

Selection of Optimal Embedding Parameters Applied to Short and Noisy Time Series from Rössler System

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Abstract

Throughout scientific research, the state space reconstruction that embeds a non-linear time series is the first and necessary step for characterizing and predicting the behavior of a complex system. This requires to choose appropriate values of time delay T and embedding dimension d_E . Three methods are applied and discussed on nonlinear time series provided by the Rössler attractor equations set: Cao's method, the C-C method developed by Kim *et al.* and the C-C-1 method developed by Cai *et al.* A way to fix a parameter necessary to implement the last method is given. Focus has been put on small size and/or noisy time series. The reconstruction quality is measured by using a criterion based on the transformation smoothness.

Keywords

Phase Space Reconstruction, Embedding Window, Rössler System, Time Series, Correlation Integral, Delay Time

1. Introduction

In many fields of science and industry, complex systems are studied through temporal time series of scalar observations of a k dimensional dynamical system [1] [2] [3] [4] [5]. In most cases, the state space dimension and the system of equations that define the system evolution and behavior in the state space are unknown. Each value in a time series results from the interaction of the state variables in the state space. The main purpose of time series analysis is to learn about the dynamics behind some time ordered measurement data. To investigate an experimental k th order dynamical system from a scalar time series, it is ne-

cessary to reconstruct a state space by using time delay or time derivative coordinates. The reconstructed trajectory is expected to have the same characteristics than the trajectory embedded in the original phase space. It can be proved through Taken's theorem that the unstable periodic orbits of a strange attractor could be recovered in an embedded state space whenever the time series is long enough with no noise [1] [2] [5] [6] [7] [8] [9]. In that case, the embedding dimension d_E and the time delay T [1] [2] [3] [10] [11] are not correlated and can be selected independently [7] [8] [9]. In the real world, time series are not infinitely long and could be hardly noisy. In that case d_E and T are correlated and an alternative approach used in the literature is to determine the time window length $t_w = (d_E - 1)T$ which is the entire time spanned by the embedding vectors [7] [8] [9]. Once t_w is determined, the time delay T should be chosen so that the serial correlation of the t_w time subseries should be minimum [7] [8]. As the essence of serial correlation is to see how sequential observations in a time series affect each other, Brock, Dechert and Scheinkman have developed a new statistic named "BDS statistic" able to test if a given data set is independently and identically distributed [12] [13]. The BDS statistic is based on the correlation integral and Brock has shown that the correlation integral behaves like the characteristic function of a time series through the fact that if the time series arises from several independent random variables, the correlation integral is the product of the correlation integrals of sub time series components. In that sense, the BDS statistic can be interpreted as the serial correlation of a nonlinear time series. State space reconstruction is necessary before developing forecasting methods and, as the quality of state space reconstruction affects significantly the accuracy in time series forecasting, the scope of this paper is threefold: i) to review test and compare, in terms of quality, three methods used for selecting state space reconstruction parameters (time delay, embedding dimension) from a nonlinear time series provided by the Rössler attractor equations set; ii) to apply these methods to small size time series and test their robustness to noise with the objective to use them for experimental data; iii) to qualify these methods by defining a criterion able to measure the quality of the state space reconstruction.

In this work, a pseudo experimental approach is considered. The equations describing the Rössler attractor are solved numerically. The numerical values obtained are assumed to be measurements. We start from a scalar time series $S = \{x(t_i)_{t_i = i\delta t, i=1, N}\}$ of N observations of the Rössler x variable with the δt sampling rate that is the shortest time between two measurements. *The method of delays* is used to embed the time series S into a set of points of a d_E dimensional space $\{y_i = (x(t_i), x(t_i + T), \dots, x(t_i + (d_E - 1)T)), i=1, p\}$ where T is the delay time given by $T = n\delta t$, d_E the embedding dimension and $p = N - (d_E - 1)n$, the number of embedded points. The reconstructed state space must be topologically equivalent to the original one, the selection of optimal values for T and d_E are very important and affect the quality of the reconstruction. Through the large number of publications dealing with this subject,

there are two main approaches of selecting d_E and T . The first approach consists in selecting d_E and T independently and is generally used in the case of sufficiently long noise free time series [7] [8] [9] [14] [15]. When time series are limited or contaminated by noise, the theorem of Takens is silent and the delay time T is observed to vary with the embedding dimension d_E . In this case, as an irrelevant partnership between T and d_E could affect the equivalence between the reconstructed space and the original one, another approach, based on the delay time window, $t_w = (d_E - 1)T$ selection, is used for the state space reconstruction [7] [8] [16] [17].

This article is structured around three sections, this work, can be considered as the preliminary step to the time series forecasting methods development, the first section is devoted to the calculation of the maximum Lyapunov exponent and the Lyapunov dimension from time series obtained by integrating numerically the Rössler differential equations set. The second section is focused on the state space reconstruction parameters selection. The main idea is to subdivide the original time series S into p sub-series, each of them representing an embedded point in a d_E dimensional space. For an optimal choice of the state space reconstruction parameters, all the embedded points form a sufficiently representative trajectory of the attractor considered.

Three methods able to select the embedding dimension d_E and the time delay T are discussed in this section. First Cao's method [14] is applied on sufficiently long noise free time series ($\approx 32,000$ values) and shows some drawbacks when applied to smaller size and/or noisy time series (≈ 4000 values). Aiming at improving these drawbacks, two other methods based on the time delay window selection are discussed: the C-C method developed by Kim *et al.* [7] and the C-C-1 method developed by Cai *et al.* [8]. These two methods are described and results obtained are compared and discussed. In the framework of the C-C-1 method, a criterion that fix the number of subseries composing the initial time series S , is suggested. The sensitivity to noise of the different techniques addressed in this section are analyzed and results obtained are discussed. The third and last section is dedicated to the measure of the reconstruction quality. In the case of long free noise time series, as Takens embedding theorem ensures a topological equivalence between the original state space and the reconstructed one, the quality of the reconstruction is measured through the conservation of invariants such as the maximum Lyapunov exponent and the correlation dimension. In the case of limited noise free data set, we have used a technic based on a statistic approach similar to the Rul'kov *et al.* test [18], by calculating the quotient F of two ratios. One ratio is the nearest-neighbor distance on the original state space to the distance on the corresponding points on the reconstructed state space. The other ratio is the nearest-neighbor distance on the reconstructed state space to the distance between the corresponding points on the original state space. For a smooth mapping between the time series, the quotient of these two ratios should be close to unity [18] [19].

This paper also constitutes a sort of set-up in order to become familiar with

these different techniques before applying them to real experimental data.

2. Calculation of Lyapunov Exponents and Lyapunov Dimension for the Rössler Attractor

The dynamical system of interest in this first part consists of the following three coupled differential equations [4] [20] [21] [22]

$$\begin{aligned} dx/dt &= -(y+z), \\ dy/dt &= x+ay, \\ dz/dt &= b+z(x-c). \end{aligned} \quad (1)$$

where a , b and c have constant values. These equations have a chaotic attractor which is displayed in **Figure 1** which was obtained from a simple numerical integrator.

The Rössler attractor is largely a product of the interaction between an attracting direction and a repelling one. The calculated trajectory starts close to a fixed point, the linear terms of the two first equations create oscillations in the variables x and y . These oscillations are amplified, which results into a spiraling-out motion. The motion in x and y is then coupled to the z variable ruled by the third equation, which contains the nonlinear term and which induces the reinjection back to the beginning of the spiraling-out motion. A very complex

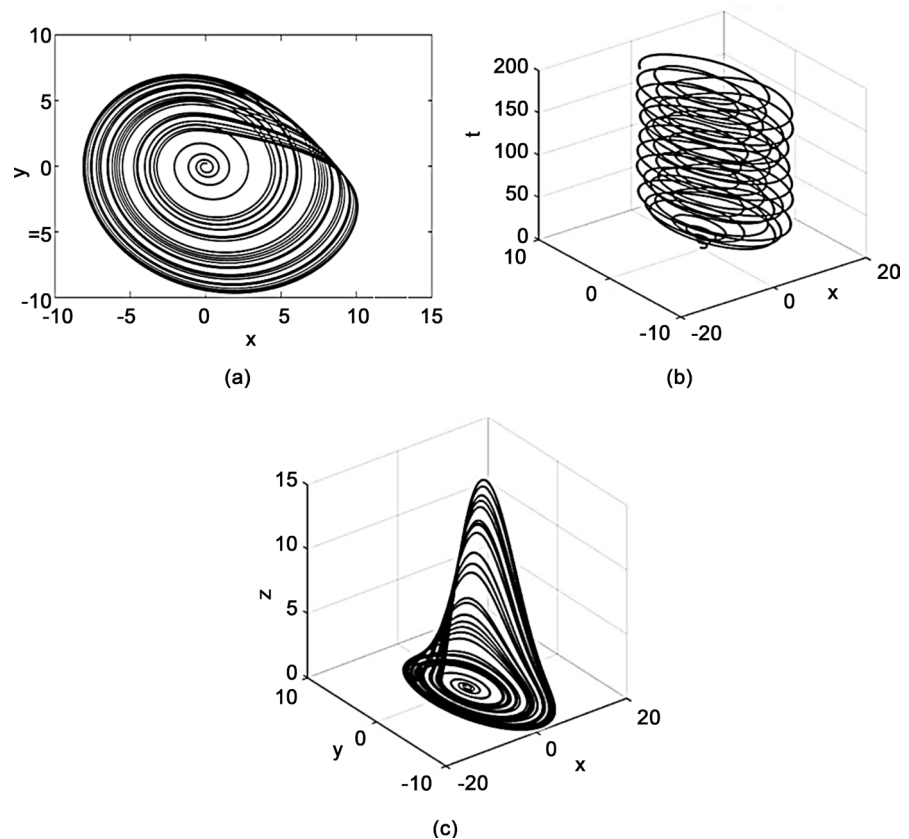


Figure 1. Rössler attractor representation from the equations set. $a = 0.2$, $b = 0.4$ and $c = 5.7$.

dynamic arises.

