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Asymptotic behaviour of cellular automata: computation and randomness

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Résumé

Les automates cellulaires sont étudiés à la fois comme des systèmes dynamiques discrets et comme un modèle de calcul massivement parallèle. L'étude empirique laisse apparaître des phénomènes d'auto-organisation, c'est-à-dire l'émergence d'un comportement structuré à partir d'une configuration initiale aléatoire. Dans le cadre de cette thèse, nous étudions l'évolution d'une mesure de probabilité initiale sous l'action d'un automate cellulaire, le comportement asymptotique typique étant décrit par la(les) mesure(s) limite(s).

Premièrement, nous caractérisons les mesures accessibles asymptotiquement par les automates cellulaires. Cette approche rejoint divers résultats récents caractérisant des paramètres de systèmes dynamiques par des conditions de calculabilité. Les résultats obtenus mettent en évidence la variété des comportements asymptotiques possibles et décrivent la puissance de calcul des automates cellulaires sur les mesures de probabilités.

Deuxièmement, nous proposons un cadre d'étude de l'auto-organisation pour des classes d'automates cellulaires pouvant être vus comme des systèmes de particules en interaction. De la dynamique des particules, nous déduisons des propriétés sur le comportement asymptotique de l'automate cellulaire et sur la vitesse de convergence de divers paramètres.

Enfin, nous étudions le problème de randomisation : trouver un automate cellulaire sous l'action duquel une large classe de mesures initiales converge vers la mesure uniforme. Nous proposons des candidats pour cette question ouverte, soutenus par des résultats expérimentaux, ainsi que quelques nouveaux résultats liés.

Mots clés : automate cellulaire, système dynamique, théorie ergodique, calculabilité, analyse calculable, système de particules en interaction, marche aléatoire, mouvement brownien, randomisation

Abstract

Cellular automata are discrete dynamical systems as well as a massively parallel model of computation. Empirical observations suggest the existence of self-organisation phenomena, that is, the emergence of an organised behaviour from an initial configuration chosen at random. In this thesis, we study the evolution of an initial probability measure under the action of a cellular automaton, the asymptotic behaviour being described by the limit measure(s).

First, we characterise measures that are asymptotically reachable by cellular automata. This approach is similar to several recent results characterising parameters of dynamical systems by computability conditions. The results reflect the variety of possible asymptotic behaviours and describe the measure-theoretical computational power of cellular automata.

Then, we introduce a framework for studying self-organisation in classes of cellular automata that can be seen as an interacting particle systems. From the particle dynamics, we deduce properties on the asymptotic behaviour of the automaton and on the rate of convergence of various parameters.

Last, we study the randomisation problem: find a cellular automata such that a large class of initial measures converge under its action towards the uniform measure. We introduce candidates for this open question, backed up by experimental evidence, as well as some new related results.

Keywords : cellular automata, dynamical system, ergodic theory, computability, computable analysis, interacting particle system, random walk, Brownian motion, randomisation

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Contents

List of Notations

Symbol	Definition	Defined in page
$\mathcal{A}, \mathcal{B}$	Finite alphabets	7
$\mathcal{A}^*$	Finite one-dimensional words on $\mathcal{A}$: $\bigcup_{n=0}^{\infty} \mathcal{A}^n$	7
$[i, j]$	Integers ranging from i to j	7
$[u]$	Cylinder corresponding to the word u	7
$d(x, y)$	Cantor distance on $\mathcal{A}^{\mathbb{Z}}$	7
$\mathrm{Freq}(u, v)$	Frequency of the word u in v	8
σ	Shift function	8
${}^{\infty}u^{\infty}$	Periodic configuration with repeated pattern u	8
$\mathcal{L}(\Sigma)$	Language of a subshift Σ	9
CA	Cellular automaton	10
$\mathcal{N}$	Neighbourhood (finite subset of $\mathbb{Z}$) used to define a cellular automaton	10
F_n	Elementary cellular automaton $\#n$	11
$\forall_\mu x \in \mathcal{A}^{\mathbb{Z}}$	For μ-almost all configurations x	13
$\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$	σ-invariant probability measures on $\mathcal{A}^{\mathbb{Z}}$	13
$F_*\mu$	Image measure of μ under the action F	13
$\mathrm{supp}(\mu)$	Support of the probability measure μ	13
$\widehat{\delta_w}$	σ-invariant measure supported by ${}^{\infty}w^{\infty}$	13
Ber_v	Bernoulli (i.i.d.) measure of parameters $(v_a)_{a\in\mathcal{A}}$	13
λ	Uniform (Bernoulli) measure	14
$d_{\mathcal{M}}(\mu, \nu)$	Distance on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$: $\sum_{n\in\mathbb{N}} \frac{1}{2^n} \max_{u\in\mathcal{A}^n} \lvert\mu([u]) - \nu([u])\rvert$	14
$\mathcal{M}_{\sigma-\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}})$	σ-ergodic probability measures on $\mathcal{A}^{\mathbb{Z}}$	14

Introduction 0

Section 0.0
Background and motivations

0.0.1 Cellular automata

Cellular automata are a model consisting in the discrete time evolution of an infinite set of cells arranged along a regular grid, each cell being in a state chosen among a finite state space. An assignment of a state to each cell is called a configuration. At each step, the state of each cell is updated synchronously depending on the state of other cells in its proximity, following a local update rule, producing a new configuration.

The model in its current form was introduced by Von Neumann [vN66] as a mechanical model exhibiting an algorithmic self-replicating behaviour, inspired by Ulam's physical model of crystal growth on a two-dimensional lattice. In the same book, he proved that cellular automata were capable of universal computation. While cellular automata originated as a physical and computational model, its uses range nowadays from biology and chemistry to massively parallel computation and cryptography.

The mathematical study of cellular automata only arose in the 60s. In particular, Hedlund's seminal result [Hed69] showed that they could be seen as space-homogeneous continuous functions of the symbolic space $\mathcal{A}^{\mathbb{Z}}$ (Curtis–Hedlund–Lyndon theorem). This gave access to the tools of the theory of dynamical systems, especially symbolic dynamics, and is at the heart of modern mathematical works on cellular automata, this thesis being no exception.

In the 80s, Wolfram undertook an empirical investigation of one-dimensional cellular automata (where the grid is $\mathbb{Z}$) with small alphabets and simple local rules [Wol84a], in contrast with physical models that often used two or more dimensions. By iterating the automata on initial configurations chosen at random, he found that some of them exhibited surprisingly complex and organised behaviours with an emergence of regular structures and patterns, a phenomenon he called self-organisation.

Wolfram suspected that the way even simple local rules could give birth to a complex global order could shed light on the emergence of complexity in nature, and also that self-organisation was related to the computational universality of the model [Wol84b].
Following this idea, his employee Cook showed that even very simple cellular automata were capable of universal computation [Coo04].

From now on, we consider one-dimensional cellular automata.

0.0.2 Self-organisation and typical asymptotic behaviour

There has been various attempts to give a rigorous content to Wolfram's intuitive notion of self-organisation. Before trying to prove rigorously empirical remarks, it is necessary to discuss the objects that are best suited to describe this notion.

Since self-organisation relates to an increase of order after an "organisational" transitional regime, it is natural to consider various objects relative to the asymptotic behaviour of the automata. Furthermore, the notion itself only makes sense when considering disordered initial states, as opposed to e.g. constant or periodic configurations. The various topological objects that have been proposed to describe the asymptotic behaviour of the automaton fail to capture this idea.

This is the reason why we consider instead that the initial configuration is drawn at random, which is consistent with Wolfram's empirical approach, and we investigate the typical asymptotic behaviour, or in other words the asymptotic properties that hold for almost every initial configuration. This is discussed in more detail in Section 0.1.3, with some examples, taking inspiration from [KM00].

The conclusion of this discussion is that the description of the typical asymptotic behaviour of a cellular automaton F that is best suited to our purposes is as follows. Starting from an initial probability measure μ, such as the uniform measure, we consider $F_*\mu$ the measure describing the distribution of configurations after one step of the automaton. Iterating this process, we obtain a sequence of measures $(F_*^t\mu)_{t\in\mathbb{N}}$ describing the distribution of configurations after t steps.

To define a notion of asymptotic measure, we use pointwise convergence topology which can be defined in the following manner. ν is defined as the limit measure of $(F_*^t\mu)_{t\in\mathbb{N}}$ if, for any finite word u, the probability that u appears in a fixed position in $F^t(x)$, where x is drawn according to μ, converges to the probability that it appears in this position in y, where y is drawn according to ν. Of course the sequence can have multiple limit points and the resulting set of measures is called the **μ-limit measures set** of F.

These measures can be seen as "physically" relevant for F in the sense that if one draws many initial configurations at random according to μ and iterates F on them many times, the resulting set of configurations will be distributed according to a law that is close to one of these measures. A similar approach is used to define SRB measures in continuous dynamical systems, which are invariant measures obtained when starting from the Lebesgue measure [You02].

Within this framework, describing the limit measure(s) has been done for only few concrete nontrivial examples. Known results are essentially of two types:

- convergence towards a simple measure, such as a measure supported by a periodic point: for example, the cyclic cellular automaton on three states introduced in [Fis90b] converges towards a linear combination of Dirac measures supported by uniform configurations [HdMS11]. Chapter 2 gives many similar examples;
- convergence towards the uniform measure: some cellular automata respecting a group structure converge in Cesàro mean towards the uniform measure [Lin84, FMMN99, MM98, PY02]. See Chapter 3 for more details.

0.0.3 Probabilistic algorithmics

We now take another, very different point of view about the interest of these limit measures.

An important part of standard algorithmics is concerned with the use of randomness, which is of practical (actual programs) and theoretical (computability, complexity analysis) interest. Having access to a source of random bits (an ideal fair coin) to take random decisions brings beneficial consequences on the behaviour of many algorithms: avoiding the existence of "pathological cases" on which the algorithm always performs at its worst, improving security through unpredictability in cryptography, etc. If the computation is deterministic, we must assume that the randomness comes from an external source (atmospheric noise, computer clock...). As far as the theoretical analysis of algorithms is concerned, we assume that the algorithm has access, in addition to its input, to a perfect source of random bits, that is:

fair probability $\frac{1}{2}$ to obtain 0 and 1;

independent the values of the different bits are independent of each other;

unbounded depending on the input, the algorithms have access to an arbitrarily large number of bits.

To formalise this notion, we assume the algorithm has access to a sequence of random bits $(x_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ sampled uniformly among all such sequences, and can use these random bits one by one as needed.

This definition raises a natural question: does considering a nonuniform source of randomness influence the computational power of our algorithms? This could be for example a biased coin or a coin with some correlations. This notion of source of randomness is described by the choice of a probability measure on $\{0,1\}^{\mathbb{N}}$. We will see that this question is linked to the ability to simulate algorithmically these nonuniform sources.

Let us define the notion of simulating a source of randomness. A probability measure on $\{0,1\}^{\mathbb{N}}$ is **computable** if there exists an algorithm[1]with an input $x \in \{0,1\}^{\mathbb{N}}$ drawn according to the uniform measure, that never stops, outputting successive bits y_i ($i \in \mathbb{N}$) such that the sequence $y = (y_i)_{i\in\mathbb{N}}$ is distributed according to μ. Notice that the algorithm does not stop, but any finite number of bits is drawn in finite time with probability 1.

Questions related to the simulation of randomness were first raised in [vN51], and practical algorithmic techniques were introduced in the seminal papers of [KY76]. The complexity of these simulations and the case where information about the source is incomplete were widely studied: see among many examples [Ueh95, PL06, DI06]. Furthermore, as long as we allow probabilistic algorithms that loop indefinitely with probability 0, it is possible to show that using any computable source of randomness does not let us solve more problems than a perfect source [DLMSS56] (we may, however, lose computational power in the case of a "bad" source).

Since cellular automata is a model of computation, we can extend these notions . In this context, it is more natural to consider $\mathbb{Z}$-indexed sequences, which does not change anything

[1]The formal definition would require to fix an ideal computational model, such as Turing machines, λ-calculus, combinatorial circuits, etc. However, we feel that we would not gain in clarity. Historically, this theory was developed using Turing machines.

about the previous definitions. If the initial configuration is taken as the input source of randomness (i.e. is drawn according to some input measure), then by analogy the limit measure can be considered as the output measure. Notice that the massively parallel nature of the model implies that computation is performed on the whole sequence of random bits at the same time, and we do not have access to "fresh" random bits after time 1.

0.0.4 Contents of this thesis

We consider these two approaches as complementary, and all results in this thesis will be presented from both points of view: a dynamical system in which an organised behaviour emerges, and a model of computation that computes probability measures.

Section 0.1 introduces general definitions on symbolic spaces, cellular automata, probability measures and typical asymptotic behaviour that are needed throughout the thesis.

In Chapter 1, we tackle the general question of which measures or sets of measures can be reached as μ-limit measures set, that is, at the limit after iteration of any cellular automaton on a simple initial measure such as the uniform measure. Our main result is that these measures and sets of measures can be entirely characterised by computability conditions. The two steps of the proof are, first, to show that for a computable initial measure μ, computability obstructions appear on the μ-limit measures set; second, to build for each such measure or set of measures an *ad hoc* cellular automaton that reaches it at the limit.

The main motivation for this question is to explain the wide variety of typical asymptotic behaviours observed in computer simulations. Since the uniform measure as well as any measure that can be sampled algorithmically is (by definition) computable, the computability hypothesis is natural. In addition, in the context of simulating sources of randomness the computational power of this model is found to be equivalent to the computational power of Turing machines.

In Chapter 2, we try to prove some self-organisation results for simple cellular automata, as opposed to the highly sophisticated constructions of the previous chapter. A family of cellular automata exhibits a similar kind of self-organisation, where regions consisting in a simple repeated pattern emerge and grow in size, while the boundaries between them can be followed from an instant to the next, moving in space and colliding, giving birth to some kind of particle-like dynamics.

We show that when these dynamics are simple enough, it is possible to deduce some properties of the typical asymptotic behaviour of the cellular automaton; more precisely, if the particles have different speeds and collisions are destructive, only one type of particle can survive asymptotically. In some cases, this approach can be refined further to obtain quantitative results on some parameters relative to the particles, or to obtain more information on the limit measure.

In Chapter 3, we consider the randomisation phenomenon, which is a kind of self-organisation phenomenon where a cellular automaton converges to the uniform measure for a large class of initial measures. Despite empirical observations and partial positive results for weaker notions of convergence, not a single cellular automaton was proven to exhibit this behaviour. The problem of finding such a cellular automaton is referred to as randomisation problem.

We performed simulations on a class of cellular automata having algebraic and dynamical properties, which led to a conjecture backed up by experimental evidence about how these properties are linked to randomisation. Nevertheless, a full proof is still out of reach. We investigated the related question of rigidity, which is the study of how the only invariant measure under the action of an automaton is the uniform measure, at the price of some additional hypotheses. This approach makes sense for randomisation candidates, that leave the uniform measure invariant, and could be the first step of a full proof.

All diagrams were made with the Sage mathematical software [S$^+$12].

Section 0.1
Definitions

In this section, we introduce general definitions that we use throughout the thesis with some illustrating examples. First, in Section 0.1.1, we introduce the symbolic space $\mathcal{A}^{\mathbb{Z}}$ and cellular automata which are particular actions on this space. Then, in Section 0.1.2, we introduce the standard measure-theoretical probabilistic framework on those spaces in order to give a rigorous content to the notion of drawing the initial configuration at random. Last, Section 0.1.3 is devoted to various notions of asymptotic behaviour in cellular automata, discussing which ones are best suited to our study of self-organisation.

0.1.1 Symbolic spaces and cellular automata

Words and configurations

In all this section, $\mathcal{A}$ and $\mathcal{B}$ are finite **alphabets** . We consider $\mathcal{A}^{\mathbb{Z}}$ the set of (one-dimensional) **configurations** on $\mathcal{A}$, and $\mathcal{A}^* = \bigcup_{n\in\mathbb{N}} \mathcal{A}^n$, the set of finite **words**. Intuitively, consider infinitely many contiguous **cells** organised along a line, and to each cell associate a letter (or colour) taken from this finite alphabet corresponding to the state of this cell.

If $u \in \mathcal{A}^n$, the **length** of u is $|u| = n$. For two words $u \in \mathcal{A}^n$ and $v \in \mathcal{A}^m$, $u \cdot v \in \mathcal{A}^{n+m}$ is the **concatenation** of u and v.

In all the following, we write $[i, j]$ for $\{i, i + 1, \dots, j\}$.

Definition 0.1.1 (Subwords)**.**
For $x \in \mathcal{A}^{\mathbb{Z}}$ and $V \subset \mathbb{Z}$ not necessarily finite, we denote $x_V = (x_i)_{i\in V}$. In particular, for $u \in \mathcal{A}^*$, we write $u \sqsubset_i x$ for $x_{[i,i+|u|-1]} = u$. We say $u \in \mathcal{A}^*$ is a **subword** of x (or **appears in** x), and we write $u \sqsubset x$, if $u \sqsubset_i x$ for some $i \in \mathbb{Z}$.

These definitions extend to define subwords u of a finite word $v \in \mathcal{A}^*$, by restricting ourselves to the cases where $i \in [0, |v| - |u|]$.

Definition 0.1.2 (Cylinders)**.**
For a finite word $u \in \mathcal{A}^*$ and $i \in \mathbb{Z}$, we define the corresponding **cylinder**:

$$[u]_i = \{x \in \mathcal{A}^{\mathbb{Z}} \ : \ u \sqsubset_i x\} \quad \text{and} \quad [u] = [u]_0.$$

and for a subset $S \subset \mathcal{A}^*$, the corresponding cylinder is $[S]_i = \bigcup_{u\in S}[u]_i$.

The notation $[u]$ proves useful when we consider parameters that are independent of the coordinate.

The set $\mathcal{A}^{\mathbb{Z}}$ is endowed with the product topology (also called **Cantor topology**), which is metrisable.

Definition 0.1.3 (Cantor distance)**.**
The **Cantor distance** between two points $x, y \in \mathcal{A}^{\mathbb{Z}}$ is defined as:

$$d(x,y) = 2^{-\Delta(x,y)} \quad \text{where } \Delta(x,y) = \min\{|k| \ : \ k \in \mathbb{Z},\ x_k \neq y_k\}.$$

Intuitively, two configurations are close to each other if they are similar on a large central finite window.

Cylinders are clopen and form a base for the Cantor topology. This topology makes $\mathcal{A}^{\mathbb{Z}}$ compact: indeed, there is only a finite number of cylinders of any given length, and from a sequence in $\mathcal{A}^{\mathbb{Z}}$ one can extract a convergent subsequence by fixing a central word of increasing length, choosing at each step a cylinder containing infinitely many members of the sequence.

Definition 0.1.4 (Frequency)**.**
The **frequency** of a finite word u in another finite word v is defined as:

$$\text{Freq}(u,v) = \frac{\text{Card}\{i \in \mathbb{Z} \ : \ u \sqsubset_i v\}}{|v| - |u| + 1} \quad (0 \text{ if not defined}).$$

The **frequency** of a finite word u in a configuration $x \in \mathcal{A}^{\mathbb{Z}}$ is defined as:

$$\text{Freq}(u,x) = \limsup_{n\to\infty} \text{Freq}(u, x_{[-n,n]}).$$

This last definition can be rewritten:

$$\text{Freq}(u,x) = \limsup_{n\to\infty} \frac{1}{(2n+1)} \text{Card}\{i \in \{-n,\dots,n\} \ : \ u \sqsubset_i x\}.$$

Those definitions extend naturally to sets of words $S \subset \mathcal{A}^*$:

$$\text{Freq}(S,v) = \frac{\text{Card}\{i \in \mathbb{Z} \ : \ \exists u \in S, u \sqsubset_i v\}}{|v| - \min_{u\in S} |u| + 1} \quad (0 \text{ if not defined}).$$

Shifts and subshifts

Definition 0.1.5 (Shift function)**.**
Define the **shift function** $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ as:

$$\forall x \in \mathcal{A}^{\mathbb{Z}}, \sigma(x)_i = x_{i-1}.$$

σ is an action of $\mathcal{A}^{\mathbb{Z}}$ on itself, and we consider the orbits and invariant subsets under the action of σ.

Definition 0.1.6 (σ-periodic configurations)**.**
A configuration $x \in \mathcal{A}^{\mathbb{Z}}$ is **σ-periodic** if, and only if there is an integer n such that $\sigma^n(x) = x$. The minimal such n is the **period** of x.

In particular, let $u \in \mathcal{A}^*$ be a finite word. It generates a σ-periodic configuration ${}^{\infty}u^{\infty}$ inductively:

- ${}^{\infty}u^{\infty}{}_i = u_i$ if $0 \leq i < |u|$;
- $\sigma^{|u|}({}^{\infty}u^{\infty}) = {}^{\infty}u^{\infty}$.

and each σ-periodic configuration x of period p can be written $x = {}^{\infty}u^{\infty}$ for some $u \in \mathcal{A}^p$.

Proposition 0.1.1. The set of σ-periodic configurations is dense in $\mathcal{A}^{\mathbb{Z}}$.

Proof. For a configuration $x \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{N}$, define $y_n = {}^{\infty}x_{[0,n]} \cdot x_{[-n,-1]}{}^{\infty}$. It is clear that x and y_n are equal on the finite window $\{-n, \dots, n\}$, and therefore $d(x, y_n) \leq 2^{-n}$. □

Definition 0.1.7 (Subshifts)**.**
A (one-dimensional) **subshift** is a closed σ-invariant set $\Sigma \subseteq \mathcal{A}^{\mathbb{Z}}$ (i.e. $\sigma(\Sigma) = \Sigma$).

Equivalently, a subshift Σ can be defined by a (not necessarily finite) **set of forbidden words** $\mathcal{F} \subset \mathcal{A}^*$ by putting:

$$\forall x \in \mathcal{A}^{\mathbb{Z}}, x \in \Sigma \underset{\text{def}}{\iff} \forall u \in \mathcal{F}, u \not\sqsubset x.$$

Definition 0.1.8 (Languages)**.**
For a subshift $\Sigma \subseteq \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{N}$, the **language** of Σ is defined as:

$$\mathcal{L}_n(\Sigma) = \{u \in \mathcal{A}^n \ : \ \exists x \in \Sigma, u \sqsubset x\} \quad \text{and} \quad \mathcal{L}(\Sigma) = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n(\Sigma).$$

A subshift Σ is entirely described by $\mathcal{L}(\Sigma)$, in the sense that $x \in \Sigma \Leftrightarrow \forall u \sqsubset x, u \in \mathcal{L}(\Sigma)$.

