Graph Algorithms

TD : Introduction

1 To begin

1. Show that a graph always has an even number of odd degree vertices.
2. Show that a graph with at least 2 vertices contains 2 vertices of equal degree.

   Hint: If G contains no isolated vertex, how many different values are possible for the degree of a vertex in G?
3. Let G be a graph of minimum degree $\delta(G) \geq 2$. Show that G contains a cycle.
4. If G is connected, and $e = uv$ is a bridge in G, how many connected components does $G \setminus e$ contain? Show that u and v are cut-vertices.

2 Dense subgraphs

1. Show that every graph of average degree d contains a subgraph of minimum degree at least $\frac{d}{2}$.

   Hint: Consider a subgraph of maximum average degree.
2. Can you find a similar relation between the maximum degree and the minimum degree? And between the maximum degree and the average degree?
3. Show that every graph of average degree d contains a bipartite subgraph of average degree at least $\frac{d}{2}$.

   Hint: Consider a maximal cut.

3 Colouring

1. What is the chromatic number of an even cycle C_{2n} ? Of an odd cycle C_{2n+1}?
2. Show that a graph is bipartite if and only if it contains no odd cycle.
3. Show that for every graph G, there exists an order on the vertices such that the greedy algorithm applied in this order returns a colouring with $\chi(G)$ colours.

4 Interval graphs

Given a set of intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$ where $I_i = [a_i, b_i]$ for every $1 \leq i \leq n$, the interval graph associated with $\mathcal{I}$ is the graph $G = (V, E)$ where $V = \{1, \ldots, n\}$ and $ij \in E$ iff I_i and I_j intersect, i.e. $a_i \leq b_j$ and $a_j \leq b_i$, for every $1 \leq i, j \leq n$.

1. Show that in an interval graph, there exists a simplicial vertex, i.e. a vertex v such that $N[v]$ induces a clique.
2. Write an algorithm that computes an optimal proper colouring of an interval graph G. What is its complexity (first assume that we know the intervals, then assume that we only know G)?

   Goal: To begin, $O(n \ln n + m)$ if we know the intervals, then $O(m)$ even if we don't.