When chaos takes place, one can observe a great sensitivity of the motion to small changes in initial conditions. Two closely neighboring trajectories diverge exponentially. Their rate of divergence is constant, and a plateau is obtained when determining the Lyapunov exponent numerically. When $a = 0.2$ and $b = 0.4$, chaos appears when c has a sufficiently high value. This is shown by calculating maximum Lyapunov exponents using Benettin's method [3] [20] [21] [23] [24] for different trajectories considering different values of parameter c .

Figure 2 shows that chaos takes place when c is between 5 and 5.2.

Then, considering $a = 0.2$, $b = 0.4$ and $c = 5.7$, a positive maximum Lyapunov exponent for this trajectory is calculated by using Benettin's method. Two trajectories with two very close initial conditions were considered and they were renormalized for every fixed time interval $\Delta\tau$. The following value was found: $\sigma \approx 7.6 \times 10^{-2}$.

The transition from simple to strange attractor proceeds via a sequence of period-doubling bifurcations [20].

Then, considering the parameters defining the Rossler attractor shown in **Figure 1** ($a = 0.2$, $b = 0.4$ and $c = 5.7$), the Lyapunov exponents spectrum is calculated. The algorithm employed was proposed Wolf *et al.* [25]. The numerical results are shown in **Figure 3**.

The values of the three Lyapunov exponents are

$$\lambda_1 \approx 7.0 \times 10^{-2}, \lambda_2 \approx 7.0 \times 10^{-4} \approx 0, \lambda_3 \approx -5.4, \quad (2)$$

as expected $\lambda_1 \approx \sigma$. On average, λ_1 is the expansion rate of the stretching process of the attractor, and λ_3 is the reduction rate of the folding process.

The number of non-negative Lyapunov exponents, $d = 2$, allows us to identify the dimension of the attractor [26]. We are going to show that its fractal dimension is a little bit greater.

The Lyapunov dimension d_L is related to the Lyapunov spectrum by [5] [25] [26] [27]

$$d_L = j + \frac{\sum_{i=1}^j \lambda_i}{|\lambda_{j+1}|}, \quad (3)$$

where j is defined by the conditions $\sum_{i=1}^j \lambda_i > 0$ and $\sum_{i=1}^{j+1} \lambda_i < 0$. Thus

$$d_L \approx 2.01, \quad (4)$$

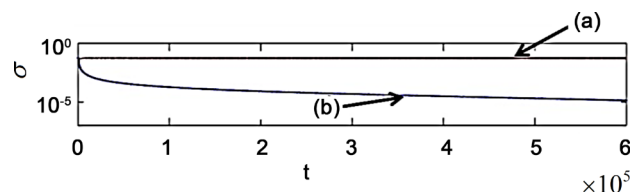


Figure 2. Influence of the c parameter on the maximum Lyapunov exponents for the trajectory. (a): $c = 5.2$, $\sigma_1 = 5.49 \times 10^{-2}$; (b): $c = 5$, $\sigma_1 \rightarrow 0$.

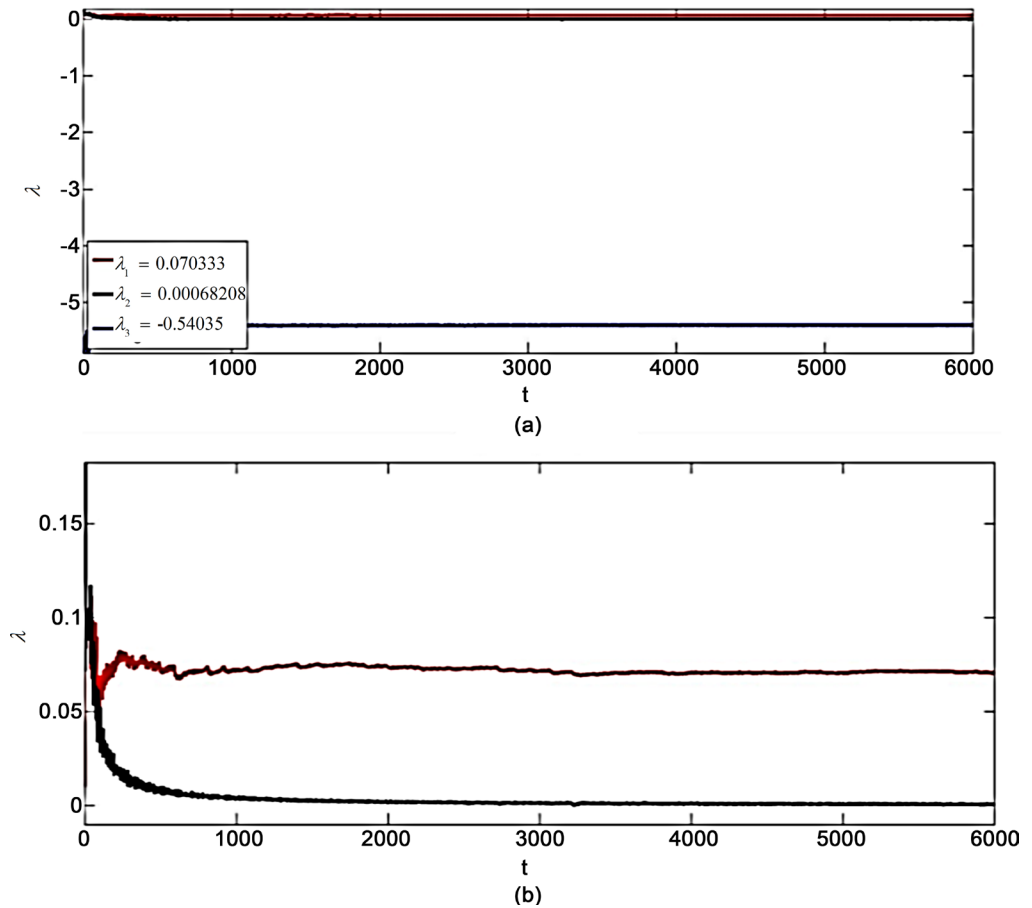


Figure 3. (a) Lyapunov spectrum for the trajectory shown in **Figure 1**; (b) magnification of (a).

which means that the attractor is close to a plane surface. The fractal dimension gives a lower bound on the number of variables needed to model the dynamics of the attractor.

3. Time Delay Reconstruction of the State Space by Sampling a Coordinate of the Rössler Attractor

3.1. Formulation of the Problem

Let us move to a pseudo experimental approach. It is assumed that we only have a single sequence of measurements obtained at different times. We seek a hidden determinism in our experimental data. Here, the “experimental results” are given by a numerical integration of Equation (1). In this section, we just try the reconstruction method provided by the tool box CDA22 [1].

It is assumed that only the x -components of the $\mathbf{X}(x, y, z)$ vector, which gives the state of the system, is measured or calculated. Then, $x(t) = G[\mathbf{X}(t)]$, where G is a scalar function of the state vector. We define what is called delay coordinate vectors such as [3] [4] [6] [7] [8]

$$\mathbf{y}_i = [x(t_i), x(t_i + T), x(t_i + 2T), \dots, x(t_i + (d_E - 1)T)], \quad (5)$$

where d_E is here a simple parameter and T is the time delay.

Packard *et al.* [4] [6] have shown that starting from the time series (5), one may reconstruct the trajectory of the attractor in a d_E -dimensional embedding space by means of vectors

$$\begin{aligned} y_1 &= [x(t), x(t+T), x(t+2T), \dots, x(t+(d_E-1)T)], \\ y_2 &= [x(t+\tau), x(t+\tau+T), x(t+\tau+2T), \dots, x(t+\tau+(d_E-1)T)], \\ &\vdots \\ y_p &= [x(t+(p-1)\tau), x(t+(p-1)\tau+T), x(t+(p-1)\tau+2T), \dots, x(t+(p-1)\tau+(d_E-1)T)], \end{aligned} \quad (6)$$

with $\tau = j \times \delta t$, where j is an integer and δt is the minimum sampling time. Time τ is the sampling interval between the first components of successive vectors y_i ($1 \leq i \leq p$). Based on (6) and assuming the time window spanned by the p embedded points is included in the time window spanned by the N values of the initial time series, it holds that $(N-1)\delta t = (p-1)\tau + (d_E-1)T$. If $T = n\delta t$ where n is an integer, it may be written $(N-1)\delta t = (p-1)\tau + (d_E-1)T$ that is: $j(p-1) + n(d_E-1) = N-1$. Thereafter, we will set $j = 1$ and we shall have

$$p + n(d_E - 1) = N. \quad (7)$$

3.2. Considerations on the Minimum Embedding Dimension

The space reconstruction requires to select values of the reconstructed space dimension and the time delay. The embedding theorem [6] [9] [10] [28] tells us that the following sufficient but not necessary condition must be verified

$$d_E > 2d_B, \quad (8)$$

where d_B is the box-counting dimension of the attractor. Considering that $d_L \geq d_B$ [2], the condition (8) is satisfied when

$$d_E > 2d_L, \quad (9)$$

that is

$$d_E > 4, \quad (10)$$

To verify the relevance of this condition, the reconstruction was achieved considering several values of d_E and using the CDA22 tool box [1]. Let $x(t)$ be a one-dimensional data set evaluated at equal increments of the variable t . The values of $x(t)$ were obtained by solving numerically Equation (1) using a fourth order Runge-Kutta scheme. These values of x play the role of experimental measurements. Using CDA22 tool box the correlation dimension d_C of the Rössler attractor was calculated for $d_E = 5$. It is reminded here that $T = n \times \delta t$ where δt is the sampling rate, we chose $\delta t = 0.1$ $n = 8$ and found $d_C = 1.911 \pm 0.027$. Then, d_C was calculated for different values of d_E . First the same values of δt and n were considered. The results are shown in **Figure 4** Very different values of δt and n were also considered, we chose $\delta t = 0.05$ and $n = 2$. **Figure 4** shows that the two curves merge.

