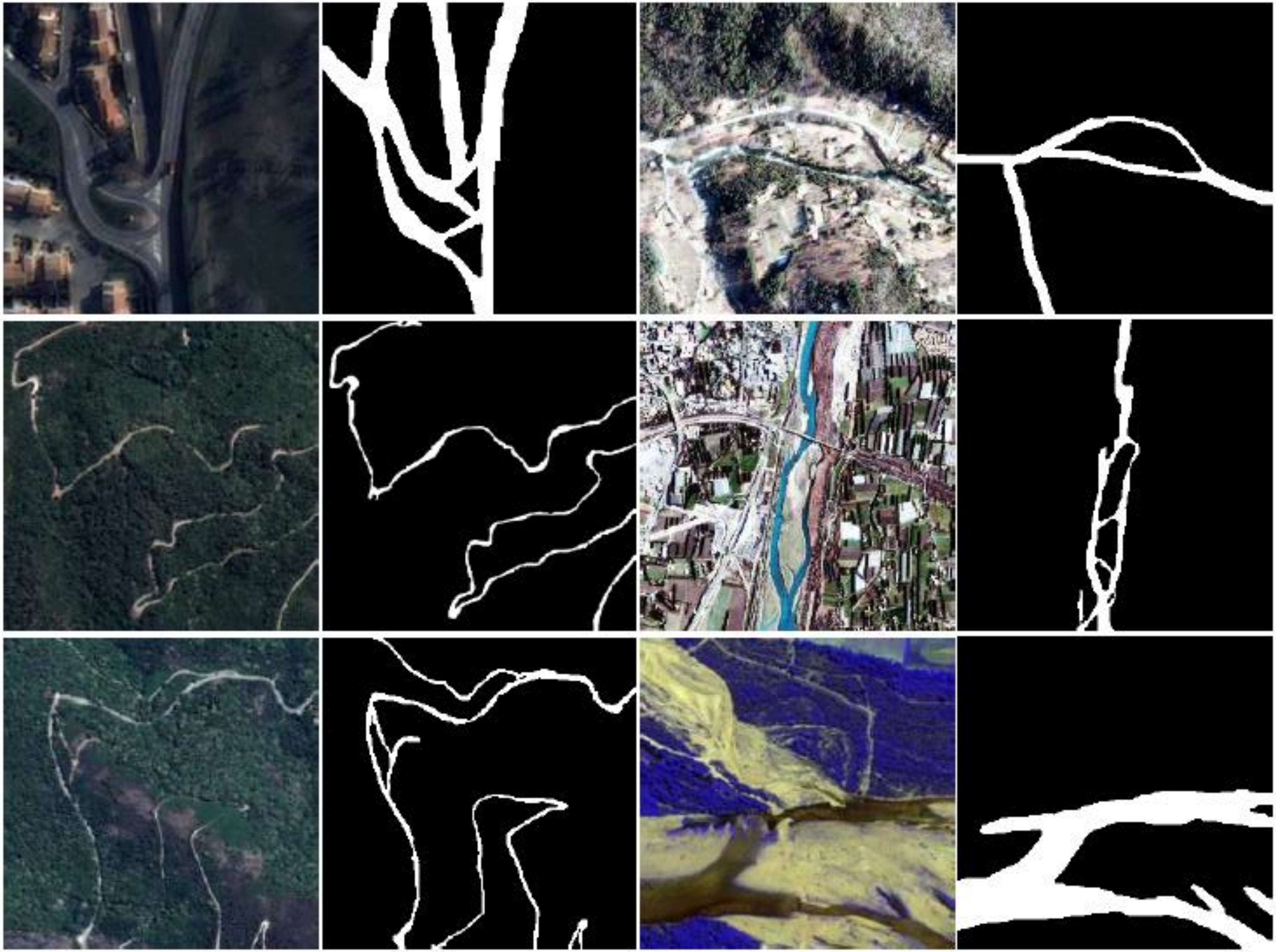
# Phase fields for network extraction from images

**Aymen El Ghoul**
Ian Jermyn
Josiane Zerubia

4th SWING meeting
May 19, 2010

# Problem: object extraction

- Ubiquitous in applications
  - In remote sensing imagery: road and hydrographic networks, trees, buildings…
  - In medical imagery: vascular networks, tumors…
- Automatic methods need to incorporate human knowledge into mathematical models.



# Difficulties?

- Appearances of the background and the foreground are similar
  - ‘shape’ (region) distinguishes between them.
- Rivers are different from roads
  - branch extremities tend to not end.
  - at junctions, a significant change of branch widths occurs.
- The presence of occlusions: gap closure.

# Problem formulation?

- Calculate a MAP estimate of the region R containing the entity:

$$\hat{R} = \arg\max_{R} \mathrm{P}(R|I,K)$$
$$\mathrm{P}(R|I,K) \propto \mathrm{P}(I|R,K)\mathrm{P}(R|K)$$

- In practice, minimize an energy:

$$\hat{R} = \arg\min_{R} E(R,I)$$
$$E(R,I) = -\ln \mathrm{P}(\mathrm{I}|\mathrm{R},\mathrm{K}) - \ln \mathrm{P}(\mathrm{R}|\mathrm{K})$$
$$= E_{\mathrm{I}}(I,R) + E_{\mathrm{P}}(R) + \mathrm{const}$$

Likelihood: will not talk about this: important, but less generic.

Prior: subject of talk

# Roadmap of the rest of talk

- Undirected network phase field HOACs
  - The model.
  - Limitations in the case of our purposes.
- Directed network phase field HOACs
  - The model: extension of the undirected network model.
  - Results on VHR remote sensing images.
- Conclusions and prospects.

# First part: undirected phase field HOACs

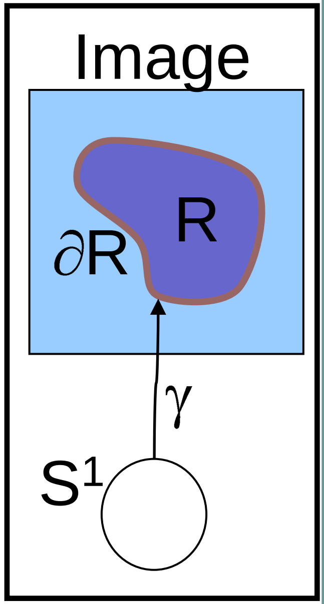
- Active contours and HOACs.
- Phase diagram of a HOAC model.
- HOACs as phase fields?

# Active contours?

- A region R is represented by its boundary, $\partial R = [\gamma]$, the 'contour'.
- Classical prior energy:
  - Length of $\partial R$ and area of R:

$$E_{C,0}(R) = \lambda_C L(R) + \alpha_C A(R)$$

  - Short-range dependencies between boundary points.
  - Describes boundary smoothness.

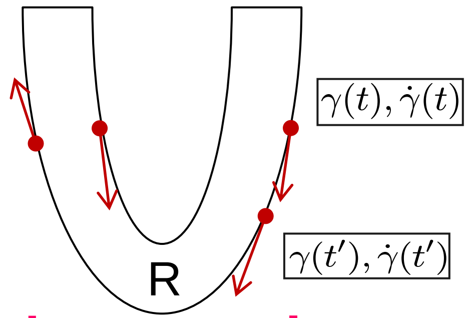


# Limitations of $E_{C,0}$?

- Remote sensing images are complex.
  - $E_{C,0}$ is insufficient for automatic solution of real problems.
- Regions of interest are distinguished by their shape.
  - But topology can be non-trivial, and unknown a priori.
- Need strong prior knowledge of shape, but without constraining topology.

# Build better shape prior: HOACs

- Incorporate prior knowledge about shape via long-range dependencies between boundary points.

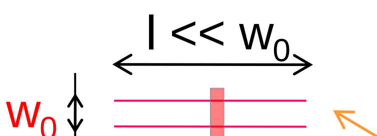



- How? Multiple integrals over the contour.
  - E.g. Euclidean invariant two-point term:

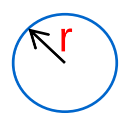
$$E_{\mathrm{C,Q}} = -\frac{\beta_{\mathrm{C}}}{2} \iint_{(S^1)^2} dt\, dt'\, \dot{\gamma}(t) \cdot \dot{\gamma}(t')\, \Psi\left(\frac{|\gamma(t) - \gamma(t')|}{d}\right)$$