Definition 0.1.9 (Subshifts of finite type, Sofic subshifts)**.**
A **subshift of finite type** (SFT for short) is a subshift that can be defined by a finite set of forbidden words. In other words, a subshift Σ is of finite type if and only if it is entirely described by $\mathcal{L}_r(\Sigma)$ for some $r \in \mathbb{N}$. The smallest such r is the **radius** of the SFT.

A **sofic subshift** is a subshift that can be defined by a rational set of forbidden words, i.e., is the language accepted by some finite automaton.

Examples.

Orbit of a configuration For any configuration $x \in \mathcal{A}^{\mathbb{Z}}$, the closure of the set $\{\sigma^i(x) : i \in \mathbb{Z}\}$ is a subshift. It is finite if, and only if, x is a σ-periodic configuration.

Checkerboard subshift The checkerboard subshift is the subshift of finite type on alphabet $\{0, 1\}$ defined by the set of forbidden words $\{00, 11\}$. It contains exactly two configurations: ${}^{\infty}01^{\infty}$ and ${}^{\infty}10^{\infty}$.

Odd subshift The odd subshift is the subshift on $\{0, 1\}$ containing all configurations whose finite clusters of 1 have odd length. It is a sofic subshift defined by the set of forbidden words $\{01^{2n}0 \ : \ n \in \mathbb{N}\}$.

Definition 0.1.10 (de Bruijn graph of a subshift)**.**
To a subshift Σ we associate its **de Bruijn graph of order** n:

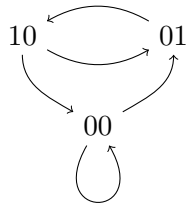


Figure 0.1: De Bruijn graph of order 2 of the SFT defined by $\mathcal{F} = \{11\}$.

- the set of vertices is $V = \mathcal{A}^n$
- the set of edges is $E = \{u_0 \dots u_{n-1} \to u_1 \dots u_n \ : \ u_0 u_1 \dots u_n \in \mathcal{L}_{n+1}(\Sigma)\}$.

For a subshift of finite type of radius less than r, there is a correspondence between configurations of the subshift and infinite paths in the de Bruijn graph of any order $n \geq r$. In other words, a subshift of finite type is entirely described by its de Bruijn graph of rank large enough.

Definition 0.1.11 (σ-transitive subshift)**.**
A subshift $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ is **σ-transitive** if there is a configuration $x \in \Sigma$ such that $(\sigma^n(x))_{n\in\mathbb{Z}}$ is dense in Σ.

Definition 0.1.12 (Minimal subshift)**.**
A subshift Σ is **minimal** if it contains no other subshift than Σ and $\emptyset$.

Cellular automata

Definition 0.1.13 (Factor and cellular automaton)**.**
A (one-dimensional) **factor** is a continuous function $\pi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ that commutes with the shift function: $\pi \circ \sigma = \sigma \circ \pi$.

A (one-dimensional) **cellular automaton** (CA for short) is a factor $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$.

Theorem 0.1.2 (Curtis–Hedlund–Lyndon theorem [Hed69])**.**
Equivalently, a cellular automaton F can be defined by the choice of a finite **neighbourhood** $\mathcal{N} \subset \mathbb{Z}$ and a **local rule** $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$, where we define:

$$\forall x \in \mathcal{A}^{\mathbb{Z}}, F(x) = (f(x_{i+\mathcal{N}}))_{i\in\mathbb{Z}},$$

and the same is true for factors. See Figure 0.2 for an example.

Proof. Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a continuous function that commutes with σ. Since $\mathcal{A}^{\mathbb{Z}}$ is compact, F is uniformly continuous, and therefore for δ large enough we have for any configurations $x, y \in \mathcal{A}^{\mathbb{Z}} : d(x, y) < 2^{-\delta} \Rightarrow F(x)_0 = F(y)_0$. This means that, for any $u \in \mathcal{A}^{2\delta+1}$, all configurations x such that $x_{[-\delta,\delta]} = u$ have the same letter in column 0. Denote this letter $f(u) \in \mathcal{A}$.

Since F commutes with σ, the same is true for any position: $F(x)_k = F(\sigma^{-k}(x))_0 =$

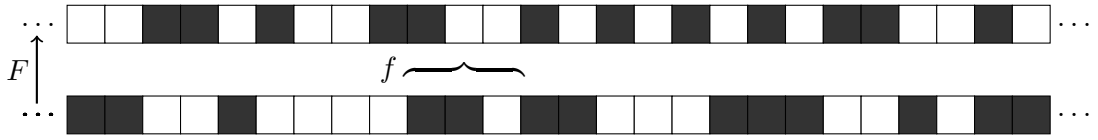


Figure 0.2: One step of a cellular automaton with alphabet $\{\square, \blacksquare\}$ and local rule f acting on neighbourhood $\{-1, 0, 1\}$. This CA is the elementary cellular automaton #26 (defined below).

$f(x_{[k-\delta,k+\delta]})$. Therefore F is defined by the local rule f acting on the neighbourhood $[-\delta, \delta]$.

Conversely, if F is defined by a local rule f acting on a neighbourhood $\mathcal{N}$, then by definition F it is σ-invariant; furthermore, if x and y are two configurations such that $d(x, y) < 2^{-\delta}$, then $x_{[-\delta,\delta]} = y_{[-\delta,\delta]}$. Taking $r > 0$ such that $\mathcal{N} \subseteq [-r, r]$, this means that $d(F(x), F(y)) \leq 2^{-\delta+r}$. In other words, F is 2^r-Lipschitz, hence continuous. □

This theorem establishes a correspondence between the topological vision of a σ-invariant continuous function and the algorithmic and combinatoric vision of a function defined "by blocks".

Factors and cellular automata can be naturally extended to higher dimensional lattices $\mathbb{Z}^d$ or even any monoid, and the previous equivalence still holds. In the context of this thesis, we only consider one-dimensional cellular automata, which is why we limited the definitions to the one-dimensional case for clarity.

Definition 0.1.14 (Space-time diagram)**.**
We represent the time evolution of a one-dimensional cellular automaton starting from an initial configuration $x \in \mathcal{A}^{\mathbb{Z}}$ by a two-dimensional **space-time diagram** $(F^t(x)_i)_{t\in\mathbb{N},i\in\mathbb{Z}}$. In our diagrams, space is horizontal, time goes from bottom to top. We also replace $0, 1, 2$ by colours, using the convention $\square = 0, \blacksquare = 1, \blacksquare = 2 \ldots$ Of course, only a finite window $0 \leq t \leq T$ and $-N \leq i \leq N$ for some $T, N \in \mathbb{Z}$ is represented. See Figure 0.4 for an example.

Definition 0.1.15 (Elementary cellular automata)**.**
A cellular automaton is **elementary** if it is defined on alphabet $\mathcal{A} = \{0, 1\}$ by a local rule acting on the neighbourhood $\mathcal{N} = \{-1, 0, 1\}$.

An elementary cellular automaton (ECA) is entirely defined by choosing an image in $\{0, 1\}$ for the local rule for each value in $\{0, 1\}^3$, for a total of 255 possibilities. Writing elements of $\{0, 1\}^3$ in decreasing lexicographic order, these images form a 8-bit binary number n (see Figure 0.3). We call the corresponding CA **Elementary cellular automaton** $\#n$, or **Rule n**, and we denote it F_n.

Examples (Some elementary cellular automata)**.**

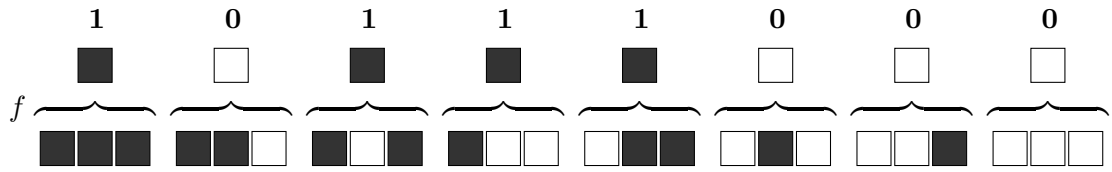


Figure 0.3: Local rule for elementary cellular automaton $\#184 = \overline{10111000}^2$

Rule 184 (Traffic automaton) F_{184} is called **traffic automaton** as it can be seen as the evolution of a (discrete) traffic jam, where the black cells (1) are cars and the white cells (0) are free space. The cars progress by one cell to the right as long as there is free space to do so.

Figure 0.4: Space-time diagram of the traffic automaton from an initial configuration drawn uniformly at random.

Rule 110 The cellular automaton F_{110} has been shown to perform universal computation [Coo04], in the following sense: starting from a finite word encoding a Turing machine number and an input, surrounded by an F_{110}-invariant background ${}^{\infty}00010011011111^{\infty}$ (corresponding to blank symbols on the tape), the time evolution of the automaton simulates the behaviour of the Turing machine.

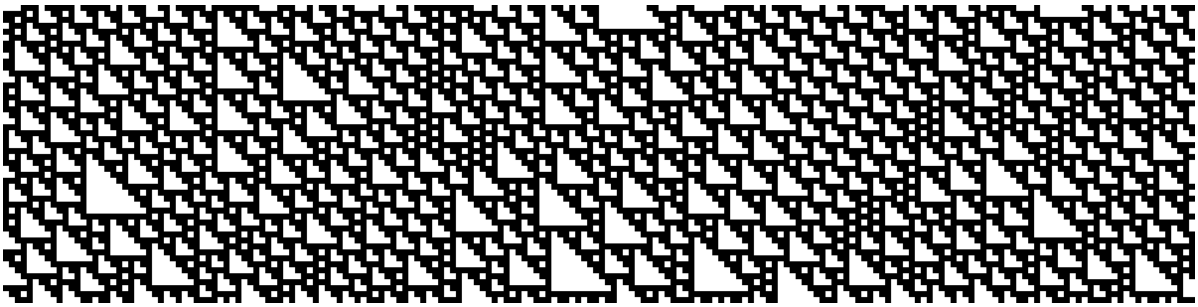


Figure 0.5: Space time diagram of the Rule 110 automaton from an initial configuration drawn uniformly at random.

This example underlines that the cellular automata model remains Turing-powerful, even with two symbols, "very local" updates and dimension one. Despite the simplicity on the definition, elementary cellular automata exhibit already highly complex behaviours.

0.1.2 Probability measures on symbolic spaces

Let $\mathfrak{B}$ be the Borel sigma-algebra of $\mathcal{A}^{\mathbb{Z}}$. Denote by $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ the set of probability measures on $\mathcal{A}^{\mathbb{Z}}$ defined on the sigma-algebra $\mathfrak{B}$. Since the cylinders $\{[u]_n \ : \ u \in \mathcal{A}^*, n \in \mathbb{Z}\}$ form a basis of the product topology on $\mathcal{A}^{\mathbb{Z}}$, a measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is entirely characterised by the values $\mu([u]_n)$.

Definition 0.1.16.
Let $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$. We say that the property P is true **μ-almost everywhere** (μ-a.e.), or **for μ-almost all configurations**, if $\mu\left(\{x \in \mathcal{A}^{\mathbb{Z}} \ : \ P(x)\}\right) = 1$. In that case, we write:

$$\forall_\mu x \in \mathcal{A}^{\mathbb{Z}}, P(x).$$

Generally we consider invariant measures under the action of σ. Since the action of a cellular automaton commutes with σ, this implies that the properties of the resulting space-time diagrams are independent from its position in space (i.e. the coordinate corresponding to the central column).

Definition 0.1.17 (Image measure)**.**
Let $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ be a measurable function and $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$. The **image measure** of μ by Φ is defined by $\Phi_*\mu(B) = \mu(\Phi^{-1}(B))$ for all $B \in \mathfrak{B}$.

If $\Phi_*\mu = \mu$, then μ is **Φ-invariant**. We denote $\mathcal{M}_\Phi(\mathcal{A}^{\mathbb{Z}})$ the set of Φ-invariant probability measures.

Throughout the thesis, the initial configuration is drawn according to a σ-invariant probability measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. In that case, the value of $\mu([u]_n)$ is independent from the choice of $n \in \mathbb{Z}$, which is why we often consider $\mu([u]) = \mu([u]_0)$.

Definition 0.1.18 (Support of a measure)**.**
The **support** of a measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, denoted $\mathrm{supp}(\mu)$, is the closure of the set of configurations $x \in \mathcal{A}^{\mathbb{Z}}$ such that any open neighbourhood of x have positive measure.

In particular, $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ has **full support** if $\mathrm{supp}(\mu) = \mathcal{A}^{\mathbb{Z}}$.

Examples.

Dirac measures The **Dirac measure** supported by $x \in \mathcal{A}^{\mathbb{Z}}$ is defined as $\delta_x(B) = \mathbf{1}_{x \in B}$ for $B \in \mathfrak{B}$. Generally δ_x is not σ-invariant.

Measures supported by a periodic orbit For a word $w \in \mathcal{A}^*$, we define the **σ-invariant measure supported by** ${}^\infty w^\infty$ by taking the mean of the Dirac measures $\delta_{\sigma^i({}^\infty w^\infty)}$ along its orbit:

$$\widehat{\delta_w} = \frac{1}{|w|} \sum_{i \in [0, |w|-1]} \delta_{\sigma^i({}^\infty w^\infty)}.$$

We call $\{\widehat{\delta_w} \ : \ w \in \mathcal{A}^*\}$ the **measures supported by a periodic orbit**.

Bernoulli measure Let $v = (v_a)_{a\in\mathcal{A}}$ be a vector of real numbers such that $0 \le v_a \le 1$ for all $a \in \mathcal{A}$ and $\sum_{a\in\mathcal{A}} v_a = 1$. The associated **Bernoulli measure** Ber_v is defined by

$$\mathrm{Ber}_v([u_0 \dots u_n]) = v_{u_0} \cdots v_{u_n} \qquad \text{for all } u_0 \dots u_n \in \mathcal{A}^*.$$

In other words, each letter is drawn in an i.i.d. manner and the letter a has a probability v_a to appear.

Uniform measure In particular, if we take $v_a = \frac{1}{|\mathcal{A}|}$ for all $a \in \mathcal{A}$, we obtain the **uniform (Bernoulli) measure** λ.

Two-step Markov measure Let $(p_{i,j})_{i,j\in\mathcal{A}^2}$ be a matrix satisfying $\sum_j p_{ij} = 1$ for all i, and $(\mu_i)_{i\in\mathcal{A}}$ an eigenvector associated with the eigenvalue 1 (the choice being unique if the matrix is irreducible). The associated **two-step Markov measure** is defined as $\mu([u]) = \mu_{u_0} p_{u_0u_1} \cdots p_{u_{|u|-2}u_{|u|-1}}$. This construction can be generalised to an n-step Markov measure.

Definition 0.1.19 (Weak* topology)**.**
We endow $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ with the **weak* topology**: for a sequence $(\mu_n)_{n\in\mathbb{N}} \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})^\mathbb{N}$ and a measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, we have $\mu_n \underset{n\to\infty}{\longrightarrow} \mu$ if, and only if:

$$\forall u \in \mathcal{A}^*, \mu_n([u]) \underset{n\to\infty}{\longrightarrow} \mu([u]).$$

If $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ and $\mathcal{M}_\sigma(\mathcal{B}^\mathbb{Z})$ are endowed with this topology, any factor $\pi : \mathcal{A}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z}$ induces a continuous function $\pi_* : \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}_\sigma(\mathcal{B}^\mathbb{Z})$.

Definition 0.1.20 (Metric $d_\mathcal{M}$ on $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$)**.**
In the weak* topology, the set $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is compact and metrisable. A metric is defined by

$$d_\mathcal{M}(\mu,\nu) = \sum_{n\in\mathbb{N}^*} \frac{1}{2^n} \max_{u\in\mathcal{A}^n} |\mu([u]) - \nu([u])|.$$

Definition 0.1.21 (Distance to a closed set)**.**
For a closed set $\mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, denote $d_\mathcal{K}(\nu) = \min_{\mu\in\mathcal{K}} d_\mathcal{M}(\mu,\nu)$.

Proposition 0.1.3. The set of measures supported by a periodic orbit is dense in $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$.

For a proof, see for example [Pet83].

Ergodicity and mixing

Definition 0.1.22 (σ-ergodic measures)**.**
A measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is ***σ*-ergodic** if, for every subset $S \subset \mathcal{A}^\mathbb{Z}$ such that $\sigma(S) = S$ μ-almost everywhere, we have $\mu(S) = 0$ or 1.
The set of σ-ergodic measures is denoted $\mathcal{M}_{\sigma-\mathrm{erg}}(\mathcal{A}^\mathbb{Z})$.

In particular, the image of a σ-ergodic measure under the action of a factor is σ-ergodic.

Theorem 0.1.4 (Birkhoff's ergodic theorem)**.**
Let $\mu \in \mathcal{M}_{\sigma-\text{erg}}(\mathcal{A}^{\mathbb{Z}})$ and let $f : \mathcal{A}^{\mathbb{Z}} \to \mathbb{R}$ be a measurable function.

$$\forall_\mu x \in \mathcal{A}^{\mathbb{Z}}, \frac{1}{n}\sum_{i=0}^{n} f(\sigma^i(x)) \underset{n\to\infty}{\longrightarrow} \int_{\mathcal{A}^{\mathbb{Z}}} f \mathrm{d}\mu.$$

See for example [Wal82]. In most cases, we will be using one of the two following corollaries:

Corollary 0.1.5.
Let $\mu \in \mathcal{M}_{\sigma-\text{erg}}(\mathcal{A}^{\mathbb{Z}})$ and $u \in \mathcal{A}^*$. Then:

$$\forall_\mu x \in \mathcal{A}^{\mathbb{Z}},\ \mathrm{Freq}(u, x) = \mu([u]).$$

Proof. Apply Birkhoff's theorem to $f = 1_{[u]}$. □

In this case, the frequency is actually a simple limit (instead of a $\limsup$).

Corollary 0.1.6.
Let $\mu \in \mathcal{M}_{\sigma-\text{erg}}(\mathcal{A}^{\mathbb{Z}})$ and $A, B \subset \mathcal{A}^{\mathbb{Z}}$ two measurable sets. Then:

$$\frac{1}{n}\sum_{k=0}^{n} \mu(A \cap \sigma^k(B)) \underset{n\to\infty}{\longrightarrow} \mu(A)\cdot\mu(B).$$

Proof. Apply Birkhoff's ergodic theorem on $1_{A\cap\sigma^k(B)}$. For μ-almost all $x \in \mathcal{A}^{\mathbb{Z}}$:

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\mu(A\cap\sigma^k(B)) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\lim_{m\to\infty}\frac{1}{m}\sum_{l=0}^{m} 1_A(\sigma^l(x))\cdot 1_B(\sigma^{k+l}(x)) \\
&\underset{(i)}{=} \lim_{m\to\infty}\frac{1}{m}\sum_{l=0}^{m} 1_A(\sigma^l(x)) \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} 1_B(\sigma^{k+l}(x)) \\
&\underset{(ii)}{=} \lim_{m\to\infty}\frac{1}{m}\sum_{l=0}^{m} 1_A(\sigma^l(x))\cdot\mu(B) \\
&= \mu(A)\cdot\mu(B).
\end{aligned}$$

(i) by dominated convergence, (ii) by using Birkhoff's theorem on $\sigma^{-l}(B)$ and σ-invariance of μ. □

This last property can be seen as a kind of mixing property, guaranteeing some sort of independence between two positions far enough apart in a single configuration. This hypothesis can be strengthened in various ways. We introduce three different mixing conditions needed in this thesis, though there are numerous other; see [Bra05] for a survey of different possible mixing conditions.

The first natural strengthening is to require simple convergence instead of Cesàro mean convergence.

Definition 0.1.23 (σ-mixing)**.**
A measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is **σ-mixing** if for any measurable sets $A, B \subset \mathcal{A}^\mathbb{Z}$,

$$\mu(A \cap \sigma^n(B)) \underset{n\to\infty}{\longrightarrow} \mu(A) \cdot \mu(B).$$

$\mathcal{M}_{\sigma-\mathrm{mix}}(\mathcal{A}^\mathbb{Z})$ is the set of σ-mixing measures on $\mathcal{A}^\mathbb{Z}$.

We can strengthen this condition further by requiring the convergence to be uniform on all events depending on the values of cells at least n coordinates apart. Let $\mathfrak{B}_{[i,j]}$ be the σ-algebra generated by the cylinders $\{[u]_i \mid u \in \mathcal{A}^{j-i}\}$. This definition extends naturally to the case where $i = -\infty$ and/or $j = +\infty$.

Definition 0.1.24 (α-mixing)**.**
The **α-mixing coefficients** of a measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ are

$$\alpha_\mu(n) = \sup\{|\mu(A \cap B) - \mu(A)\mu(B)| : A \in \mathfrak{B}_{]-\infty,0]}, B \in \mathfrak{B}_{[n,+\infty[}\}.$$

A measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is **α-mixing** if $\alpha_\mu(n) \underset{n\to\infty}{\longrightarrow} 0$.

Definition 0.1.25 (ψ-mixing)**.**
The **ψ-mixing coefficients** of a measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ are defined as:

$$\psi_\mu(n) = \sup\left\{\left|\frac{\mu(A \cap B)}{\mu(A)\mu(B)} - 1\right| : A \in \mathfrak{B}_{]-\infty,0]},\ B \in \mathfrak{B}_{[n,\infty[},\ \mu(A) \cdot \mu(B) > 0\right\}.$$

A measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is **ψ-mixing** if $\psi_\mu(n) \underset{n\to\infty}{\longrightarrow} 0$.

For α and ψ-mixing, one can make more precise statements; for example, a measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is **exponentially α-mixing** if there is a constant $C > 1$ such that $\alpha_\mu(n) = o(C^{-n})$, and so on.

Proposition 0.1.7. Those properties form the following hierarchy:

$$\psi\text{-mixing} \Rightarrow \alpha\text{-mixing} \Rightarrow \sigma\text{-mixing} \Rightarrow \sigma\text{-ergodic}.$$

More precisely, $\alpha_\mu(n) \leq \frac{1}{4}\psi_\mu(n)$. See [Bra05] for a proof.

0.1.3 Asymptotic behaviour and self-organisation

In this section, we aim at finding an object that describes asymptotic behaviour of cellular automata in a way that corresponds to our visual intuition of self-organisation. Let us consider a very simple example.