Figure 4 shows the curves d_C versus d_E obtained with two set of parameters $(\delta t, n)$. The full line is obtained with $\delta t = 0.1$ and $n = 8$, squares are ob-

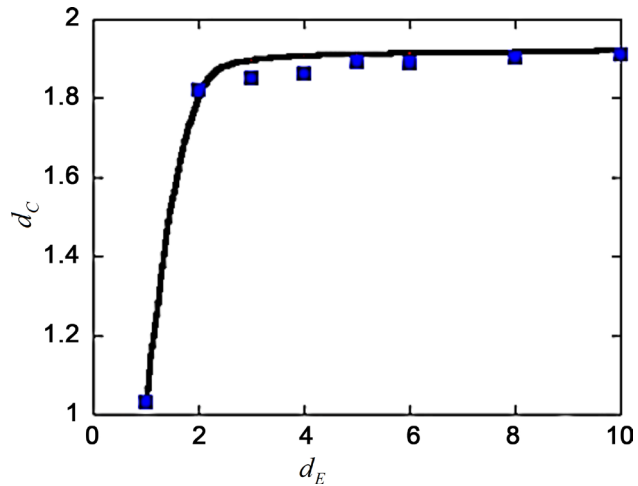


Figure 4. The correlation dimension versus the embedding dimension. The full line was obtained for $\delta t = 0.1$ and $n = 8$, the squares were obtained when $\delta t = 0.05$ and $n = 2$.

tained with $\delta t = 0.05$ and $n = 2$. When $d_E \geq 5$, squares are very closed to the full line curve and the plateau seen in d_C corroborates condition (10) which means that, for the long limited noise free data sequence considered here, Takens's theorem is satisfied. The correlation dimension d_C becomes an invariant on the attractor when the embedding dimension used for the computation increases. Then, the optimal embedding dimension is reached [11]. Still, **Figure 4** shows that $d_E \geq 5$ is a sufficient but not necessary condition, $d_E \geq 3$ would be quite suitable. The typical problem with this way to determine the embedding dimension is that it is very time-consuming for computation.

Figure 4 also shows that the results obtained for d_C are not, at least in this range of values, sensitive to n and consequently to the time delay T . Because the initial data set contains about 32,000 values and is noise-free, results obtained are in good agreement with Takens [6] [7] [9]. In this case, the existence of a diffeomorphism between the original attractor and the reconstructed image exists for almost any choice of time delay and a sufficiently high embedding dimension.

It is known that $d_B \geq d_C$ [1], then, as $d_L \geq d_B$, one must have $d_L \geq d_C$ [29] which in good agreement with our numerical results.

3.3. Cao's Method for Determining the Embedding Dimension

The optimal embedding dimension has also been calculated by using Cao's algorithm which is much less time-consuming [9] [14]. According to the IIIa paragraph notations and (7), Cao defines $a(i, d_E)$ as a function of $y_i = (x_i, x_{i+1}, \dots, x_{i+(d_E-1)n})$ (with $i = 1, 2, \dots, p = N - (d_E - 1)n$) the i^{th} reconstructed vector and d_E the embedding dimension, $a(i, d_E)$ is written as

$$a(i, d_E) = \frac{\|y_i(d_E + 1) - y_{\tilde{n}(i, d_E)}(d_E + 1)\|}{\|y_i(d_E) - y_{\tilde{n}(i, d_E)}(d_E)\|}, \quad (11)$$

where $\|\dots\|$ denotes the sup-norm, *i.e.*, $\|\mathbf{y}_k(m) - \mathbf{y}_l(m)\| = \max_{0 \leq j \leq m-1} |x_{k+jn} - x_{l+jn}|$ and $\mathbf{y}_i(d_E + 1)$ is the i^{th} reconstructed vector in the $d_E + 1$ dimensional reconstructed space. Subscript $\tilde{n}(i, d_E)$ refers to the $\mathbf{y}_{\tilde{n}(i, d_E)}$ which is the nearest neighbor of $\mathbf{y}_i(d_E)$ in the d_E dimensional reconstructed space. Integer $\tilde{n}(i, d_E)$ depends on i and d_E . If d_E is qualified as an embedding dimension by the embedding theorem [1] [2] [3] [6] [27], then any two points which stay close in a d_E dimensional reconstructed space will be still close in a $d_E + 1$ dimensional reconstructed space. Such a pair of points are called true neighbors, otherwise, they are called false neighbors. To qualify two points to be false neighbors, $a(i, d_E)$ must be larger than a threshold value which depends on the i state point chosen. To avoid this problem, the quantity $E(d_E)$ is defined as the mean value of all $a(i, d_E)$'s

$$E(d_E) = \frac{1}{N - d_E n} \sum_{i=1}^{N - d_E n} a(i, d_E), \quad (12)$$

with $T = n\delta t$, so that $E(d_E)$ is only dependent on the embedding dimension and the time delay. To investigate the variation of E between d_E and $d_E + 1$, we define $E_1(d_E)$ as $E_1(d_E) = E(d_E + 1)/E(d_E)$. If $E_1(d_E)$ stops changing when $d_E > d_{EM}$, then $d_{EM} + 1$ is the minimum embedding dimension we are looking for. When meaningful predictions from chaotic time sequence cannot be made, data appears to come from a random system. Considering that $E_1(d_E)$ provided by a random set of numbers will never attain a saturation value as d_E increases, it is necessary to distinguish deterministic chaotic from random data sequences. In most cases, it is difficult to resolve whether $E_1(d_E)$ is slowly increasing or has stop changing if d_E is sufficiently large. In fact, since available observed data samples are limited in number, it may happen that $E_1(d_E)$ stops changing at some d_{E_0} value although the time series is random. To solve this problem Cao [14] has suggested to consider the quantity $E^*(d_E)$ which is useful to make distinction between deterministic signals from stochastic ones. Let us consider the following quantity

$$E^*(d_E) = \frac{1}{N - d_E n} \sum_{i=1}^{N - d_E n} |x_{i+d_E n} - x_{\tilde{n}(i, d_E)+d_E n}|, \quad (13)$$

where the meaning of $\tilde{n}(i, d_E)$ is the same as above. As for $E(d_E)$, to study the $E^*(d_E)$ variations, an $E_2(d_E)$ quantity is defined as $E_2(d_E) = E^*(d_E + 1)/E^*(d_E)$. For random data, since the future values are independent of the past values, $E_2(d_E)$ will equal unity for any d_E . In the case of deterministic data $E_2(d_E)$ is certainly related to d_E and cannot be a constant for all d_E . In other words, there must exist some d_E values such that $E_2(d_E) \neq 1$. Cao recommends to calculate both $E_1(d_E)$ and $E_2(d_E)$ for determining the minimum embedding dimension of a scalar time series. **Figure 5** shows results obtained with Cao's method applied to an about 32,000-data sequence by using CDA22 tool box.

Figure 5 shows that $E_2(d_E)$ is related to d_E and is not a constant for all

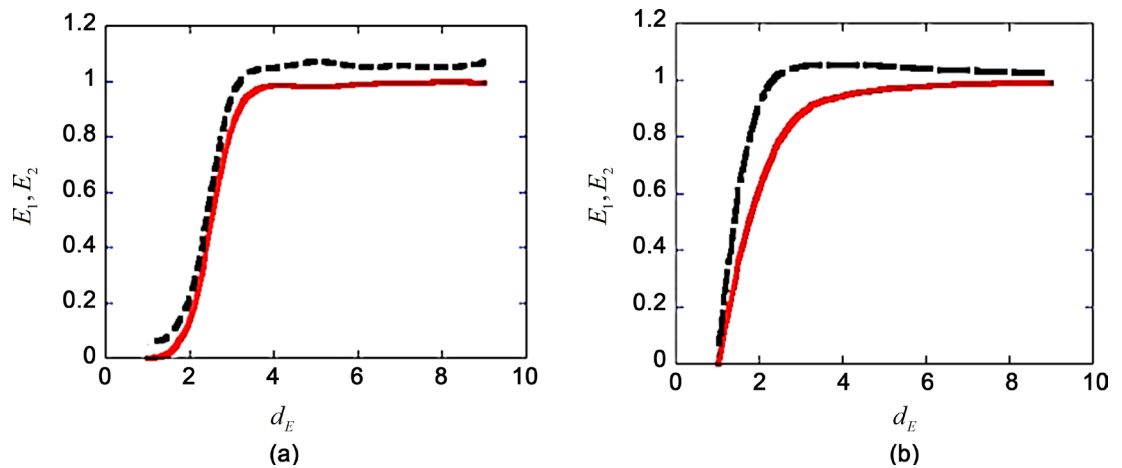


Figure 5. E_1 (solid red line) and E_2 (black long dashed line) graph values as a function of the embedding dimension d_E from Rössler attractor time series data. (a) $n = 15$, (b) $n = 1$.

d_E . This is in good agreement with the fact that the data used are deterministic. When $n = 15$, the minimum embedding dimension is $d_E = 4$. When $n = 1$, the same value is found for d_E . This means that, when enough points are considered and when no noise is considered, the minimum value for d_E is almost independent of T .