# Energy minimization

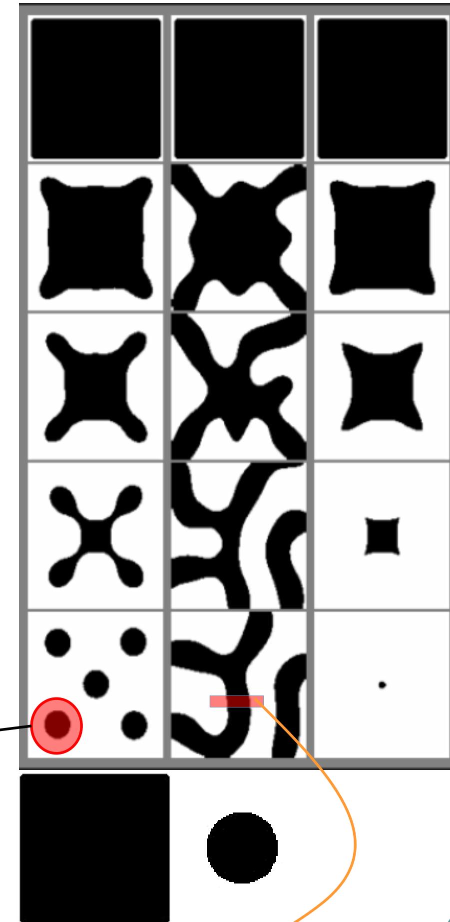
- Total energy (4 parameters):
  $E_{C,P}(R) = E_{C,0}(R) + E_{C,Q}(R)$
- Problem: different stable configurations for some PSs.
- Solution: stability analysis
  $\Rightarrow$ parameter constraints




Analysis of a long bar




Analysis of a circle



# Stability analysis

- Taylor series expansion up to second order around $\gamma_o$ (circle, bar):

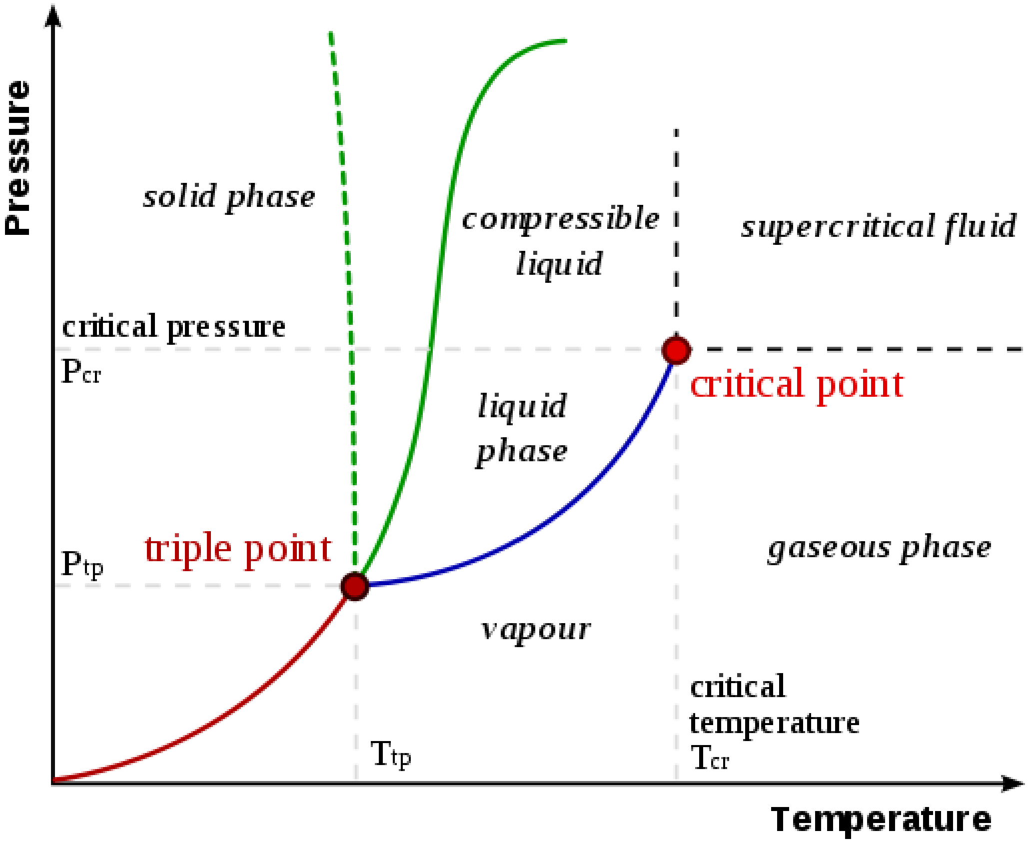
$$E_G^{(2)}(\gamma) = E_G^{(2)}(\gamma_0 + \delta\gamma) \triangleq E_G(\gamma_0) + \langle \delta\gamma | \frac{\delta E_G}{\delta\gamma}\Big|_{\gamma_0} \rangle + \frac{1}{2}\langle \delta\gamma | \frac{\delta^2 E_G}{\delta\gamma^2}\Big|_{\gamma_0} |\delta\gamma\rangle$$

Energy of $\gamma_o$ — First order term — Second order term

$E_G$ has a **minimum** $\gamma_0$ ⟺ $\frac{\delta E_G}{\delta\gamma}\Big|_{\gamma_0} = 0$ and $\frac{\delta^2 E_G}{\delta\gamma^2}\Big|_{\gamma_0} > 0$

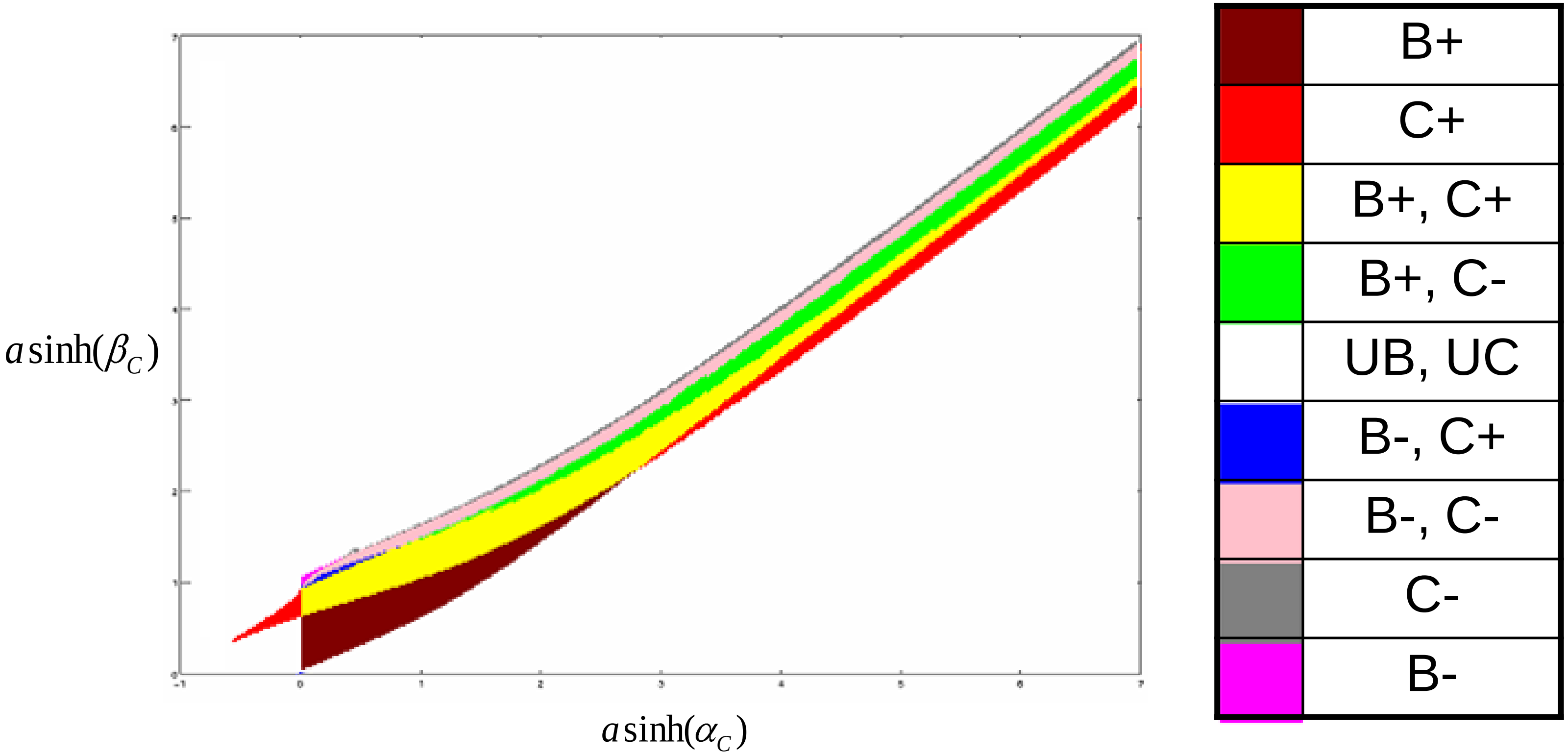
# Result: Phase diagram.

Typical phase diagram of water. ©Wikipedia.