We introduce the mod 2 product automaton, which corresponds to the elementary cellular automaton #128. On alphabet $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$, the local rule acting on the neighbourhood $\mathcal{N} = \{-1, 0, 1\}$ is defined as $f(x_{-1}, x_0, x_1) = x_{-1} \cdot x_0 \cdot x_1$ (product mod 2). As we can see in Figure 0.6, when the initial configuration is drawn according to a nondegenerate Bernoulli measure, the visual intuition is that this cellular automaton exhibits a "trivial" self-organising behaviour where white cells invade the whole space; formally, for any configuration that does not contain an infinite cluster of black cells, any finite window becomes eventually entirely white.

First, we can describe asymptotic behaviour in a purely topological manner by considering the set of configurations that appear arbitrarily late in the space-time diagram.

Figure 0.6: The product automaton F_{128}. The initial configuration is drawn according to $\mathrm{Ber}_{(\frac{1}{10},\frac{9}{10})}$.

Definition 0.1.26 (ω-limit set)**.**

The **ω-limit set** of a cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ (sometimes simply called **limit set**) is defined as:

$$\Omega_F = \bigcup_{T\in\mathbb{N}} \bigcap_{t\geq T} F^t(\mathcal{A}^{\mathbb{Z}}).$$

In particular, Ω_F is a subshift and thus it would be equivalent to define an ω-limit language.

Let us determine $\Omega_{F_{128}}$. Since $F_{128}(^{\infty}\blacksquare^{\infty}) = {}^{\infty}\blacksquare^{\infty}$, this configuration is included in the ω-limit set; more generally, one can check that:

$$\Omega_{F_{128}} = \Big\{ \cdots \underbrace{\square\square\square\square\square\square\square}_{m}\underbrace{\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare}_{n}\square\square\square\square\square\cdots \;:\; m \leq n \in \mathbb{Z} \cup \{-\infty, +\infty\} \Big\}$$

Indeed, denoting $x_{m,n}$ the configuration that is white everywhere except for $[m, n-1]$, we have $x_{m,n} = F^t_{128}(x_{m-t,n+t})$. Furthermore, any configuration with a finite white region of length l cannot appear after l steps or more.

From this example we can see that the ω-limit measure set does not accurately represent the typical asymptotic behaviour that is observed on simulations, if only because the monochromatic configurations $^{\infty}\square^{\infty}$ and $^{\infty}\blacksquare^{\infty}$ are both members of the ω-limit set even though they play very different roles in the asymptotic behaviour of the automata. Making a distinction according to the number of preimages by F^t_{128} for all t does not solve this problem, since all $x_{m,n}$ with $m \neq -\infty, n \neq +\infty$ have uncountable sets of preimages.

Therefore, to describe the empirically observed behaviour, we have to take into account the fact that the initial configuration is drawn according to some probability measure. This is the reason why Hurley introduced the notion of μ-attractor in [Hur90]:

Definition 0.1.27 (μ-attractor)**.**

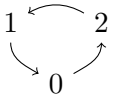
Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ an initial probability measure. $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ is a **μ-attractor** if:

$$\mu\left(\left\{x \in \mathcal{A}^{\mathbb{Z}} \;:\; d(F^t(x), \Sigma) \underset{t\to\infty}{\longrightarrow} 0\right\}\right) > 0.$$

Of course $\mathcal{A}^{\mathbb{Z}}$ is always a μ-attractor and does not describe well the asymptotic behaviour, but we could consider some notion of minimal μ-attractor. In the previous example, ${}^{\infty}\square^{\infty}$ is the unique minimal μ-attractor for any nonatomic measure, which seems satisfying. However, let us consider another example.

Example (3-state cyclic automaton)**.**
The 3-state cyclic automaton C_3, sometimes informally referred to as the "rock-paper-scissors" automaton, is defined on the alphabet $\mathbb{Z}/3\mathbb{Z}$ by the local rule f acting on the neighbourhood $\mathcal{N} = \{-1, 0, 1\}$ defined as:

$$f(x_{i-1}, x_i, x_{i+1}) = \begin{cases} x_i + 1 & \text{if } x_{i-1} = x_i + 1 \text{ or } x_{i+1} = x_i + 1, \\ x_i & \text{otherwise.} \end{cases}$$



Starting from any Bernoulli measure μ, the visual intuition is that monochromatic regions

Figure 0.7: The 3-state cyclic "rock-paper-scissors" automaton C_3 iterated on a configuration drawn uniformly at random.

grow larger and larger, eventually encompassing the whole space, while the probability of seeing a border tends to 0. However, for μ-almost every configuration, borders appear in the central column infinitely often, which means that $\{{}^{\infty}0^{\infty}, {}^{\infty}1^{\infty}, {}^{\infty}2^{\infty}\}$ is not a μ-attractor.

This is the reason why Kůrka and Maass introduced instead the notion of μ-limit set [KM00]. The intuition is that asymptotic behaviour corresponds to finite patterns whose probability to appear does not tend to 0 as time tends to infinity.

By the definition of an image measure, any continuous action $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ commuting with σ can be extended to a continuous action $\Phi_* : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \to \mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$. We consider in particular the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ of iterated images of μ by F_*.

Definition 0.1.28 (μ-limit set)**.**
Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. The **$\mu$-limit set** of F is the subshift defined by:

$$u \in \mathcal{L}(\Lambda_\mu(F)) \iff F_*^t\mu([u]) \underset{t\to\infty}{\not\to} 0.$$

In other words, it is the subshift whose set of forbidden patterns is the set of words whose probability to appear tends to 0 as time tends to infinity.

Since $\Lambda_\mu(F)$ is a subshift, it is equivalent to define it by its language. Note that any μ-limit set is included in the corresponding ω-limit set.

The notion of μ-limit set is a natural conversion of the notion of ω-limit set in a measure-theoretical framework. Another way to describe typical asymptotic behaviour would be to consider directly the limit(s) of the sequence $F_*^t\mu$ with regards to the topology defined earlier.

Definition 0.1.29 (μ-limit measures set)**.**
Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. The ***μ*-limit measures set** of F, denoted $\mathcal{V}(F,\mu)$, is the set of **limit points** of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$.

The μ-limit measures set contains more information than the μ-limit set, since it also describes the asymptotic probability of appearance for each pattern.

Proposition 0.1.8 (Link between the μ-limit sets)**.**

$$\Lambda_\mu(F) = \overline{\bigcup_{\nu\in\mathcal{V}(F,\mu)} \operatorname{supp}(\nu)}.$$

Proof. Let $x \in \Lambda_\mu(F)$. For any $n \in \mathbb{N}$, we have by definition $F_*^t\mu([x_{[-n,n]}]) \underset{t\to\infty}{\nrightarrow} 0$. Therefore there is an accumulation point ν_n of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ satisfying $\nu_n([x_{[-n,n]}]) > 0$. This means that $[x_{[-n,n]}] \cap \operatorname{supp}(\nu_n) \neq \emptyset$ for every n, with $\nu_n \in \mathcal{V}(F,\mu)$ by definition. By closure, we have $x \in \overline{\bigcup_{\nu\in\mathcal{V}(F,\mu)} \operatorname{supp}(\nu)}$. The converse is easy. □

Let us return to the previous examples.

Product automaton Take an initial measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ such that $\mu(^\infty 1^\infty) = 0$. For μ-almost every initial configuration, the white cells eventually invade the whole space, and the probability to see a black cell tends to 0. Thus:

$$\Lambda_\mu(F_{128}) = \{^\infty 0^\infty\} \qquad \text{and} \qquad \mathcal{V}(F,\mu) = \widehat{\delta_0}.$$

3-state cyclic automaton Following the intuition given above, we have

$$\Lambda_\mu(C_3) = \{^\infty 0^\infty, {}^\infty 1^\infty, {}^\infty 2^\infty\}.$$

This is a consequence of Theorem 2.1.2, although it is not difficult to prove it by hand.

In the last case, this is not enough information to determine $\mathcal{V}(F,\mu)$: since any limit measure ν is supported by $\Lambda_\mu(F)$, we can only conclude that it can be written as a convex combination of $\widehat{\delta_0}, \widehat{\delta_1}$ and $\widehat{\delta_2}$. In the case where μ is the uniform Bernoulli measure λ, a symmetry argument shows that $\mathcal{V}(F,\lambda) = \left\{\frac{1}{3}\widehat{\delta_0} + \frac{1}{3}\widehat{\delta_1} + \frac{1}{3}\widehat{\delta_2}\right\}$. Determining $\mathcal{V}(F,\lambda)$ for any Bernoulli measure is actually much more difficult and is done in Section 2.3.

From those examples, we can see that the μ-limit measures set is the object that corresponds to our visual intuition of self-organisation in cellular automata. This is why we use it throughout the thesis, and we consider the μ-limit set instead when not enough information is available to fully describe the limit measures.

When $(F_*^t\mu)_{t\in\mathbb{N}}$ does not converge, it can be useful to consider convergence in Cesàro mean instead:

Definition 0.1.30 (Cesàro mean)**.**
The **Cesàro mean** of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ at time $t \in \mathbb{N}$ is defined by:

$$\varphi_t^F(\mu) = \frac{1}{t+1}\sum_{i=0}^{t} F_*^i\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}).$$

Definition 0.1.31 (μ-limit measures set in Cesàro mean)**.**
Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. The ***μ*-limit measures set in Cesàro mean** of F, denoted $\mathcal{V}'(F,\mu)$, is the set of **limit points** of the sequence $(\varphi_t^F(\mu))_{t\in\mathbb{N}}$.

Of course one can also define a μ-limit set in Cesàro mean (set of configurations), but this notion is not used in this thesis and has not been considered historically to our knowledge.

Proposition 0.1.9 (Topological remarks on μ-limit measures sets)**.**
For any initial measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$,

(i) $\mathcal{V}(F,\mu)$ is a nonempty, compact set;

(ii) $\mathcal{V}'(F,\mu)$ is a nonempty, compact, connected set;

(iii) $\mathcal{V}'(F,\mu)$ is included in the convex hull of $\mathcal{V}(F,\mu)$.

Proof. By compacity of $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, any sequence admits a nonempty set of accumulation points, which is also closed hence compact. Furthermore, the set of limit points of a Cesàro mean sequence is always connected, and we prove it in this particular case. For all $u \in \mathcal{A}^*$,

$$\begin{aligned}|\varphi_{t+1}^F(\mu)([u]) - \varphi_t^F(\mu)([u])| &\leq \left|\sum_{i=0}^{t}\left(\frac{1}{t+1}-\frac{1}{t}\right)F_*^i\mu([u]) + \frac{1}{t+1}F_*^{t+1}\mu([u])\right| \\ &\leq \frac{1}{t} + \frac{1}{t+1} \underset{t\to\infty}{\longrightarrow} 0\end{aligned}$$

and so $d_{\mathcal{M}}(\varphi_{t+1}^F(\mu), \varphi_t^F(\mu)) \to 0$. Now let $\mathcal{K} \subset \mathcal{V}(F,\mu)$ be a clopen set (not $\emptyset$ or $\mathcal{V}(F,\mu)$), so that $\mathcal{V}(F,\mu)\backslash\mathcal{K}$ is also clopen. If both of them are nonempty, we have $\min_{\mu\in\mathcal{K}} \min_{\nu\in\mathcal{V}(F,\mu)\backslash\mathcal{K}} d_{\mathcal{M}}(\mu,\nu) = r > 0$, and by taking T large enough we have:

- $\forall t \geq T,\ d_{\mathcal{M}}(\varphi_{t+1}^F(\mu), \varphi_t^F(\mu)) < \frac{r}{3}$;
- $\forall t \geq T,\ d_{\mathcal{V}(F,\mu)}(\varphi_t^F(\mu)) = d_{\mathcal{M}}(\varphi_t^F(\mu), \mathcal{V}(F,\mu)) < \frac{r}{3}$.

Now, if we assume that $d_{\mathcal{K}}(\varphi_t^F(\mu)) < \frac{r}{3}$, then it is impossible that $d_{\mathcal{V}(F,\mu)\backslash\mathcal{K}}(\varphi_{t+1}^F(\mu)) < \frac{r}{3}$ by the first point; therefore $d_{\mathcal{K}}(\varphi_{t+1}^F(\mu)) < \frac{r}{3}$, and by induction the sequence can never return close to $\mathcal{V}(F,\mu)\backslash\mathcal{K}$. If $\mathcal{K} \neq \mathcal{V}(F,\mu)$, this is in contradiction with the definition of a set of limit points. The other case is symmetrical.

Let us prove the third point. For any $\nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, let $p\nu$ an arbitrary point of $\mathcal{V}(F,\mu)$ realising the minimum distance $d_{\mathcal{M}}(\nu, p\nu) = d_{\mathcal{V}(F,\mu)}(\nu)$. Take any $\varepsilon > 0$ and let $T \in \mathbb{N}$ be

large enough that $d_{\mathcal{V}(F,\mu)}(F_*^t\mu) \leq \varepsilon$ for all $t \geq T$. When $t \geq \frac{T}{\varepsilon}$, we have:

$$\begin{aligned} d_{\mathcal{M}}\left(\frac{1}{t+1}\sum_{i=0}^{t} F_*^i\mu,\ \frac{1}{t+1}\sum_{i=0}^{t} pF_*^i\mu\right) &= \frac{1}{t+1}\sum_{n\in\mathbb{N}}\max_{u\in\mathcal{A}^*}\left|\sum_{i=0}^{t}(F_*^i\mu([u]) - pF_*^i\mu([u]))\right| \\ &\leq \frac{1}{t+1}\sum_{i=0}^{t} d_{\mathcal{M}}(F_*^i\mu, pF_*^i\mu) \\ &\leq \frac{1}{t+1}\left(\sum_{i=0}^{T-1} d_{\mathcal{M}}(F_*^i\mu, pF_*^i\mu) + \sum_{i=T}^{t} d_{\mathcal{M}}(F_*^i\mu, pF_*^i\mu)\right) \\ &\leq \frac{T}{t} + \varepsilon \leq 2\varepsilon \end{aligned}$$

and since by definition $\frac{1}{t+1}\sum_{i=0}^{t} pF_*^i\mu$ is included in the convex hull of $\mathcal{V}(F,\mu)$, the distance between $\varphi_t^F(\mu)$ and the convex hull of $\mathcal{V}(F,\mu)$ tends to 0 as $t \to \infty$. □

We now add a last example where $\mathcal{V}(F,\mu)$ and $\mathcal{V}'(F,\mu)$ are very different.

Figure 0.8: The elementary CA F_{102}, starting from an initial configuration drawn randomly according to a Bernoulli measure of parameters $(\frac{1}{3}, \frac{2}{3})$.

Example (Addition mod 2 automaton)**.**

The elementary CA F_{102} performs addition mod 2 on the neighbourhood $\{0,1\}$, that is, it is defined by the local rule $f : (\mathbb{Z}/2\mathbb{Z})^2 \to \mathbb{Z}/2\mathbb{Z}$ defined by $f(x_0, x_1) = x_0 + x_1$.

Take an initial measure $\mu = \mathrm{Ber}_{(\frac{1}{3},\frac{2}{3})}$. On the one hand, $\mathcal{V}'(F_{102},\mu) = \{\lambda\}$, where λ is the uniform measure [Lin84]. On the other hand, $\mathcal{V}(F_{102},\mu)$ contains λ as well as countably many other measures such as $\mathrm{Ber}_{(\frac{5}{9},\frac{4}{9})}$, $\mathrm{Ber}_{(\frac{41}{81},\frac{40}{81})}$, etc. See Section 3.1.1, and in particular Proposition 3.1.2, for more information.

Characterisation of typical asymptotic behaviours

1

Section 1.0
Introduction

In the previous chapter, we concluded that the intuitive notion of typical asymptotic behaviour for a cellular automaton F is well described by taking an initial probability measure, for example the uniform measure, and considering the set of limit points of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ (its **μ-limit measures set**). Seen from another angle, this set is also a natural way to define the output if one considers that the cellular automaton is simulating a source of randomness. In this chapter, we aim to characterise which measures or sets of measures can be obtained as limit points in this way.

This study is motivated by the two following questions:

- explaining the variety of typical asymptotic behaviours in cellular automata, and in particular the way complex behaviours emerge from disordered initial states;
- understanding the computational power of this model in the specific setting of simulating probability measures.

Obviously, any measure can be reached by iterating the identity cellular automaton on itself. Therefore, a more interesting approach is to start from some simple measure, such as the uniform Bernoulli measure. This corresponds to "physically" relevant distributions of configurations for F in the sense that, if we iterate F on a uniformly distributed set of configurations and we observe the result after a long time, the set will be distributed according to one of these measures. This also makes sense from a computational point of view, since we obtain the sources of randomness that can be simulated by a CA having access to a simple source.

Exploring the computational content of dynamical systems is not restricted to cellular automata, and is often linked with such questions as robustness to noise or undecidability of some properties. See for example [Moo90] for piecewise linear maps of the unit interval and the unit square, [KCG94] and [AMP95] for systems with piecewise constant derivatives, [AB01] for larger classes of dynamical systems, or [DKB06] for various symbolic systems (such as cellular automata).

Recently, this approach has led to a series of results where some objects or parameters of the system are fully characterised by computability conditions. For subshifts of finite type, possible entropies [HM10], possible growth-type invariants [Mey11] and possible sub-actions [Hoc09, AS13] were characterised in this way.

Concerning cellular automata, previous works focused on the ω-limit sets [Hur87, Maa95] and μ-limit sets [BPT06, BDS10]. In each case, the authors tried to construct examples or classes of very (computationally) complex sets, and our construction is inspired from these works. For ω-limit sets, a characterisation was recently obtained [BCV14].

Here, we want to characterise all sets that can be reached as $\mathcal{V}(F,\mu)$ (the μ-limit measures set of F starting from μ) or $\mathcal{V}'(F,\mu)$ (the Cesàro mean μ-limit measures set of F starting from μ) by any cellular automata F and any "simple" initial measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.

If the initial measure is arbitrary, the only property we can deduce about the μ-limit measures set is that it is a nonempty compact set. There are different ways of restricting the class of initial measures, and we consider the consequences after a finite number of steps and at the limit. Starting from a Bernoulli measure or a Markov measure, we obtain after a finite number of steps a hidden Markov chain which is well understood [BP11]. More generally, we can define the notion of computable probability measure, which means that there is a probabilistic algorithm (having access to a perfect source of randomness) whose output is distributed according to this measure. For example, a Bernoulli or Markov measure is computable if and only if its parameters are computable real numbers.

If the initial measure μ is computable, it is easy to see that $F_*^t\mu$ is also computable. However, even a single limit measure is not necessarily computable since the speed of convergence is not known. This is why we extend these notions in Section 1.1 by defining higher order computability on probability measures, and we also introduce computability on sets of probability measures so as to handle μ-limit measures set that are not reduced to a singleton. Thus, we are able to exhibit the following necessary computational obstructions:

Theorem 1.0.1 (Computable obstructions - Theorems 1.1.3 and 1.1.5)**.**
Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton.
If $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is a computable measure, then the μ-limit measures set $\mathcal{V}(F, \mu)$ is a Π_2-computable set. In particular, if $F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu$, then ν is a limit-computable measure.

Let us consider this last statement for clarity. It essentially means that a cellular automaton is not more than Turing-powerful for simulating a probability measure. This is not a surprising result since a cellular automaton is performing computation locally.

In Section 1.2, we tackle the main problem of proving a reciprocal. More precisely, given a measure or set of measures satisfying the computational obstructions, we construct a cellular automaton which, starting on any simple initial measure, reaches exactly this set asymptotically. Since this construction is initialised on a random configuration, this requires to self-organise the space, in the same spirit as the probabilistic cellular automaton of [Gác01] performing reliable computation by correcting the random perturbations.

A first difficulty is to reach (sets of) probability measures without access to a source of randomness. Lemma 1.1.6 states that sets satisfying the computability obstruction can be described as the accumulation points of a computable polygonal path of measures supported by periodic orbits. From there, it is sufficient to prove the following theorem:

Theorem 1.0.2 (Theorem 1.2.1)**.**
Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly computable sequence of words of $\mathcal{B}^*$, where $\mathcal{B}$ is a finite alphabet. There is a cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A} \supset \mathcal{B}$, such that for any ψ-mixing measure μ with full support, the μ-limit measures set of F is exactly the set of accumulation points of the polygonal path drawn by $(\widehat{\delta_{w_n}})_{n\in\mathbb{N}}$.

The hypothesis of shift-mixing on the initial measure may be relaxed to an ergodicity condition when the set of accumulation points is reduced to a singleton, i.e. when $(\widehat{\delta_{w_n}})_{n\in\mathbb{N}}$ converges to a limit.

First of all, the cellular automaton divides the initial configuration in segments, and formats each segment using a method similar to the one developed in [DPST11]. Computation takes place in a negligible part of each segment and the result is copied periodically on the rest of the segment. However, the computation may require an arbitrarily large space; to ensure that the computation space is eventually large enough, we merge segments progressively in a controlled manner. The difficulty of the construction is to synchronise all operations to ensure convergence.

Section 1.3 focuses in adapting this construction in the case where auxiliary states are not allowed, i.e., the cellular automaton can only use the same alphabet as the limit measure(s) (that is, $\mathcal{A} = \mathcal{B}$ in the previous theorem). This is only possible, however, when a word $u \in \mathcal{A}^*$ does not appear in any of the w_n, which corresponds to the case where the limit measure does not have full support.

In Section 1.4, we present various theorems resulting from the construction of Section 1.2. For any measure μ fixed in a large class of measures, we obtain:

- characterisation of shift-invariant measures ν such that there exists a cellular automaton F which verifies $F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu$ (Corollary 1.4.2);
- characterisation of connected subsets of shift-invariant measures $\mathcal{K}$ such that there exists a cellular automaton F which verifies $\mathcal{V}(F,\mu) = \mathcal{K}$ (Corollary 1.4.3);
- characterisation of subsets of shift-invariant measures $\mathcal{K}'$ such that there exists a cellular automaton F which verifies $\mathcal{V}'(F,\mu) = \mathcal{K}'$ (Corollary 1.4.6);
- characterisation of connected subsets of shift-invariant measures $\mathcal{K}' \subset \mathcal{K}$ such that there exists a cellular automaton F which verifies $\mathcal{V}(F,\mu) = \mathcal{K}$ and $\mathcal{V}'(F,\mu) = \mathcal{K}'$ (Corollary 1.4.7).
- Rice theorem for shift-invariant measures and connected subsets of shift-invariant measures reached by a cellular automaton (Corollaries 1.4.10, 1.4.11 and 1.4.12).

Furthermore, all these results have another version where auxiliary states are not allowed, using the construction of Section 1.3. This entails additional hypotheses on the support of the target limit measure(s).