Then, Cao's algorithm has been applied on a much smaller data sequence of about 4000 values, E_1 and E_2 were calculated. It was shown that $E_2(d_E)$ can be different from zero in all the cases. It confirms the deterministic character of the data. It also shows that when $n = 50$ or when $n = 10$, the minimum embedding dimension is close to $d_E = 4$. A saturation value $E_1(d_E)$ as d_E increases is more difficult to discern when $n = 1$. Still, we can conclude that for this relatively small number of data, the minimum embedding dimension is still almost independent of n .

Cao's method has been applied then to the 4000 values data-sequence with noise added. **Figure 6** show results obtained when a white Gaussian noise with variance one is added to the 4000 values data-sequence.

In **Figure 6**, when $n = 50$ and $n = 10$, $E_2(d_E)$ can be clearly different from unity, $E_1(d_E)$ reaches a constant value for about the same value of d_E : $d_E = 10$. For $n = 1$, $E_2(d_E)$ remains close to unity, our data do not appear to be deterministic.

Then, a white Gaussian noise with variance five was added. **Figure 7** show that a high value of n is necessary to determine a minimum embedding dimension. The following values for E_1 and E_2 were found

In this case $E_2(d_E)$ is clearly different from a constant only for $n = 100$.

In summary, it was shown that for noise-free data of very long length, the reconstruction is valid for any time delay as far as the embedding dimension is high enough. When going to small number noisy data samples, the time delay used to determine the minimum embedding dimension cannot have any value.

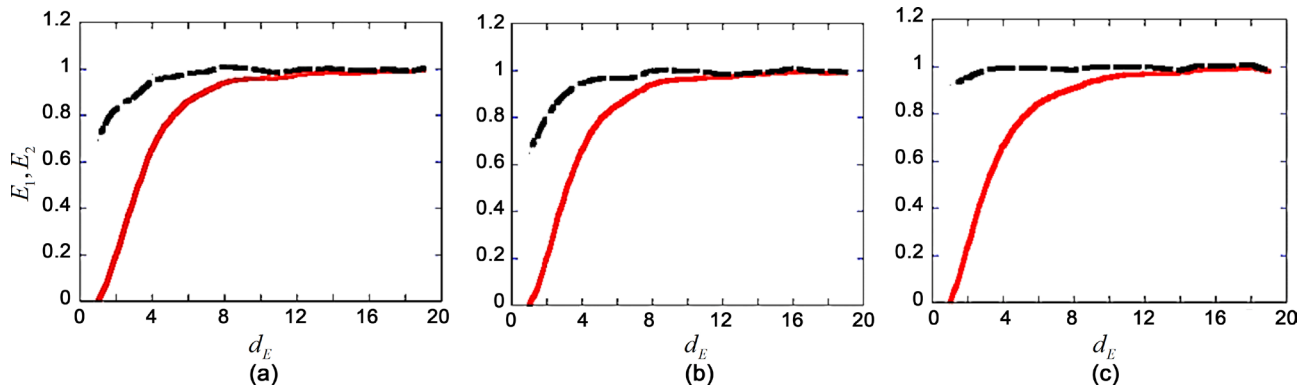


Figure 6. E_1 (solid red line) and E_2 (black long dashed line) graph values as a function of the embedding dimension d_E from time series data noisy with a white Gaussian noise with variance one. (a) $n = 50$, (b) $n = 10$, (c) $n = 1$.

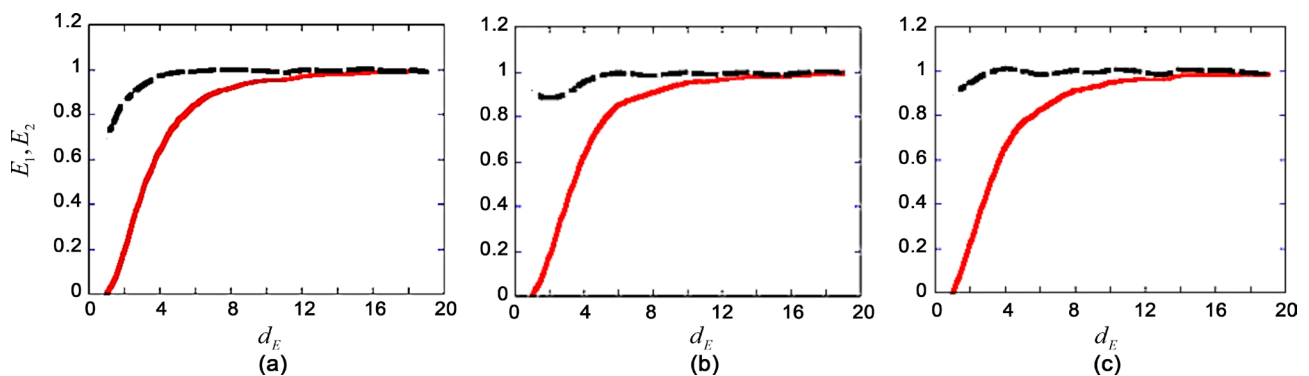


Figure 7. E_1 , E_2 graph values from Rössler attractor time series noisy with a white Gaussian noise with variance five (a) $n = 100$ (b) $n = 10$, (c) $n = 1$.

3.4. Simultaneous Determination of the Embedding Dimension and Time Delay by Using the C-C Method

The Cao's method [14], used to determine the optimal embedding dimension d_E , has shown that for a sufficiently long noise free data set, the time delay T and the minimum embedding dimension are almost independent and the delay time T can be set arbitrarily. However, when white Gaussian noise is added, T varies with d_E and an irrelevant partnership between T and d_E will directly impact the equivalence between the original state space and the reconstructed one. Moreover, in the real world, measurements data set are finite and noisy, and in this case, a more useful quantity to estimate the embedding dimension is the delay time window $t_w = (d_E - 1)T$ which is the entire time spanned by the y_i vectors. Martinerie *et al.* [17] have shown that t_w is an essential factor for estimating the correlation dimension. In their paper, they have demonstrated that first, t_w determines the correlation integral characteristics and second, the correlation integral is very sensitive to the t_w values. H.S. Kim *et al.* [7] and Wei-Dong Cai *et al.* [8] have developed a method which uses the correlation integral and is based on a statistic similar to the BDS statistic that Brock *et al.* [12] [13] used for their development for distinguishing random time series from chaotic or nonlinear stochastic time series. By using the notations of Equations

((6) and (7)), the BDS statistic applied to the time series writes [7] [8]

$$S(d_E, N, r, n) = C(d_E, N, r, n) - C^{d_E}(1, N, r, n), \quad (14)$$

where $C(d_E, N, r, n)$ is the correlation integral

$$C(d_E, N, r, n) = \frac{2}{p(p-1)} \sum_{1 \leq i < j \leq p} H(r - \|y_i - y_j\|), \quad (15)$$

r is a search radius and H is the Heaviside function: $H(a) = 0$ if $a < 0$ and $H(a) = 1$ if $a \geq 0$. N is still the size of the data set, n is the index lag, $p = N - n(d_E - 1)$ is the number of embedded points in the d_E dimensional space and $\|\cdot\|$ still denotes the sup-norm. $C(d_E, N, r, n)$ measures the fraction of pair of points whose sup-norm distance is not greater than r . Brock *et al.* have proved that if the stochastic process $\{y_i\}$ is independent and identically distributed (iid) then, $C(d_E, r) = C^{d_E}(1, r)$ for all d_E and r . The density of points in a hypersphere of radius r scales like r^{d_E} . It means that the correlation integral of the process $\{y_i, i = 1, p\}$ behaves like the one of an independent random variables product which is the product of the correlation integral of each random variable y_i . This leads to interpret the statistic S as a dimensionless quantity which highlights determinism. A significant nonzero value of S is evidence for determinism in the time series. The technique developed by Kim *et al.* called the C-C method consists in subdividing the time series $\{x(t_i) = x_i\}, i = 1, N$ into n disjoint time series $\{y_s, s = 1, n\}$ each one of N/n values. One has

$$y_s = \left\{ x_s, x_{s+n}, x_{s+2n}, \dots, x_{\left(\frac{N}{n}-1\right)n+s} \right\}, s = 1, n. \quad (16)$$

The average of the statistical quantity given by Equation (14) is defined as

$$S_1(d_E, N, r, n) = \frac{1}{n} \sum_{s=1}^n \left[C_s \left(d_E, \frac{N}{n}, r, n \right) - C_s^{d_E} \left(1, \frac{N}{n}, r, n \right) \right], \quad (17)$$

when $N \rightarrow \infty$, S_1 can be rewritten in the following way

$$S_1(d_E, r, n) = \frac{1}{n} \sum_{s=1}^n \left[C_s(d_E, r, n) - C_s^{d_E}(1, r, n) \right]. \quad (18)$$

The locally optimal times may be either the zero crossing of $S_1(d_E, r, n)$ for all r or the times at which $S_1(d_E, r, n)$ shows the least variation with r , since this indicates a nearly uniform distribution of points. From several representative values r_j , we define the quantity

$$\Delta S_1(d_E, n) = \max_j \{S_1(d_E, r_j, n)\} - \min_j \{S_1(d_E, r_j, n)\}, \quad (19)$$

which measures the variations of $S_1(d_E, r, n)$ with r .

