Distance-Based Shape Statistics

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joint work with O. Faugeras, R. Keriven, P. Maurel and J.-P. Pons

Shapes and Shape Metrics

Set of Shapes

A shape: a smooth, closed manifold of $\mathbb{R}^n$.

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- **Shape Metrics**

  Explicit

$$d_H(\Gamma_1, \Gamma_2) \; = \; \max\left\{ \sup_{\mathbf{x}\in\Gamma_1} d_{\Gamma_2}(\mathbf{x}), \; \sup_{\mathbf{x}\in\Gamma_2} d_{\Gamma_1}(\mathbf{x}) \right\}$$

Shapes and Shape Metrics

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  Explicit - **Implicit**

$$d_{W^{1,2}}(\Gamma_1, \Gamma_2)^2 = \left\| \tilde{d}_{\Gamma_1} - \tilde{d}_{\Gamma_2} \right\|^2_{L^2(\Omega,\mathbb{R})} + \left\| \nabla \tilde{d}_{\Gamma_1} - \nabla \tilde{d}_{\Gamma_2} \right\|^2_{L^2(\Omega,\mathbb{R}^n)}$$

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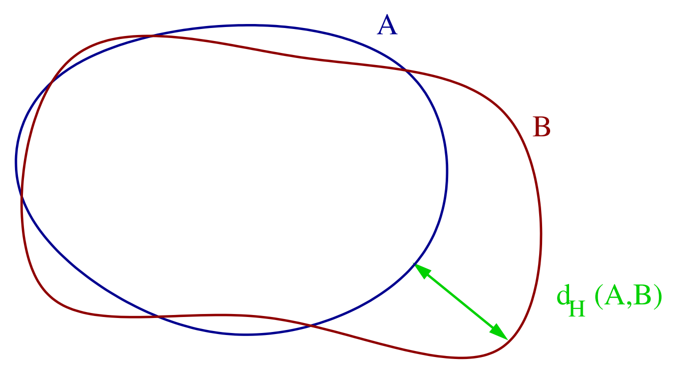
Shape Metrics

Explicit - Implicit - **Path-based**

$$\underset{v,\ \substack{v(0,\cdot)=\Gamma_1 \\ v(1,\cdot)=\Gamma_2}}{\arg\min} \int_t \|v(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} dt$$

- Hausdorff distance:
$$d_H(\Gamma_1,\Gamma_2) = \max\left\{\sup_{\mathbf{x}\in\Gamma_1} d_{\Gamma_2}(\mathbf{x}), \sup_{\mathbf{x}\in\Gamma_2} d_{\Gamma_1}(\mathbf{x})\right\}$$



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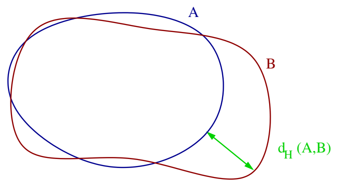

- smooth, differentiable approximation

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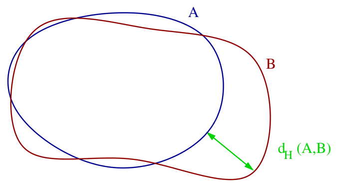
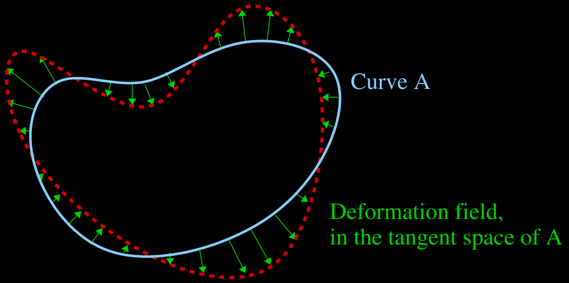


- smooth, differentiable approximation
- build geodesics : minimize $d_H(\Gamma_1, \Gamma_2)$ with respect to Γ_1.

Variational Shape Warping

Shape Gradient

Directional derivative: $\mathcal{G}_\Gamma\left(E(\Gamma), \mathbf{v}\right) = \lim_{\varepsilon \to 0} \frac{E(\Gamma + \varepsilon\, \mathbf{v}) - E(\Gamma)}{\varepsilon}$



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Usual tangent space: $F = L^2$:

$$\langle f | g \rangle_{L^2} = \int_\Gamma f(\mathbf{x}) \cdot g(\mathbf{x})\, d\Gamma(\mathbf{x})$$

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- Build minimizing path:

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- F as a prior on the minimizing flow

Generalized Gradients: Spatially Coherent Flows

- L^2 inner product

$$\langle f \,|\, g \rangle_{L^2} \;=\; \int_\Gamma f(x) \cdot g(x) \, d\Gamma(x)$$

Generalized Gradients: Spatially Coherent Flows

- L^2 inner product
- H^1 inner product

$$\langle f \,|g \rangle_{H^1} \;=\; \langle f \,|g \rangle_{L^2} + \langle \partial_x f \,|\partial_x g \rangle_{L^2}$$