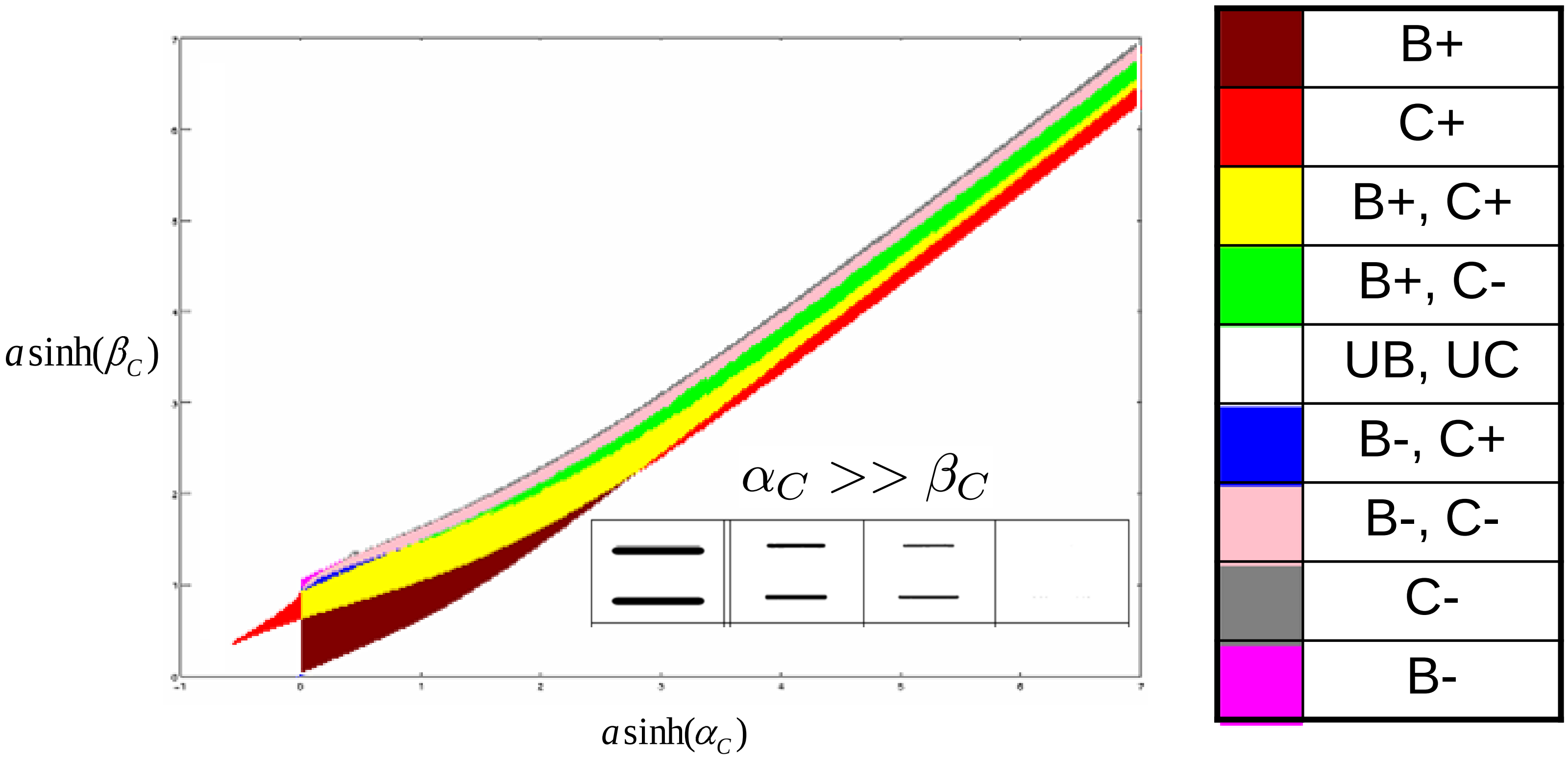
# Result: Phase diagram.

$$E(w_0, l) \approx l e(w_0)$$



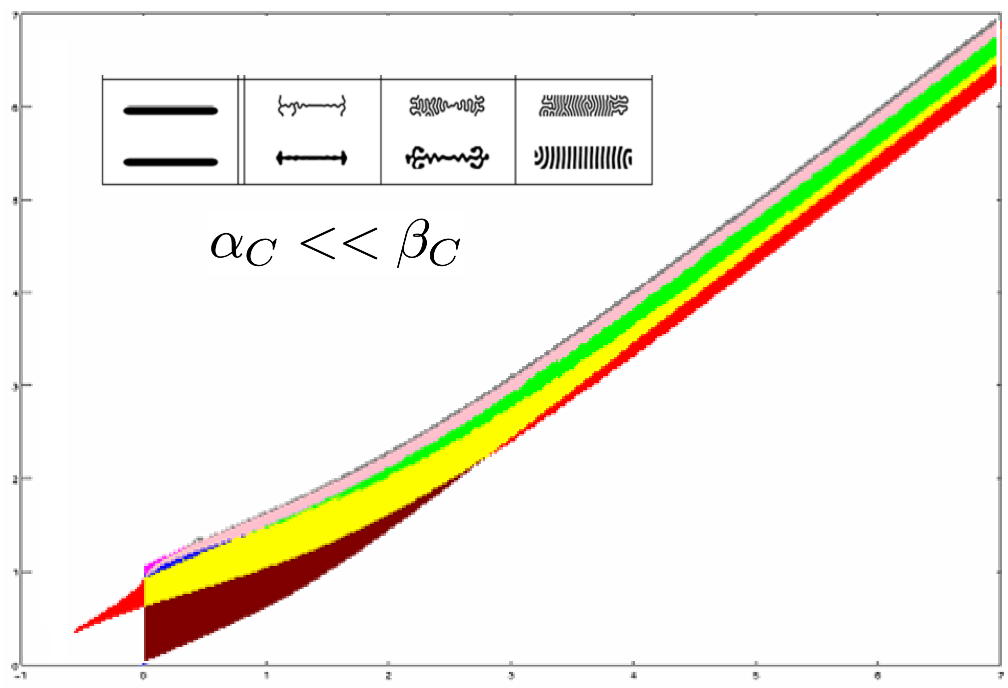
# Result: Phase diagram.