For data set with finite sample sizes N , appropriate choices for d_E , r and n should be in agreement with the BDS statistic. For example, when applied to a data set with a sequence of about 4000 values, d_E varies in the range $[2, 7]$, n varies in the range $[1, 200]$ and $r = k \hat{\sigma}/2$ varies in the range $[\sigma/2, 2\sigma]$ with $k = 1, 2, 3, 4$ and $\hat{\sigma}$ is the standard deviation of the time series. We then define

the average of quantities given by Equations ((17) and (19)).

$$\overline{S}_1(n) = \frac{1}{24} \sum_{d_E=2}^7 \sum_{k=1}^4 S_1(d_E, r_k, n), \quad (20)$$

and

$$\Delta \overline{S}_1(n) = \frac{1}{6} \sum_{d_E=2}^7 \Delta S_1(d_E, n). \quad (21)$$

As locally optimal times are either zero crossing of $\overline{S}_1(n)$ or times at which $\Delta \overline{S}_1(n)$ shows the least variation with r , we look for the first zero crossing of $\overline{S}_1(n)$ or the first local minimum of $\Delta \overline{S}_1(n)$ to find the optimal times for data independence which will gives T . The optimal time is the time for which $\overline{S}_1(n)$ and $\Delta \overline{S}_1(n)$ are both closest to zero. As the two quantities $\Delta \overline{S}_1(n)$ and $|\overline{S}_1(n)|$ may not be minimum at the same time (see **Figure 8**), we may look at the minimum of the quantity [7] [8]

$$S_{1,CORR}(n) = \Delta \overline{S}_1(n) + |\overline{S}_1(n)|, \quad (22)$$

which gives the delay time window t_w . T is in a sense the minimum value of t_w , it is determined as the minimum of the curve $S_{1,CORR}(n)$ versus n running from 1 to 200. The C-C method has been programmed in R by using the packages “nonlinearTseries and tseriesChaos”. An organigram of the C-C method used in this work is presented in **Appendix 1** and the results obtained on low size data sequence of about 4000 values are presented in **Figure 8 & Figure 9**.

The first local minimum of $\Delta \overline{S}_1(n)$ (**Figure 8**) occurs when $n = 12$ and represents the optimal delay time T .

The minimum of $S_{1,CORR}(n)$ (**Figure 9**) occurs when $n = 157$ which is the optimal time embedding window t_w . Then, the embedding dimension is given by $\text{int}((t_w/T)+1)+1=15$, where the function $\text{int}()$ represents the integer part.

White Gaussian noises with different variances σ ($\sigma = 1, \sigma = 5, \sigma = 10$) has been added to the noise free original signal of 4000 data sequence and the C-C method has been applied to the time series $x_B = x + \alpha \mathcal{N}(\sigma)$ where x is the

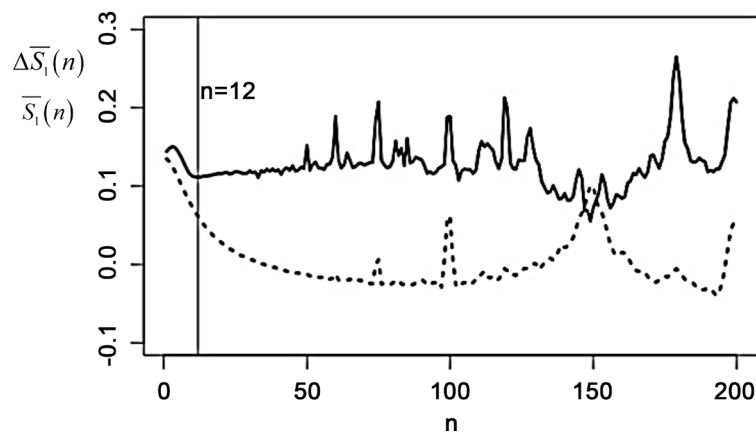


Figure 8. Graphic representations of $\overline{S}_1(n)$ (dashed black line) and $\Delta \overline{S}_1(n)$ (solid black line).

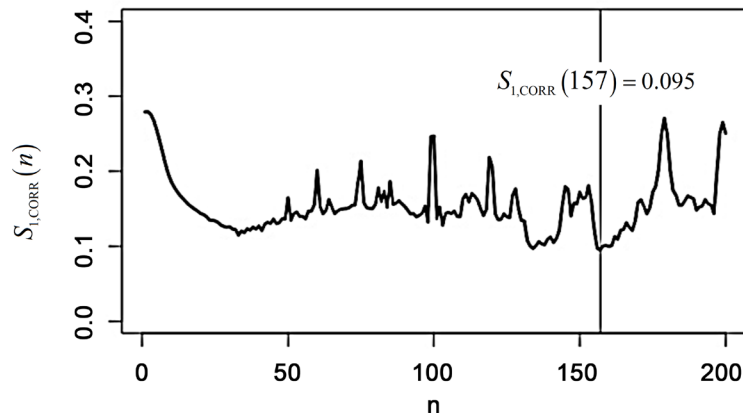


Figure 9. $S_{L,CORR}(n)$.

Table 1. T and t_w variations as a function of different white Gaussian noise variances and strength.

α	$\sigma = 1$			$\sigma = 5$			$\sigma = 10$		
	d_E	T	t_w	d_E	T	t_w	d_E	T	t_w
0.2	15	12	157	14	13	157	15	12	157
0.3	15	12	157	14	13	157	14	12	156
0.5	15	12	157	16	11	157	20	8	147
0.7	15	12	157	20	8	147	39	4	150
1.0	13	12	138	77	2	151		2	1

noise free original signal, $\mathcal{N}(\sigma)$ is a white Gaussian noise with zero mean and a variance σ , α is the strength of the noise and represents the level of noise in percentage (20%, 30%, 50%, 70%, 100%). The C-C method is performed for each σ and α values and the variations of T and t_w compared to the reference values obtained from the original 4000 noise free data set, are shown in **Table 1**. The σ and α values ensuring the stability of the C-C method when applied to noisy data set are those for which $|\Delta d_E| \leq 1$.

The parameter α is linked to the signal to noise ratio SNR which can be defined as $SNR = \overline{x_B} / \sqrt{\sigma_B}$ where $\overline{x_B}$ is the mean value of the noisy data sequence x_B and $\sqrt{\sigma_B}$ is the standard deviation of the noisy part of x_B that is $\alpha \mathcal{N}(\sigma)$. As the standard deviation of $\alpha \mathcal{N}(\sigma)$ may be written $\alpha \sqrt{\sigma}$ with $\alpha \in [0.1, 1]$ and $\sigma = 1, 5, 10$, one has:

$$SNR = \frac{\overline{x_B}}{\alpha \sqrt{\sigma}}, \quad (23)$$

then, the C-C method should be stable against the noise for values of α such as $SNR \geq 1$.

In **Table 1**, we observe that for $\sigma = 1$, the C-C method is stable against the noise when $\alpha \leq 70\%$. For $\sigma = 5$, it is stable when $\alpha \leq 50\%$, and for $\sigma = 10$, it is stable when $\alpha \leq 30\%$.

In summary, the C-C method is a relatively simple method easy to implement

that can be used for relatively small data set to determine both the time delay T and the time delay window t_w . This method is robust against low and intermediate noise level.

3.5. An Optimization of the C-C Method

In their paper, W.D Cai *et al.* [8] pointed out some problems that limit to the C-C method. The first one is that there are local minimal points whose values are very close to the minimum of $S_{1,CORR}(n)$, and they disturb the $S_{1,CORR}(n)$ minimum estimation (see **Figure 9**). The second one is that $\overline{\Delta S_1}(n)$ shows high frequencies oscillations, increasing with n , that can affect the estimation of the first local minimum of $\overline{\Delta S_1}(n)$ (see **Figure 8**). Based on these remarks, W.D. Cai *et al.* have developed in their paper [8] a new method called C-C-1 different from the C-C method calculating $S(d_E, N, r, n)$ with another average method.

Starting from the N values initial data set and according to (7), the number of the embedded points calculated from the delay time T in the d_E dimensional reconstructed space is $p = N - (d_E - 1)n$. A positive integer q independent of the delay time T is selected as a constant, to subdivide the embedded points series $\{y_i, i = 1, p\}$ into q subseries $\{Y(i), i = 1, q\}$, each with $\text{int}(p/q)$ embedded points where the “int” function means integer part.

$$\begin{aligned} Y(1) &= \left\{ y_1, y_{1+q}, \dots, y_{\left(\frac{p-1}{q}\right)q+1} \right\}, \\ Y(2) &= \left\{ y_2, y_{2+q}, \dots, y_{\left(\frac{p-1}{q}\right)q+2} \right\}, \\ &\vdots \\ Y(q) &= \left\{ y_q, y_{q+q}, \dots, y_{\left(\frac{p-1}{q}\right)q+q} \right\}, \end{aligned} \quad (24)$$

as each component Y_i is composed of d_E components, Equation (24) can be rewritten as

$$\begin{aligned} Y(1) &= \left\{ x_1(1), x_1(2), \dots, x_1(d_E), x_{1+q}(1), \dots, x_{1+q}(d_E), \dots, x_{\left(\frac{p-1}{q}\right)q+1}(d_E) \right\}, \\ Y(2) &= \left\{ x_2(1), x_2(2), \dots, x_2(d_E), x_{2+q}(1), \dots, x_{2+q}(d_E), \dots, x_{\left(\frac{p-1}{q}\right)q+2}(d_E) \right\}, \\ &\vdots \\ Y(q) &= \left\{ x_q(1), x_q(2), \dots, x_q(d_E), x_{q+q}(1), \dots, x_{q+q}(d_E), \dots, x_{\left(\frac{p-1}{q}\right)q+q}(d_E) \right\}. \end{aligned} \quad (25)$$

Kim *et al.* have shown in their paper [7] that, when using the BDS statistic on time series, the sample data size N should be appropriate relatively to the values of d_E , r and n . They have shown that for finite time series of size $N \geq 500$ the

statistic $S(d_E, N, r, n)$ represents the true correlation of the time series. The parameter q can be adjusted so that the size of the time subseries $\{Y(i), i = 1, q\}$ will not be too short.

