every connected component in $G \setminus e$ contains either u or v, otherwise it would be a connected component in G disjoint from that which contains u; this contradicts the fact that G is connected. We conclude that $G \setminus e$ contains exactly two connected components, the one that contains u, C_u, and the one that contains v, C_v. If $|C_u| > 1$, then the connected components of $G \setminus u$ are C_v and the connected components of $G[C_u \setminus \{u\}]$; in particular G is disconnected and hence u is a cut-vertex. The same holds for v, by symmetry.

2. *Show that a graph G is a tree if and only if there exists a unique path from u to v in G, for every pair of vertices $u, v \in G$.*

   We first show the implication, by proving the converse. Assume that there are two vertices u and v that are linked by two different paths in G. Let P_0 and P_1 be two different paths from u to v. Let x be the last vertex in which the beginning of the paths P_0 and P_1 coincide, and let y be the next common vertex between P_0 and P_1. Then the union of the two subpaths $P_0[x, y]$ and $P_1[x, y]$ is a cycle, so G is not a tree.

   We now show the reverse implication, again by proving the converse. Assume that G is connected and not a tree, so G contains a cycle C. Let x, y be two consecutive vertices on C; there are two differents paths from u to v in G, namely the path u—v, and the path $C \setminus uv$.

3. *Let T a BFS tree of a graph G. Show that every edge of G is contained either within a layer of T, or between two consecutive layers of T.*

   Let $(T_i)_i$ be the layers of T, and let v_0 be the root vertex. Assume for the sake of contradiction that there is an edge $uv \in E(G)$ such that $u \in X_i$, $v \in X_j$, and $j \geq i+2$. Then by definition there is a path P_u of length i from v_0 to u. Together with the edge uv, this forms a path of length $i+1$ from v_0 to v, which contradicts the fact that $\mathrm{dist}(v_0, v) = j > i + 1$ (by definition of the layers).

4. *Let T be a DFS tree of a graph G. Show that, for every edge $e \in E(G)$, there is a branch of T that contains both extremities of e.*

   Assume for the sake of contradiction that there is an edge $uv \in E(G)$ such that u and v and unrelated in T, and let us assume without loss of generality that u has been added to T before v during the DFS. Let w be the last common ancestor of u and v in T, and let w' be the child of w in T in the same branch as that of v (we could have $w' = v$). Let i be the step at which w' has been added to the tree T_i in order to obtain the tree T_{i+1}. It holds that $v \notin V(T_i)$, so $N(u) \setminus V(T_i)$ is non-empty; hence the DFS should consider adding the edge uv before adding the edge ww', a contradiction.