Estimating suitable metrics for an empirical manifold of shapes with application to human silhouettes

Guillaume Charpiat

Projet Pulsar

Shape WorkIN'Group

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Map

- **Introduction**
  - Motivation
  - Issues
- **Searching for solutions**
  - Main existing approaches and their limitations
  - Main idea
- **The approach**
  - Shape matching
  - Transport
  - Metric estimation (statistics on deformations)
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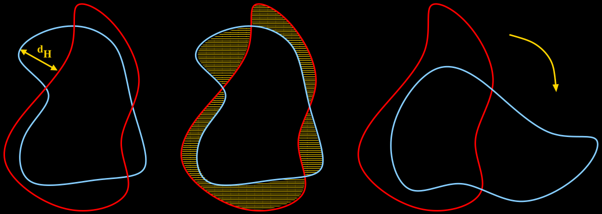
Motivation

- Shape spaces : which metric ?
  (to define similarity/distance between shapes)
  - Hausdorff distance
  - Symmetric difference area
  - Quotients by transformation groups (rotation, translation, scaling, affine...)



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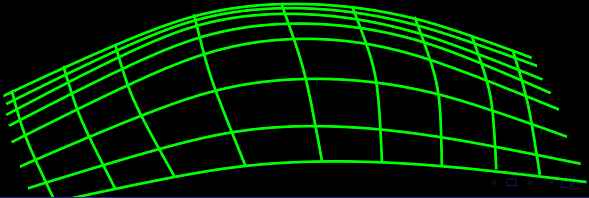

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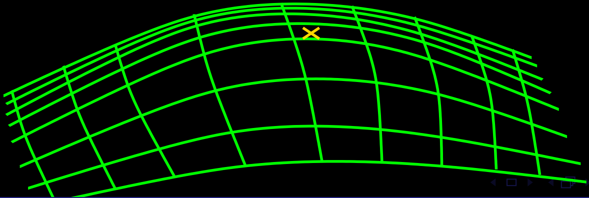

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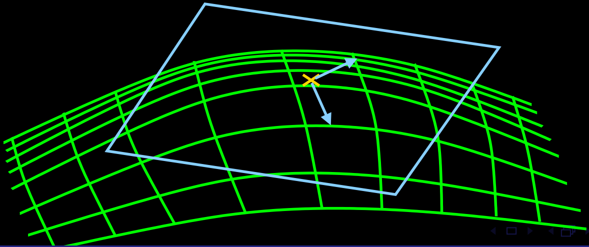

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  - rigid motion more probable $\Longrightarrow$ associated metric

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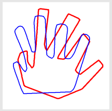
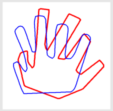
L^2 inner product  vs.  rigidifying inner product

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  - rigid motion more probable $\Longrightarrow$ associated metric
- $\Longrightarrow$ learn the suitable metric from examples (datasets of shapes)

Issues

- Sparse sets of highly varying shapes
  - e.g. human silhouettes
  - high intrinsic dimension ($\geqslant 30$)
  - $\Longrightarrow$ no dense training set

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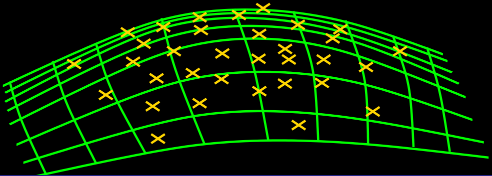

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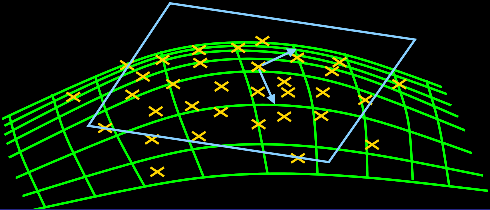

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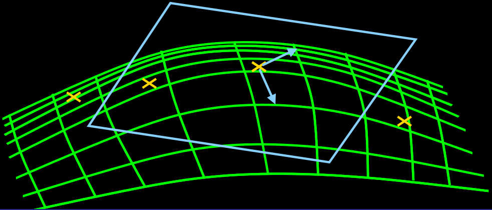