In Section 1.5, we consider the case where the set of limit points depends on the initial measure. From a computational point of view, this corresponds to computing nonconstant functions on probability measures. Computational constraints appear on functions $\mu \longmapsto \mathcal{V}(F,\mu)$ that can be realised in this way. Indeed, the computational complexity of the initial measure (used as an oracle) limits the complexity of the set of limit points. By modifying the construction of Section 1.2, we manage to build a set of limit points that depends on the density of a special state; however, we do not obtain a complete characterisation.

This chapter is a more detailed version of [HdMS13], which has been accepted in *Ergodic Theory and Dynamical Systems*.

Section 1.1

Computability and computable analysis

In this section, we introduce some general notions of computability and computable analysis, in order to exhibit the computable obstructions on the measures and sets of measures that can be reached as the limit set of a cellular automaton. We begin by introducing classical computability definitions on functions between countable sets in Section 1.1.1, then we extend these definitions as needed to larger sets in Sections 1.1.2 and 1.1.3. The main results are the computability obstructions shown in Theorems 1.1.3 and 1.1.5. We also prove along the way some technical results that are needed for the next section.

Definition 1.1.1 (Turing machine)**.**
A **Turing machine** $\mathcal{TM} = (Q, \Gamma, \#, q_0, \delta, Q_F)$ is defined by:

- Γ a finite alphabet, with a blank symbol $\# \notin \Gamma$. Initially, a one-sided infinite memory tape is filled with $\#$, except for a finite prefix (the input), and a computing head is located on the first letter of the tape;
- Q the finite set of states of the head; $q_0 \in Q$ is the initial state;
- $\delta : Q \times (\Gamma \cup \#) \to Q \times (\Gamma \cup \#) \times \{\leftarrow, \rightarrow\}$ the transition function. Given the state of the head and the letter it reads on the tape — depending on its position — the head can change state, replace the letter and move by one cell at most.
- $Q_F \subset Q$ the set of final states — when a final state is reached, the computation stops and the output is the value currently written on the tape.

Turing machines are an idealised model of computation with unbounded memory and no side effects. Thus they are a robust tool, though not the only one (λ-calculus, combinatorial circuits...) to define computability of mathematical operations.

Definition 1.1.2 (Computable functions on Γ^*)**.**
A function $f : \Gamma^* \to \Gamma^*$ is **computable** if there exists a Turing machine working on an alphabet including Γ that, on any input $w \in \Gamma^*$, eventually stops and outputs $f(w)$.

1.1.1 Computability of functions mapping countable sets

Encodings

Before extending this definition to other countable sets, we need to introduce the notion of **encoding**. An encoding for a countable set X is the choice of a finite alphabet Γ_X, a subset $V_X \subset \Gamma_X^*$ of **valid encodings** and a surjection $e_X : V_X \twoheadrightarrow X$. Intuitively, a word in Γ^* represents an element of X, but an element can have several valid encodings and not all elements of Γ^* need to be a valid encoding.

Strictly speaking, all these definitions depend on the chosen encoding, even though all reasonable choices lead to the same notion of computability. Since all countable sets considered in this thesis are Γ^* (for some finite alphabet Γ), $\mathbb{N}$, $\mathbb{Z}$ $\mathbb{Q}$ and their products, we fix canonical encodings for the latter three cases that are valid throughout the thesis.

$\mathbb{N}$ or $\mathbb{Z}$: $\Gamma = \{0, 1\}$ and to each binary number we assign the corresponding integer (removing initial zeroes), with the first bit encoding the sign for $\mathbb{Z}$;

$\mathbb{Q}$: $\Gamma = \{0, 1, |\}$ and to $p|q$ ($p \in \mathbb{N}^*$ and $q \in \mathbb{Z}$ being written in binary) we assign the rational $\frac{p}{q}$;

$X \times Y$: If Γ_X, e_X and Γ_Y, e_Y are the encodings fixed for X and Y, respectively, we put $\Gamma = \Gamma_X \cup \Gamma_Y \sqcup \{|\}$ (disjoint union, i.e. assume $|$ is a fresh symbol), and to $x|y$ we assign $(e_X(x), e_Y(y))$.

In all the following, X and Y are two countable sets and we suppose encodings Γ_X, e_X and Γ_Y, e_Y have been fixed. We now introduce the standard definition of a computable function between two countable sets.

Definition 1.1.3 (Computable functions between countable sets)**.**
A function $f : X \to Y$ is **computable** if there exists a Turing machine working on an alphabet containing $\Gamma_X \cup \Gamma_Y$ that, on any input $x \in \Gamma_X^*$, eventually stops and outputs $y \in \Gamma_Y^*$, where $f(e_X(x)) = e_Y(y)$.

Remark. The set of computable functions $X \to Y$ is countable.

Even among noncomputable functions, we can distinguish various levels of computational complexity, taking the form of a hierarchy.

Higher order computability: the arithmetical hierarchy

We now present the arithmetical hierarchy of functions mapping countable sets. This hierarchy was introduced in [ZW01] on real numbers by analogy with Kleene's arithmetical hierarchy on decision problems [Kle43], with $\exists$ and $\forall$ quantifiers being replaced by $\sup$ and $\inf$ operators. This hierarchy applies to functions mapping countable sets when the codomain is ordered, considering reals as a function mapping n to the nth digit of the real. From now on Y is assumed ordered.

Definition 1.1.4 (Computable sequence of functions between countable sets)**.**
A sequence of functions $(f_n : X \to Y)_{n\in\mathbb{N}}$ is **uniformly computable** if the function $(n, x) \mapsto f_n(x)$ is computable. This notion generalises naturally to multi-indices sequences.

Notice that this is a stronger statement than to say that all functions f_n are computable.

Definition 1.1.5 (Arithmetical hierarchy of functions between countable sets)**.**
Let $n \in \mathbb{N}$. A function $f : X \to Y$ is **$\boldsymbol{\Sigma_n}$-computable** (resp. **$\boldsymbol{\Pi_n}$-computable**) if there exists a uniformly computable sequence of functions $(f_{i_1,\dots,i_n} : X \to Y)_{i_1,\dots,i_n\in\mathbb{N}}$ such that:

$$f = \sup_{i_1\in\mathbb{N}} \inf_{i_2\in\mathbb{N}} \sup_{i_3\in\mathbb{N}} \cdots f_{i_1,\dots,i_n} \qquad \left(\text{resp. } f = \inf_{i_1\in\mathbb{N}} \sup_{i_2\in\mathbb{N}} \inf_{i_3\in\mathbb{N}} \cdots f_{i_1,\dots,i_n}\right).$$

f is **$\boldsymbol{\Delta_n}$-computable** if it is both Σ_n-computable and Π_n-computable.

Remark.
Δ_1-computability is equivalent to computability.

A Δ_2-computable function is also called **limit-computable**. $f : X \to Y$ is a limit-computable function if, and only if, there exists a uniformly computable sequence of functions $(f_n : X \to Y)_{n\in\mathbb{N}}$ such that $f = \lim_{n\to\infty} f_n$.

Inclusions between these sets are shown in Figure 1.1.

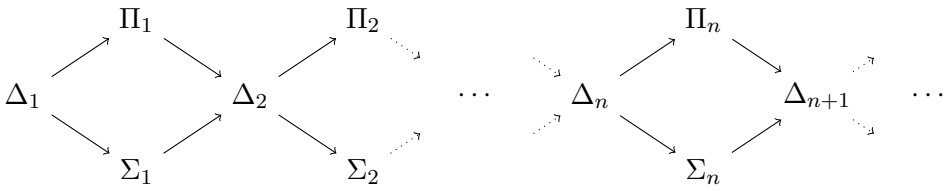


Figure 1.1: Representation of the arithmetical hierarchy. Arrows indicate strict inclusion relations.

Definition 1.1.6 (Computability of subsets of countable sets)**.**
A subset $U \subset X$ is said to be $\Pi_n(\Sigma_n)$**-computable** if its characteristic function $\mathbf{1}_U : X \to \{0, 1\}$ is $\Pi_n(\Sigma_n)$-computable.

1.1.2 Computability of probability measures

Different points of view can be used to define the computability of a probability measure. Similarly to $\mathbb{R}$, $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is a metric space with a countable dense set $\{\widehat{\delta_w} : w \in \mathcal{A}^*\}$. First, it would be natural to define a computable measure in a similar fashion as a computable real ([ZW01]), that is, as the limit of a uniformly computable sequence of elements in this countable dense set ($\mathbb{Q}$ and $\{\widehat{\delta_w} : w \in \mathcal{A}^*\}$, respectively) with a computable rate of convergence.

Probability measures

Definition 1.1.7 (Computability of probability measures)**.**
A measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is **computable** if there exists a computable function $f : \mathbb{N} \to \mathcal{A}^*$ such that $d_{\mathcal{M}}\left(\mu, \widehat{\delta_{f(n)}}\right) \leq 2^{-n}$ for all $n \in \mathbb{N}$.

It is **limit-computable** if there exists a computable function $f : \mathbb{N} \to \mathcal{A}^*$ such that $\lim\limits_{n\to\infty} \widehat{\delta_{f(n)}} = \mu$.

We denote $\mathcal{M}_\sigma^{\text{comp}}(\mathcal{A}^{\mathbb{Z}})$ the set of σ-invariant computable measures and $\mathcal{M}_\sigma^{\text{s-comp}}(\mathcal{A}^{\mathbb{Z}})$ the set of σ-invariant limit-computable measures.

Examples.

- any measure supported by a periodic orbit is computable;

- any Bernoulli measure or Markov measure with Σ_n-computable (resp. Π_n-computable) parameters is Σ_n-computable (Π_n-computable)[1];
- if an effective subshift (defined by a Σ_1 - i.e recursively enumerable - set of forbidden patterns) admits a unique ergodic measure, then this measure is limit-computable. This is the case for any subshift obtained by a primitive substitution or the orbit of a Sturmian word with computable slope.

We provide a proof of this last statement. Notice that since σ-ergodic measures are the extremal points of the set of σ-invariant measures, there is a unique σ-invariant measure supported by Σ.

Proof of the last example. Let Σ be a subshift defined by a recursively enumerable set of forbidden patterns $\mathcal{F}$, which means that there exists a computable function $f : i, w \mapsto \{0, 1\}$ such that $w \in \mathcal{F} \Leftrightarrow \sup_i f(i, w) = 1$.

We consider the following algorithm: on input $j \in \mathbb{N}$, it computes the set $\mathcal{F}_j = \{w \in \mathcal{A}^* \ : \ \exists i \leq j, f(i, w) = 1\}$. It then builds the de Bruijn graph of order $\max_{w\in\mathcal{F}_j} |w|$ corresponding to the subshift of finite type defined by forbidding words of $\mathcal{F}_j$. To any configuration of Σ corresponds an infinite path in this graph, since this configuration cannot contain a subword in $\mathcal{F}_j$. This infinite path contains a cycle, and any such cycle yields a word $w_j \in \mathcal{A}^*$ such that ${}^\infty w_j^\infty$ does not contain any word of $\mathcal{F}_j$.

The algorithm then outputs w_j. Thus we obtain a computable function $j \mapsto w_j$.

Let ν be an accumulation point of $(\widehat{\delta_{w_j}})_{j\in\mathbb{N}}$. If $u \in \mathcal{F}$, then it is contained in all $\mathcal{F}_i$ for i large enough, and this means that $\widehat{\delta_{w_i}}([u]) = 0$. Therefore $\nu([u]) = 0$, and $\mathrm{supp}(\nu) \subset \Sigma$. This shows that ν can only be the unique measure supported on Σ, and it is limit-computable as the single limit of the sequence $(\widehat{\delta_{w_j}})_{j\in\mathbb{N}}$. □

For more details about primitive substitutions and Sturmian words, see for example [FM10].

Functions with countable domain, metric codomain with countable dense set

Before stating the first computability obstruction, we spend the next few pages considering other natural definitions for the computability of a probability measure in $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, and showing that these notions are equivalent.

We extend the definitions of the previous sections to uncountable codomains. To gain in clarity, we write definitions for functions $\mathcal{A}^* \to \mathbb{R}$ as needed in this thesis, even though they can be extended to any countable domain and any metric codomain with a countable dense set (such as $\mathbb{R}$ or $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, in which the measures supported by periodic word are dense).

Definition 1.1.8 (Computable function - codomain with countable dense set)**.**
A function $f : \mathcal{A}^* \to \mathbb{R}$ is **computable** if there exists a uniformly computable sequence of functions $(f_n : \mathcal{A}^* \to \mathbb{Q})_{n\in\mathbb{N}}$ such that:

$$|f(u) - f_n(u)| < 2^{-n} \qquad \text{for all } u \in \mathcal{A}^* \text{ and } n \in \mathbb{N}.$$

[1] As mentioned in the previous section, the computability of a real is defined as the computability of the function that maps n to the nth digit of the real, and similarly for k-tuples of reals.

It is **limit-computable** if there exists a uniformly computable sequence of functions $(f_n : \mathcal{A}^* \to \mathbb{Q})_{n\in\mathbb{N}}$ such that:

$$\lim_{n\to\infty} f_n(u) = f(u) \qquad \text{for all } u \in \mathcal{A}^*.$$

Remark. $f : \mathcal{A}^* \to \mathbb{R}$ is a limit-computable function if, and only if, there is a uniformly computable sequence of functions $(f_n : \mathcal{A}^* \to \mathbb{Q})_{n\in\mathbb{N}}$ such that $f = \lim_{n\to\infty} f_n$.

Proposition 1.1.1 (Equivalence of the definitions)**.**
A measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is computable, resp. limit-computable, if the function $u \mapsto \mu([u])$ is computable, resp. limit-computable.

Proof. Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ such that $u \mapsto \mu([u])$ is computable. Given some $n \in \mathbb{N}$, we can enumerate words in $\mathcal{A}^*$ in a computable manner until we find a word $f(n)$ such that $|\mu([u]) - \widehat{\delta_{f(n)}}([u])| < 2^{-n-2}$ for all $u \in \mathcal{A}^{\leq n+1}$. Such a word exists since the set $\left\{\widehat{\delta_w} : w \in \mathcal{A}^*\right\}$ is dense in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, and it can be found since μ and $\widehat{\delta_{f(n)}}([u])$ are computable. Therefore

$$d_{\mathcal{M}}(\mu, \widehat{\delta_{f(n)}}) = \sum_{i\in\mathbb{N}} \frac{1}{2^i} \max_{u\in\mathcal{A}^i} |\mu([u]) - \widehat{\delta_{f(n)}}([u])| \leq \frac{1}{2^{n+1}} + \sum_{i\geq n+2} \frac{1}{2^i} \leq \frac{1}{2^n},$$

which means that $\mu \in \mathcal{M}_\sigma^{\text{comp}}(\mathcal{A}^{\mathbb{Z}})$. The reciprocal is obvious since $(n, u) \mapsto \widehat{\delta_{f(n)}}([u])$ is computable and we have

$$\forall u \in \mathcal{A}^*, \left|\mu([u]) - \widehat{\delta_{f(n)}}([u])\right| \leq 2^{|u|} d_{\mathcal{M}}\left(\mu, \widehat{\delta_{f(n)}}\right).$$

Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ such that $u \mapsto \mu([u])$ is limit-computable. There exists a uniformly computable sequence of functions $(f_n : \mathcal{A}^* \to \mathbb{Q})_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n(u) = \mu([u])$ for all $u \in \mathcal{A}^*$. For each n, consider the following function :

$$\begin{array}{ccc} \mathcal{A}^{\leq n} & \longrightarrow & \mathbb{Q} \\ w & \longmapsto & \displaystyle\sum_{i\leq n} \frac{1}{2^i} \max_{u\in\mathcal{A}^i} |f_n(u) - \widehat{\delta_w}([u])|. \end{array}$$

Let w_n be the word that realises the minimum of this function. Given an integer n, one can compute w_n by enumerating all elements of $\mathcal{A}^{\leq n}$ and computing the above function which is a composition of computable functions. Therefore $(w_n)_{n\in\mathbb{N}}$ is a uniformly computable sequence of words.

For every n:

$$d_{\mathcal{M}}(\mu, \widehat{\delta_{w_n}}) \leq \sum_{i\leq n} \frac{1}{2^i} \max_{u\in\mathcal{A}^i} |\mu([u]) - f_n(u)| + \sum_{i\leq n} \frac{1}{2^i} \max_{u\in\mathcal{A}^i} |f_n(u) - \widehat{\delta_{w_n}}([u])| + \frac{1}{2^n}.$$

The first term tends to 0 by dominated convergence. Now consider the function $v \mapsto d_{\mathcal{M}}(\mu, \widehat{\delta_v})$ for $v \in \mathcal{A}^{\leq n}$ and let v_n be a word minimising this function. Then:

$$\sum_{i \leq n} \frac{1}{2^i} \max_{u \in \mathcal{A}^i} |f_n(u) - \widehat{\delta_{v_n}}([u])| \leq d(\mu, \widehat{\delta_{v_n}}) + \sum_{i \in n} \frac{1}{2^i} \max_{u \in \mathcal{A}^i} |\mu([u]) - f_n(u)| \underset{n \to \infty}{\longrightarrow} 0.$$

Since w_n realises the minimum of the left-hand term, we see that $\sum_{i \leq n} \frac{1}{2^i} \max_{u \in \mathcal{A}^i} |f_n(u) - \widehat{\delta_{w_n}}([u])| \underset{n \to \infty}{\longrightarrow} 0$ and we conclude.

The reciprocal is similar to the reciprocal of the first point. □

An advantage of this definition is that it can be extended to define an arithmetical hierarchy of probability measures, even though we do not use it in this thesis.

Definition 1.1.9 (Arithmetical hierarchy of functions - codomain with countable dense set)**.**

Let $n \in \mathbb{N}$. A function $f : \mathcal{A}^* \to \mathbb{R}$ is Σ_n**-computable** (resp. Π_n**-computable**) if, and only if, there is a uniformly computable sequence of functions $(f_{i_1,\dots,i_n} : \mathcal{A}^* \to \mathbb{Q})_{i_1,\dots,i_n \in \mathbb{N}}$ such that:

$$f = \sup_{i_1 \in \mathbb{N}} \inf_{i_2 \in \mathbb{N}} \sup_{i_3 \in \mathbb{N}} \cdots f_{i_1,\dots,i_n} \qquad \left(\text{resp. } f = \inf_{i_1 \in \mathbb{N}} \sup_{i_2 \in \mathbb{N}} \inf_{i_3 \in \mathbb{N}} \cdots f_{i_1,\dots,i_n} \right).$$

f is $\boldsymbol{\Delta_n}$**-computable** if it is both Σ_n-computable and Π_n-computable.

Now we consider the notion of simulability of a measure: that a probabilistic algorithm is able to draw a configuration distributed according to this measure. Again this definition, that we gave as an intuition in the introduction, is equivalent to the previous one.

Definition 1.1.10 (Simulability of probability measures)**.**
A measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is **simulable** if, and only if, there is a probabilistic Turing machine (in other words, a Turing machine having access to a source of independent, fair random bits), working on an alphabet $\Gamma \supset \mathcal{A}$ such that, on an empty input:

1. The machine never stops;
2. A letter written on the output tape can never be overwritten;
3. The word on the output tape converges to a limit configuration distributed according to μ.

A measure is **limit-simulable** if only conditions 1 and 3 are satisfied. In other words, the machine can write over a letter, but to ensure convergence this can happen only a finite number of times for each tape cell.

Proposition 1.1.2 (Equivalence between computability and simulability)**.**

(i) A measure is computable if, and only if, it is simulable.

(ii) A measure is limit-computable if, and only if, it is limit-simulable.

See [KY76] for a proof of this result and an overview of simulating probability measures by Turing machines.

We now state the first computability obstruction for single limits of the sequence $(F_*^t \mu)_{t \in \mathbb{N}}$.

Proposition 1.1.3 (First computability obstruction)**.**
Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton. If $\mu \in \mathcal{M}_\sigma^{\text{comp}}(\mathcal{A}^{\mathbb{Z}})$, then $(u \mapsto F_*^t\mu([u]))_{t\in\mathbb{N}}$ is a uniformly computable sequence of functions. In particular, if $F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu$ then $\nu \in \mathcal{M}_\sigma^{\text{s-comp}}(\mathcal{A}^{\mathbb{Z}})$.

Proof. By definition, there is a computable function $f : \mathcal{A}^* \times \mathbb{N} \to \mathbb{Q}$ such that $|\mu([u]) - f(u,n)| \leq 2^{-n}$ for all $u \in \mathcal{A}^*$. Because F is defined locally, $F^t(x)_{[0,k]}$ depends only on $x_{[lt,rt+k]}$ where $l = \min \mathbb{U}_F$ and $r = \max \mathbb{U}_F$. In other words, for all $u \in \mathcal{A}^k$, there is a set $\mathbf{Pred}_t(u) \subset \mathcal{A}^{[lt,rt+k]}$ such that $F^{-t}([u]) = \cup_{v\in\mathbf{Pred}_t(u)}[v]$. Now consider the function

$$f' : (u,n,t) \mapsto \sum_{v\in\mathbf{Pred}_t(u)} f(v, |u| + n + (r-l)t).$$

It is computable by enumerating elements of $\mathcal{A}^{k+(r-l)t}$ and checking if $F^t([v]_{-lt}) \subset [u]$ by iterating the local rule on v. Finally:

$$\begin{aligned}
|F_*^t\mu([u]) - f'(u,n,t)| &= \left|\mu\left(\bigcup_{v\in\mathbf{Pred}_t(u)}[v]\right) - \sum_{v\in\mathbf{Pred}_t(u)} f(v,|u|+n+(r-l)t)\right| \\
&\leq \sum_{v\in\mathbf{Pred}_t(u)} |\mu([v]) - f(v,|u|+n+(r-l)t)| \\
&\leq 2^{|u|+(r-l)t} \cdot 2^{-|u|-n-(r-l)t} = 2^{-n}
\end{aligned}$$

which means that $(u \mapsto F_*^t\mu([u]))_{t\in\mathbb{N}}$ is a uniformly computable sequence of functions. □

1.1.3 Computability of sets of probability measures

Computable analysis

In the previous section, we defined the notion of computability of probability measures and exhibited a computability obstruction on single limit measures. However, in general, the set of adherence values of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ is not restricted to single limit measures. Thus we need to extend the notion of computability to compact subsets of $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.

For a closed set $\mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, computing the characteristic $1_{\mathcal{K}} : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \to \{0,1\}$ is problematic because the possible inputs for a Turing machine form a countable set. Even if this function restricted to a countable dense set of the codomain (here $\{\widehat{\delta_u} : u \in \mathcal{A}^*\}$) is computable, we would need some notion of continuity to get an approximation of the result for any measure in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, which is not possible for a codomain $\{0,1\}$.