Generalized Gradients: Spatially Coherent Flows

- L^2 inner product
- H^1 inner product
- Set S of prefered transformations (rigid motion)
  Projection on S: P
  Projection orthogonal to S: $Q \quad (P + Q = Id)$

$$\langle f \,|g \rangle_S \;=\; \langle P(f)\,|P(g)\rangle_{L^2} + \alpha\,\langle Q(f)\,|Q(g)\rangle_{L^2}$$

Generalized Gradients: Spatially Coherent Flows

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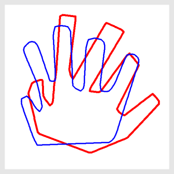
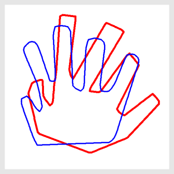
- Example: two different geodesics for the Hausdorff distance



usual



rigidified

Mean and Modes of Variation

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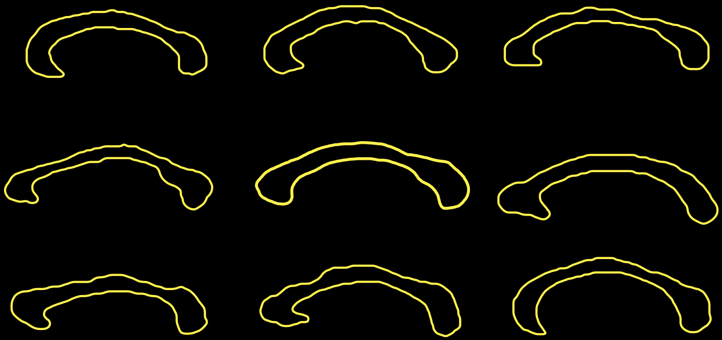
- PCA on instantaneous deformation fields β_i:
  diagonalize $\Lambda \implies$ characteristical modes m_k

Modes: example

Example: set of 2D corpi callosi contours



First characteristic modes of deformation: 1 2 3

Graph Laplacian method

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$$W_{i,j} = \delta_{i \sim j} \; e^{-\frac{d(\Gamma_i,\Gamma_j)^2}{2\sigma^2}}$$

where

$$\delta_{i \sim j} = \begin{cases} 1 & \text{if} \quad i \in N^j \quad \text{or} \quad j \in N^i \\ 0 & \text{otherwise} \end{cases}$$

Graph Laplacian method

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- Approximation of the Laplacian operator: $L = W - D$

where

$$D_{i,j} \;=\; \sum_i W_{i,j}\, \delta_{i \sim j}$$

Graph Laplacian method

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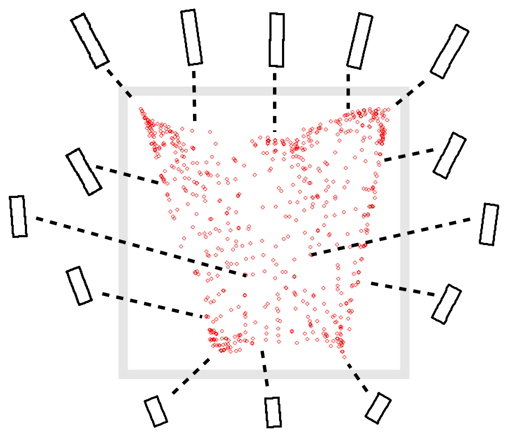
- Approximation of the Laplacian operator: $L = W - D$
- Eigenvector F_k of L: associates to each shape a real value

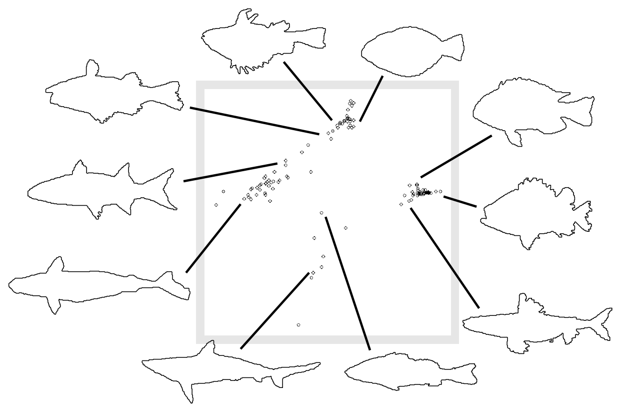
First eigenvectors $\implies$ best coordinate system: $\Gamma_i \mapsto (F_k(\Gamma_i))$.

Examples



Map from the graph Laplacian method for a set of rectangles whose length and orientation have been chosen randomly ($K = 15$).

Examples



Two first coordinates for a set of 111 fish from different classes. The elements from each family are got together into clusters ($K = 25$).

Summary

- Some distances on the set of shapes
- Warping through a gradient descent
  Importance of the inner product (priors on minimizing flows)
- Warping $\implies$ Mean and characteristic modes of deformation (first and second order statistics)
- Without warping: graph methods
  coordinate system, maps.

References:

- *Approximations of shape metrics and application to shape warping and empirical shape statistics*, in *Foundations of Computational Mathematics*, Feb. 2005.
- *Designing spatially coherent minimizing flows for variational problems based on active contours*, in *ICCV 2005*.