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- to compare quantities defined on different shapes : need for correspondences
  - match shape with different topologies ?
  - very frequent topological changes

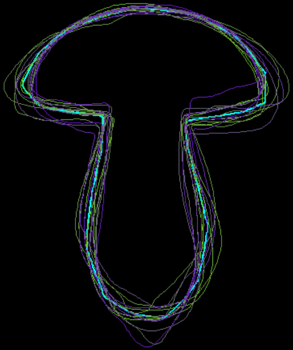
Searching for solutions

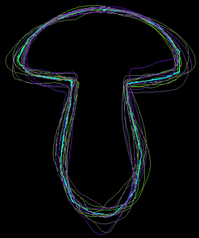
Main existing approaches and their limitations

Approach 1 : *mean* + *modes* model

↪ e.g. my PhD thesis

Automatic alignment
⟶
and average shape computation



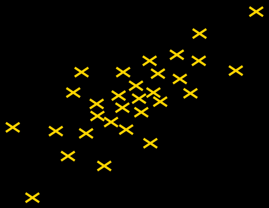


Statistics (PCA) on deformation fields
$\longrightarrow$
between the mean shape and each sample

modes of deformation
= deformation prior
= Gaussian probabilistic model

More details on PCA/modes/Gaussian distributions/inner product

- Mean M, shapes S_i, warpings $W_{M \to S_i}$



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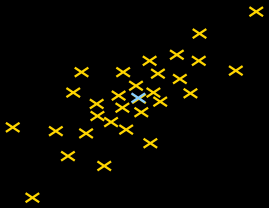

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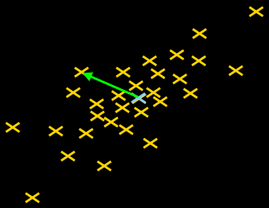

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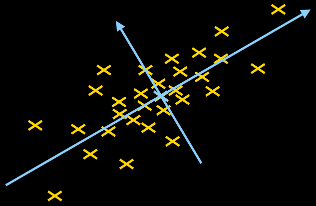
- Mean M, shapes S_i, warpings $W_{M \to S_i}$
- PCA : diagonalize correlation matrix C : $C_{ij} = \left\langle W_{M \to S_i} \middle| W_{M \to S_j} \right\rangle$
  $\Longrightarrow$ eigenmodes e_k with eigenvalues λ_k : best coordinate system



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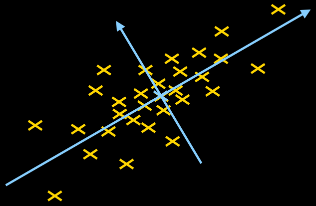
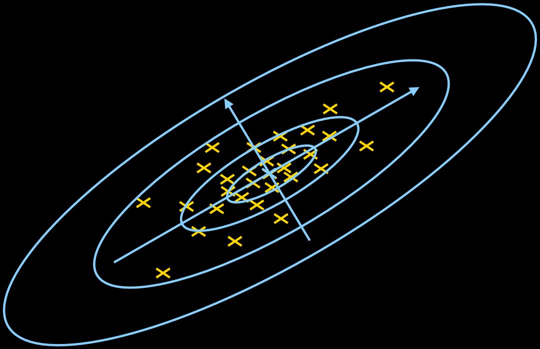
- any new deformation W of M :
  $W = \sum_k \alpha_k e_k +$ noise