$$E(w_0, l) \approx l e(w_0)$$



# Result: Phase diagram.

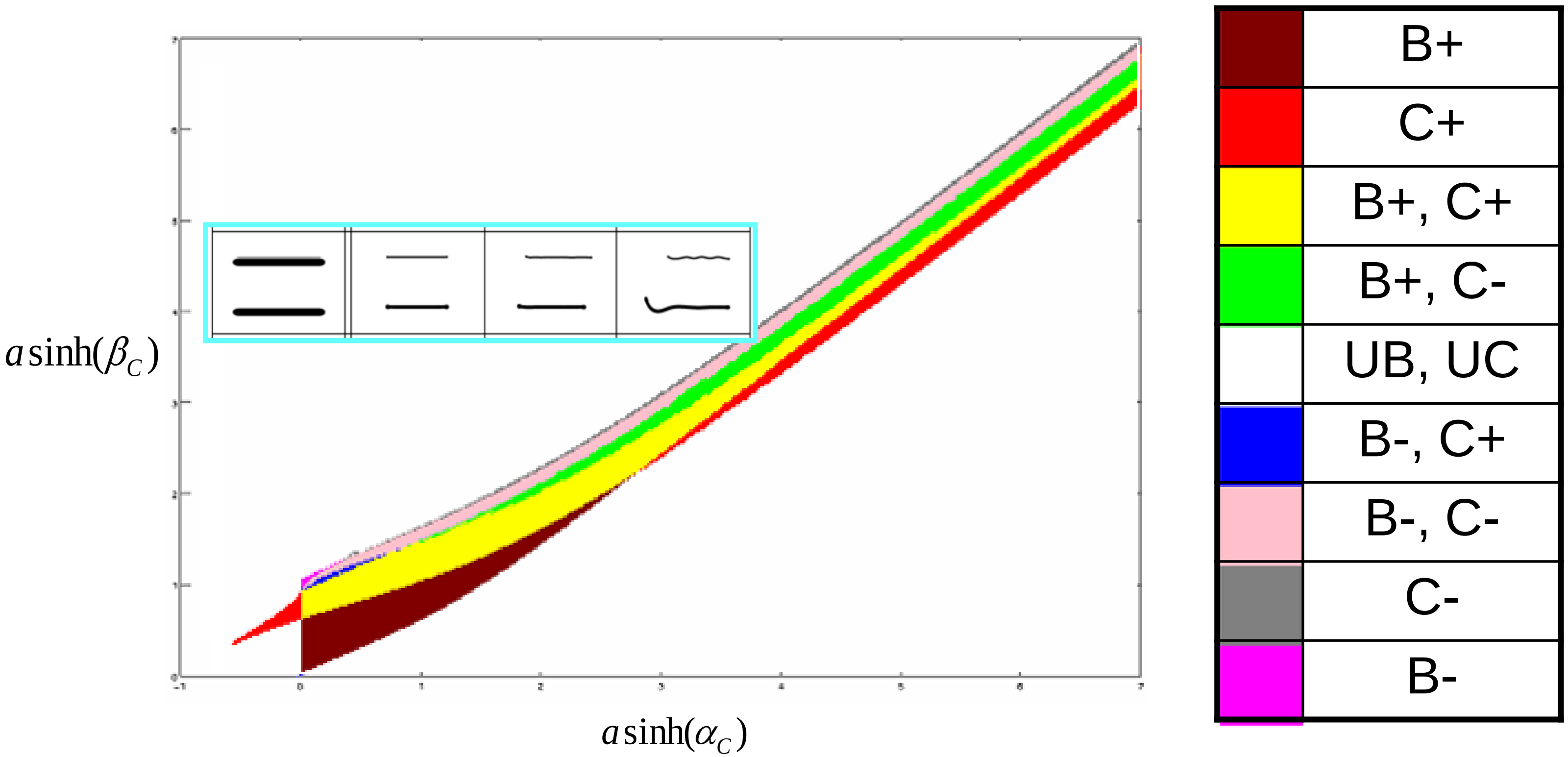
$$E(w_0, l) \approx le(w_0)$$




- B+
- C+
- B+, C+
- B+, C-
- UB, UC
- B-, C+
- B-, C-
- C-
- B-

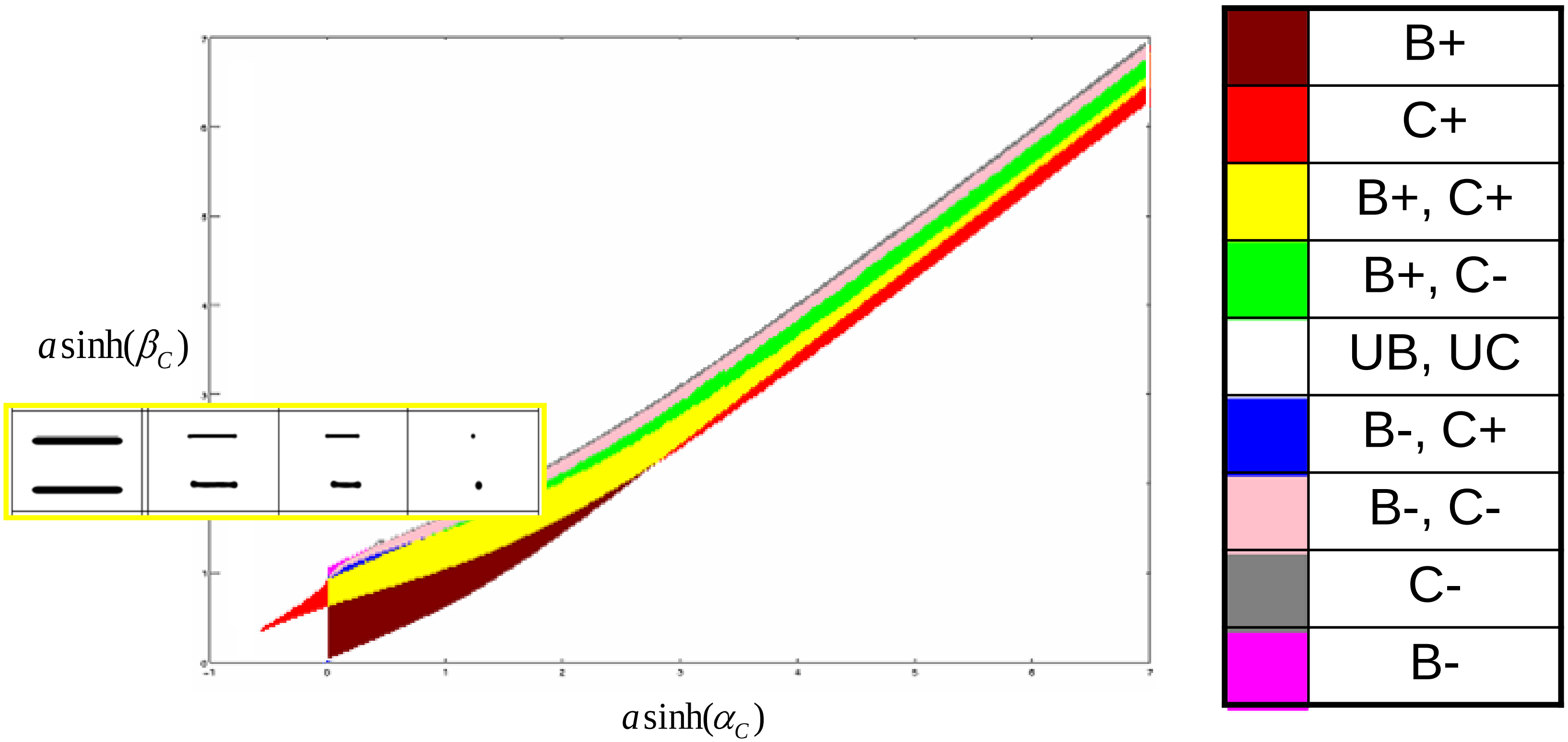
# Result: Phase diagram.

$$E(w_0, l) \approx le(w_0)$$



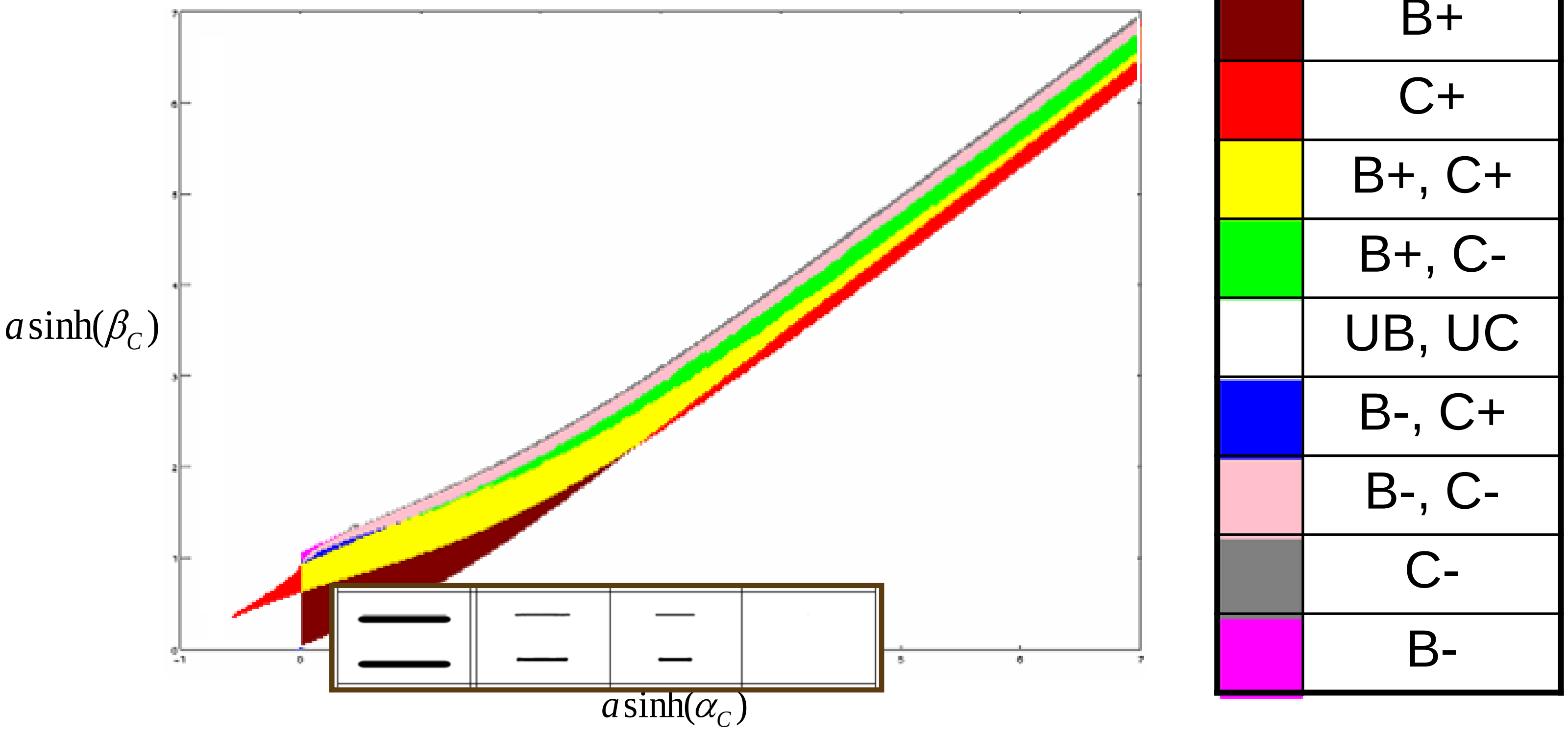
# Result: Phase diagram.

$$E(w_0, l) \approx l e(w_0)$$



# Result: Phase diagram.

$$E(w_0, l) \approx le(w_0)$$

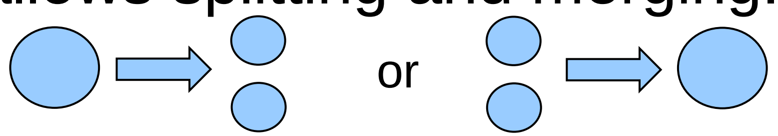


# Difficulties of HOACs?

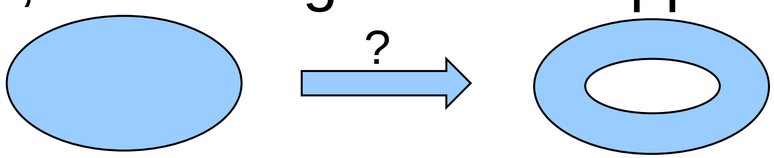
- Model:
  - Complex topologies require many contours.
- Algorithm:
  - Implementation of $\partial E_Q$ is very complex.
  - Execution is slow, especially with long boundary.
  - Not enough topological freedom.