The average of the statistical quantity given by Equation (14) is defined as follows:

$$S_2(d_E, N, r, n) = \frac{1}{q} \sum_{s=1}^q C_s(d_E, N, r, n) - \left[\frac{1}{q} \sum_{s=1}^q C_s(1, N, r, n) \right]^{d_E}. \quad (26)$$

The definitions of $\overline{S_2}, \Delta \overline{S_2}, S_{2,CORR}$ are given formally by Equations (20)-(22). The first local minimum of $\Delta \overline{S_2}$ is the optimal delay time T .

If we define the mean orbital period P of a chaotic system as the mean period generated by the oscillations of the chaotic attractor in the phase space orbits, an optimal value for t_w would coincide with the first period of the N values initial time series S . Cai has shown in his paper that with the new statistical quantity average he defines in (26), the peak values of $S_{2,CORR}$ corresponds to the orbital period P values of S and all the points that bring this values are the minima of $S_{1,CORR}$.

Therefore, a new quantity S_{CORR} is defined by

$$S_{CORR}(n) = S_{1,CORR}(n) - S_{2,CORR}(n). \quad (27)$$

By looking for the minimum of S_{CORR} , we estimate the optimal time window t_w corresponding both to the minima of $S_{1,CORR}$ and to the first period P of the initial N values time series S .

The C-C-1 method has been programmed in R language by using the same packages as with the C-C method. An organigram of the C-C-1 method is presented in **Appendix 2** and results obtained the 4000-values data sequence with $q = 19$ are shown in **Figure 10**.

High frequencies oscillations occurring when applying the C-C method (red dashed line) have disappeared in the C-C-1 method (black solid line) (see **Figure 10**). In **Figure 10**, the $\Delta \overline{S_2}$ first local minimum occurs when $n = 10$ while it

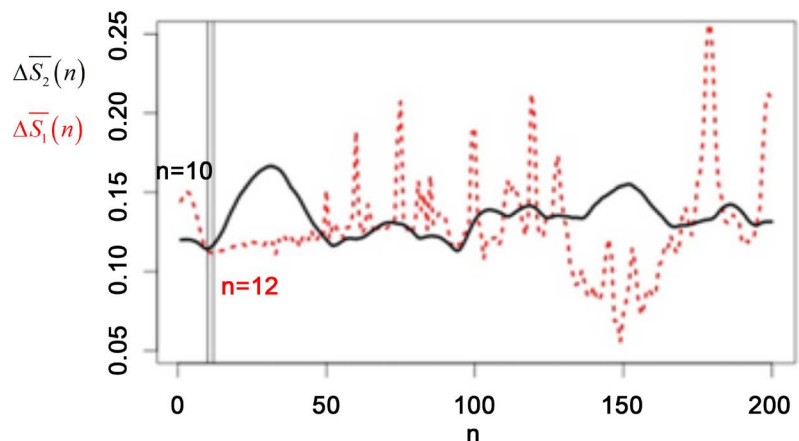


Figure 10. $\Delta \overline{S_2}(n)$ versus n obtained by using the C-C-1 method and $q = 19$ (black solid line), $\Delta \overline{S_1}(n)$ versus n obtained through the C-C method (red dashed line).

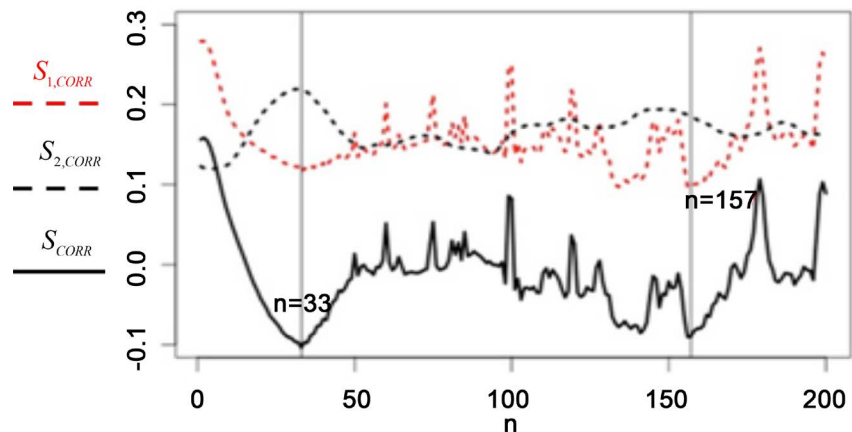


Figure 11. $S_{1,CORR}$ (red dashed line) obtained with the C-C method, $S_{2,CORR}$ (black dashed line) and S_{CORR} (black solid line) obtained with the C-C-1 method.

was $n = 12$ for the C-C method. $n = 10$ represents the optimal delay time T .

Figure 10 shows that the estimate of the optimal delay time T in the C-C-1 method ($n = 10$) is quite the same as with the C-C method ($n = 12$). The first local minimum of $S_{CORR}(n)$ coincide with the first period P of the chaotic time series S , and gives the optimal delay time window t_w . The graph of S_{CORR} in **Figure 11** (black solid line) enables to distinguish clearly the S_{CORR} first local minimum from the other local minima. The estimate of the optimal delay time window t_w occurs for $n = 33$ when $S_{1,CORR}$ is minimum, or with the first peak value of $S_{2,CORR}$ [7] and is different from the optimal delay time window estimated in the C-C method. The optimal embedding dimension d_E is given by $d_E = \text{int}(t_w/T + 1) + 1 = 5$ and is closer to the estimation given by the Cao's method (paragraph III c) and agrees with the results presented in paragraph IIIb.

An optimization of the C-C-1 method should be to define a criterion for the optimal selection of the q parameter value. Based on the results obtained from the C-C method, a criterion is suggested here to select optimally the q parameter value. We define first the quantity

$$S'_{1,CORR}(d_E, n) = \Delta S_1(d_E, n) + \left| \overline{S'_1}(d_E, n) \right|, \quad (28)$$

where ΔS_1 is given by Equation (19) and $\overline{S'_1}(d_E, n) = \sum_{k=1}^4 S_1(d_E, r_k, n)$. The optimal choice of the q parameter value should coincide with the first value of n at which $S'_{1,CORR}(d_E, n)$ shows the least variation with d_E . This requires to define the quantity

$$Q(n) = \left[\max_{d_E} \{S'_1(d_E, n)\} - \min_{d_E} \{S'_1(d_E, n)\} \right] \quad (29)$$

Figure 12 shows the evolution of $Q(n)$ as a function of n .

As the value of the q parameter should be chosen so that the time subseries $\{Y(i), i = 1, q\}$ will not be too short. The optimal q value may be chosen as the first value of n for which $S'_{1,CORR}$ shows a minimum variation with d_E , that is $q = 19$. An organigram for obtaining the graph of the variable Q as a function of

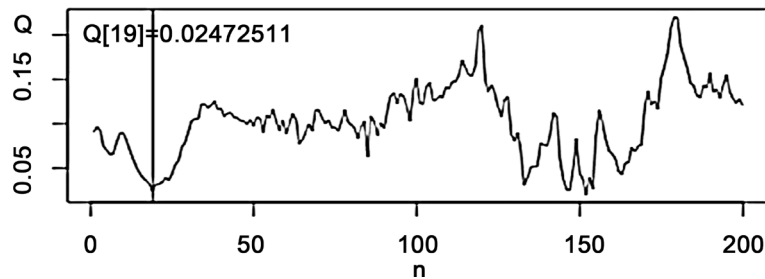


Figure 12. Graphic representation of Q as a function of n .

Table 2. T and t_w variations as a function of different white Gaussian noise variances and strengths.

α	$\sigma = 1$			$\sigma = 5$			$\sigma = 10$		
	d_E	T	t_w	d_E	T	t_w	d_E	T	t_w
0.2	5	10	33	5	10	33	5	9	31
0.3	5	10	33	5	9	31	5	8	26
0.5	5	9	29	5	8	25	5	5	17
0.7	5	9	31	5	5	19	18	9	150
1.0	5	8	26	10	18	151	26	6	149

n is presented in **Appendix 3**.

The C-C-1 method has been applied to the time series $x_B = x + \alpha \mathcal{N}(\sigma)$ where x is the noise free original signal obtained from the about 4000 values data set, $\mathcal{N}(\sigma)$ is a white Gaussian noise with zero mean and a variance σ ($\sigma = 1, 5, 10$), α is the strength of the noise and represents the level of noise in percentage (20%, 30%, 50%, 70%, 100%). T and t_w variations with the different values of σ and α are shown in **Table 2**.

We observe that the C-C-1 method gives stable results for $\sigma = 1$, for $\sigma = 5$ since $\alpha \leq 70\%$, and for $\sigma = 10$ since $\alpha \leq 50\%$.

In conclusion, the C-C-1 method is an improvement of the C-C method. The original time series is subdivided by setting a parameter q which is independent of the time delay T . Tests performed on this method show that it gives more reliable and stable estimates of the optimal delay time T and the optimal time delay window t_w . Tests show also the robustness of this method in presence of noise as the embedding dimension d_E remains equal when noise free data set is degraded with white Gaussian noises with variances respectively equal to 1, 5, 10.