More details on PCA/modes/Gaussian distributions/inner product

- Mahalanobis distance : $d(M+W,(S)) = \sum_k \frac{\alpha_k^2}{\lambda_k^2}$



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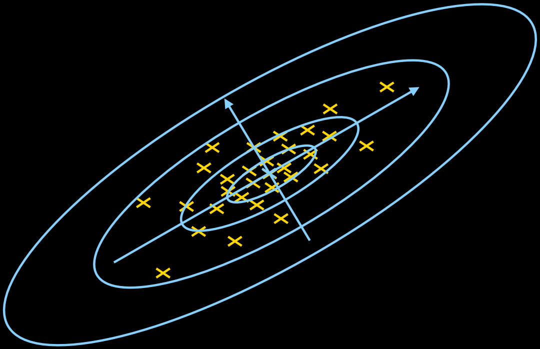
- Mahalanobis distance : $d(M+W,(S)) = \sum_k \frac{\alpha_k^2}{\lambda_k^2}$
- probability $p(W) \propto exp(-\sum_k \frac{\alpha_k^2}{2\lambda_k^2})$: Gaussian distribution



More details on PCA/modes/Gaussian distributions/inner product

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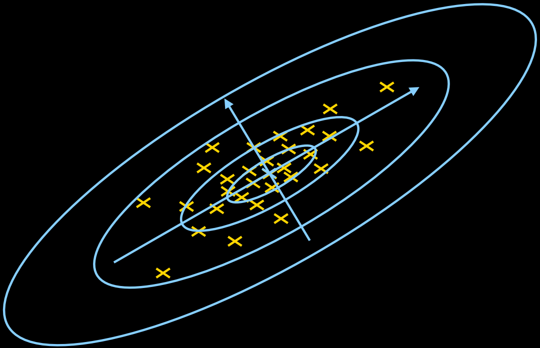
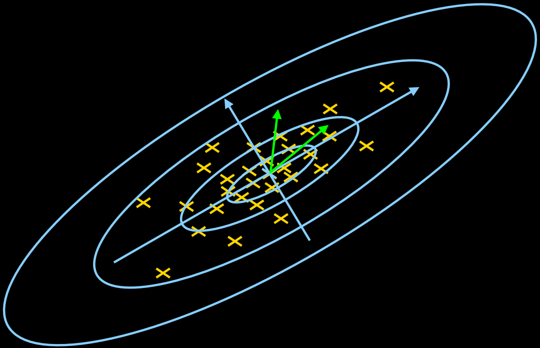


More details on PCA/modes/Gaussian distributions/inner product

- defines a Gaussian shape prior
- associated inner product on deformations, in the tangent space of M:
  $\langle W_1 \,|\, W_2 \rangle = \sum_k \frac{1}{\lambda_k^2} \alpha_{1,k}\alpha_{2,k}$



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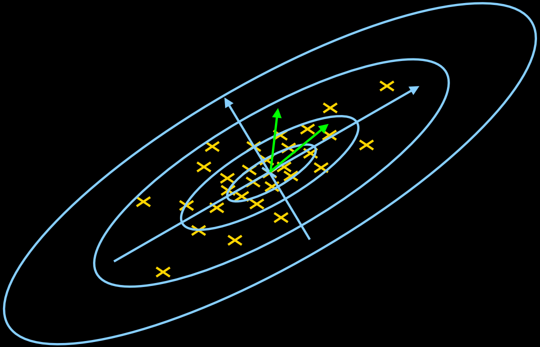
  $$\langle W_1 | W_2 \rangle = \sum_k \frac{1}{\lambda_k^2} \alpha_{1,k} \alpha_{2,k}$$
- defines a deformation cost $\|W\|^2 = \langle W | W \rangle$



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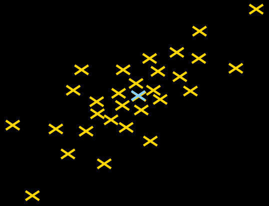
- Empirical distribution : $\mathcal{D}_{emp} = \sum_i \delta_{W_{M \to S_i}}$

  (possibly smoothed by a kernel)