A standard reference book of the theory of computable analysis on metric spaces is [Wei00], but this theory is widely applied in the context of invariant measures (see for example [GHR11]). Computability in a general metric space is defined according to a countable dense subset, $(\widehat{\delta_w})_{w\in\mathcal{A}^*}$ in the case of $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.

Sets of probability measures

Definition 1.1.11 (Computability of closed subsets of metric spaces)**.**
A closed set $\mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is **computable** (resp. Π_n, Σ_n, Δ_n-computable) if the set:

$$\left\{(w, r) \in \mathcal{A}^* \times \mathbb{Q} \ : \ \overline{\mathbf{B}(\widehat{\delta_w}, r)} \cap \mathcal{K} \neq \emptyset\right\}$$

is computable (resp. Π_n, Σ_n, Δ_n-computable).

Examples.

- $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is a computable set.
- the set $\mathcal{M}_\sigma(\Sigma)$ of shift-invariant measures supported by any effective subshift $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ is a Π_2-computable compact set;
- let $K \subset [0,1]$ be a closed $\Sigma_n(\Pi_n)$-computable set[2] and denote $\lambda_p = Ber(p, 1-p) \in \{0,1\}^{\mathbb{Z}}$. The set $\{\lambda_p : p \in K\}$ is a $\Sigma_n(\Pi_n)$-computable compact set of $\mathcal{M}_\sigma(\{0,1\}^{\mathbb{Z}})$ and is connected if and only if K is. Furthermore $\{\alpha\lambda_p + (1-\alpha)\lambda_q : p, q \in K \text{ and } \alpha \in [0,1]\}$ is a $\Sigma_n(\Pi_n)$-computable compact connected set of $\{0,1\}^{\mathbb{Z}}$. This example extends naturally to larger alphabets and Markov measures;
- denote $\mu_\alpha \in \mathcal{M}_\sigma(\{0,1\}^{\mathbb{Z}})$ the measure supported by the Sturmian subshift of slope α. The set $\{\mu_\alpha : \alpha \in K\}$, where K is a Σ_n-computable closed subset of $[0,1]$, is a Σ_n-computable compact set of $\mathcal{M}_\sigma(\{0,1\}^{\mathbb{Z}})$ and is connected if and only if K is.

We consider connectedness in these examples, since our characterisation of μ-limit measures sets is only valid for connected sets - see Corollary 1.4.3. Let us prove the second example.

Proof of the second example. Let Σ be an effective subshift, which means that it is defined by a set of forbidden patterns $\mathcal{F}$ and there exists a computable function $f : i, u \mapsto \{0,1\}$ such that $u \in \mathcal{F} \Leftrightarrow \sup_i f(i,u) = 1$. Denote $\mathcal{F}_i = \{u \in \mathcal{A}^* \ : \ \sup_{j \leq i} f(j,u) = 1\}$.

Now define:

$$\mathbf{W}_{i,j} = \left\{ w \in \mathcal{A}^{\leq i} \ : \ \sum_{\ell \in \mathbb{N}} \frac{1}{2^\ell} \max_{u \in \mathcal{F}_i \cap \mathcal{A}^\ell} \widehat{\delta_w}([u]) \leq \frac{1}{j} \right\},$$

where the maximum is worth 0 when the set is empty, which means that the sum has ℓ terms where ℓ is the maximum length of a word in $\mathcal{F}_i$.

Let A be the algorithm that, on input $(u, r) \in \mathcal{A}^*$ and $i, j \in \mathbb{N}^2$,

1. computes all elements of $\mathcal{F}_i$ (evaluating a computable function over a finite set of arguments);
2. computes all $w \in \mathbf{W}_{i,j}$ (a finite number of tests, and $u \mapsto \widehat{\delta_w}([u])$ is a function $\mathcal{A}^* \to \mathbb{Q}$ which can be evaluated exactly);
3. computes $d_i(\widehat{\delta_w}, \widehat{\delta_u}) = \sum_{n=0}^{i} \frac{1}{2^n} \max_{v \in \mathcal{A}^n} |\widehat{\delta_w}([v]) - \widehat{\delta_u}([v])|$ for all $w \in \mathbf{W}_{i,j}$;

[2] The computability of a closed set of real numbers is defined similarly to the computability of a closed set of probability measures.

4. outputs 1 if $\max_{w\in\mathbf{W}_{i,j}} d_i(\widehat{\delta_w},\widehat{\delta_u}) \leq r+\frac{1}{j}$, 0 otherwise.

We prove the correctness of this algorithm, that is, we show that

$$\inf_{j\in\mathbb{N}}\sup_{i\geq j} A(u,r,i,j)=1 \Leftrightarrow \overline{\mathbf{B}(\widehat{\delta_u},r)}\cap\mathcal{M}_\sigma(\Sigma)\neq\emptyset.$$

Notice that for every sequence $(w_j)_{j\in\mathbb{N}}$ satisfying for all j $w_j\in\mathbf{W}_{i,j}$ for some $i\geq j$, any accumulation point ν of the sequence $(\widehat{\delta_{w_j}})_{j\in\mathbb{N}}$ satisfies $\nu([u])=0$ for all $u\in\mathcal{F}$, and therefore $\nu\in\mathcal{M}_\sigma(\Sigma)$.

$$\begin{aligned}\inf_{j\in\mathbb{N}}\sup_{i\geq j} A(u,r,i,j)=1 &\Leftrightarrow \forall j,\exists i\geq j,\exists w_j\in\mathbf{W}_{i,j}, d_i(\widehat{\delta_{w_j}},\widehat{\delta_u})\leq r+\frac{1}{j}\\ &\Rightarrow \forall j,\exists i\geq j,\exists w_j\in\mathbf{W}_{i,j}, d_{\mathcal{M}}(\widehat{\delta_{w_j}},\widehat{\delta_u})\leq r+\frac{1}{2^i}+\frac{1}{j}\\ &\Rightarrow \text{For any accumulation point } \nu \text{ of } (\widehat{\delta_{w_j}})_{j\in\mathbb{N}}, d_{\mathcal{M}}(\nu,\widehat{\delta_u})\leq r\\ &\Rightarrow \overline{\mathbf{B}(\widehat{\delta_u},r)}\cap\mathcal{M}_\sigma(\Sigma)\neq\emptyset.\end{aligned}$$

Conversely, let $\nu\in\overline{\mathbf{B}(\widehat{\delta_w},r)}\cap\mathcal{M}_\sigma(\Sigma)$. There exists a sequence $(w_n)_{n\in\mathbb{N}}$ such that $\widehat{\delta_{w_n}}\to\nu$. For any $j\in\mathbb{N}$, take n large enough that $d_{\mathcal{M}}(\nu,\widehat{\delta_{w_n}})\leq\frac{1}{j}$.

Since $\nu\in\mathcal{M}_\sigma(\Sigma)$,

$$\begin{aligned}\frac{1}{j}\geq d_{\mathcal{M}}(\nu,\widehat{\delta_{w_n}}) &\geq \sum_{\ell\in\mathbb{N}}\max_{u\in\mathcal{F}\cap\mathcal{A}^\ell}\widehat{\delta_{w_n}}([u])\\ &\geq \sum_{\ell\in\mathbb{N}}\max_{u\in\mathcal{F}_i\cap\mathcal{A}^\ell}\widehat{\delta_{w_n}}([u])\quad\text{for all } i\in\mathbb{N},\end{aligned}$$

which means that $w_n\in\mathbf{W}_{i,j}$ for all $i\geq\max(j,|w_n|)$. Furthermore,

$$d_i(\widehat{\delta_u},\widehat{\delta_{w_n}})\leq d_{\mathcal{M}}(\widehat{\delta_u},\widehat{\delta_{w_n}})\leq r+\frac{1}{j}$$

and $A(u,r,i,j)=1$ for all $i\geq j$. This is true for all j, so $\inf_{j\in\mathbb{N}}\sup_{i\geq j}A(u,r,i,j)=1$. We conclude that $\mathcal{M}_\sigma(\Sigma)$ is a Π_2-computable set. □

Functions mapping metric spaces with countable dense sets

Before stating the second computability obstruction, we give several other equivalent definitions, which requires first to introduce computability of functions mapping metric spaces with countable dense sets, for which the simplest example is functions $\mathbb{R}\to\mathbb{R}$. Computability of real functions was considered as early as 1957 [Grz57]. The equivalence of the various possible definitions, and the link between computability and continuity has been well studied (see for example [Ko91] and [Zie05]), giving rise to the field that is known as computable analysis, encompassing other objects such as computable (uncountable) sets from a topological viewpoint.

In this section, to gain in clarity, we write definitions for functions $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})\to\mathbb{R}$ as needed in this thesis, even though they can be extended to any pair of metric spaces with a countable

dense set.

Definition 1.1.12 (Computable function between metric spaces)**.**
A function $f : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \longrightarrow \mathbb{R}$ is **computable** if:
- there exists $a : \mathbb{N} \times \mathcal{A}^* \longrightarrow \mathbb{Q}$ computable such that $\left|f(\widehat{\delta_w}) - a(n,w)\right| \leq \frac{1}{n}$ for every $w \in \mathcal{A}^*$ and $n, m \in \mathbb{N}$;
- there exists $b : \mathbb{N} \longrightarrow \mathbb{Q}^+$ computable such that $d_{\mathcal{M}}(\mu,\nu) < b(m)$ implies $|f(\mu) - f(\nu)| \leq \frac{1}{m}$ for all $n, m \in \mathbb{N}$ (computable uniform continuity).

The first condition is similar to Definition 1.1.8, applied on the countable dense set of the codomain. However, to approximate the value of $f(\mu)$ when μ is not a measure supported by a periodic orbit, we need a notion of continuity with a known computable rate of convergence at each point. In this particular case, since $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is compact, the second condition of computable uniform continuity is equivalent to computable continuity; in the general case, the function b may depend on the neighbourhood.

Definition 1.1.13 (Computable sequence of functions between metric spaces)**.**
A sequence of functions $(f_n : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \longrightarrow \mathbb{R})_{n\in\mathbb{N}}$ is **uniformly computable** if:
- there exists $a : \mathbb{N} \times \mathbb{N} \times \mathcal{A}^* \longrightarrow \mathbb{Q}$ computable such that $\left|f_n(\widehat{\delta_w}) - a(n,m,w)\right| \leq \frac{1}{m}$ for every $w \in \mathcal{A}^*$ and $n, m \in \mathbb{N}$ (sequential computability);
- there exists $b : \mathbb{N} \longrightarrow \mathbb{Q}^+$ computable such that $d_{\mathcal{M}}(\mu,\nu) < b(m)$ implies $|f_n(\mu) - f_n(\nu)| \leq \frac{1}{m}$ for all $n, m \in \mathbb{N}$ (computable uniform equicontinuity).

Definition 1.1.14 (Arithmetical hierarchy of functions between metric spaces)**.**
A function $f : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \longrightarrow \mathbb{R}$ is Σ_n**-computable** (resp. Π_n**-computable**) if there exists a uniformly computable sequence of functions $\left(f_{i_1,\dots,i_n} : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \longrightarrow \mathbb{R}\right)_{i_1,\dots,i_n\in\mathbb{N}}$ such that:

$$f = \sup_{i_1\in\mathbb{N}} \inf_{i_2\in\mathbb{N}} \sup_{i_3\in\mathbb{N}} \cdots f_{i_1,\dots,i_n} \qquad \left(\text{resp. } f = \inf_{i_1\in\mathbb{N}} \sup_{i_2\in\mathbb{N}} \inf_{i_3\in\mathbb{N}} \cdots f_{i_1,\dots,i_n}\right).$$

f is $\boldsymbol{\Delta_n}$**-computable** if it is both Σ_n-computable and Π_n-computable.

Proposition 1.1.4 (Equivalent definitions of computability of closed sets)**.**
Let $\mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be a closed set. The following are equivalent:

1. $\mathcal{K}$ is Π_n-computable;
2. $\{(w,r) \in \mathcal{A}^* \times \mathbb{Q} \ : \ \overline{\mathbf{B}(\widehat{\delta_w}, r)} \cap \mathcal{K} = \emptyset\}$ is a Σ_n-computable set;
3. $d_{\mathcal{K}} : \begin{array}{ccc} \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) & \to & \mathbb{R}^+ \\ \mu & \mapsto & \min_{\nu\in\mathcal{K}} d_{\mathcal{M}}(\mu,\nu) \end{array}$ is a Σ_n-computable function;
4. $\mathcal{K} = \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \setminus \bigcup_{(w,r)\in S} \mathbf{B}(\widehat{\delta_w}, r)$ where $S \subset \mathcal{A}^* \times \mathbb{Q}$ is a Σ_n-computable set;
5. $\mathcal{K} = f^{-1}(0)$, where $f : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \to \mathbb{R}$ is a Π_{n-1}-computable function.

Proof. For clarity, we prove this equivalence in the case $n = 2$. The general case can be deduced by introducing additional indices in the proof as needed, or by assuming the computable functions in the proof have access to a Δ_{n-2}-complete oracle.

1 ⇔ 2 Obvious.

2 ⇒ 3 Assume there is a computable function $f : \mathbb{N}^2 \times \mathcal{A}^* \times \mathbb{Q} \longrightarrow \{0,1\}$ such that, for every $w \in \mathcal{A}^*$ and $r \in \mathbb{Q}$, $\overline{\mathbf{B}(\widehat{\delta_w}, r)} \cap \mathcal{K} = \emptyset \Leftrightarrow \sup_i \inf_j f(i,j,w,r) = 1$. Consider the sequence $\left(d_{i,j,w,r} : \mu \mapsto f(i,j,w,r) \max\left(0, r - d_{\mathcal{M}}(\widehat{\delta_w}, \mu)\right)\right)_{(i,j,w,r) \in \mathbb{N}^2 \times \mathcal{A}^* \times \mathbb{Q}}$. The function $(i,j,w,r,w') \mapsto d_{i,j,w,r}(\widehat{\delta_{w'}})$ is computable as a product of computable functions and every $d_{i,j,w,r}$ is 1-Lipschitz, hence this sequence is computably equicontinuous. We now show that $d_{\mathcal{K}} = \sup_{w,r} \sup_i \inf_j d_{i,j,w,r}$.

For any (w,r) such that $\sup_i \inf_j f(i,j,w,r) \neq 0$, we have $d_{\mathcal{K}}(\widehat{\delta_w}) > r$, and thus:

$$\sup_i \inf_j d_{i,j,w,r}(\mu) = \max\left(0, r - d_{\mathcal{M}}(\widehat{\delta_w}, \mu)\right) \leq \max\left(0, d_{\mathcal{K}}(\widehat{\delta_w}) - d_{\mathcal{M}}(\widehat{\delta_w}, \mu)\right) \leq d_{\mathcal{K}}(\mu)$$

for all $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. If $\mu \in \mathcal{K}$, we conclude that $\sup_{i,w,r} \inf_j d_{i,j,w,r}(\mu) = 0 = d_{\mathcal{K}}(\mu)$.

Now let $\mu \notin \mathcal{K}$. For all $\varepsilon > 0$, there exists w such that $\overline{d_{\mathcal{M}}(\widehat{\delta_w}, \mu)} \leq \varepsilon$. Let $r \in \mathbb{Q}$ be such that $0 < d_{\mathcal{K}}(\widehat{\delta_w}) - r < \varepsilon$, which implies that $\overline{\mathbf{B}(\widehat{\delta_w}, r)} \cap \mathcal{K} = \emptyset$ and so $\sup_i \inf_j f(i,j,w,r) \neq 0$. Furthermore $d_{\mathcal{K}}(\mu) \leq d_{\mathcal{K}}(\widehat{\delta_w}) + d_{\mathcal{M}}(\widehat{\delta_w}, \mu) \leq r + 2\varepsilon$, and $\sup_i \inf_j d_{i,j,w,r}(\mu) = r - d_{\mathcal{M}}(\widehat{\delta_w}, \mu) > d_{\mathcal{K}}(\mu) - 3\varepsilon$. The latter is true for every $\varepsilon > 0$, and we deduce that $\sup_{i,w,r} \inf_j d_{i,j,w,r}(\mu) = d_{\mathcal{K}}(\mu)$.

3 ⇒ 4 Let $(d_{i,j} : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \to \mathbb{R})_{(i,j) \in \mathbb{N}^2}$ be a uniformly computable sequence of functions such that $d_{\mathcal{K}} = \sup_{i \in \mathbb{N}} \inf_{j \in \mathbb{N}} d_{i,j}$. This means in particular that there is a computable function a such that $|d_{i,j}(\widehat{\delta_w}) - a(i,j,w,n)| \leq \frac{1}{n}$. Now define:

$$g_{i,j,n} : \begin{array}{ccc} \mathcal{A}^* \times \mathbb{Q} & \to & \{0,1\} \\ w, r & \mapsto & 1 \text{ if } a(i,j,w,n) > r - \frac{1}{n},\ 0 \text{ otherwise.} \end{array}$$

Now, defining S by $(w,r) \in S \Leftrightarrow \sup_{i \in \mathbb{N}} \inf_{j \in \mathbb{N}} \inf_{n \in \mathbb{N}}, g_{i,j,n}(w,r) = 1$, we have:

$$\begin{aligned} (w,r) \in S &\Rightarrow \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, d_{i,j}(\widehat{\delta_w}) > r \\ &\Rightarrow d_{\mathcal{K}}(\widehat{\delta_w}) > r \end{aligned}$$

Reciprocally, if $d_{\mathcal{K}}(\mu) > r$, then $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, d_{i,j}(\mu) > r$. By density, take $w \in \mathcal{A}^*$ such that $d_{\mathcal{M}}(\mu, \widehat{\delta_w}) \leq \min\left(\frac{r}{2}, b\left(\left\lceil \frac{2}{r} \right\rceil\right)\right)$, where b is defined is the definition of a uniformly computable sequence of functions $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \to \mathbb{R}$. By definition, this ensures that for all $i,j \in \mathbb{N}$ we have $|d_{i,j}(\mu) - d_{i,j}(\widehat{\delta_w})| \leq \frac{r}{2}$, and consequently, for all n, $a(i,j,w,n) > \frac{r}{2} - \frac{1}{n}$.

Therefore, for all i,j,n, we have $g_{i,j,n}\left(w, \frac{r}{2}\right) = 1$, and so $\left(w, \frac{r}{2}\right) \in S$. Since $\mu \in \overline{\mathbf{B}(\widehat{\delta_w}, \frac{r}{2})}$, we conclude.

4 ⇒ 5 Let $f_S : \mathbb{N}^2 \times \mathcal{A}^* \times \mathbb{Q} \longrightarrow \{0,1\}$ be the computable function satisfying $(w,r) \in S \Leftrightarrow \sup_i \inf_j f_S(i,j,w,r) = 1$. Fix an enumeration of all words $k \mapsto w_k$ and of all positive

rationals $l \mapsto r_l$, and consider the function:

$$f : \begin{array}{ccc} \mathcal{A}^* \times \mathbb{N}^2 \times \mathcal{A}^* \times \mathbb{Q} & \to & \mathbb{R}^+ \\ w', j & \mapsto & \displaystyle\sum_{i,k,l \in \mathbb{N}^3} \frac{1}{2^{i+k+l}} \max\left(0, r - d(\widehat{\delta_{w_k}}, \widehat{\delta_{w'}})\right) f_S(i,j,w_k,r). \end{array}$$

This function is computable. Indeed, to compute $f(w', j)$ up to precision $\frac{1}{n}$, computing a finite number of terms is sufficient, and each of these terms is simply a product of functions that are computable from their parameters.

$$\begin{aligned} w' \notin \mathcal{K} &\Leftrightarrow \exists (w,r) \in S, \widehat{\delta_{w'}} \in \mathbf{B}(\widehat{\delta_w}, r) \\ &\Leftrightarrow \exists (w,r), \exists i, \inf_j f_S(i,j,w,r) = 1 \text{ and } r - d(\widehat{\delta_w}, \widehat{\delta_{w'}}) > 0 \\ &\Leftrightarrow \sum_{i,k,l \in \mathbb{N}^3} \frac{1}{2^{i+k+l}} \inf_j \max(0, r_l - d(\widehat{\delta_{w_k}}, \widehat{\delta_{w'}})) f_S(i,j,w_k,r_l) > 0 \\ &\Leftrightarrow \inf_j \sum_{i,k,l \in \mathbb{N}^3} \frac{1}{2^{i+k+l}} \max(0, r_l - d(\widehat{\delta_{w_k}}, \widehat{\delta_{w'}})) f_S(i,j,w_k,r_l) > 0 \\ &\Leftrightarrow \inf_j f(j, w') > 0. \end{aligned}$$

where the fourth equivalence is obtained by dominated convergence. Therefore $f_{\mathcal{K}} : w' \mapsto \inf_j f(w', j)$ is a Π_1-computable function and $\mathcal{K} = f_{\mathcal{K}}^{-1}(\{0\})$.

5 ⇒ 2 Let $(f_n : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \to \mathbb{R})_{n \in \mathbb{N}}$ be a uniformly computable sequence of functions such that $f = \inf_{n \in \mathbb{N}} f_n$. We assume w.l.o.g that the sequence is decreasing. For $i \in \mathbb{N}$, we note:

$$d_i(\mu, \nu) = \sum_{n=1}^{i} \frac{1}{2^n} \max_{u \in \mathcal{A}^n} |\mu([u]) - \nu([u])|,$$

so that $0 \leq d_{\mathcal{M}}(\mu, \nu) - d_i(\mu, \nu) \leq \frac{1}{2^i}$. For any $q \in \mathbb{Q}, w' \in \mathcal{A}^*$ and $i \in \mathbb{N}$, define

$$K_{q,w',i} = \left\{ (w,r) \in \mathcal{A}^* \times \mathbb{Q} \ : \ d_i(\widehat{\delta_w}, \widehat{\delta_{w'}}) \leq r \Rightarrow |f_i(\widehat{\delta_{w'}})| > q \right\}.$$

(notice that $K_{q,w',i}$ is either $\emptyset$ or $\mathbf{B}(\widehat{\delta_{w'}}, r)$). The function $(q, w', i, w, r) \mapsto \mathbf{1}_{(w,r) \in K_{q,w',i}}$ is computable and thus the characteristic functions $\mathbf{1}_{K_{q,w',i}}$ are uniformly computable. Define:

$$K = \bigcup_{\substack{q \in \mathbb{Q}^+ \\ n \in \mathbb{N}}} \bigcap_{\substack{w' \in \mathcal{A}^* \\ i \geq n}} K_{q,w',i} \qquad \text{and thus} \qquad \mathbf{1}_K = \sup_{\substack{q \in \mathbb{Q}^+ \\ n \in \mathbb{N}}} \inf_{\substack{w' \in \mathcal{A}^* \\ i \geq n}} \mathbf{1}_{K_{q,w',i}}.$$

We prove that $K = \{(w,r) \in \mathcal{A}^* \times \mathbb{Q} \ : \ \overline{\mathbf{B}(\widehat{\delta_w}, r)} \cap \mathcal{K} = \emptyset\}$.