4. Reconstruction Qualification

How can we measure the quality of a reconstruction? Time-delay embedding provides a diffeomorphic representation of the original state space. This means that the mapping between the original and the reconstructed state space is a smooth one. As the optimality of the reconstruction is based on minimizing the distortion of the original attractor when applying the reconstruction map [30],

an appropriate measure of the quality of a reconstruction would be to measure the smoothness of the transformation between the original and the reconstructed space [18]. Once such a transformation is achieved, a good evaluation of invariants such as the Lyapunov exponent and the fractal dimensions of the attractors is required.

In the case of a large size of noise-free data (about 32,000 values), Takens time-delay embedding ensures a topological equivalence between the original and reconstructed space and a way to assess this equivalence is to check whether the fractal dimensions of the attractors are preserved [9] [18] [31]. The maximum Lyapunov exponent was calculated using the CDA22 tool box, we found $\lambda \approx 7.3 \times 10^{-2} \pm 3 \times 10^{-3}$ for $\delta t = 1$, $n = 2$ and $d_E = 5$. The value found when using Benettin's method in paragraph II is in the range defined by this error bar. Considering the same parameters, the correlation dimension, d_C , was also calculated with CDA22. We found $d_C = 2. \pm 0.013$ which is very close to the Lyapunov dimension, d_L , calculated directly by integrating Equation (1).

Moreover, considering for $\delta t = 0.1$, $n = 8$ and $d_E = 5$, the correlation dimension calculated with CDA22, d_C , was found again to be very close to the Lyapunov dimension, d_L . In this case the Lyapunov exponent was not calculated with enough accuracy because the number of data which can be used with CDA22 is limited. In each case, at least one invariant is conserved. Then, one can conclude that the reconstruction was achieved satisfactorily [18] [32].

In the case of lower size of noise free data (about 4000 values), Takens time delay embedding does not ensure the optimality of the reconstruction and requires to measure the smoothness of the mapping with the embedding parameters values determined by using the C-C-1 method ($d_E = 5, n = 10$). We have calculated the factor F based on the statistic Rul'kov test as explained in the introduction [18] [19]. Let be X_i the i th point in the original three dimensional state space $X_i = (x_i, y_i, z_i)$ and V_{X_i} , a neighborhood of X_i with a radius r $V_{X_i} = \{X'_i \text{ such that } |X_i - X'_i| \leq r\}$. Let be $y_i = f(X_i)$ the mapping of X_i in the five-dimensional reconstructed space. The y_i components in transformed coordinates may be written as [18] [33]

$$y_i = \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ x_{i,3} \\ x_{i,4} \\ x_{i,5} \end{bmatrix} = f \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}. \quad (30)$$

Let be V_{y_i} , a neighborhood of y_i with the same radius r as for V_{X_i} . Let be f^{-1} the inverse mapping from the five-dimensional reconstructed space on the original three-dimensional state space. To establish that the mapping f could be able to produce a diffeomorphic representation of the original state space, it can be shown that neighbors of X_i in V_{X_i} may be kept by the $f^{-1} * f$ transformation. Let X_{iNO} be the nearest neighbor of X_i such as $y_{iNO} = f(X_{iNO})$, the corresponding mapping point in the reconstructed space would be a neighbor of

y_i in V_{y_i} . Let y_{iNR} be the nearest neighbor of y_i such as $X_{iNR} = f^{-1}(y_{iNR})$, the corresponding point in the original space would be a neighbor of X_i in V_{X_i} . The factor F used to measure the mapping smoothness is given by [18] [19]

$$F = \frac{1}{p} \sum_{i=1}^p \frac{|y_i - y_{iNO}|}{|X_i - X_{iNO}|} \frac{|X_i - X_{iNR}|}{|y_i - y_{iNR}|}, \quad (31)$$

with $p = N - (d_E - 1)T$ is the number of embedded points, d_E is the embedding dimension and T the time delay. The factor F should be closed to unity so that it would be able to characterize an ideal diffeomorphic mapping. Thus, the closer the F value is to unity, the better is the reconstruction. The F factor has been calculated for the lower size noise free data set studied in paragraphs IIId and IIIf by using the R script presented in **Appendix 4**. The embedding parameters found for this data set by using the C-C-1 method were $d_E = 5, T = 10$ and the estimated value of the F factor is 0.958. The correlation dimension d_C has been calculated for $d_E = 5, T = 10$ and we found $d_C = 1.87785$. The maximum Lyapunov exponent has been calculated for the same embedding parameters values and we found $\lambda = 0.062$. **Figure 13(a)** show a trajectory in the (x, y, z) three-dimensional space obtained by integrating numerically Equation (1). **Figure 13(b)** shows a trajectory in a three-dimensional space obtained by considering three delay coordinates in the reconstruction space. **Figure 13(a), Figure 13(b)** show similarities.

5. Conclusions

This paper provides an overview of methods for embedding parameters optimal selection applied to the Rössler strange attractor reconstruction through chaotic time series. Two main approaches are used whether times series are sufficiently long free noise data set [5] [6] [7] [8] [9] or finite and noisy data set [7] [8] [33]. In the first case the embedding parameters d_E and T can be determined independently and the theorem of Takens allows recreating the underlying dynamics. When data set are finite and/or noisy, the theorem of Takens is silent and parameters d_E and T would appear correlated and as an irrelevant partnership between them could affect the quality of the reconstruction, the delay time window $t_w = (d_E - 1)T$ should be a more useful parameter to determine. Along to these two approaches, three methods have been presented. Coherence between the different results is discussed and robustness of all these three methods is tested. Results obtained with the Cao's method [14] show that for noise-free data of very long length, the reconstruction is valid for any time delay as far as the embedding dimension is high enough. When going to small number noisy data samples, the time delay used to determine the minimum embedding dimension cannot have any value.

The C-C method developed by Kim *et al.* [7] has been applied to finite data sequence of about 4000 values and the robustness of this method has been studied when the original data set is degraded white Gaussian noise with different

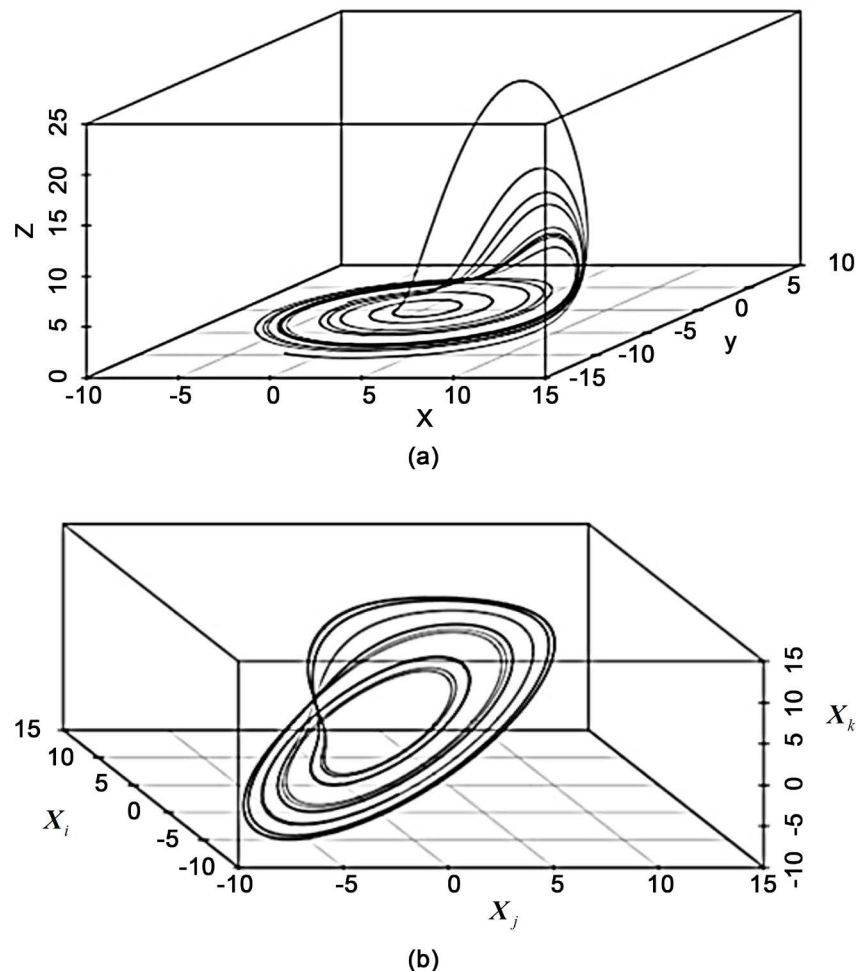


Figure 13. Similarity between the initial state space (a) and the reconstructed one (b).

variance σ and different strength α . Results are summarized in **Table 1** and discussed. The C-C-1 method suggested by Cai *et al.* [8] improve some drawbacks of the C-C method and has been tested on the same time series of about 4000 values and show d_E , T and t_w estimates in line with Cao's method results. Results on the C-C-1 method robustness against noise are summarized in **Table 2**, and shows that the C-C-1 method is an improvement of the C-C method. A criterion for determining the C-C-1 method q parameter is suggested on paragraph IIIId and improves the implementation of the C-C-1 method. A technic based on the statistic Rul'kov test is proposed in paragraph IV to measure the state space reconstruction quality [18] [19] [33].

