Graph Algorithms

TD2 : Graph colouring

1 Some properties of colouring

1. *What is the chromatic number of an even cycle C_{2n} ? Of an odd cycle C_{2n+1}?*

   Let $G = C_n = v_1, \ldots, v_n$ be an even cycle. Then $\chi(G) \geq \omega(G) = 2$. On the other hand, let $c(v_i) \coloneqq i \bmod 2$ for every $i \in [n]$; c is a proper 2-colouring of G and hence $\chi(G) = 2$.

   Let $G = C_n = V_1, \ldots, v_n$ be an odd cycle. Let $c(v_n) \coloneqq 2$, and $c(v_i) \coloneqq i \bmod 2$ for $i \in [n-1]$. Then c is a proper 3-colouring of G, hence $\chi(G) \leq 3$. In order to prove that $\chi(G) = 3$, we need the following intermediate result.

   Claim 1 Let $H = P_n = x_1, \ldots, x_n$ be a path on n vertices, and c a proper 2-colouring of H. Then $c(v_i) = c(v_j)$ iff $i \equiv j \mod 2$ (this is proved easily by induction on $|i-j|$).

   Let us assume for the sake of contradiction that there exists a proper 2-colouring c of G. In particular, it induces a proper 2-colouring of the path $G \setminus v_n$. By Claim 1, $c(v_1) \neq c(v_{n-1})$ since n is odd. The two neighbours of v_n in G have different colours, hence there is no available colour for v_n, a contradiction.

2. *Show that a graph is bipartite if and only if it contains no odd cycle.*

   If a graph G contains an odd cycle C, then $\chi(G) \geq \chi(C) = 3$, so G is not bipartite.

   Reciprocally, let G be a graph that contains no odd cycle, and let T be a spanning tree of G. We can find a proper 2-colouring c of T by rooting T in an arbitrary vertex $r \in V(G)$, and letting $c(v) = \mathrm{dist}_T(v, r) \bmod 2$ for every $v \in V(G)$. We now argue that c is a proper colouring of H. Assume otherwise that there exists an edge $e = uv \in E(H)$ such that $c(u) = c(v)$. Since c is a proper colouring of T, $e \notin E(T)$. Let P be the path from u to v in T, of length at least 2. Then P is properly 2-coloured by c, and since $c(u) = c(v)$, Claim 1 implies that it contains on odd number of vertices. Then $P + uv$ is an odd cycle in G, a contradiction.

3. *Show that for every graph G, there exists an order on the vertices such that the greedy algorithm applied in this order returns a colouring with $\chi(G)$ colours.*

   Let c be a proper k-colouring of G, with $k = \chi(G)$. Let $v_1, \ldots, v_n$ be an ordering of $V(G)$ such that $c(v_i) \leq c(v_j)$ whenever $i \leq j$. Let us consider a run of the greedy colouring with that order on the vertices, and let c^* be the returned colouring. Let us prove by induction on i that for each vertex v_i, $c^*(v_i) \leq c(v_i)$. When $i = 1$, this is obvious since v_1 is the first coloured vertex, with colour 1. When $i \geq 2$, observe that for every neighbour v_j of v_i such that $j < i$, it holds that $c(v_j) < c(v_i)$ (since $c(v_j) \neq c(v_i)$), and so by the induction hypothesis $c^*(v_j) < c(v_i)$. So the colour $c(v_i)$ is available for v_i when the greedy algorithm assigns the colour $c^*(v_i)$, and hence $c^*(v_i) \leq c(v_i)$.

4. *Prove that $\chi(G) \geq |V(G)|/\alpha(G)$, for every graph G.*

   Let c be a proper k-colouring of G, with $k = \chi(G)$, and let $V_1, \ldots, V_k$ be its colour classes. Each V_i is an independent set, hence $|V_i| \leq \alpha(G)$. On the other hand, the sets $(V_i)_{i \in [k]}$ partition $V(G)$, hence $n \coloneqq |V(G)| = \sum_{i=1}^{k} |V_i| \leq k \cdot \alpha(G)$. We conclude that $\chi(G) = k \geq n/\alpha(G)$.

2 Interval graphs

Given a set of intervals $\mathcal{I} = \{I_1, \dots, I_n\}$ where $I_i = [a_i, b_i]$ for every $1 \le i \le n$, the interval graph associated with $\mathcal{I}$ is the graph $G = (V, E)$ where $V = \{1, \dots, n\}$ and $ij \in E$ iff I_i and I_j intersect, i.e. $a_i \le b_j$ and $a_j \le b_i$, for every $1 \le i, j \le n$.

1. *Show that in an interval graph, there exists a simplicial vertex, i.e. a vertex v such that $N[v]$ induces a clique.*

   Let i_0 be such that b_{i_0} is minimal; we show that i_0 is a simplicial vertex. Let i_1, i_2 be two neighbours of i. By definition, and since b_{i_0} is minimal, we have $a_{i_1} \le b_{i_0} \le b_{i_2}$ and $a_{i_2} \le b_{i_0} \le b_{i_1}$. So $i_1 i_2 \in E$, whence $G[N(v)]$ is complete.