Indeed, let (w,r) be such that $\overline{\mathbf{B}(\widehat{\delta_w}, r)} \cap \mathcal{K} = \emptyset$. Let $\varepsilon = \min\{|f(\mu)| \ : \ \mu \in \overline{\mathbf{B}(\widehat{\delta_w}, r)}\} > 0$. For any $\mu \in \overline{\mathbf{B}(\widehat{\delta_w}, r)}$, there is a rank $n_\varepsilon(\mu)$ such that for all $n \geq n_\varepsilon(\mu), f_n(\mu) > \frac{3\varepsilon}{4}$. By taking $r_\varepsilon \in \mathbb{N}$ such that $r_\varepsilon > \frac{4}{\varepsilon}$, we have $f_n(\nu) > \frac{\varepsilon}{2}$ for all $\nu \in \overline{\mathbf{B}(\mu, b(r_\varepsilon))}$ and all

$n \geq n_\varepsilon(\mu)$, where b is given in the definition of the computable uniform equicontinuity of $(f_n)_{n\in\mathbb{N}}$. Since $\overline{\mathbf{B}(\widehat{\delta_w}, r)}$ is compact, it can be covered by a finite number of balls of radius $b(r_\varepsilon)$, and we take n_ε the maximal value of $n_\varepsilon(\mu)$ on all the ball centers.

The previous paragraph shows that for all w' such that $\widehat{\delta_{w'}} \in \overline{\mathbf{B}(\widehat{\delta_w}, r)}$, $|f_i(\widehat{\delta_{w'}})| > \frac{\varepsilon}{2}$ for all $i \geq n_\varepsilon$; and if $\widehat{\delta_{w'}} \notin \overline{\mathbf{B}(\widehat{\delta_w}, r)}$, $d_i(\widehat{\delta_{w'}}, \widehat{\delta_w}) > r$ for i large enough. Thus $(w, r) \in K$ by taking any $q \leq \frac{\varepsilon}{2}$.

Conversely,

$$\begin{aligned}(w,r) \in K &\Rightarrow \exists q \in \mathbb{Q}^+, \exists n \in \mathbb{N}, \forall w' \in \mathcal{A}^*, \forall i \geq n,\ d_{\mathcal{M}}(\widehat{\delta_w}, \widehat{\delta_{w'}}) \leq r \Rightarrow |f_i(\widehat{\delta_{w'}})| > q \\ &\Rightarrow \forall \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_{\mathcal{M}}(\widehat{\delta_w}, \nu) \leq r \Rightarrow |f(\nu)| > q \\ &\Rightarrow \mathcal{B}(\widehat{\delta_w}, r) \cap \mathcal{K} = \emptyset\end{aligned}$$

where the second line is obtained by density of $(\widehat{\delta_w})_{w\in\mathcal{A}^*}$ and because $f = \lim_i f_i$. □

We now state the second computability obstruction for subsets of $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ reachable as limit sets of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ (μ-limit measures sets).

Proposition 1.1.5 (Second computability obstruction)**.**
Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton and $\mu \in \mathcal{M}_\sigma^{\text{comp}}(\mathcal{A}^{\mathbb{Z}})$. Then $\mathcal{V}(F, \mu)$ and $\mathcal{V}'(F, \mu)$ are nonempty Π_2-computable compact sets. Furthermore, $\mathcal{V}'(F, \mu)$ is connected.

Remark. If a Π_2-computable closed set of measures is a single measure, then the measure is limit-computable. Thus Proposition 1.1.5 implies Proposition 1.1.3.

Proof. We use the third definition in Proposition 1.1.4. Let $f_n : \nu \mapsto d_{\mathcal{M}}(F_*^n\mu, \nu)$. Since μ is computable, $(f_n)_{n\in\mathbb{N}}$ is uniformly computable. Moreover $|f_n(\nu) - f_n(\nu')| = |d_{\mathcal{M}}(F_*^n\mu, \nu) - d_{\mathcal{M}}(F_*^n\mu, \nu')| \leq d_{\mathcal{M}}(\nu, \nu')$ so $(f_n)_{n\in\mathbb{N}}$ is computably uniformly equicontinuous. The result follows from the fact that

$$d_{\mathcal{V}(F,\mu)}(\nu) = \liminf_{n\to\infty} d_{\mathcal{M}}(F_*^n\mu, \nu) = \sup_m \inf_{n>m} f_n(\nu).$$

The same reasoning holds for $\mathcal{V}'(F, \mu)$. The second point was proved in Proposition 0.1.9. □

Π_2-computable compact set of measures can be described as the limit points of a sequence $(\widehat{\delta_{w_n}})_{n\in\mathbb{N}}$ corresponding to some uniformly computable sequence of words $(w_n)_{n\in\mathbb{N}}$. However, for technical reasons, the μ-limit measures set of the construction presented in Section 1.2 corresponds to the limit set of an infinite polygonal path composed of segments of the form $\left[\widehat{\delta_u}, \widehat{\delta_v}\right] = \left\{t\widehat{\delta_u} + (1-t)\widehat{\delta_v} : t \in [0,1]\right\} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ where $u, v \in \mathcal{A}^*$, and is in particular connected. This is why we describe in the following proposition how compact, Π_2-computable, connected sets can be covered by a polygonal path corresponding to a uniformly computable sequence of words.

Definition 1.1.15 (Limit points of a polygonal path)**.**

Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of words of A^*. Denote $\mathcal{V}((w_n)_{n\in\mathbb{N}})$ the **limit points of the polygonal path** defined by the sequence of measures $(\widehat{\delta_{w_n}})_{n\in\mathbb{N}}$:

$$\mathcal{V}((w_n)_{n\in\mathbb{N}}) = \bigcap_{N>0} \overline{\bigcup_{n\geq N} \left[\widehat{\delta_{w_n}}, \widehat{\delta_{w_{n+1}}}\right]}.$$

Proposition 1.1.6 (Technical characterisation of Π_2-CCC sets)**.**
Let $\mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be a non-empty Π_2-computable, compact, connected set (Π_2-CCC for short). Then there exists a uniformly computable sequence of words $(w_n)_{n\in\mathbb{N}}$ such that $\mathcal{K} = \mathcal{V}((w_n)_{n\in\mathbb{N}})$.

Proof. By Proposition 1.1.4 there is a uniformly computable sequence of functions $(f_n)_{n\in\mathbb{N}}$ satisfying $\mathcal{K} = f^{-1}(\{0\})$ where $f = \lim_{n\in\mathbb{N}} f_n$. Let $a : \mathbb{N}\times\mathbb{N}\times\mathcal{A}^* \to \mathbb{Q}$ and $b : \mathbb{N} \to \mathbb{Q}^+$ be the computable functions given by Definition 1.1.14. Without loss of generality, we can assume that b is a decreasing function and $b(i) \underset{i\to\infty}{\longrightarrow} 0$. For $k \in \mathbb{N}$, define:

$$\alpha_k^t = \min\{\ell \leq t \ : \ \forall u \in \mathcal{A}^{\leq t}, \exists w \in \mathcal{A}^{\leq \ell}, d_{\mathcal{M}}(\widehat{\delta_u}, \widehat{\delta_w}) \leq b(k)\}$$

$$\alpha_k = \min\left\{ l \in \mathbb{N} \ : \ \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) = \bigcup_{u\in\mathcal{A}^{\leq l}} \mathbf{B}(\widehat{\delta_u}, b(k))\right\}$$

$$\mathbf{V}_k^t = \left\{ w \in \mathcal{A}^{\leq \alpha_k^t} : \exists n \in [k,t] \text{ such that } a(n, 2k, w) < \frac{2}{k}\right\}$$

$$\mathbf{V}_k = \left\{ w \in \mathcal{A}^{\leq \alpha_k} : \exists n \geq k \text{ such that } a(n, 2k, w) < \frac{2}{k}\right\}$$

Claim 1: *$\mathbf{V}_k^t$ is increasing w.r.t. t and $\exists T_k, \mathbf{V}_k = \mathbf{V}_k^{T_k}$. Furthermore, the function $(k,t,w) \to 1_{\mathcal{V}_k^t}(w)$ is computable.*

Proof. Because the periodic measures are dense in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, we have $\alpha_k^t = \alpha_k$ when t is large enough.

For all k and t, $\mathbf{V}_k^t \subset \mathbf{V}_k^{t+1}$. Furthermore, if $w \in \mathbf{V}_k$, then $w \in \mathbf{V}_k^t$ for t large enough. Since $\mathbf{V}_k$ is finite, there is a T_k such that $\mathbf{V}_k = \mathbf{V}_k^{T_k}$.

The conditions for being included in $\mathbf{V}_k^t$ can be checked by computing computable functions over a finite range of values, so $(k,t,w) \mapsto 1_{\mathbf{V}_k^t}(w)$ is computable. ◊ Claim 1

Notice that T_k is not necessarily computable, which means that even though each $\mathbf{V}_k$ is finite, there is not necessarily a way to know when the enumeration is finished. The algorithm for computing the sequence $(w_n)_{n\in\mathbb{N}}$ is the following:

Algorithm.

$n \leftarrow 0$.
For $t \in \mathbb{N}$, by increasing order: (1)
  For $k \leq t$, by increasing order: (2)
    For each element $w \in \mathbf{V}_k^t$: (3)
      If $n = 0$:

$w_0 \leftarrow w$ and $n \leftarrow 1$. Go to the next element of $\mathbf{V}_k^t$.
Else if $t > 0$ and $w \in \mathbf{V}_k^{t-1}$:
Go to the next element of $\mathbf{V}_k^t$.
Else:
For each $i \leq k$, by decreasing order: (4)
Enumerate all finite sequences without repetition $u_1, \dots, u_{l-1} \in \mathbf{V}_i^t$.
If a path $w_{n-1} = u_0, u_1, \dots, u_l = w$ with $d_{\mathcal{M}}(u_k, u_{k+1}) \leq 4b(i)$ is found:
$w_n \leftarrow u_1, \dots, w_{n+l} \leftarrow w$ and $n \leftarrow n + l + 1$.
Go to the next element of $\mathbf{V}_k^t$.
If no such path was found for any i:
$w_n \leftarrow w$. Go to the next element of $\mathbf{V}_k^t$.

Notice that in the fourth loop, if a path is found, then it corresponds to the largest $i \leq k$ for which such a path exists.
Now we prove the correctness of this algorithm.

Claim 2: *If $\mu \in \mathcal{K}$, then $\mu \in \mathcal{V}((w_n)_{n\in\mathbb{N}})$.*

Proof. By definition of α_n, there is a sequence of words $(u_n)_{n\in\mathbb{N}}$ such that $u_n \in \mathcal{A}^{\leq \alpha_n}$ and $d_{\mathcal{M}}(\widehat{\delta_{u_n}}, \mu) < b(n)$ for all $n \in \mathbb{N}$; by equicontinuity of f, since $f(\mathcal{K}) = \{0\}$ by definition, one has $f(\widehat{\delta_{u_n}}) < \frac{1}{n}$. Thus, there is a $t > |u_n|$ such that $f_t(\widehat{\delta_{u_n}}) < \frac{3}{2n}$. One deduces that $a(t, 2n, u_n) \leq f_t(\widehat{\delta_{u_n}}) + \frac{1}{2n} < \frac{2}{n}$, which means that $u_n \in \mathbf{V}_n$ for every n, and by construction it appears at some point in the sequence $(w_n)_{n\in\mathbb{N}}$. We conclude that $\lim_{n\to\infty} \widehat{\delta_{u_n}} = \mu \in \mathcal{V}((w_n)_{n\in\mathbb{N}})$. ◊ Claim 2

Claim 3: $\forall \varepsilon > 0, \exists k_\varepsilon, \forall k \geq k_\varepsilon, w \in \mathbf{V}_k \Rightarrow d_{\mathcal{M}}(\widehat{\delta_w}, \mathcal{K}) \leq \varepsilon$.

Proof. By compacity, there exists $\delta_\varepsilon > 0$ such that $f(\widehat{\delta_w}) \leq \delta_\varepsilon \Rightarrow d_{\mathcal{M}}(\widehat{\delta_w}, \mathcal{K}) \leq \varepsilon$.

Now let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be any measure such that $f(\mu) \geq \delta_\varepsilon$. There is a $n_\varepsilon(\mu) \in \mathbb{N}$ such that $f_n(\mu) \geq \frac{2\delta_\varepsilon}{3}$ for any $n \geq n_\varepsilon(\mu)$. By taking $r_\varepsilon \in \mathbb{N}$ such that $\frac{1}{r_\varepsilon} < \frac{\delta_\varepsilon}{3}$, we have by computable uniform equicontinuity of $(f_n)_{n\in\mathbb{N}}$ $f_n(\nu) > \frac{\delta_\varepsilon}{3}$ for all $\nu \in \mathbf{B}(\mu, b(r_\varepsilon))$ and all $n \geq n_\varepsilon(\mu)$.

Since $\{\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) : f(\mu) \geq \delta_\varepsilon\}$ is compact, we can cover it with a finite number of balls of radius $b(r_\varepsilon)$, and we define n_ε the maximum value of $n_\varepsilon(\mu)$ on ball centers. Thus, $\forall n > n_\varepsilon, \forall \mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), f(\mu) > \delta_\varepsilon \Rightarrow f_n(\mu) > \frac{\delta_\varepsilon}{3}$.

To conclude, taking $k_\varepsilon \geq \max(n_\varepsilon, \frac{9}{\delta_\varepsilon})$, we have for all $k \geq k_\varepsilon$: $w \in \mathbf{V}_k \Rightarrow f_k(\widehat{\delta_w}) \leq \frac{2}{k} + \frac{1}{2k} \leq \frac{\delta_\varepsilon}{3} \Rightarrow f(\widehat{\delta_w}) \leq \delta_\varepsilon \Rightarrow d_{\mathcal{M}}(\widehat{\delta_w}, \mathcal{K}) \leq \varepsilon$. ◊ Claim 3

Claim 4: *For every $\varepsilon > 0$, there exists a t_ε such that in the previous algorithm, if $t' \geq t \geq t_\varepsilon$, $w_n \in \mathbf{V}_k^{t+1} \backslash \mathbf{V}_k^t$ and $w \in \mathbf{V}_{k'}^{t'+1} \backslash \mathbf{V}_{k'}^{t'}$, then the path $w_n = u_0, \dots w = u_l$ built in the fourth loop satisfies $\forall \nu \in \bigcup_{0\leq i<l} [\widehat{\delta_{u_i}}, \widehat{\delta_{u_{i+1}}}], d_{\mathcal{M}}(\nu, \mathcal{K}) \leq \varepsilon$.*

Proof. Let $K_1 \geq k_{\frac{\varepsilon}{2}}$ (where k_i is defined in the previous claim) be large enough such that $b(i) \leq \frac{\varepsilon}{4}$ for any $i \geq K_1$, and put $K_2 = k_{b(K_1)}$. Let $t_\varepsilon = \max_{0\leq i\leq K_2}(T_i)$ and

assume $w_n \in \mathbf{V}_k^{t+1} \backslash \mathbf{V}_k^t$ and $w \in \mathbf{V}_{k'}^{t'+1} \backslash \mathbf{V}_{k'}^{t'}$ with $t' \geq t \geq t_\varepsilon$. By definition of the T_i, we have $\mathbf{V}_k^{t_\varepsilon} = \mathbf{V}_k^t$ for all $k \leq K_2$. Consequently $k \geq K_2$ and $k' \geq K_2$.

For each element $\mu \in \mathcal{K}$ there is an element $u_{K_1} \in \mathcal{A}^{\leq \alpha_{K_1}}$ such that $d_{\mathcal{M}}(\mu, \widehat{\delta_{u_{K_1}}}) \leq b(K_1)$, and therefore $f(\widehat{\delta_{u_{K_1}}}) \leq \frac{1}{K_1}$ so $u_{K_1} \in \mathbf{V}_{K_1}$. In other words,

$$\mathcal{K} \subset \bigcup_{u \in \mathbf{V}_{K_1}} \mathbf{B}\left(\widehat{\delta_u}, b(K_1)\right).$$

Since $w_n \in \mathbf{V}_k$ with $k \geq K_2 = k_{b(K_1)}, d_{\mathcal{M}}(\widehat{\delta_{w_n}}, \mathcal{K}) \leq b(K_1)$ and the same is true for w. Therefore $\bigcup_{u \in \mathbf{V}_{K_1}} \mathbf{B}\left(\widehat{\delta_u}, 2b(K_1)\right)$ contains $\widehat{\delta_{w_n}}$ and $\widehat{\delta_w}$ as well as $\mathcal{K}$ in a single connected component, since $\mathcal{K}$ is connected. This means that in the fourth loop of the algorithm, a path can be found with $i \geq K_1$. Since the path is entirely included in $\bigcup_{u \in \mathbf{V}_{K_1}} \mathbf{B}\left(\widehat{\delta_u}, 2b(i)\right)$ with $b(i) \leq \frac{\varepsilon}{4}$, and since $u \in \mathbf{V}_{K_1} \Rightarrow d(\widehat{\delta_u}, \mathcal{K}) \leq \frac{\varepsilon}{2}$, the result follows. ♢ Claim 4

Claim 5: *If $\mu \notin \mathcal{K}$, then $\mu \notin \mathcal{V}((w_n)_{n \in \mathbb{N}})$.*

Proof. Take any $\varepsilon > 0$, and wait that the first loop reaches the value $t = t_\varepsilon$ where t_ε is defined in Claim 4. At some point, a new element w_n will be found in the third loop and it will be added to the sequence already built (with a path of words before it). By construction, $w_n \in \mathbf{V}_k^t$ for some $t \geq t_\varepsilon$, and the same is true for any element found in the third loop from now on.

By Claim 4, this means that any pair of elements (w_k, w_{k+1}) with $k \geq n$ added in the sequence from now on satisfies $\forall \nu \in [\widehat{\delta_{w_k}}, \widehat{\delta_{w_{k+1}}}],\ d_{\mathcal{M}}(\nu, \mathcal{K}) \leq \varepsilon$. This is true for all $\varepsilon > 0$, so any accumulation point of the polygonal path $\overline{\bigcup_{n \geq N} [\widehat{\delta_{w_n}}, \widehat{\delta_{w_{n+1}}}]}$ is included in $\mathcal{K}$. ♢ Claim 5

□

Section 1.2

A cellular automaton realising a set of limit measures

In this section, we prove a reciprocal to the computability obstructions of Proposition 1.1.3 and a partial reciprocal to Proposition 1.1.5 using Proposition 1.1.6. Given a uniformly computable sequence of words $(w_n)_{n\in\mathbb{N}}$ in $\mathcal{B}^*$, we construct a cellular automaton realising $\mathcal{V}((w_n)_{n\in\mathbb{N}})$ as its μ-limit measures set. We remind that $\mathcal{V}((w_n)_{n\in\mathbb{N}})$ is defined as the set of limit points of the polygonal path defined by the sequence of measures $(\widehat{\delta_{w_n}})_{n\in\mathbb{N}}$:

$$\mathcal{V}((w_n)_{n\in\mathbb{N}}) = \bigcap_{N>0} \overline{\bigcup_{n\geq N} \left[\widehat{\delta_{w_n}}, \widehat{\delta_{w_{n+1}}}\right]}.$$

Theorem 1.2.1 (Realisation of a computable polygonal path of measures)**.**
Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly computable sequence of words of $\mathcal{B}^*$, where $\mathcal{B}$ is a finite alphabet. Then there is a finite alphabet $\mathcal{A} \supset \mathcal{B}$ and a cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that:

- for any measure $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$, $\mathcal{V}(F,\mu) = \mathcal{V}((w_n)_{n\in\mathbb{N}})$;
- if $\mathcal{V}((w_n)_{n\in\mathbb{N}}) = \{\nu\}$, then for any measure $\mu \in \mathcal{M}^{\text{full}}_{\sigma-\text{erg}}(\mathcal{A}^{\mathbb{Z}})$, $F^t_*\mu \underset{t\to\infty}{\longrightarrow} \nu$.

Furthermore we get an explicit bound for the convergence rate in the first point of the theorem. Assume that w_n is computable in space $O(\sqrt{n})$ (by repeating elements of the sequence $(w_n)_{n\in\mathbb{N}}$ if necessary), one has:

$$d_{\mathcal{M}}(F^t_*\mu, \mathcal{V}((w_n)_{n\in\mathbb{N}})) = O\left(\frac{1}{\log(t)}\right) + \sup\left\{ d_{\mathcal{M}}(\nu, \mathcal{V}((w_n)_{n\in\mathbb{N}})) \ : \ \nu \in \bigcup_{n\geq C(\log t)^2} [\widehat{\delta_{w_n}}, \widehat{\delta_{w_{n+1}}}] \right\}$$

for some constant $C > 0$. The first term of the upper bound corresponds to the intrinsic limitations of the construction, the second term depends on the speed of convergence of the polygonal path defined by $\widehat{\delta_{w_n}}$ towards $\mathcal{V}((w_n)_{n\in\mathbb{N}})$, which is intuitively the quality of the approximation of $\mathcal{V}(F,\mu)$ by a computable path.

In the rest of the section, we detail the construction of this cellular automaton and prove the correctness of the construction.

1.2.1 Overview of the construction

In this section, we present a sketch of the construction of the alphabet $\mathcal{A}$ and the cellular automaton F. $\mathcal{A}$ contains a symbol $\boxed{\text{W}}$ (for **wall**) persisting in time, except under special circumstances, defining independent areas of computation (**segments**).

Independently in each segment, three tasks are performed in parallel:

Formatting the initial contents of the segment are erased;

Computation and copy each word w_i is successively computed and concatenated copies of it are written on the whole segment;

Merging the length of the segment is checked at regular intervals, and it merges with the segment to its right if it is too small.

The key task is the second, since the goal of the construction is that $F_*^t\mu$ gets close to each measure $\widehat{\delta_{w_i}}$ successively. This requires that the computation is performed synchronously between all segments, so that each segment contains copies of w_i at the same instant. To do this, we define another symbol $\boxed{\text{I}}$ (**init**), which appears only in the initial configuration, creating a wall and initialising all these processes. This process is detailed in Section 1.2.2.