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- Empirical distribution : $\mathcal{D}_{emp} = \sum_i \delta_{W_{M \to S_i}}$
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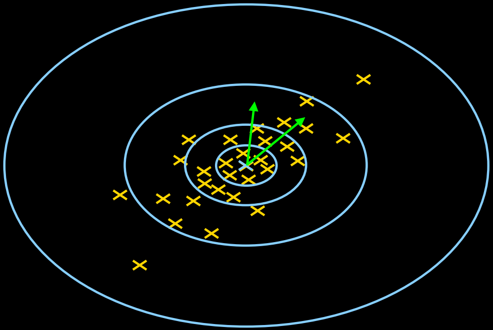
- Any inner product $< | >_P$ in tangent space of the mean
  $\implies$ Gaussian distribution $\mathcal{D}_P(W) \propto \exp(-\|W\|_P^2)$



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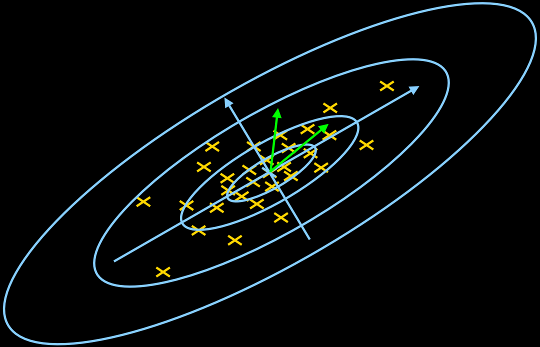
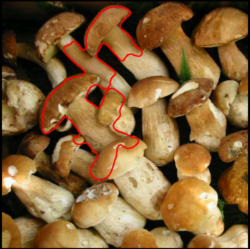
  $\Longrightarrow$ Gaussian distribution $\mathcal{D}_P(W) \propto \exp(-\|W\|_P^2)$
- Best P for Kullback-Leibler$(\mathcal{D}_P|\mathcal{D}_{emp})$: PCA!



Example of application : image segmentation with shape prior



without shape prior

with shape prior

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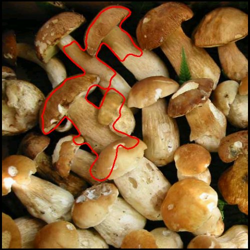

without shape prior

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- requires a mean shape (does not always make sense, e.g. person walking)

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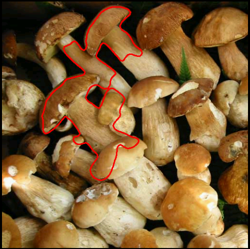


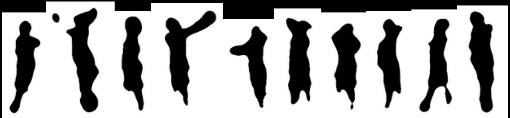
without shape prior

with shape prior

- requires a mean shape (does not always make sense, e.g. person walking)
- requires all deformations between the mean and samples :
  $\Longrightarrow$ relatively similar sample shapes (otherwise, not reliable)

Approach 1 : mean + modes model

$\hookrightarrow$ example 2 : level-set means and modes



mean + first modes

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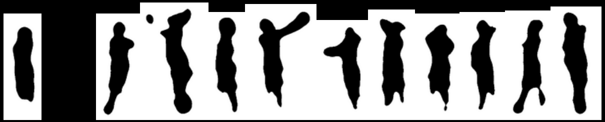

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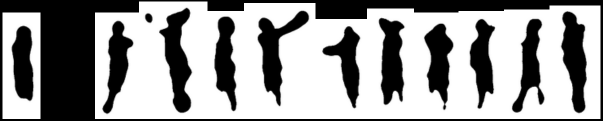
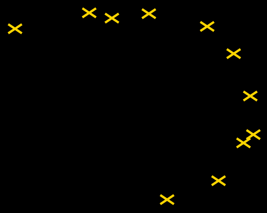


mean + first modes

- mean shape makes no sense
- level-set differences to express deformations : do not handle thin parts

Approach 2 : distance-based methods (e.g. kernel methods)