2. *Write an algorithm that computes an optimal proper colouring of an interval graph G. You may assume that we know the intervals. The goal complexity is $O(n \ln n + m)$.*

   Let $v_1, \dots, v_n$ be an order obtained by successively extracting simplicial vertices from G. If we know the intervals, we can simply order the vertices i increasingly with respect to b_i, in time $O(n \ln n)$ (where $n = |V(G)|$). Otherwise, finding a simplicial vertex in G can be done in time $O(m)$ (where $m = |E(G)|$), so construct that ordering can be done in time $O(nm)$.

   Run the greedy colouring algorithm on G with the reverse order. When v is coloured, its coloured neighbours form a clique (of size $k \le \omega(G) - 1$), and so its colour is at most $k + 1 \le \omega(G)$. In the end, the number of colours introduced is at most $\omega(G) \le \chi(G)$. The complexity of the greedy colouring algorithm is $O(m)$, so the final complexity is either $O(n \ln n + m)$ or $O(nm)$.

3. *We now want to write an algorithm which computes a proper colouring of any graph G, and uses $\chi(G)$ colours if G is an interval graph (so in particular we don't know the intervals if this is the case). Show that this can be done with the greedy colouring algorithm applied with a reverse degeneracy ordering.*

   We argue that in an interval graph G, $\delta^*(G) + 1 \le \omega(G)$, which means that we can instead use a degeneracy ordering, which can be computed in time $O(m)$. Let H be a subgraph of G of such that $\delta(H) = \delta^*(G)$. Since H contains a simplicial vertex v, $\omega(H) \ge \deg(v)+1 \ge \delta(H)+1 \ge \delta^*(G)+1$. The conclusion follows since $\omega(G) \ge \omega(H)$.

3 Mycielski graphs

Given the graph M_i, we decompose $V(M_i)$ into $V_i = V(M_{i-1})$, V_i' the set of copies, and w_i the vertex linked to V_k'.

1. *Let G be a k-chromatic graph, and c a proper k-colouring of G. Show that for every colour i, there exists a vertex $v \in V(G)$ such that $c(v) = i$ and all the other colours appear in its neighbourhood.*

   Let G be a graph of chromatic number k, and let c be a proper k-colouring of G. Let us assume for the sake of contradiction that, for some colour i, every vertex v such that $c(v) = i$ misses a colour $a_v \in [k] - i$ in its neighbourhood. Let us recolour every such vertex v with colour a_v, thus creating some colouring c' of G. Since the recoloured vertices where the ones coloured with i by c, they form an independent set, and so after the recolouring process the colours in their neighbourhood remain unchanged. So c' is a proper colouring of G, and c' uses colours only from $[k] - i$, so $k - 1$ different ones. This contradicts the fact that $\chi(G) = k$.

2. *Show that for all $i \ge 2$, the graph M_i contains no triangle (i.e. a copy of the complete graph K_3).*

We prove the result by induction on i. The base case $i = 2$ is trivial. Assume for the sake of contradiction that M_i contains a triangle $T = (u_1, u_2, u_3)$ for some $i \geq 3$. Since the neighbourhood of w_i is an independent set, w_i is contained in no triangle, hence $w_i \notin T$. Since V_i' is an independent set, $|T \cap V_i'| \leq 1$. By the induction hypothesis, M_{i-1} contains no triangle, hence T is not entirely contained in $V_i = V(M_{i-1})$. We infer that $T \cap V_i'$ contains exactly one vertex, say u_1. Let $u_1' \in V_i$ be the vertex of which u_1 is a copy, then $u_2, u_3 \in N(u_1')$. So (u_1', u_2, u_3) is a triangle entirely contained in V_i, a contradiction.

3. *Show by induction that $\chi(M_i) \leq i$, for all $i \geq 2$.*

   We know already that $\chi(M_2) = \chi(K_2) = 2$. Let $i \geq 3$, and let c be a proper $(i-1)$-colouring of M_{i-1} obtained by induction. We let $c_i(v) \coloneqq c_{i-1}(v)$ for every $v \in V_i = V(M_{i-1})$, and we let $c_i(v') \coloneqq c_i(v)$ for the copy $v' \in V_i'$ of v. Finally, we let $c_i(w_i) \coloneqq i$. It is straightforward that c_i is a proper i-colouring of M_i.

4. *Show that $\chi(M_i) \geq i$, for all $i \geq 2$.*

   Assume for the sake of contradiction that there exists a proper $(i-1)$-colouring c of M_i. In particular, c induces a proper $(i-1)$-colouring of M_{i-1} on V_i. By the result of Question 3.1, for every colour $j \in [i-1]$, there exists some vertex $v_j \in V_i$ such that $c(v_j) = j$ and $c(N_{M_{i-1}}(v_j)) = [i-1] \setminus \{j\}$. Let $v_j' \in V_i'$ be the copy of v_j. Then the colour of v_j' is forced to be j, since $N_{M_{i-1}}(v_j) \subset N_{M_i}(v_j')$. We conclude that $c(V_i') = [i-1]$, so there remains no colour available for w_i, a contradiction.