SHAPE METRICS, WARPING AND STATISTICS

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ABSTRACT

We propose to use approximations of shape metrics, such as the Hausdorff distance, to define similarity measures between shapes. Our approximations being continuous and differentiable, they provide an obvious way to warp a shape onto another by solving a Partial Differential Equation (PDE), in effect a curve flow, obtained from their first order variation. This first order variation defines a normal deformation field for a given curve. We use the normal deformation fields induced by several sample shape examples to define their mean, their covariance "operator", and the principal modes of variation. Our theory, which can be seen as a nonlinear generalization of the linear approaches proposed by several authors, is illustrated with numerous examples. Our approach being based upon the use of distance functions is characterized by the fact that it is intrinsic, i.e. independent of the shape parametrization.

1. INTRODUCTION

Learning shape models from examples, using them to recognize new instances of the same class of shapes are fascinating problems that have attracted the attention of many scientists for many years. Central to this problem is the notion of a random shape which in itself has occupied people for decades. Frechet [1] is probably one of the first mathematicians to develop some interest for the analysis of random shapes, i.e. curves. He was followed by Matheron [2] who founded with Serra the French school of mathematical morphology and by David Kendall [3, 4, 5] and his colleagues. In addition, and independently, a rich body of theory and practice for the statistical analysis of shapes has been developed by Bookstein [6], Dryden and Mardia [7], Carne [8], Cootes, Taylor and colleagues [9]. Except for the mostly theoretical work of Frechet and Matheron, the tools developed by these authors are very much tied to the point-wise representation of the shapes they study: objects are represented by a finite number of salient points or landmarks. This is an important difference with our work which deals explicitly with curves as such, independently of their sampling or even parameterization.

In effect, our work bears more resemblance with that of several other authors. Like in Grenander's theory of patterns [10, 11], we consider shapes as points of an infinite dimensional manifold but we do not model the variations of the shapes by the action of Lie groups on this manifold, except in the case of such finite-dimensional Lie groups as rigid displacements (translation and rotation) or affine transformations (including scaling). For infinite dimensional groups such as diffeomorphisms [12, 13] which smoothly change the objects' shapes previous authors have been dependent upon the choice of parameterizations and origins of coordinates [14, 15, 16, 17, 18]. For these authors, warping a shape onto another requires the construction of families of diffeomorphisms that use these parameterizations. Our approach, based upon the use of the distance functions, does not require the arbitrary choice of parameterizations and of origins. From our viewpoint this is already very nice in two dimensions but becomes even nicer in three dimensions and higher where finding parameterizations and tracking origins of coordinates can be a real problem: this is not required in our case. Another piece of related work is that of Soatto and Yezzi [19] who tackle the problem of jointly extracting and characterizing the motion of a shape and its deformation. In order to do this they find inspiration in the above work on the use of diffeomorphisms and propose the use of a distance between shapes (based on the set-symmetric difference). This distance poses a number of problems that we address in the same section where we propose two other distances which we believe to be more suitable.

Some of these authors have also tried to build a Riemannian structure on the set of shapes, i.e. to go from an infinitesimal metric structure to a global one. The infinitesimal structure is defined by an inner product in the tangent space (the set of normal deformation fields) and has to vary continuously from point to point, i.e. from shape to shape. This is dealt with in the work of Trouvé and Younes [14, 15, 13, 16] and, more recently, in the work of Klassen and Srivastava [20], again at the cost of working with parameterizations. The problem with these approaches, beside that of having to deal with parameterizations of the shapes, is that there exist global metric structures on the set of shapes which are useful and relevant to the problem of the comparison of shapes but that do not derive from an infinitesimal structure.

Our approach can be seen as taking the problem from exactly the opposite viewpoint from the previous one: we start with a global metric on the set of shapes and build smooth functions (in effect smooth approximations of these metrics) that are dissimilarity measures, or energy functions; we then minimize these functions using techniques of the calculus of variation by computing their gradient and performing infinitesimal gradient descent.

In this article we revisit the problem of defining statistics on sets of 2D shapes and propose a new approach by combining several notions such as topologies on set of shapes, calculus of variations, and some measure theory. Our theory is intrinsic and graciously extends to higher dimensions.

2. SHAPES AND SHAPE TOPOLOGIES

To define fully the notion of a shape is beyond the scope of this article in which we use a limited, i.e purely *geometric*, definition. Due to lack of place, we refer the reader to [21] for a more detailed and rigorous report.

In our context we define a shape to be a regular bounded subset of $\mathbb{R}^2$. Since we are driven by image applications we also assume that all our shapes are contained in a hold-all regular open bounded subset of $\mathbb{R}^2$ which we denote by D. We note Ω any shape, i.e. any regular bounded subset of D, and Γ or $\partial\Omega$ its boundary, a smooth curve of $\mathbb{R}^2$. To be independent of any particular parameterization of the shape, we represent a shape by its distance function: $d_\Omega(x) = \inf_{y\in\Omega} \mid y - x \mid= \inf_{y\in\Omega} d(x, y)$ (see [21] for other ways of representing a shape).