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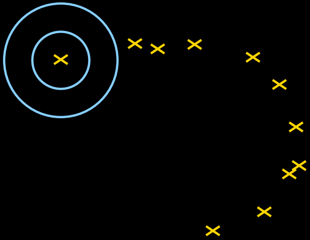

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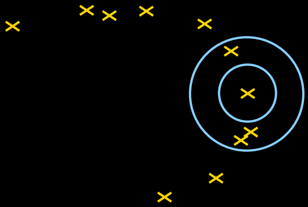

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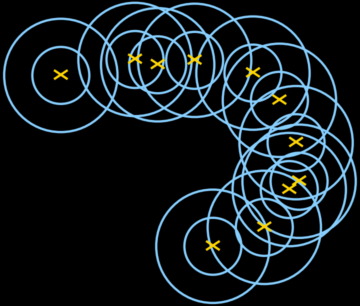

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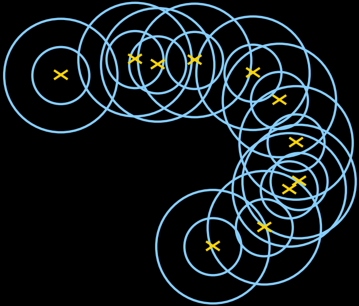

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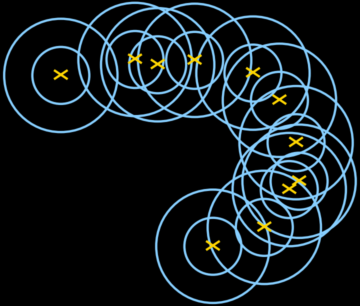

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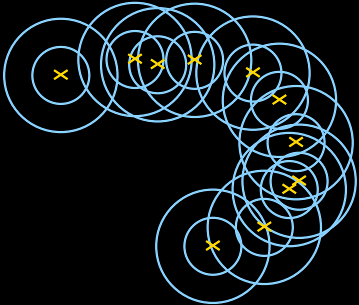

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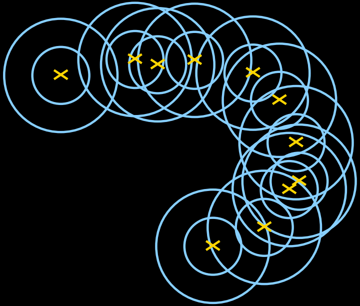

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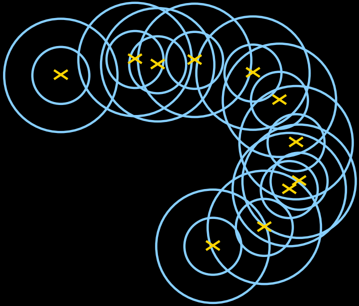

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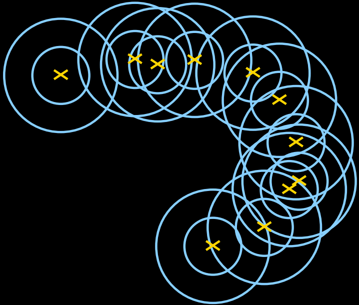

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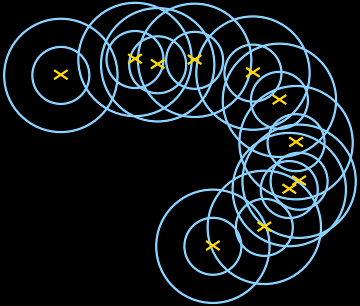

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- distances are not reliable/meaningful for far shapes
- $\Longrightarrow$ needs for a representative neighborhood, i.e. a high dataset density
- in a high-dimensional manifold, all distances are similar, and all points are on the boundary of the manifold
- $\Longrightarrow$ cannot work, need for more information than distances

Main idea

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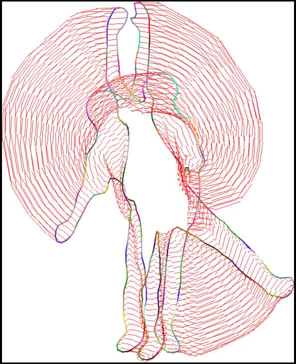