# 'Automatic' topological freedom?

- Contour (explicit) representation
  - No change of topology.
- Level set (implicit) representation
  - Constrained to be a distance function.
    - allows splitting and merging.



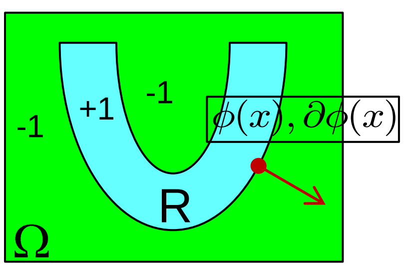

    - but, not enough for our application.

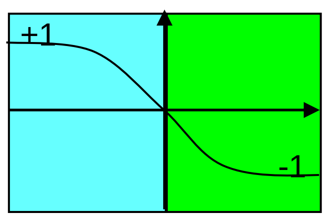



# Solution: phase fields

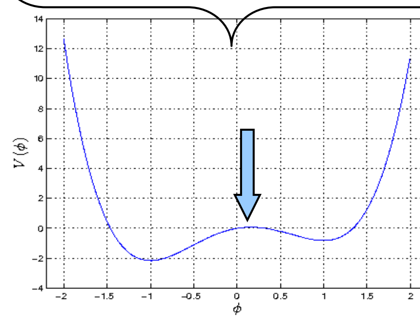
- A phase field $\phi: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a function.
- It defines a region in the image domain $\Omega$: R = {x : $\phi$(x) > z}.
- Basic, local energy:

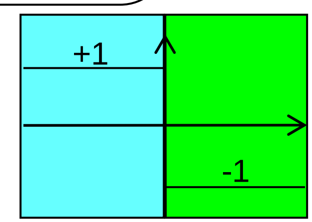
$$E_0^s(\phi) = \int d^2x \left\{ \frac{D}{2} \partial\phi \cdot \partial\phi + \lambda \left( \frac{\phi^4}{4} - \frac{\phi^2}{2} \right) + \alpha \left( \phi - \frac{\phi^3}{3} \right) \right\}$$




Smoothed charact-eristic function.



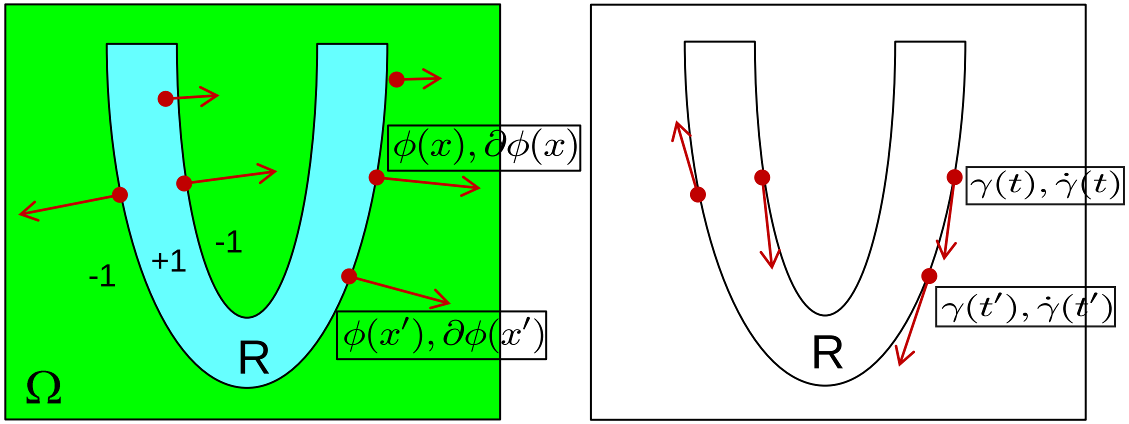

$\lambda \rightsquigarrow \infty$ + discretization $\Rightarrow$ Ising model.

# Shape information?

- Write tangent vector $\dot{\gamma}$ in terms of $\partial\phi$

$$E_{\mathrm{C,Q}}(\gamma) = -\frac{\beta_{\mathrm{C}}}{2} \iint_{(S^1)^2} dt\, dt'\, \dot{\gamma}(t) \cdot \dot{\gamma}(t')\, \Psi\left(\frac{|\gamma(t) - \gamma(t')|}{d}\right)$$

$$E_{\mathrm{NL}}(\phi) = -\frac{\beta}{2} \iint_{\;\;2} d^2x\, d^2x'\, \partial\phi(x) \cdot \partial\phi(x')\, \Psi\left(\frac{|x - x'|}{d}\right)$$



# Phase fields as HOACs?

- Total prior energy:

$$E^s_{\mathrm{P}}(\phi) = E^s_0(\phi) + E_{\mathrm{NL}}(\phi)$$

- For a given region R, one can show:

$$E^s_0(\phi_R) \approx \lambda_{\mathrm{C}} L(R) + \alpha_{\mathrm{C}} A(R) \triangleq E_{\mathrm{C},0}(R)$$

$$E_{\mathrm{NL}}(\phi_R) \propto E_{\mathrm{C,Q}}(R)$$

- Result: one can use phase fields instead of HOACs.

# Overview of the first part

- Contribution: phase diagram of a HOAC model.
- Limitations:
  - The undirected network model does not allow large range of stable branch widths
    - works very well for roads but not for rivers.
  - Lack of connectivity: presence of gaps.
- Solution: directed network models.

# Preview of the second part

- Directed networks (e.g. rivers) carry 'flow' through their branches.
- Desiderata:
  - large range of branch widths, but
  - width changes must be slow, except
  - at junctions, $\sum_i w_i = 0$.

- Goal: build priors which favor network regions with these geometric properties.

# Second part: directed phase field HOACs

- The model.
- Stability analysis.
- Results on real images.

# The proposed model?

- Introduce a local phase field model incorporating two phase field functions:
  - a scalar field $\phi$ representing a region by its smoothed characteristic function, and
  - a vector field $v$ representing the 'flow' through the network branches.
- Total prior energy:

$$E_{\mathrm{P}}(\phi, v) = E_0(\phi, v) + E_{\mathrm{NL}}(\phi)$$

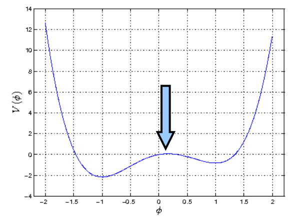
# The local term?

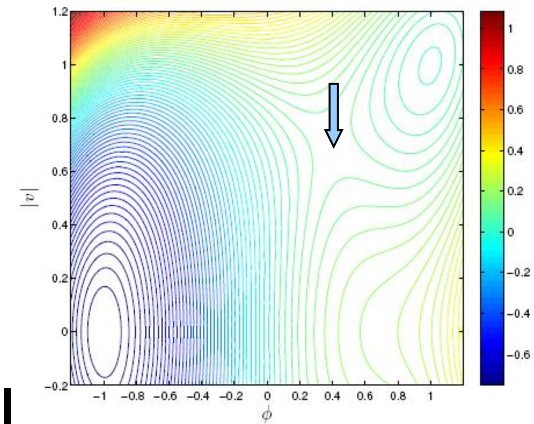
- We require the vector field $v$ to be:
  - 0 outside the network and |v| = 1 inside,
  - smooth,
  - parallel to the region boundary ∂R, and
  - divergence-free.
- The proposed local phase field energy:

$$E_0(\phi, v) = \int d^2x \Big\{ \underbrace{\frac{D}{2}\, \partial\phi \cdot \partial\phi}_{Smoothness} + \underbrace{\frac{D_v}{2}\, (\partial \cdot v)^2}_{Divergence} + \underbrace{\frac{L_v}{2}\, \partial v : \partial v}_{Smoothness} + \underbrace{W(\phi, v)}_{Potential} \Big\}$$

# The potential W($\phi$, $v$)?

- The potential must have 2 minima
  - ($\phi$, |v|) = (-1, 0) for the exterior
  - ($\phi$, |v|) = (1, 1) for the interior






- E.g.: the simplest, a fourth order polynomial of $\phi$ and $|v|$

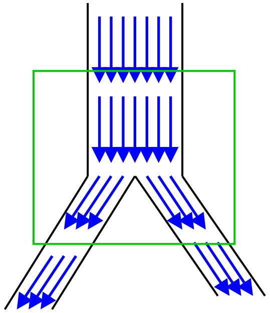
$$W(\phi,|v|)=\frac{|v|^4}{4}+(\lambda_{22}\frac{\phi^2}{2}+\lambda_{21}\phi+\lambda_{20})\frac{|v|^2}{2}+\lambda_{04}\frac{\phi^4}{4}+\lambda_{03}\frac{\phi^3}{3}+\lambda_{02}\frac{\phi^2}{2}+\lambda_{01}\phi$$

# Intuitions?

- Large range of stable branch widths:
  - We choose $\Psi$ to be the Bessel function $K_0$.
- Branch width changes must be slow:
  - low divergence + transition of $|v|$ from 0 to 1 across $\partial R \Rightarrow$ // to the boundary,
  - // to $\partial R$ + smooth $\Rightarrow$ // in the interior,
  - // in the interior + low divergence + $|v| = 1$ $\Rightarrow$ slow width changes and branches prolong.