The next question we want to address is that of the definition of the similarity between two shapes. This question is closely connected to that of metrics of sets of shapes which in turn touches that of what is known as shape topologies. Among the possible choices discussed in [21], we will only use here the well-known Hausdorff metric

$$\rho_H(\Omega_1, \Omega_2) = \max \quad \sup_{x\in\Omega_2} d_{\Omega_1}(x), \ \sup_{x\in\Omega_1} d_{\Omega_2}(x) \qquad (1)$$

3. APPROXIMATIONS OF SHAPE TOPOLOGIES

The problem of continuously deforming a shape so that it turns into another is central to this paper. It can be seen as an instance of the warping problem: given two shapes Ω_1 and Ω_2, how do I warp Ω_1 onto Ω_2? The applications in the field of medical image processing and analysis are immense (see for example [22, 23]).

We assume that we are given a function $E : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$, called the energy, which is smooth for one of the shape topologies of interest. This energy can also be thought of as a measure of the dissimilarity between the two shapes. Using the Euler-Lagrange equation, we can derive a gradient $\nabla E(\Gamma, \Gamma_2)$ of $E(\Gamma, \Gamma_2)$ (see [21]). Smoothly deforming a curve Γ_1 onto a curve Γ_2 can be stated as finding a family of shapes $\Gamma(t),\ t \geq 0$, solution of the following PDE

$$\Gamma_t \;= -\nabla E(\Gamma, \Gamma_2)\mathbf{n} \quad \Gamma(0) = \Gamma_1 \qquad (2)$$

where $\mathbf{n}(p)$ the unit normal at the point $\Gamma(p)$ of Γ.

The problem we are faced with is that the Hausdorff distance between two shapes is not differentiable despite the fact that it is a very good candidate for an energy function E. The goal here is to provide a smooth approximation of this distance, i.e. an approximation that admits a gradient.

Let Γ be a given shape. We denote by $\langle f\rangle_\Gamma$ the average of f along the curve Γ. For real positive integrable functions f, and for any one to one function φ from $\mathbb{R}^+$ or $\mathbb{R}^{+*}$ we define the φ-average of f along Γ as

$$\langle f\rangle_\Gamma^\varphi = \varphi^{-1} \quad \frac{1}{|\Gamma|}\int_\Gamma \varphi(f(x))\, d\Gamma(x) \qquad (3)$$

The discrete version of this is also useful: let $a_i, i = 1, \ldots, n$ be n positive numbers, we define their φ-mean by:

$$\langle a_1, \cdots, a_n\rangle^\varphi = \varphi^{-1} \quad \frac{1}{n}\sum_{i=1}^{n} \varphi(a_i) \Bigg), \qquad (4)$$

The key idea is to use theses means to approximate extrema. For instance, consider a continuous strictly monotonously decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^{+*}$ and $a \in \mathbb{R}^{+*}$. We have (see [21]):

$$d_\Gamma(y) = \lim_{a\to+\infty} \varphi^{-1} \qquad \frac{1}{|\Gamma|}\int_\Gamma \varphi^a(d(y, x))\, d\Gamma(x) \quad ^{\frac{1}{a}} \Bigg).$$

Using this idea extensively, we finally get the following:

Definition 1 *Let $\tilde\rho$ be defined by*

$$\tilde\rho_H(\Gamma, \Gamma') = \quad \langle d(\cdot, \cdot)\rangle_{\Gamma'}^{\varphi_a} \ {}^{p_b}_{\Gamma}, \ \ \langle d(\cdot, \cdot)\rangle_{\Gamma}^{\varphi_a} \ {}^{p_b}_{\Gamma'} \ {}^{p_c} \qquad (5)$$

where p_a is the power function ($p_a(x) = x_a$). $\tilde\rho$ is an approximation of the Hausdorff distance (see [21])

This approximation has many "nice" properties, the most important being:

Proposition 1 *The approximation $\tilde\rho_H(\Gamma, \Gamma_0)$ is differentiable with respect to Γ. Its gradient at any point y of Γ is proportional to $\alpha(y)\kappa(y) + \beta(y)$ where $\kappa(y)$ is the curvature of Γ at point y (see [21] for the values of α and β).*

4. "HAUSDORFF WARPING"

In this section we show a number of examples of solving equation (2) with the gradient mentioned in proposition (1). Starting from Γ_1, we follow this gradient and smoothly converge to the curve Γ_2 (see [21] for discussion and justification). We call the resulting warping technique the *Hausdorff* warping.