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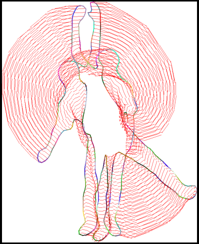

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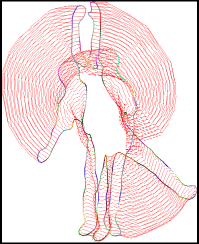

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- but shape matching reliable only for close shapes

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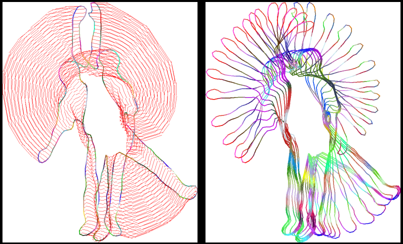
- prior chosen : transported deformations make sense,
  i.e. a deformation observed on one shape can be applied to other shapes



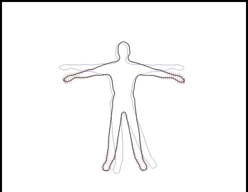
- transport requires correspondences
- but shape matching reliable only for close shapes
- $\Longrightarrow$ compute correspondences between close shapes only, and combine small steps of reliable correspondences to build longer-distance correspondences

Map

- Close shape matching
- Transport
- Metric estimation (statistics on transported deformations)
- Theoretical justifications

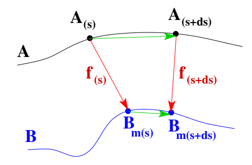
Shape matching

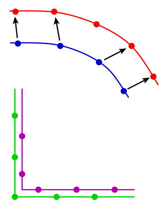
Simple case : two shapes, A and B, with one connected component



$$\inf_{f:A\to B} \int_A \|f\|^2 + \alpha \|\nabla f\|^2 dA$$

- shape sampling
- dynamic time warping
- theory & experiments :
  higher sampling rate on target





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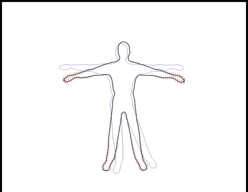

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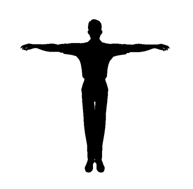
Usual case : random topologies



Usual cases = more complex
(more than 10 connected components in this silhouette)
but
one connected component $\rightarrow \bigcup_i$ connected components
= the same

Further possible improvements

- as such, allows appearing points (mismatches)
- allows disappearing points : matching to $\varnothing$ with a fixed high cost
- pb : better matchings, but energy value loses meaning

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Drawbacks

- specific to planar curves
- not symmetric : $m_{A \to B} \neq m^{-1}_{B \to A}$

Transport

Local transport

- Set of shapes $(S_i)_{i \in I}$ (e.g. from a video segmentation)

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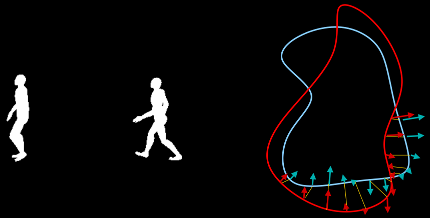

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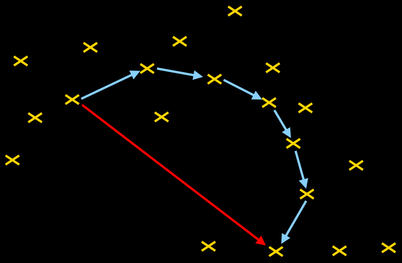

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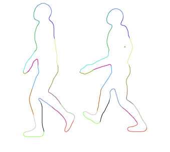
- reliability : $w^G_{i \to j} = \prod_k w^L_{i_k \to i_{k+1}}$
- use transport to propagate information



Example : colored walker

Metric estimation (statistics on deformations)