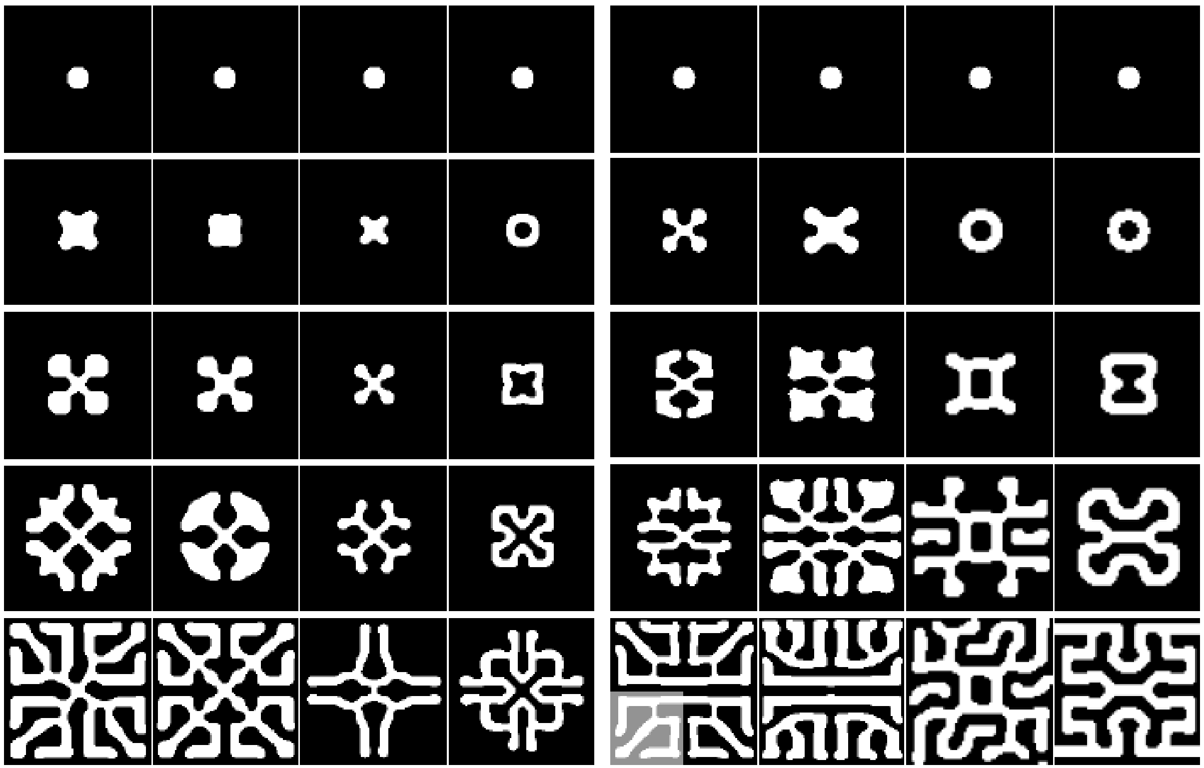
# Intuitions?

- At junctions, the total incoming widths equal total outgoing widths ('flow conservation' of v):
  - // in the interior + low divergence + $|v| = 1$ $\Rightarrow$ branch width conservation.
- The preferred configuation:



# Geometeric evolutions

Undirected network model    Directed network model




# Difficulties?

- Parameter learning: the model has many free parameters (9).
- Many stable configurations for some parameter ranges:
  - e.g.: circular structures, line network structures.
- Solution: constrain the parameter values to favor stable networks.

# Turing's stability

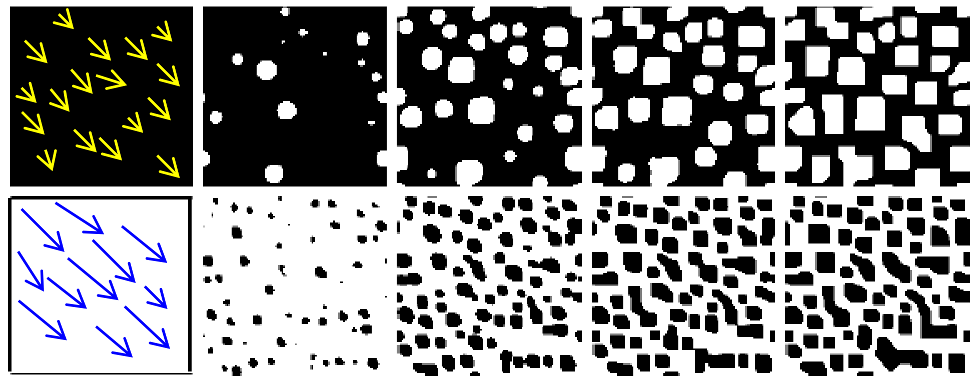
- Consider the energy without gradient (stabilizing) terms: $E(W(\phi, v))$.
- Let $(\phi(x), v(x)) = (\phi_0, v_0)$ $\forall x \in \Omega$ being a uniform phase of the system.
- $(\phi_0, v_0) = (-1, 0)$ and $(1, 1)$ are two stable uniform phases i.e. $(\phi_0, v_0)$ minimum of $E(W(\phi, v))$.
- Adding stabilizing terms to the energy, the uniform phase $(\phi_0, v_0)$ must remain stable:
  - $(\phi_0, v_0)$ is stable to infinitesimal perturbations $(\delta\phi(k), \delta v(k))$ $\Leftrightarrow$ the 3x3 Hessian matrix $H(\phi_0, v_0)$ of $E_P$ is positive definite.

$\Rightarrow$ lower and upper bounds on parameter values.

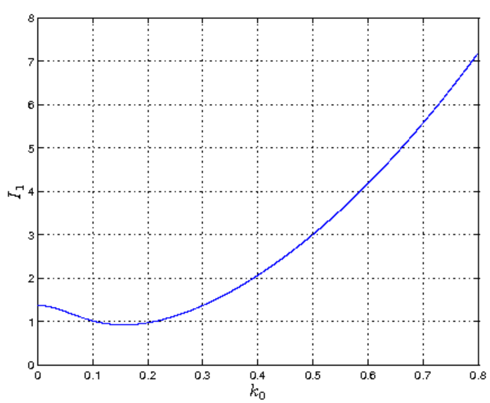
# Why? Turing's instabilities…

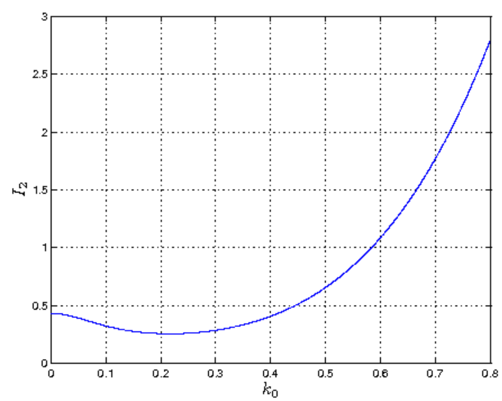
$(\phi, v) = $ $(-1, 0)$ $+ (\delta\phi, \delta v)$

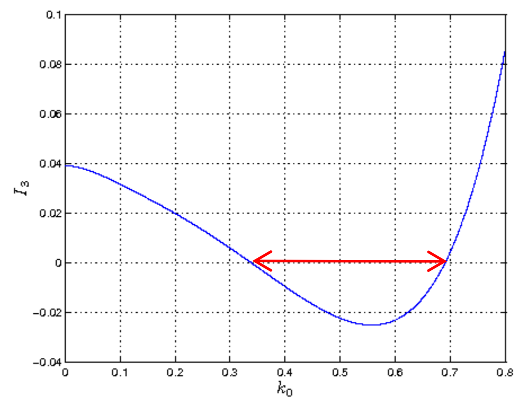
$(\phi, v) = $ $(1, 1)$ $+ (\delta\phi, \delta v)$



- The 3 invariants ($\Leftrightarrow$ eigenvalues) of H:

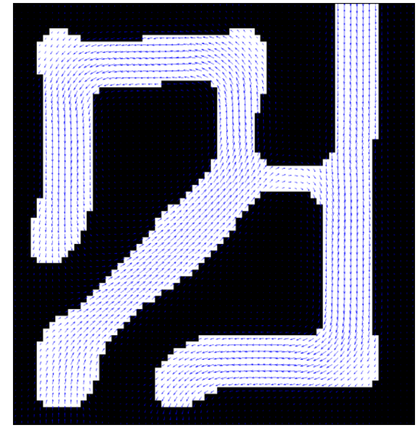


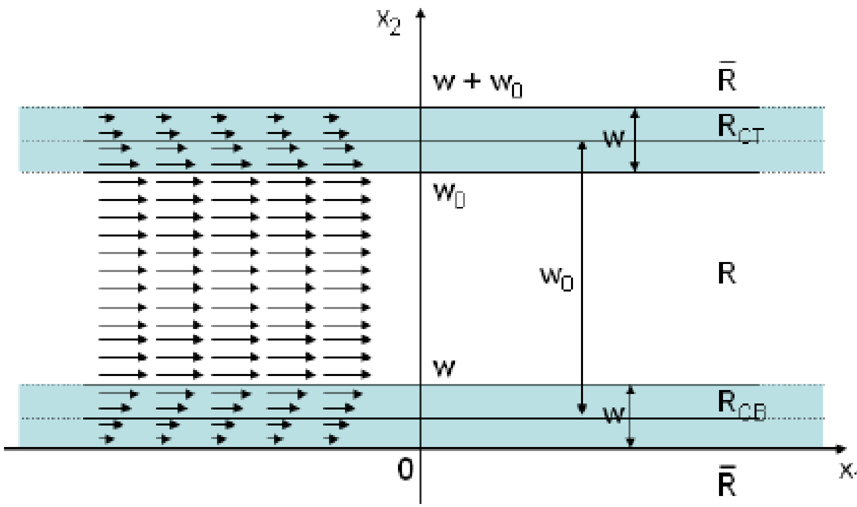



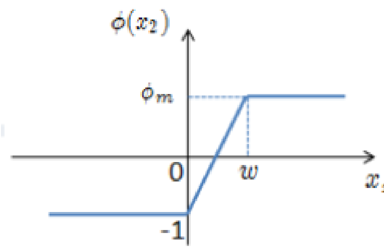



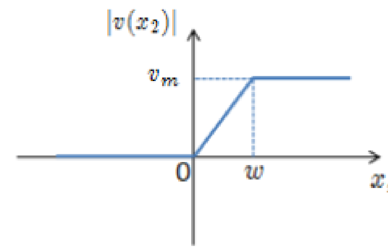
# Network modelling: bar *ansatz*?

- A ‘network’ is thought of as a set of $\approx$ straight, long bars.





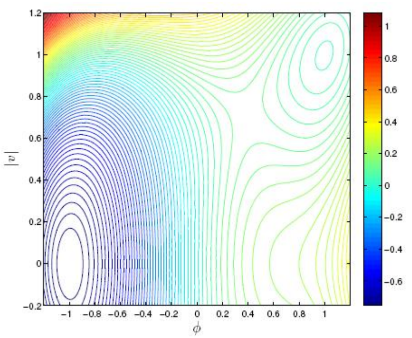

- The bar is defined by 4 physical parameters: $w_0$, w, $\phi_m$ and $v_m$.

# Energy of the bar

- Bar energy per unit length:

$$e_{\mathrm{P}}(\hat{w}_0, \hat{w}, \phi_m, v_m) = \hat{w}_0 \nu(\phi_m, v_m) + \hat{w}\mu(\phi_m, v_m) - \beta(\phi_m + 1)^2 G_{00}(\hat{w}_0, \hat{w}) + \frac{\hat{D}(\phi_m + 1)^2 + \hat{L}_v v_m^2}{\hat{w}}$$

  - $\nu = W(\phi_m, v_m) - W(-1, 0)$: energy gap between the background and the foreground,
  - $\nu > 0$ to favor pixels belonging to backgound rather than foreground (area force).



# Stability conditions of the bar?

- First order

$$\left.\begin{aligned}\frac{\partial e_{\mathrm{P}}(\hat{w}_0,\hat{w},\phi_m,v_m)}{\partial \hat{w}_0} &= 0\\ \frac{\partial e_{\mathrm{P}}(\hat{w}_0,\hat{w},\phi_m,v_m)}{\partial \hat{w}} &= 0\\ \frac{\partial e_{\mathrm{P}}(\hat{w}_0,\hat{w},\phi_m,v_m)}{\partial \phi_m} &= 0\\ \frac{\partial e_{\mathrm{P}}(\hat{w}_0,\hat{w},\phi_m,v_m)}{\partial v_m} &= 0\end{aligned}\right\} \xrightarrow{(\hat{w}_0,\hat{w},1,1)}$$

$$\begin{aligned}&\rho^\star - G(\hat{w}_0,\hat{w}) = 0\\ &\beta = \frac{\nu^\star}{4G_{10}(\hat{w}_0,\hat{w})}\\ &\hat{D} = \frac{\hat{w}}{2}\left[\frac{\nu^\star G_{00}(\hat{w}_0,\hat{w})}{2G_{10}(\hat{w}_0,\hat{w})} - \hat{w}\mu_\phi^\star\right] > 0\\ &\hat{L}_v = -\frac{\hat{w}^2\mu_v^\star}{2} > 0\end{aligned}$$

- Second order:

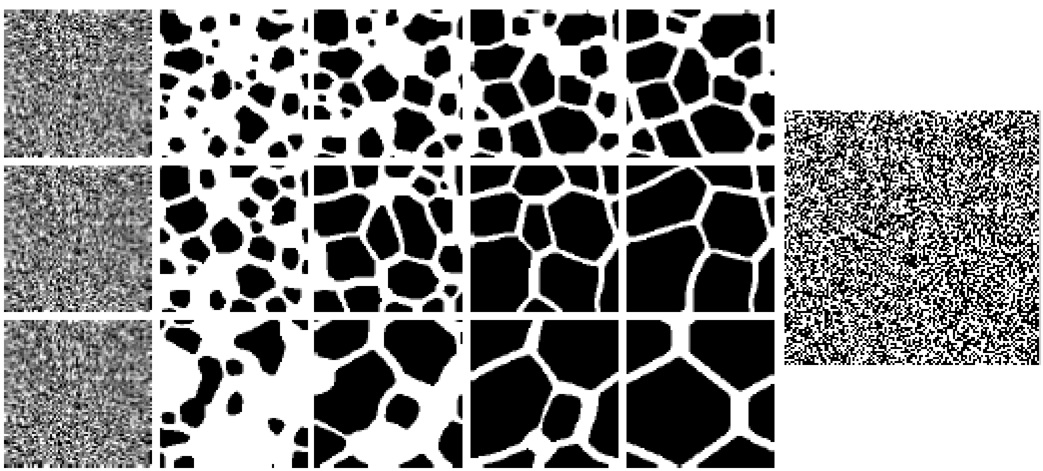
$$H = \begin{pmatrix} \frac{\partial^2 e_{\mathrm{P}}}{\partial \hat{w}_0^2} & \frac{\partial^2 e_{\mathrm{P}}}{\partial w \partial \hat{w}_0} & \frac{\partial^2 e_{\mathrm{P}}}{\partial \phi_m \partial \hat{w}_0} & \frac{\partial^2 e_{\mathrm{P}}}{\partial v_m \partial \hat{w}_0} \\ \frac{\partial^2 e_{\mathrm{P}}}{\partial w \partial \hat{w}_0} & \frac{\partial^2 e_{\mathrm{P}}}{\partial w^2} & \frac{\partial^2 e_{\mathrm{P}}}{\partial \phi_m \partial w} & \frac{\partial^2 e_{\mathrm{P}}}{\partial v_m \partial w} \\ \frac{\partial^2 e_{\mathrm{P}}}{\partial \phi_m \partial \hat{w}_0} & \frac{\partial^2 e_{\mathrm{P}}}{\partial \phi_m \partial w} & \frac{\partial^2 e_{\mathrm{P}}}{\partial \phi_m^2} & \frac{\partial^2 e_{\mathrm{P}}}{\partial v_m \partial \phi_m} \\ \frac{\partial^2 e_{\mathrm{P}}}{\partial v_m \partial \hat{w}_0} & \frac{\partial^2 e_{\mathrm{P}}}{\partial v_m \partial w} & \frac{\partial^2 e_{\mathrm{P}}}{\partial v_m \partial \phi_m} & \frac{\partial^2 e_{\mathrm{P}}}{\partial v_m^2} \end{pmatrix}_{(\hat{w}_0,\hat{w},1,1)}$$

H is positive definite $\rightarrow$ lower and upper bounds on parameter values.