Definition 1.2.1. Let $x \in \mathcal{A}^{\mathbb{Z}}$. $[i, j]$ is a **segment at time 0** if x_i and x_j are two consecutive $\boxed{\text{I}}$ symbols, and a **segment at time** t if $F^t(x)_i$ and $F^t(x)_j$ are two consecutive initialised walls $\boxed{\text{W}}$. Define the **length** of this segment as $j - i - 1$.

This means that walls not issued from a symbol $\boxed{\text{I}}$ (**uninitialised walls**) are not considered as valid segment borders, but rather as unwanted symbols to be erased.

Apart from $\boxed{\text{I}}$ and $\boxed{\text{W}}$, the new alphabet $\mathcal{A}$ is divided in different layers: the **main layer** where the words w_n are output and recopied, and **auxiliary layers** where computation and other processes take place. This allows to perform all tasks in parallel.

Formatting Since we have no control over the initial contents of each segment, we first want to **format** the segment, that is, to erase uninitialised walls and uninitialised contents of the auxiliary layers (i.e. not issued from an $\boxed{\text{I}}$ symbol).

Most processes defined below are designed to self-destroy when they are not initialised. This is detailed as each new process is introduced. The difficult task is to distinguish uninitialised walls from initialised walls.

To do that, each initialised wall sends to its right a signal on a specific layer progressing at speed one (**formatting counter** - see Section 1.2.2), that keeps track of its age using a binary counter. Meanwhile, each initialised walls also keeps track of its age under the form of a binary counter on another layer, to its left, incrementing at each step (**time counter** - see Section 1.2.2).

Time and formatting counters already present in the initial configuration (uninitialised) have a nonnegative value at time 0, whereas those created by an $\boxed{\text{I}}$ symbol (initialised) have value 0 at time 1, and they increment at the same rate. Thus, uninitialised walls have older time counters, and by comparing time counters and formatting counters as they cross, we can erase older counters and uninitialised walls. Figure 1.2 is an overview of those processes.

Computation and copy Meanwhile, on another layer, a Turing machine is simulated in the space delimited by the time counter. This machine successively computes each w_n and writes concatenated copies on the main layer of the segment to its left (see Section 1.2.3). For each w_n, this copy happens synchronously on the whole configuration, so as to approach the measure $\widehat{\delta_{w_n}}$.

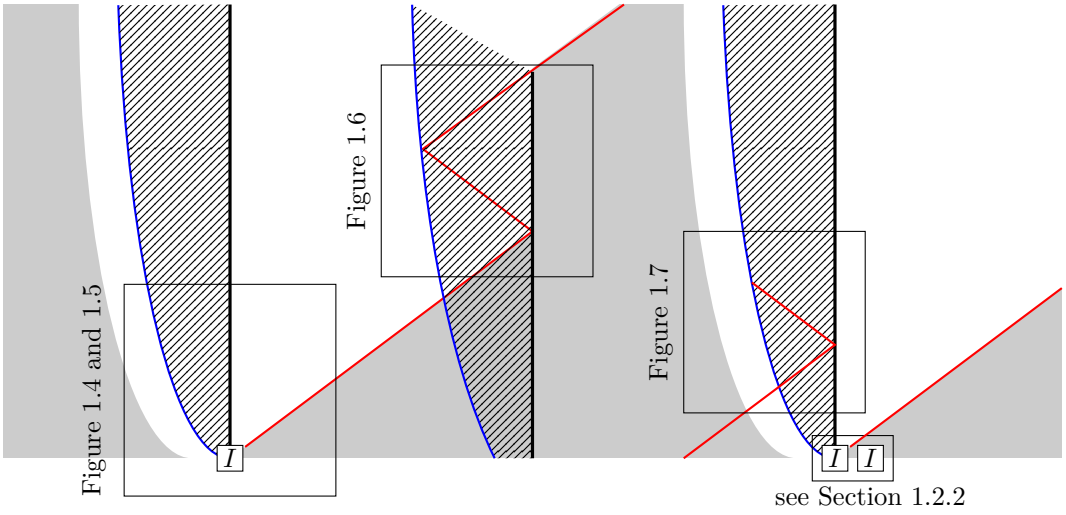


Figure 1.2: Sketch of the bootstrapping and formatting processes. Vertical lines are walls. Dashed parts contain time counters (section 1.2.2) and Turing machines (section 1.2.3). Slanted lines are formatting counters (section 1.2.2), white and grey areas are respectively formatted and non-formatted.

Merging Synchronously, segments of a given length are merged with their left neighbour in order to enlarge computational space and decrease the density of cells with nonempty auxiliary layers (see Section 1.2.4). To determine the length of its right segment, each wall sends a signal to the right on a dedicated layer that bounces off the next wall and counts the return time. Figure 1.3 is an overview of copy and merging processes.

Alphabet We obtain an enlarged alphabet $\mathcal{A} = \left\{\boxed{\text{I}}, \boxed{\text{W}}\right\} \cup \mathcal{A}_{\texttt{main}} \times \mathcal{A}_{\texttt{comp}} \times \mathcal{A}_{\texttt{time}} \times \mathcal{A}_{\texttt{format}} \times \mathcal{A}_{\texttt{copy}} \times \mathcal{A}_{\texttt{merge}}$. All those alphabets contain a symbol $\#$ (blank) representing the absence of information.

- $\boxed{\text{I}}$ and $\boxed{\text{W}}$ are the two above-mentioned symbols;
- $\mathcal{A}_{\texttt{main}} = \mathcal{B} \cup \{\#\}$ is the layer on which w_n is output and then recopied;
- $\mathcal{A}_{\texttt{comp}}$ is the layer where Turing machines are simulated to compute w_n and other processes;
- $\mathcal{A}_{\texttt{time}}$ is the layer on which time counters are incremented;
- $\mathcal{A}_{\texttt{format}}$ is the layer on which formatting counters move and are incremented, and where comparisons are performed;
- $\mathcal{A}_{\texttt{copy}}$ is a layer used in the process of writing copies of the output on the main layer;
- $\mathcal{A}_{\texttt{merge}}$ is a layer used in the process of merging two segments.

We have $\mathcal{B} \subset \mathcal{A}$ up to the identification $b \mapsto (b, \#, \#, \#, \#, \#)$. If $u \in \mathcal{A}$, denote $main(u)$, resp. $comp(u)$, $time(u)$... the projections on each layer (the result being $\#$ on $\boxed{\text{I}}$ and $\boxed{\text{W}}$).

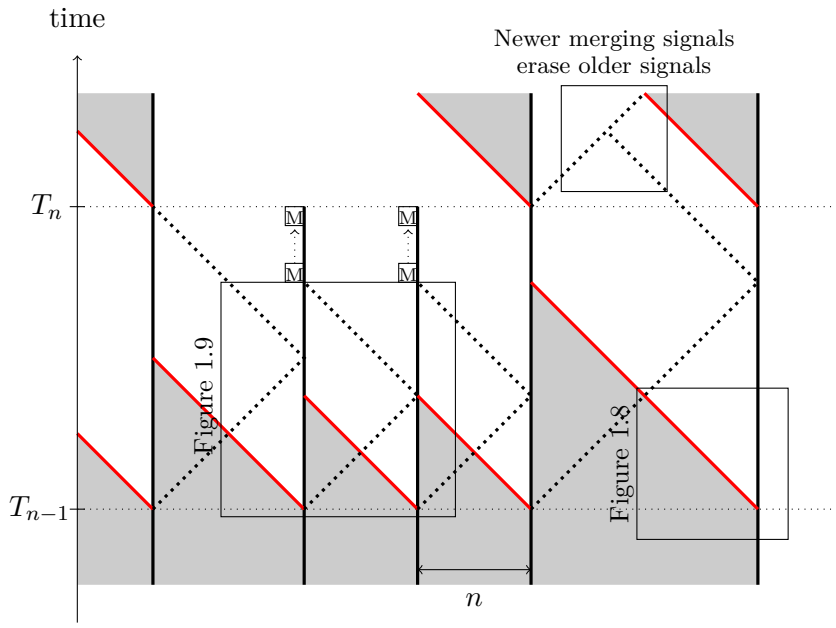


Figure 1.3: Sketch of the copying and merging processes. Here all walls are initialised. Slanted thick lines are copy processes (see Section 1.2.3), slanted dotted lines are merging signals (see Section 1.2.4).

We detail the different alphabets in the following sections. As we will see, our construction needs interactions at a distance at most three, so we can take $\mathcal{N}_F = \{-3, \dots, 3\}$ as the neighbourhood of the local rule of F.

1.2.2 Formatting the segments

Bootstrapping

If two symbols I are separated by two cells or less, the rightmost one is destroyed. Otherwise, all I symbols turn into W, erasing the contents of three cells to their right and left (including walls), initialising on its left a computation process and a time counter, and on its right a formatting counter. No more I or W symbols can be created. Walls, counters and computing areas created in this way are **initialised**, by opposition to those already present at time 0.

Walls persist over time and are only destroyed under two circumstances:

- if it can be determined that it is uninitialised (e.g. it is without a time counter to its left);
- by the merging process detailed in Section 1.2.4.

As the only exception, if a segment is of length three at time 0, then the leftmost I prevents the creation of a time counter for the rightmost wall at time 1 and the wall itself is destroyed at time 2. Thus segments have minimum length four from time 2 onwards.

Counters

All counters are binary in a redundant basis, so that they can be incremented by one at each step (keeping track of current time) in a local manner. Notice that in the following two definitions, the indexing of the letters is inverted.

Definition 1.2.2 (Redundant binary basis)**.** Let $u = u_0 \dots u_{n-1} \in \{0,1,2\}^*$. The **value** of u is

$$val(u) = \sum_{i=0}^{n-1} u_i 2^i.$$

Since the basis is redundant, different counters can have the same value.

Definition 1.2.3 (Incrementation)**.** The incrementation operation $inc : \{0,1,2\}^* \mapsto \{0,1,2\}^*$ is defined in the following way. If $u_{|u|-1} = 2$, then $|inc(u)| = |u| + 1$, $|u|$ otherwise, and:

$$inc(u)_i = \begin{cases} 1 & \text{if } i = |u| \text{ and } u_{|u|-1} = 2; \\ u_i \mod 2 + 1 & \text{if } i = 0 \text{ or } u_{i-1} = 2; \\ u_i \mod 2 & \text{otherwise.} \end{cases}$$

Intuitively, the counter is increased by one at the rightmost bit and 2 behaves as a carry propagating along the counter. If the most significant bit was a carry, the length of the counter is increased by one. Thus:

Fact 1. $val(inc(u)) = val(u) + 1$.

This operation is defined locally and can be seen as the local rule of a cellular automaton.

Time

We use the alphabet $\mathcal{A}_{\texttt{time}} = \{0,1,2,\#\}$. In a configuration, a time counter is a word of maximal length containing no $\#$ in the time layer. A time counter is **attached** if it is bounded on its right by a wall $\boxed{\text{W}}$, **detached** otherwise.

At each step, attached counters are incremented by one while detached counters have their rightmost bit deleted (see Figure 1.4). Indeed, detached counters are uninitialised and can be safely deleted. Formally,

- if $u_1 = \boxed{\text{W}}$, then $time(F(u)_0) = time(u_0) \mod 2 + 1$;
- if $time(u_1) = \#$, then $time(F(u)_0) = \#$;
- otherwise, follow the incrementation definition (Definition 1.2.3).

When a counter increases in length, it may erase a wall by overwriting it. However, this is not a problem, as we shall see in Facts 2 and 6.

Fact 2. An initialised wall cannot be erased by a detached time counter.

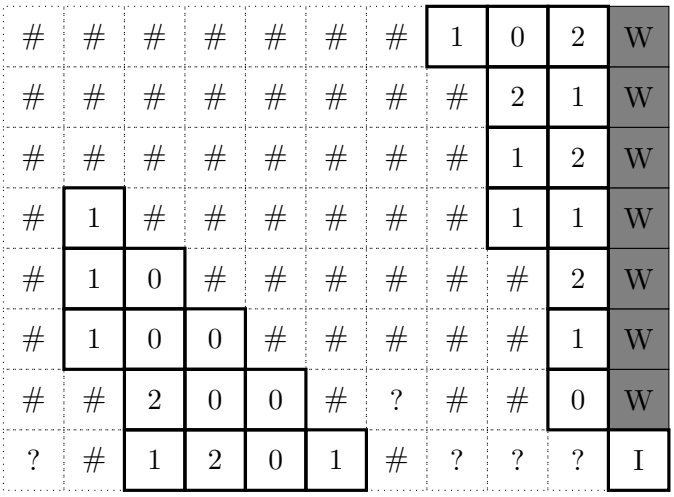


Figure 1.4: A detached time counter, and a time counter attached to an initialised wall. Only the time layer is represented. ? cells have arbitrary values.

Proof. A detached time counter is not incremented and can extend by one cell at most because of the carries initially present in the word. But $\boxed{\text{I}}$ symbols erase two cells to their right at initialisation. □

Fact 3. Let $x \in \mathcal{A}^{\mathbb{Z}}$ be the initial configuration. Each attached time counter u in $F^t(x)$ satisfies $val(u) \geq t - 1$, the equality being attained if this counter is attached to an initialised wall.

Proof. No time counter is created except at $t = 1$ (by $\boxed{\text{I}}$). Therefore such a counter was present either in the initial configuration (with a nonnegative value), or was created at $t = 1$ by a $\boxed{\text{I}}$ symbol. It is incremented by one at each step in both cases. □

Thus we can use time counters to tell apart initialised walls from uninitialised walls, which is the object of the next section.

Formatting and comparisons

Formatting counters are defined and incremented at each step in a similar way as time counters, but they have a range of different behaviours. The formatting layer is decomposed into two layers $\mathcal{A}_{\texttt{state}}$ and $\mathcal{A}_{\texttt{value}}$. A formatting counter is a word of maximal length of state different than #. The possible states of each bit are:

"Go" state The counter progresses at speed one to the right.

"Stop" state Once a wall is encountered, the counter progressively (right to left) stops.

Comparison states Once the whole counter has stopped, we locally compare the formatting counter and the time counter, left to right, with a method we describe later.

The wall is destroyed if the formatting counter is strictly younger, and the formatting counter is destroyed otherwise (see Figures 1.6 and 1.7). In the former case, the counter progressively returns to the "Go" state.

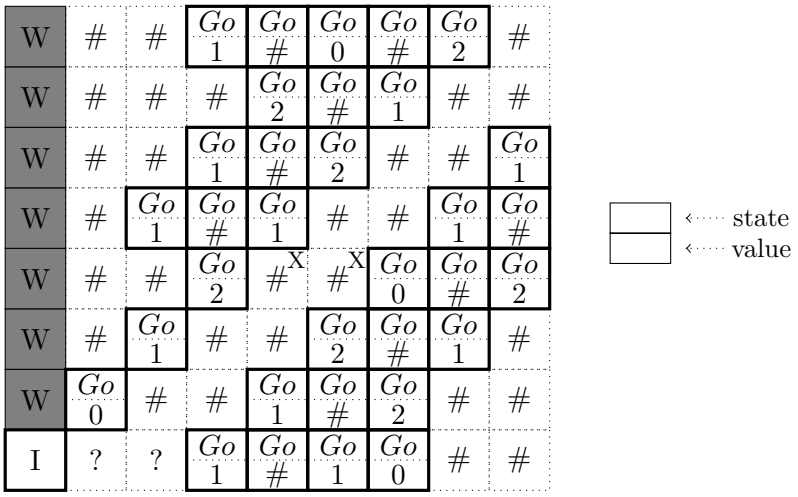


Figure 1.5: One initialised and one uninitialised formatting counter. X symbols mark the cells where values are prevented to appear to avoid merging: the right counter is dominated. Only the formatting layer is represented.

Changing state takes some time to propagate the information along the counter. Therefore, counters passing from a "Go" state to a "Stop" state are temporarily in a situation where the left part of the counter progresses whereas the right part has stopped. To avoid erasing information, counters in a "Go" state have **buffers**, i.e. the value of the counter is only written on half the cells, the other half containing $(Go, \#)$ (see Figure 1.5).

When its length increase, a counter never merges with another counter, erasing bits from the right-hand counter instead in order to avoid merging: we say the right-hand counter is **dominated**. Notice that it is impossible for a counter located to the right of another counter to be initialised, and so it is safe to erase bits of it.

Fact 4. Let $x \in \mathcal{A}^{\mathbb{Z}}$ be the initial configuration. Any non-dominated formatting counter u of $F^t(x)$ satisfies $val(u) \geq t - 1$, the equality being attained if the counter is initialised.

Proof. Similar to Fact 3. □

Thus we guarantee that an initialised (hence non-dominated) formatting counter is strictly younger than any uninitialised wall, and symmetrically. Uninitialised formatting counters can only progress to the right to be destroyed by the nearest initialised wall. We will see that dominated counters, whose value is arbitrary, are not a problem since they are erased before any comparison takes place.

Definition 1.2.4 (Comparison method)**.** Let $u = u_0u_1\dots$ and $v = v_0v_1\dots$ be two counters in redundant binary basis (adding zeroes so that $|u| = |v|$). Let us note $sign(u - v)$ the result of the comparison between u and v, that is, $+, 0$ or $-$.

Case 1 if $|u| = |v| = 1$, $sign(u - v) = sign(u_0 - v_0)$;

Case 2 if $u_0 + \lfloor u_1/2 \rfloor > v_0 + \lfloor v_1/2 \rfloor + 1$, then $sign(u - v) = +$,
and symmetrically;

Case 3 if $u_0 + \lfloor u_1/2 \rfloor = v_0 + \lfloor v_1/2 \rfloor + \varepsilon$ (for some $\varepsilon \in \{-1, 0, 1\}$), then $sign(u - v) = sign((u'_1 + 2\varepsilon)u_2 \cdots - v'_1 v_2 \dots)$,
where $u'_1 = u_1 \mod 2$ and $v'_1 = v_1 \mod 2$.

In other words, we do a bit-by-bit comparison starting from the most significant bit, considering that $\# = 0$, and taking into account the carry propagation "in advance", so that the incrementation and carry propagation can continue during the comparison. When the "local difference" ε is too small, the result cannot be determined locally and a remainder is carried (consider a comparison between $120\cdots0$ and $11\cdots12$).

Formally, for each pair of bits (u_n, v_n), we add 1 to each bit if the following bit of the corresponding counter is 2, and depending on the value of $u_n - v_n + 2\varepsilon$:

result	< -1	-1	0	$+1$	$> +1$
new state	$-$	$=_-$	$=$	$=_+$	$+$

If the result can be determined locally (cases 1 and 2), the state is changed to $+$ or $-$, and the result propagates to the right without further comparisons. Otherwise (case 3), the state changes to $=$, which means future bit comparisons will decide the result in the same way. If there is a remainder ε, it is remembered for the next comparison by having three states $=_-, =_+, =$. See Figure 1.7 for an example.

#	#	#	#	=	Go	Go	Go
#	#	#	#	=	−	Go	W
#	#	#	#	=	−	−	W
#	#	#	#	=	−	Stop	W
#	#	#	#	=	Stop	Stop	W
#	#	#	#	Stop	Stop	Stop	W
#	#	#	Go	Go	Stop	Stop	W
#	#	Go	Go	Go	Go	Stop	W
#	Go	Go	Go	Go	Go	#	W

Figure 1.6: A younger formatting counter encountering an older wall, which is destroyed. Only the state layer of $\mathcal{A}_{\texttt{format}}$ is represented, with greyed words for buffers.

After the comparison, two cases are possible:

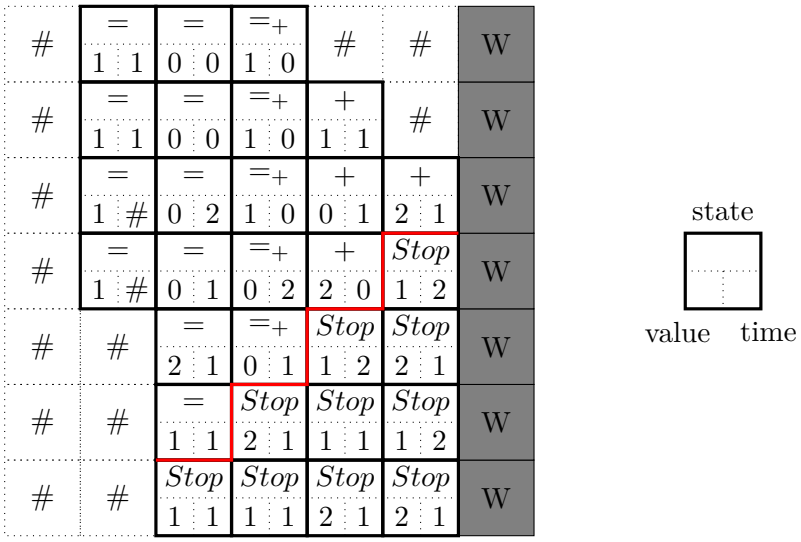


Figure 1.7: The comparison process in detail. Here the formatting counter is older than the time counter and is destroyed. Only the layers $\mathcal{A}_{\texttt{time}}$ and $\mathcal{A}_{\texttt{format}}$ are represented.

- if the state of the rightmost bit is $-$ or $=_-$, the wall is strictly older than the counter. The wall is destroyed and the state of the rightmost bit becomes "Go". The counter then progressively returns to the "Go" state.
- if the state of the rightmost bit is $+$, $=_+$ or $=$, the wall is younger than the counter. The rightmost bit is erased, and the rest of the counter is progressively erased in a similar way as a detached time counter.

The second case covers the case where both the counter and the wall are initialised (result $=$), which means that the formatting counter has finished formatting its segment and may be safely erased. Also, if the counter is dominated, then its leftmost bit is erased at each step, preventing the comparison to start, until the counter is entirely erased.

Finally, $\mathcal{A}_{\texttt{format}} = \{\#\} \cup (\{Go\} \times \{0,1,2,\#\}) \cup (\{Stop,+,-,=,=_+,=_-\} \times \{0,1,2\})$.

When a formatting counter reaches the right wall of the segment, the segment is said to be **formatted**. This implies that the segment contains no more uninitialised walls. Uninitialised merging counters are destroyed in exactly the same way as uninitialised time counters. To prevent uninitialised merging signals from disturbing a merging process, any right merging signal $\rightarrow$ erase incoming left merging signals $\leftarrow$. Merging signals arriving to a wall outside of a merging process is simply ignored and destroyed.

Fact 5. At time $k(1+\lceil \log k \rceil)$, all segments of length k (for $k > 3$) are formatted.