A simple example is shown in figure 1. We have borrowed the next example from the database of fish silhouettes collected by the researchers of the University of Surrey at the VSSP center. A few steps of the result of warping one of these silhouettes onto another are shown in figure 2. Another interesting example is shown in figures 3 and 4. In both cases the two shapes can be described as the union of a large horizontal rectangle and a small vertical one. In the first case, the two vertical rectangles being really close, the behaviour of the warping algorithm is the expected one: it just translates one vertical rectangle horizontally until it coincides with the other. In the second case, the two vertical rectangles being further away, the warping algorithm prefers to "deflate" one of them (on the right) and to grow another one from scratch on the left. This is also a rather "natural" behaviour.

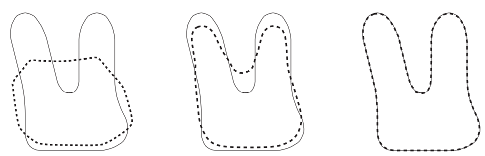

Fig. 1. Hausdorff warping of the shape in dotted line onto the one in continuous line.

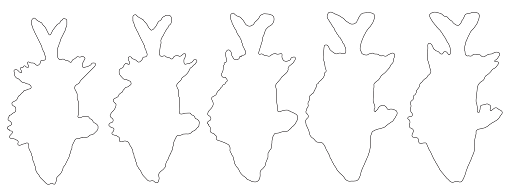

Fig. 2. Hausdorff warping of a fish onto another.

As a last example, we show in figure 5 that, indeed, our warping is not bothered by open curves.

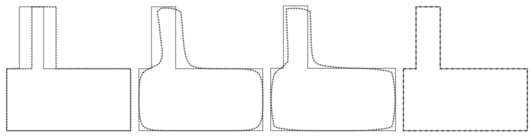

Fig. 3. Hausdorff warping of boxes (i): A translation-like behaviour.

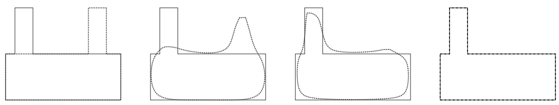

Fig. 4. Hausdorff warping of boxes (ii): A different behaviour: a detail disappears while another one appears.

5. MEAN, COVARIANCE AND MODES OF VARIATION

We have now developed the tools for defining several concepts relevant to a theory of stochastic shapes as well as providing the means for their effective computation. The first obvious concept is that of the mean of a set of shapes.

Definition 2 *Given $\Gamma_1, \cdots, \Gamma_N$, N shapes, we define their **mean** as any shape $\hat{\Gamma}$ that minimizes the function $\mu : \mathcal{S} \to \mathbb{R}^+$ defined by $\mu(\Gamma, \Gamma_1, \cdots, \Gamma_N) = \frac{1}{N} \sum_{i=1,\cdots,N} E(\Gamma, \Gamma_i)$*

An algorithm for computing approximations to a mean readily follows from the previous section: start from an initial shape and solve the PDE: $\Gamma_t = -\nabla\mu(\Gamma, \Gamma_1, \cdots, \Gamma_N)\,\mathbf{n}$. We show some examples in figures 6 and 7.

We can now define the covariance of N shapes and their modes of variation (see [21] for justifications).

Definition 3 *Given N shapes Γ_i, we note β_i the normal velocity fields of the gradients of the functions $\Gamma \to \tilde{d}_H(\Gamma, \Gamma_i)$ and $\hat{\beta} = \frac{1}{N} \sum_{i=1}^{N} \beta_i$ their mean. The **covariance** operator of these N shapes for their mean $\hat{\Gamma}$ is the linear continuous operator of $L^2(\hat{\Gamma})$ defined by $\Lambda(\beta) = \sum_{i=1,N} < \beta, \beta_i - \hat{\beta} >_{\hat{\Gamma}} (\beta_i - \hat{\beta})$,*

Definition 4 *Let $\hat{\Lambda}$ be the $N \times N$ symmetric semi positive definite matrix $\hat{\Lambda}$ defined by $\hat{\Lambda}_{ij} =< \beta_i - \hat{\beta}, \beta_j - \hat{\beta} >_{\hat{\Gamma}}$. Let $p \leq N$ be its rank, $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_p^2 > 0$ its positive eigenvalues and $\mathbf{u}_1, \cdots, \mathbf{u}_N$ the corresponding eigenvectors. Let u_{ij} be the ith coordinate of the vector $\mathbf{u}_j$ and v_j be defined by $v_j = \frac{1}{\sigma_j} \sum_{i=1}^{N} u_{ij}(\beta_i - \hat{\beta})$. The velocities v_k, $k = 1, \cdots, p$ can be interpreted as **modes of variation** of the shapes and the σ_k^2's as variances for these modes. Looking at how the shape varies for the kth mode is equivalent to solving the PDEs $\Gamma_t = \pm v_k(\Gamma)\,\mathbf{n}$ with $\Gamma(0, .) = \hat{\Gamma}(.)$.*

Examples of theses modes for the cases of the fingers and of the ten fishes is shown in figure 8.