- set of shapes (S_i), local deformations $m_{i \to j}$, transport $T^G_{i \to k}$

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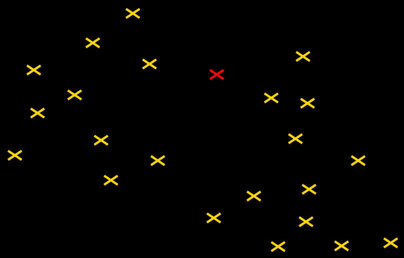

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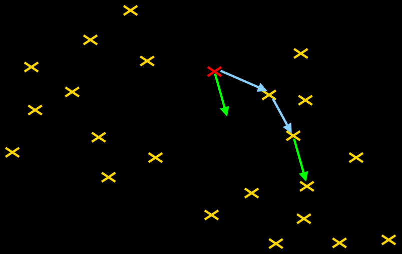

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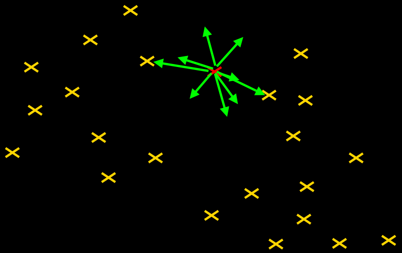

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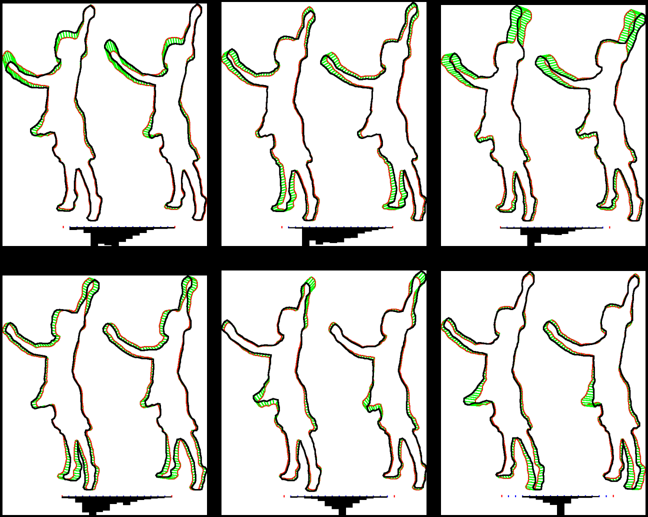
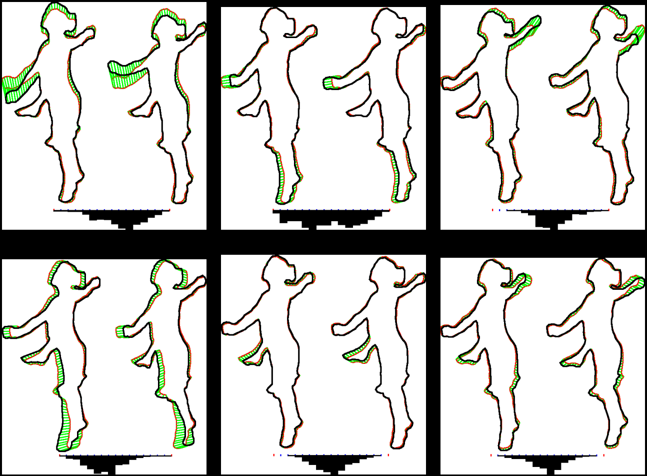
- $\Longrightarrow$ defines an inner product P_k = metric in the tangent space of the shape S_k
- P_k varies smoothly as a function of k