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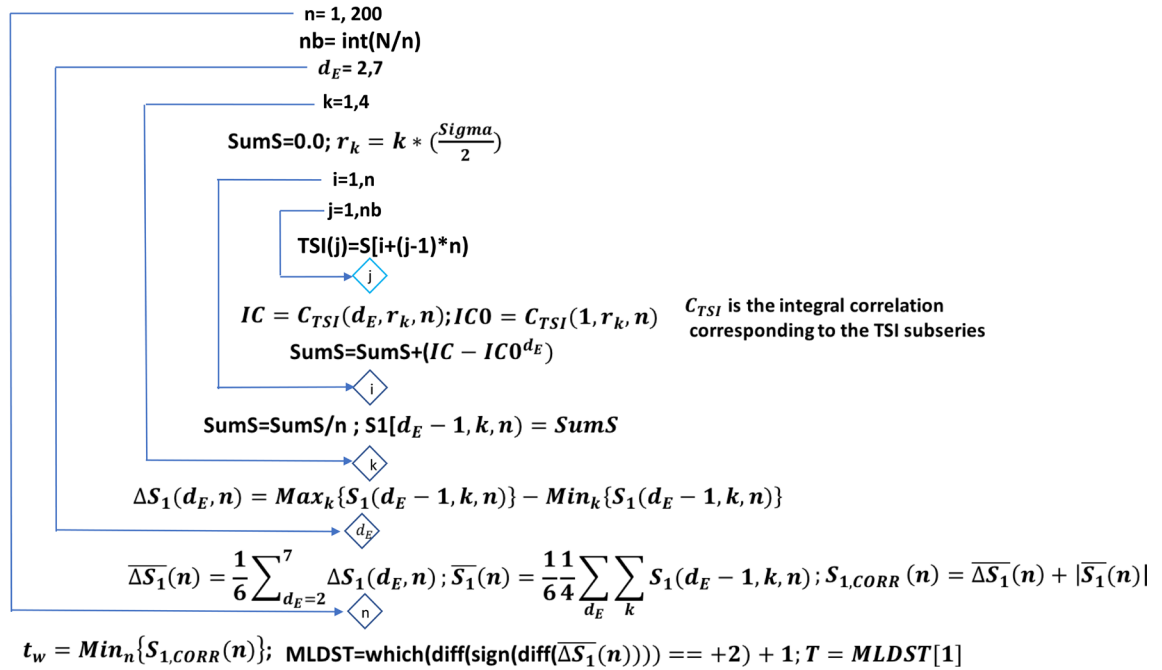
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Appendix 1

The C-C method organigram

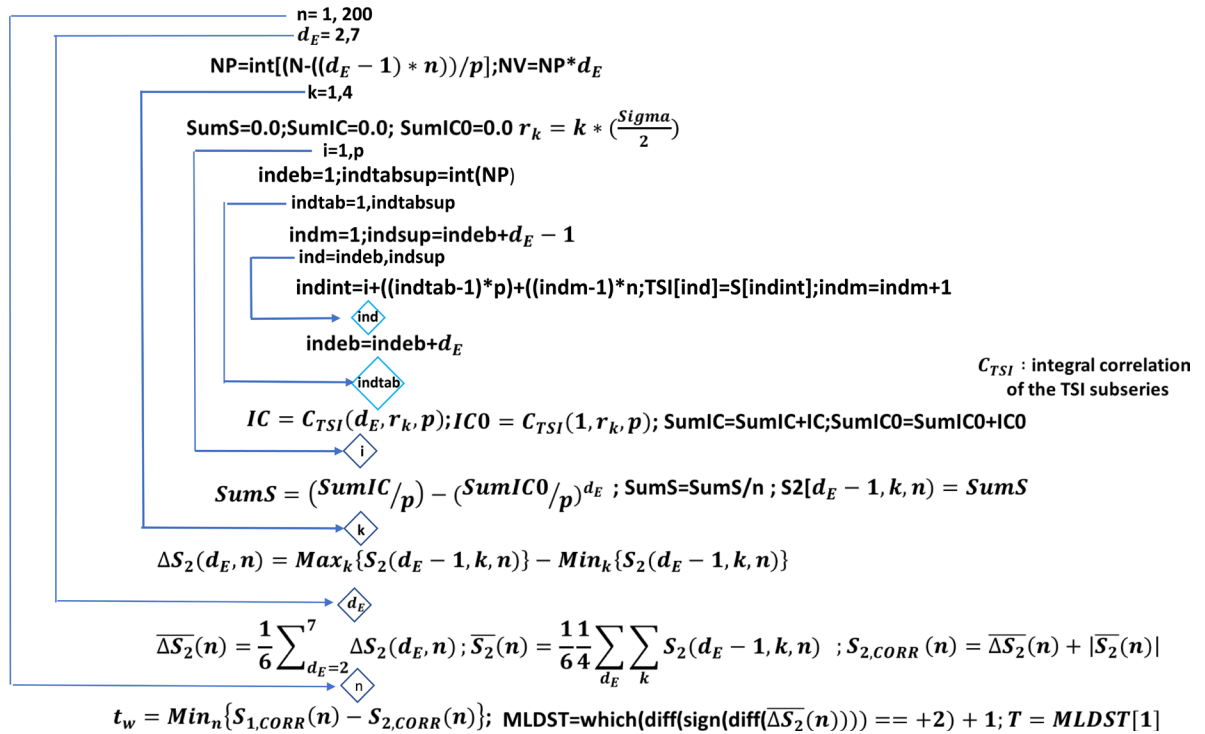
S: Initial time series → N= length (S); Sigma=standard-deviation(S)



Appendix 2

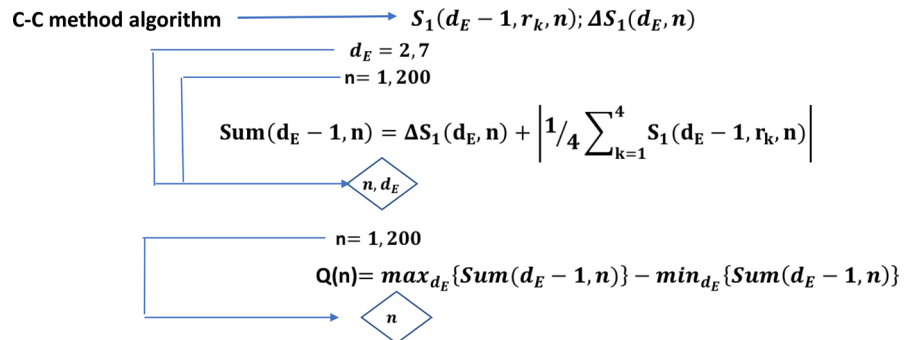
The C-C-1 method organigram

S: Initial time series → N= length (S); Sigma=standard-deviation(S); p=19



Appendix 3

Organigram for obtaining $Q[n]$



Appendix 4

Smoothness mapping F factor calculation R script

Input data: Initial time series S : ts4xnew, T (time delay) = 10, m (embedding dimension) = 5

Output data: F : F factor

$X0 = x(t_0), Y0 = y(t_0), Z0 = z(t_0)$ are the coordinates of the initial point of the trajectory obtained by solving numerically Equation (1) with a time step Δt

```

library(stats)
library(scatterplot3d)
library(nonlinearTseries)
library(tseriesChaos)
X0=2.4099243
X1=2.2247949
Y0=4.0068145
Y1=4.0693054
a=0.2
b=0.4
c=5.7
deltat=(Y1-Y0)/(X0+(a*Y0))
Z0=Y0+((X1-X0)/deltat)
ro-
sor=rossler(a=0.2,b=0.4,w=5.7,start=c(X0,Y0,Z0),time=seq(1,79.82,by=deltat))
N=length(rosor$x)
m=5
T=10
NP=N-((m-1)*T)
rosrec2=buildTakens(ts4xnew,m,T)
MATOR<-matrix(data=0.0,nrow=NP,ncol=3)
Vecsum<-vector(mode="numeric",length = NP)
for(irow in 1:NP){

```

```

MATOR[irow,1]=rosor$x[irow]
MATOR[irow,2]=rosor$y[irow]
MATOR[irow,3]=rosor$z[irow]
}
for(i in 1:NP){
  XI=c(MATOR[i,1],MATOR[i,2],MATOR[i,3])
  YI=c(rosrec2[i,1],rosrec2[i,2],rosrec2[i,3],rosrec2[i,4],rosrec2[i,5])
  nno=neighbourSearch(MATOR,i,0.7)
  nnr=neighbourSearch(rosrec2,i,0.7)
  VO<-nno[[2]]
  VR=nnr[[2]]
  Vinto=intersect(VO,VR)
  INNO=Vinto[1]
  XINNO= c(MATOR[INNO,1],MATOR[INNO,2],MATOR[INNO,3])
  YINNO=
c(rosrec2[INNO,1],rosrec2[INNO,2],rosrec2[INNO,3],rosrec2[INNO,4],rosrec2
[INNO,5])
  vecxinno=XI-XINNO
  deno=max(vecxinno)
  vecyinno=YI-YINNO
  numo=max(vecyinno)
  var1=numo/deno
  Vintr=intersect(VR,VO)
  INNR=Vintr[1]
  XINNR= c(MATOR[INNR,1],MATOR[INNR,2],MATOR[INNR,3])

YINNR=c(rosrec2[INNR,1],rosrec2[INNR,2],rosrec2[INNR,3],rosrec2[INNR,4]
,rosrec2[INNR,5])
  vecyinnr=YI-YINNR
  vecxinnr=XI-XINNR
  denr=max(vecyinnr)
  numr=max(vecxinnr)
  var2=numr/denr
  var=abs(var1*var2)
  Vecsum[i]=var
}
result=sum(Vecsum)/NP

```



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