6. IMPLEMENTATION

The Hausdorff distance was approximated with $(a, b, c) = (4, 4, 2)$. For closed curves, we used a level set method [24] to implement the evolution $\Gamma_t = \beta\mathbf{n}$. While β is only known on the curve Γ itself, a "narrow banded" velocity extension scheme like in [25] was required. Figure 9 shows a typical "level set" behaviour where a curve splits while evolving toward another one. For open curves, a straight Lagrangian approach and polygonal approximations were used as a first step toward more refined methods like the ones described in [26].

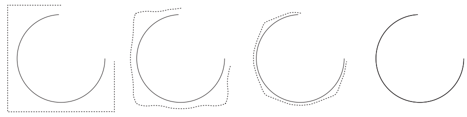

Fig. 5. Hausdorff warping of an open curve onto another one.

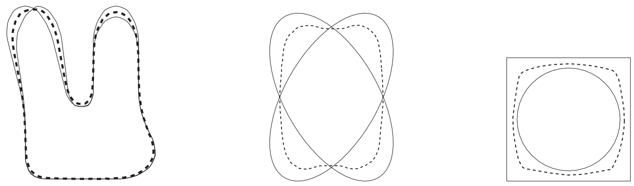

Fig. 6. Examples of means of two curves.

7. CONCLUSION

We have presented in section 1 the similarities and dissimilarities of our work with that of others. We would like to add to this presentation the fact that ours is an attempt to generalize to a nonlinear setting the work that has been done in a linear one by such people as Cootes, Taylor and their collaborators [9] and by Leventon who, like us, proposed to use distance functions to represent shapes in a statistical framework but did it sort of the wrong way by assuming that the set of distance functions was a linear manifold [27] which of course it is not. Our work shows that dropping the incorrect linearity assumption is possible at reasonable costs, both theoretical and computational. Comparison of results obtained in the two frameworks is a matter of future work.

In this respect we would also like to emphasize that in our framework the process of linear averaging shape representations has been more or less replaced by the linear averaging of the normal deformation fields which are tangent vectors to the manifold

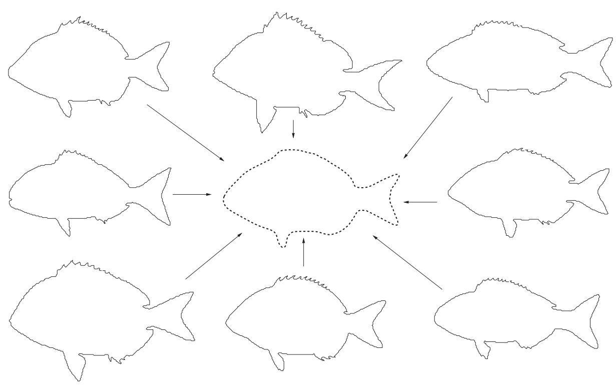

Fig. 7. The mean of ten fishes (only eight shown).

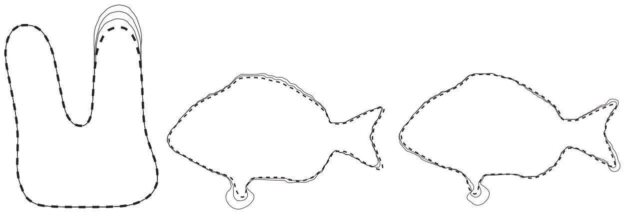

Fig. 8. The first mode of variation in the finger case and the first two ones for the ten sample shapes and their mean shown in figure 7 .

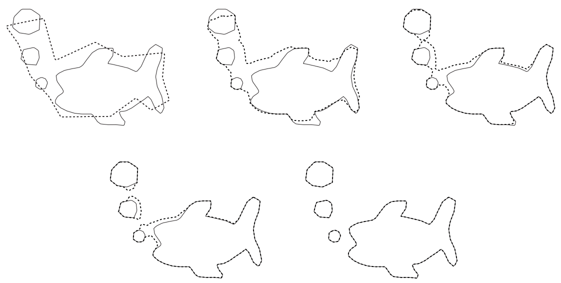

Fig. 9. Splitting while Hausdorff warping curves.

of all shapes (see the definition of the covariance operator in section 5 and in [21]) and by solving a PDE based on these normal deformation fields (see the definition of a mean) and of the deformation modes.

Another advantage of our viewpoint is that it apparently extends graciously to higher dimensions thanks to the fact that we do not rely on parameterizations of the shapes and work intrinsically. This is clearly also worth pursuing in future work.

8. REFERENCES

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