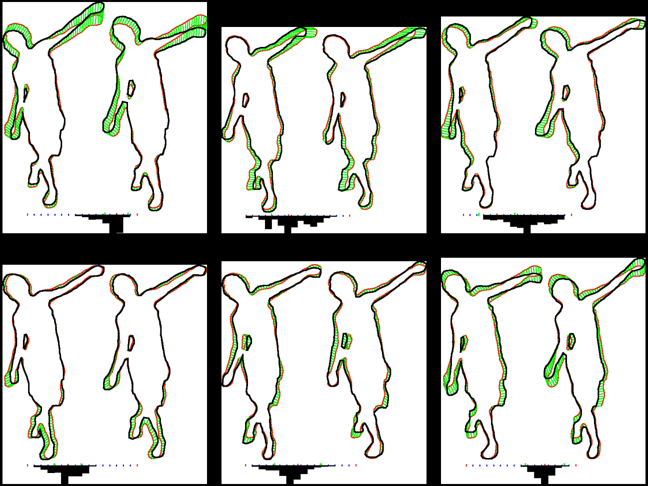
Example of results : dancing sequence (9s, 24Hz), shape 1



Example of results : shape 2



Example of results : shape 3



Weighted PCA with H^1 norm

- PCA = find the best axes (to project data on this subspace)
- Minimize projection error :

$$\inf_{\left\langle \mathbf{e}_n \middle| \mathbf{e}_{n'} \right\rangle_{H^1_\alpha} = \delta_{n=n'}} \sum_{i,j} w^k_{i\to j} \left\| \mathbf{f}^k_{i\to j} - \sum_n \left\langle \mathbf{f}^k_{i\to j} \middle| \mathbf{e}_n \right\rangle_{H^1_\alpha} \mathbf{e}_n \right\|^2_{H^1_\alpha}$$

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  where $F = \sum_{i,j} w^k_{i\to j} \mathbf{f}^k_{i\to j} \otimes \mathbf{f}^k_{i\to j}$ = weighted covariance matrix,
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- Change of variables: $\mathbf{d}_n = H^{1/2} \mathbf{e}_n$: $\sup_{\langle \mathbf{d}_n | \mathbf{d}_{n'} \rangle_{L^2} = \delta_{n=n'}} \sum_n \mathbf{d}_n \, H^{1/2} F H^{1/2} \, \mathbf{d}_n$

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- eigenvectors :

$$\mathbf{e}_n = \sum_{ij} \gamma_n^{(i,j)} \sqrt{w^k_{i\to j}}\, \mathbf{f}^k_{i\to j}$$

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The best metric ?

Searching for principal modes of deformations which vary smoothly (as a function of the shape S_k) ?

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- $\implies$ best metric for another criterion involving transport & L^2-norm of deformations.

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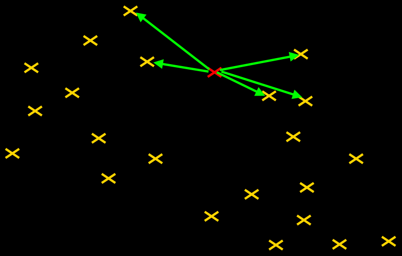

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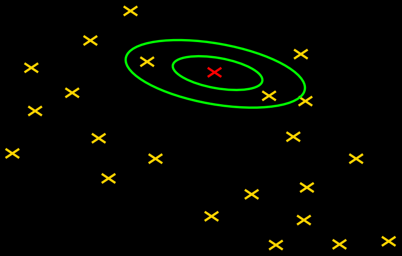

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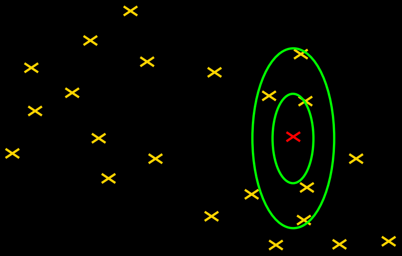

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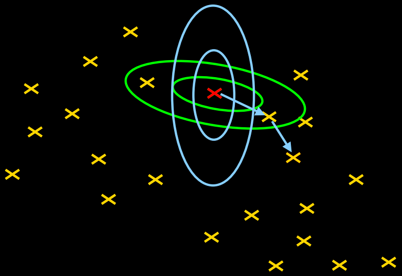

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