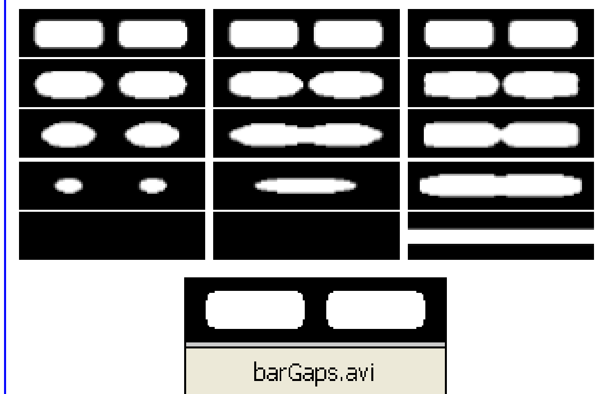
# Geometric evolutions…

- of a bar:

- of a random configuration:



- for gap closure:



# Likelihood?

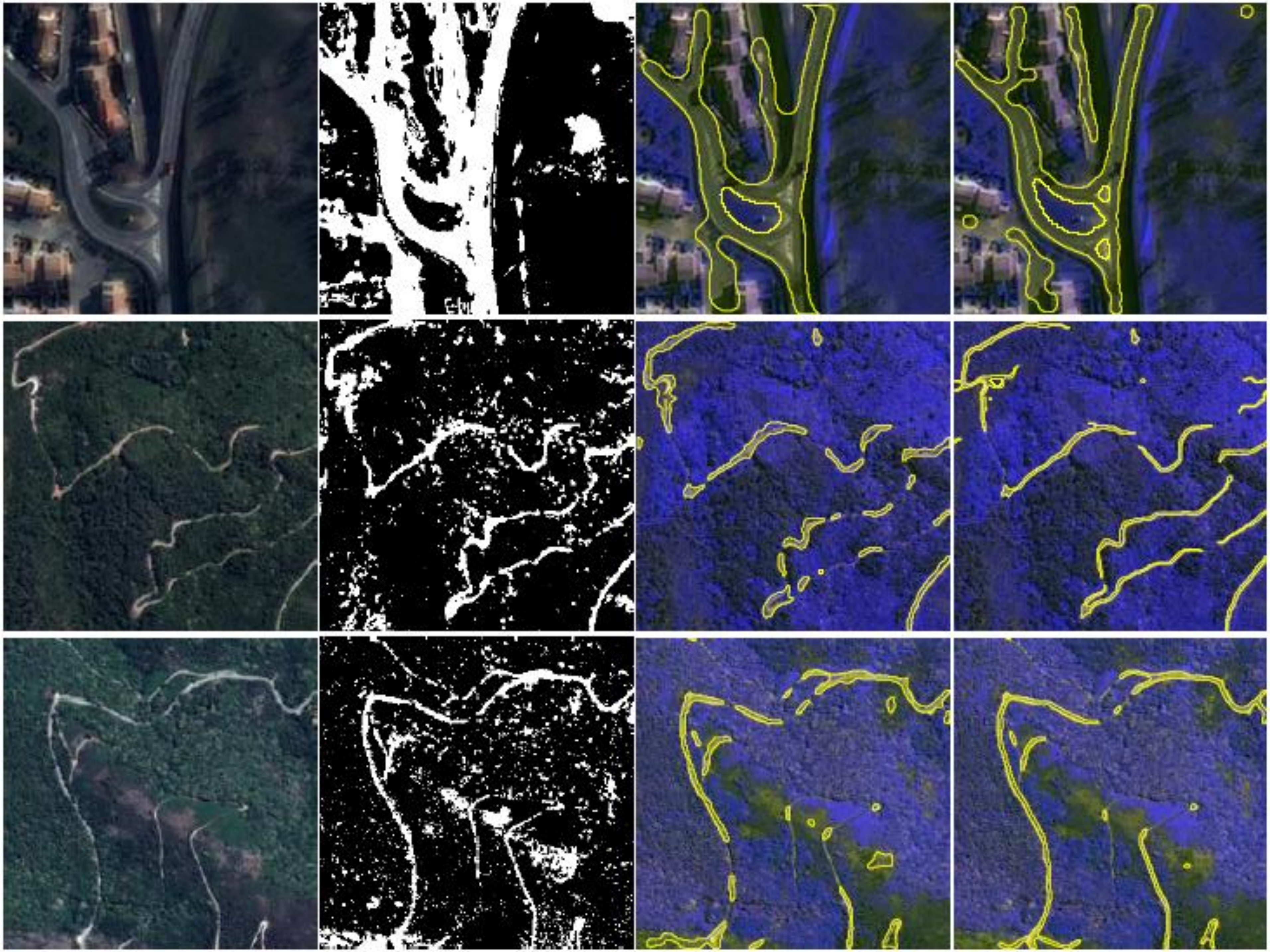
- Multispectral Quickbird VHR images.

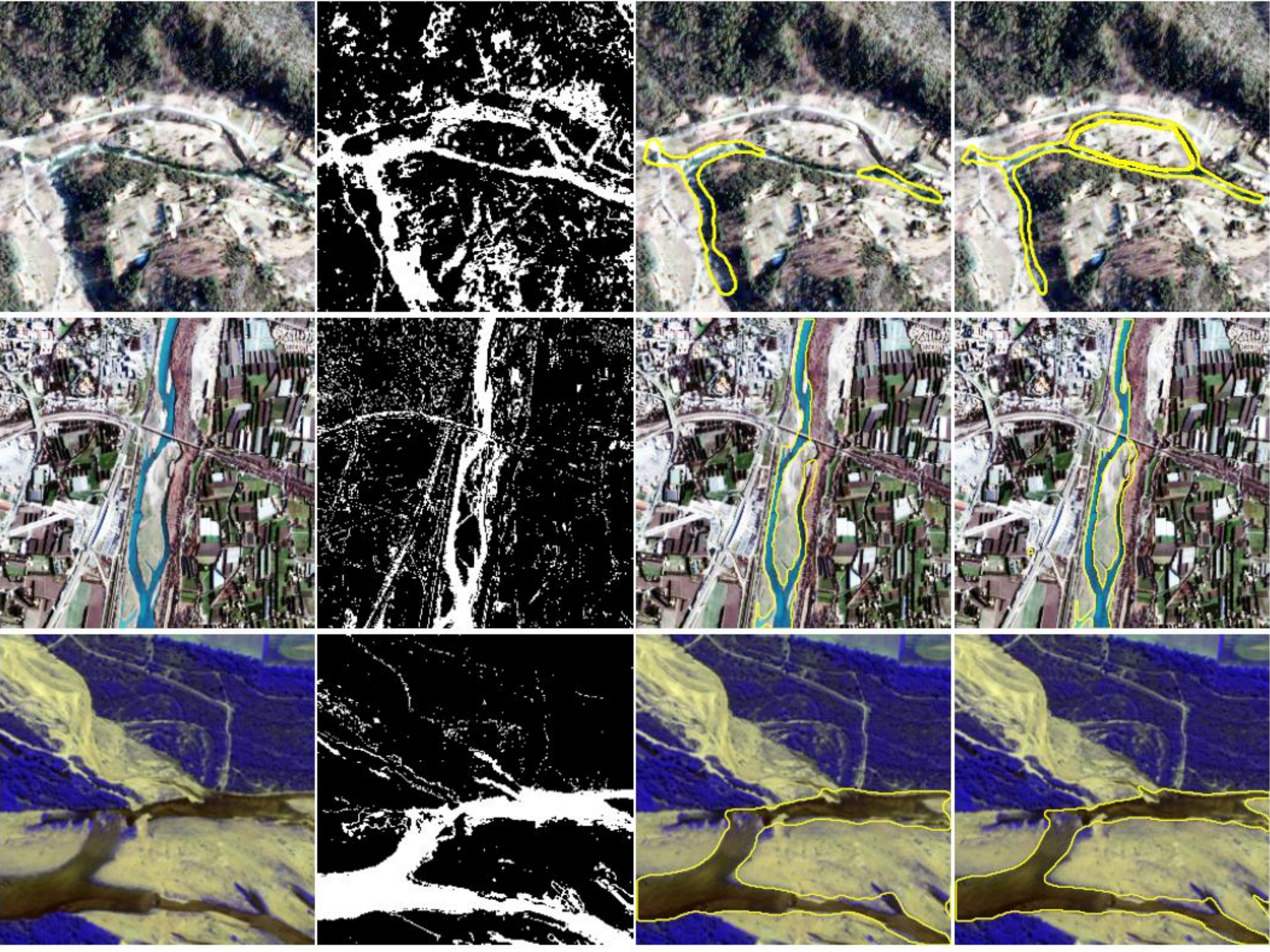
  

- Likelihood term $E_I(\phi)$:
  - Multivariate mixture of two Gaussians for the background and foreground:

$$E_{\mathrm{I}}(\phi) = -\frac{1}{2}\int dx \left\{ \ln \sum_{i=1}^{2} p_i |2\pi\Sigma_i|^{-1/2} e^{-\frac{1}{2}(I(x)-\mu_i)^t \Sigma_i^{-1}(I(x)-\mu_i)} - \ln \sum_{i=1}^{2} \bar{p}_i |2\pi\bar{\Sigma}_i|^{-1/2} e^{-\frac{1}{2}(I(x)-\bar{\mu}_i)^t \bar{\Sigma}_i^{-1}(I(x)-\bar{\mu}_i)} \right\} \phi(x)$$

- Total energy: $E(\phi, v) = E_P(\phi, v) + E_I(\phi)$.





# Conclusions and prospects

- Conclusions:
  - The stability analysis reduces the parameter tuning difficulties.
  - The directed network model outperforms the undirected network model.
- Prospects:
  - Parameter estimation.
  - Global optimization algorithm (simulating annealing, …)