> *Proof.* As long as $t \leq k(1+\lceil \log k \rceil)$, any initialised formatting counter has length $\lceil \log t \rceil \leq 2\lceil \log k \rceil$ (excluding the buffers) since it is in base 2. The counter progresses at speed one except when it meets another wall. Each comparison takes a time equal to twice the current length of the counter (again excluding the buffers). Furthermore, two consecutive walls

are separated by three cells at least (cf. Section 1.2.2). Thus, the segment is formatted in less than $k + \frac{k}{4} \cdot 2 \cdot 2\lceil \log k \rceil$ steps, which is coherent with our first assumption. □

Fact 6. An initialised wall cannot be erased by a time counter attached to a uninitialised wall.

Proof. Consider two walls, the left being initialised and the right uninitialised. As explained in Section 1.2.2, we can assume they are separated by $k > 3$ cells. Since the value of the time counter attached to the right wall cannot exceed 2^{k-3} at $t = 1$ (since $\boxed{\text{I}}$ erases three cells to its right), it takes more than $2^k - 2^{k-3}$ steps before the left wall is erased. According to Fact 5, the right wall is destroyed in less than $k(1 + \lceil \log k \rceil)$ steps, and the time counter takes at most k more steps to be erased.

For $k \geq 5$, $k(1 + \log k) + k \leq 2^k - 2^{k-3}$, so the counter is erased before it reaches the left wall. For $k = 4$, any wall between them is destroyed at time 1, so the destruction time is actually less than $k + 2 \log k + k \leq 2^k - 2^{k-3}$. □

1.2.3 Computation and copy

Simulating a Turing machine in a cellular automaton

Let $\mathcal{TM} = (Q, \Gamma, \#, q_0, \delta, Q_F)$ be a Turing machine. We simulate this machine in a cellular automaton F on the alphabet $(\Gamma \cup \#) \times (Q \cup \#)$. The left part contains the content of the tape; the right part contains the state of the machine for the cell where the head is located, and # everywhere else.

The local rule of F is governed by the rules of the machine. That is, for all $u \in ((\Gamma \cup \#) \times (Q \cup \#))^{\mathbb{Z}}$, and writing _ to denote an arbitrary value:

- if the head is on u_0 and $\delta(u_0) = (q, \gamma, \_)$, then $F(u)_0 = (\gamma, q)$;
- if the head is on u_1, $\delta(u_1) = (q, \_, \leftarrow)$ and $u_0 = (\gamma', \#)$, then $F(u)_0 = (\gamma', q)$;
- similarly if the head is on u_{-1} and $\delta(u_{-1}) = (q, \_, \rightarrow)$;
- otherwise, $F(u)_0 = u_0$.

When starting from a configuration filled with $(\#, \#)$ everywhere except for a finite window with only one head, the time evolution of the cellular automaton matches the time evolution the Turing machine. When the machine has stopped (the state being in Q_F), the local rule is the identity function.

Computation

Computation takes place to the left of each initialised wall. $\mathcal{A}_{\texttt{comp}}$ is divided into three layers, on which three Turing machines are simulated, using the alphabet $\mathcal{A}_{\texttt{comp}} = \bigotimes_{i=1}^{3} (\Gamma_i \cup \#) \times (Q_i \cup \#)$. We adapt the simulation so that these Turing machines can read input from or write output to another layer (when indicated).

We now describe the operations to be performed during the time interval $[T_{n-1}, T_n]$. Assume that, at time T_{n-1}, n is already written on the layer 1 and T_{n-1} on layer 3. The machines:

1. replace n by $n + 1$ on layer 1 and stops;

2. compute w_n on layer 2, outputting it on the main layer, and stops;
3. compute T_n on layer 3, and stops;

When $t = T_n$ (t being read from the time layer), the copying process triggers and the next computation starts, except when merging occurs; see next subsections.

All these operations must be performed in less than $T_n - T_{n-1}$ steps. We now fix the value of T_n so that it is indeed possible.

A Turing machine with tape alphabet Γ and set of states Q and using only a computational space S_n stops in time $S_n \cdot |\Gamma|^{S_n} \cdot |Q|$ (number of possible states); otherwise, the same state would be reached twice, entering a loop.

Therefore there exists a constant $q > 0$ large enough that the operations on layers 1 and 2 can be performed in space $\lfloor\sqrt{n}\rfloor \log_2 q$ and time $O(q^{\lfloor\sqrt{n}\rfloor})$. Furthermore, the function $(r, n) \mapsto r^{\lfloor\sqrt{n}\rfloor}$ is computable in space $\lfloor\sqrt{n}\rfloor \log_2 r$ (length of the output) and time $O(n^{3/2}(\log r)^2)$ (compute $\lfloor\sqrt{n}\rfloor$ in time $O(n)$, then perform $\lfloor\sqrt{n}\rfloor$ multiplications between numbers of length $\lfloor\sqrt{n}\rfloor \log_2 r$ at most in time $O((\sqrt{n}\log_2 r)^2)$).

In other words, if we fix

$$T_n - T_{n-1} = q^{\lfloor\sqrt{n}\rfloor},$$

then the operation on layer 3 can be performed in space $\lfloor\sqrt{n}\rfloor \log_2 q$ and time $O(q^{\lfloor\sqrt{n}\rfloor})$. However, we need an upper bound on the time at each step and not only an asymptotic bound. This is solved by the linear speedup theorem for Turing machines: we can divide the computational time by any fixed constant C by replacing each machine M_i by a new machine M_i', such that M_i' performs C computational steps of M_i at each step, increasing the radius as necessary.

Remark. We fixed T_n so that the computation space is of size $\sqrt{n}$ at time T_n and constitutes an asymptotically negligible fraction of its segment. We could choose instead of $\sqrt{n}$ any other easily computable function which is $o(n)$.

Uninitialised computation states self-destroy, similarly to time counters, whenever they find an empty computational layer to their right (instead of a wall). This requires that the Turing machines are adapted so that they never write $(\#, \#)$ is the middle of a computation.

Copying

At time T_n ($n \geq 0$), w_n has been output on the main layer, followed by a $\#$. If the segment is not in the process of merging, repeated copies of w_n have to be written over the main layer. The Turing machine triggers the copying process by copying the rightmost letter of w_n from the main layer to the copy layer.

First phase As long as it has not met a $\#$, the word on the copy layer progresses at speed -2, and at each step a letter is copied from the main layer to the tail of the word;

Second phase The word keeps progressing at speed -2 but the head loses one letter at each step and copies it on the main layer. The tail keeps copying letters from the main layer.

Intuitively, the cellular automaton performs a caterpillar-like movement between the copy and main layers (see Figure 1.8 for an example). The process ends when it meets a wall. Thus, $\mathcal{A}_{\mathsf{copy}} = \mathcal{B} \cup \{\#\}$.

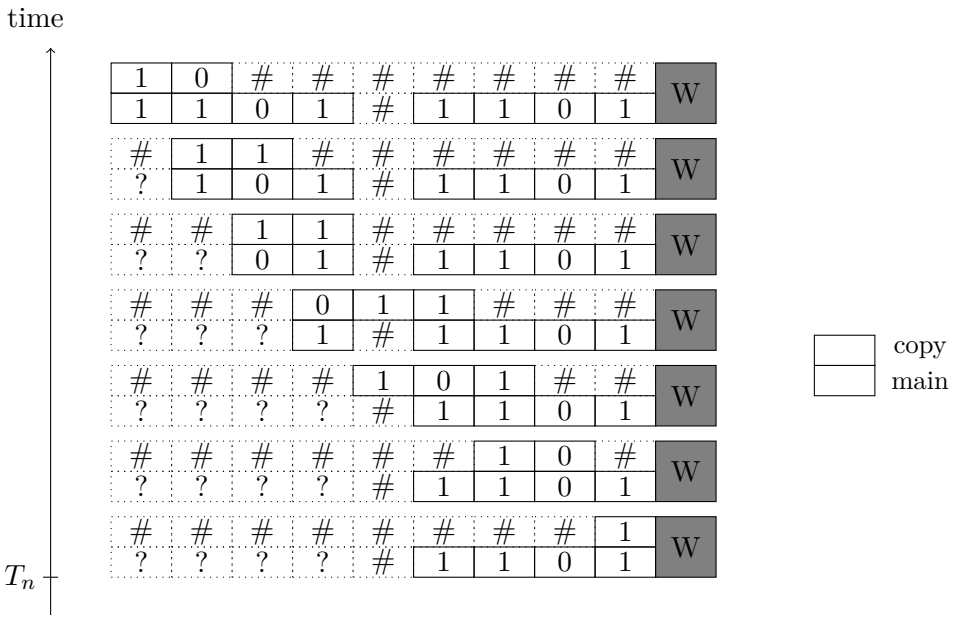


Figure 1.8: Beginning of the copying process, with $w_n = 1101$. Only the layers $\mathcal{A}_{\texttt{copy}}$ and $\mathcal{A}_{\texttt{main}}$ are represented.

Uninitialised copying processes may write arbitrary words on the main layer, but they progress to the left at speed one and are destroyed by the nearest wall in this direction.

1.2.4 Merging

At time T_n, all segments of length n are forced to merge with their left neighbour, so that the density of walls tends to 0. This means that merging is performed at time T_n between a segment larger than n to the left, and any number of consecutive segments of length n to the right. To determine the length of each segment, a signal is sent to the right and bounces off the right wall, and its return time is measured.

To do so, a **merging counter** of value $2n$ is initialised at time T_{n-1} on the merge layer. The value of n is copied from the first computing layer to the merge layer (with an additional 0 at the end), using an auxiliary state $\boxed{\text{C}}$ (**copy**). This counter is decrementing at each step, similarly to incrementing counters except it uses -1 as negative carry. See Figure 1.9 for an example of this process.

If the signal returns at the end of the decrementation, a symbol $\boxed{\text{I}}$ (**merge**) is created on the merge layer, to indicate that the wall will be destroyed at the next T_n; otherwise, the output is copied in the main layer as described above. Thus $\mathcal{A}_{\texttt{merge}} = \{-1, 0, 1, \boxed{\text{I}}, \boxed{\text{C}}\} \times \{\rightarrow, \leftarrow\} \cup \{\#\}$.

Fact 7. Left walls of segments of length ℓ are erased at time $T'_\ell = \min(T_\ell, 2^\ell + \ell)$.

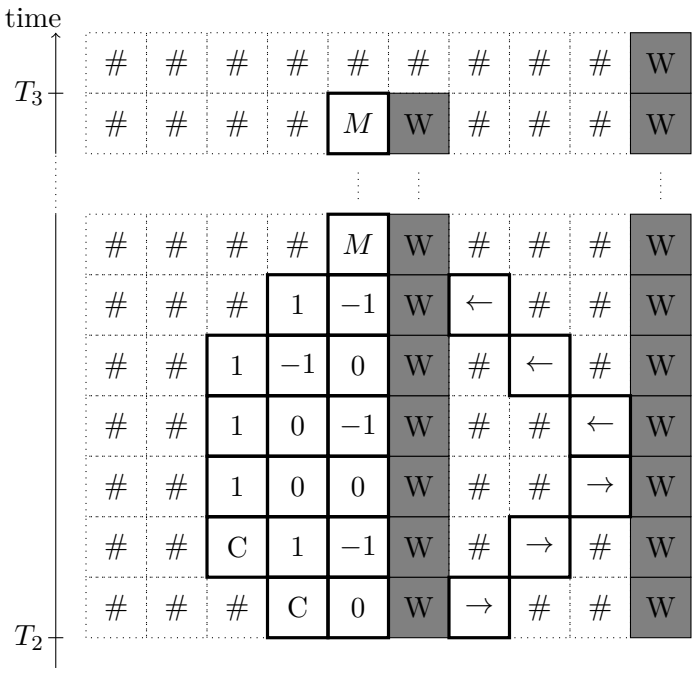


Figure 1.9: Determination of the length of a segment. Here the right segment is of length 3 and the two segments merge at time T_3. Only the merging layer is represented, with the counter of the right segment omitted for clarity.

Proof. Except for the situation described above, the only other way for an initialised wall to be erased is a time counter attached to an initialised wall, see Facts 2 and 6. A redundant binary counter whose initial value is 0 reaches length ℓ at time $2^\ell + \ell$ (carry propagation). □

Uninitialised merging counters are destroyed in exactly the same way as uninitialised time counters. To prevent uninitialised merging signals from disturbing a merging process, any right merging signal $\rightarrow$ erase incoming left merging signals $\leftarrow$. Merging signals arriving to a wall outside of a merging process is simply ignored and destroyed.

1.2.5 Correctness of the construction

To sum up, we have two time sequences $(T_n)_{n\in\mathbb{N}}$ and $(T'_n)_{n\in\mathbb{N}}$ such that:

- at time T_n, the computation of w_n is finished and the copy starts;
- at time T'_n, the segments of length n merge with their left neighbour.

Furthermore, those sequences are equal for n large enough.

The computation, copy and merging processes described in the previous section have to be performed between time T_n and time T_{n+1}, which requires that the segments are not too large. In this section, we control the length of segments at time T_n.

Proposition 1.2.2. $T_n = \Theta(\lfloor\sqrt{n}\rfloor q^{\lfloor\sqrt{n}\rfloor})$ where q is defined in Section 1.2.3.

Proof. $T_n = \sum_{k=1}^{n} T_k - T_{k-1}$. Since $T_{k+1} - T_k = q^{\lfloor\sqrt{k}\rfloor}$, and:

$$(2\lfloor\sqrt{n}\rfloor-1)q^{\lfloor\sqrt{n}\rfloor-1} \leq \sum_{k=1}^{\lfloor\sqrt{n}\rfloor-1}(2k+1)q^k \leq \sum_{k=1}^{n} q^{\lfloor\sqrt{k}\rfloor} \leq \sum_{k=1}^{\lfloor\sqrt{n}\rfloor}(2k+1)q^k \leq (2\lfloor\sqrt{n}\rfloor+1)q^{\lfloor\sqrt{n}\rfloor+1},$$

the proposition follows. □

Acceptable segments

Definition 1.2.5. Denote:

$$\Gamma^t_{l,k} = \left\{x \in \mathcal{A}^{\mathbb{Z}} : [0,l] \text{ is included in a segment of } F^t(x) \text{ of length } k\right\},$$

$$\Gamma^t_{l,\geq k} = \bigcup_{i\geq k} \Gamma^t_{l,i} \qquad \text{and} \qquad \Gamma^t_l = \Gamma^t_{l,\geq 1}.$$

Proposition 1.2.3 (Lower bound)**.**
Let $\mu \in \mathcal{M}^{\text{full}}_{\sigma-\text{erg}}(\mathcal{A}^{\mathbb{Z}})$. For all $l \in \mathbb{N}$, one has $\mu(\Gamma^{T_n}_{l,\geq n}) \underset{n\to\infty}{\longrightarrow} 1$.

Proof. $T_n = T'_n$ for n large enough, so we do the proof for T'_n. At time T'_n, no configuration can contain a segment smaller than n. Since μ has full support,

$$\mu\left(x \in \mathcal{A}^{\mathbb{Z}} : x_0 = \boxed{\text{I}} \text{ and } x_i \neq \boxed{\text{I}} \text{ for all } i \in [1,n]\right) \neq 0.$$

By σ-ergodicity, this means segments of length larger than n exist at $t = 0$ for μ-almost all configurations, and those segments survive up to time T'_n by construction.

Therefore, the cell 0 is μ-almost surely included in a segment at time T'_n. Since this segment has length larger than n and by σ-invariance, the probability that $[0,l]$ crosses a border of the segment tends to 0 as n tends to infinity. □

Definition 1.2.6.
Let $x \in \mathcal{A}^{\mathbb{Z}}$, $[i,j]$ a segment at time $t \in [T_n, T_{n+1}]$. It is **acceptable** if $j - i - 1 \leq K_n = \sqrt{T_{n+1} - T_n}$. For n large enough, $K_n = q^{\frac{\lfloor\sqrt{n}\rfloor}{2}}$.

Proposition 1.2.4 (Upper bound)**.**
Let $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$. One has $\mu(\Gamma^{T_n}_{l,\geq K_n}) \underset{n\to\infty}{\longrightarrow} 0$, that is to say:

$$\mu(\{x \in \mathcal{A}^{\mathbb{Z}} : [0,l] \text{ is in an acceptable segment of } F^t(x)\}) \underset{t\to\infty}{\longrightarrow} 1$$

and the rate of convergence is exponential.

Proof. Again, $T_n = T'_n$ for n large enough, so we do the proof for T'_n. Any segment at time T'_n corresponds to a segment larger than n merged with 0 or more consecutive segments of length n at time T'_{n-1} (only the left wall of segments of size n are destroyed at time T'_n). See Figure 1.2.4 for an illustration of this decomposition. For $l \leq n$, define:

$\Delta^t_{n,\alpha} = \{x \in \mathcal{A}^{\mathbb{Z}} : \text{starting from 0 there is a strip of } \alpha \text{ consecutive segments of size } n \text{ in } F^t(x)\}$

Suppose $[0, l]$ is included in a segment longer than k at time T'_n. This segment is issued from the merging of one segment with 0 or more segments of length $n-1$ at time T'_{n-1}. Take any $L > 2n$ and distinguish the two following cases concerning the segments at time $T'_n - 1$ it is issued from:

- There were less than $\left\lfloor \frac{L}{n} \right\rfloor$ segments of length n: then the other segment is larger than $k - L$. By shifting the configuration by $L - l$ cells at most, we can ensure that $[0, l]$ is included in this segment at time T'_{n-1}.
- There were more than $\left\lfloor \frac{L}{n} \right\rfloor$ segments of length n. Therefore there is a strip of $\left\lfloor \frac{L}{n} \right\rfloor$ segments of length n starting at some $j \in [-k, k]$.

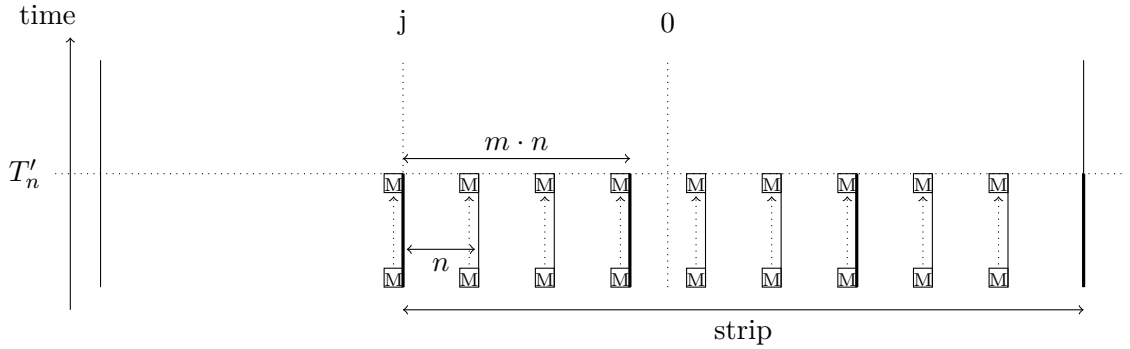


Figure 1.10: Illustration of the proof of Proposition 1.2.4 with $\alpha = 9$ and $m = 3$.

In other words,

$$\Gamma^{T'_n}_{l,\geq k} \subset \bigcup_{i=-L+l}^{0} \sigma^i \left(\Gamma^{T'_{n-1}}_{l,\geq k-L} \right) \cup \bigcup_{j=-k+1}^{k-1} \sigma^j \left(\Delta^{T'_{n-1}}_{n,\lfloor \frac{L}{n} \rfloor} \right)$$

$$\mu \left(\Gamma^{T'_n}_{l,\geq k} \right) \leq L\mu \left(\Gamma^{T'_{n-1}}_{l,\geq k-L} \right) + 2k\mu \left(\Delta^{T'_{n-1}}_{n,\lfloor \frac{L}{n} \rfloor} \right) \tag{1.1}$$

Thus we try to bound the value of $\mu(\Delta^t_{n,\alpha})$. If $x \in \Delta^t_{n,\alpha}$, then $x_k = \boxed{\text{I}}$ for all $k \in [0, \alpha n]$ such that $n|k$. For any $m > 0$, by considering one symbol out of every m:

$$\begin{aligned} \mu \left(\Delta^t_{n,\alpha} \right) &\leq \mu \left(\bigcap_{i=0}^{\alpha} \sigma^{in} \left(\left[\boxed{\text{I}} \right] \right) \right) \\ &\leq \mu \left(\bigcap_{i=0}^{\lfloor \frac{\alpha}{m} \rfloor} \sigma^{in \cdot m} \left(\left[\boxed{\text{I}} \right] \right) \right) \\ &\leq (1 + \psi_\mu(mn))^{\lfloor \frac{\alpha}{m} \rfloor} \mu \left(\left[\boxed{\text{I}} \right] \right)^{\lfloor \frac{\alpha}{m} \rfloor + 1}. \end{aligned} \tag{1.2}$$

Now take any $M \geq n$. Using (1.2) with $m = \left\lceil \frac{M}{n} \right\rceil$ inside equation (1.1):

$$\begin{aligned}\mu\left(\Gamma^{T'_n}_{l,\geq k}\right) &\leq L\mu\left(\Gamma^{T'_{n-1}}_{l,\geq k-L}\right) + 2k\left[1+\psi_\mu\left(n\cdot\left\lceil\frac{M}{n}\right\rceil\right)\right]^{\frac{L}{M}}\mu\left(\left[\boxed{\text{I}}\right]\right)^{\frac{L}{M}+1}\\ &\leq L\mu\left(\Gamma^{T'_{n-1}}_{l,\geq k-L}\right) + 2k\left[(1+\psi_\mu(M))\mu\left(\left[\boxed{\text{I}}\right]\right)\right]^{\frac{L}{M}}\end{aligned}$$

Now, if $k \geq nL$, we obtain by induction:

$$\mu\left(\Gamma^{T'_n}_{l,\geq k}\right) \leq L^n\mu\left(\Gamma^0_{l,\geq k-nL}\right) + 2kn\left[(1+\psi_\mu(M))\mu\left(\left[\boxed{\text{I}}\right]\right)\right]^{\frac{L}{M}} \tag{1.3}$$

For the left-hand term, we have:

$$\begin{aligned}\mu\left(\Gamma^0_{l,\geq k-nL}(x)\right) &\leq \mu\left(\mathcal{A}^{\mathbb{Z}}\Big\backslash \bigcap_{j=-k+nL}^{-1}\bigcup_{i=0}^{k-nL}\left[\boxed{\text{I}}\right]_{j+i}\right)\\ &\leq \mu\left(\bigcup_{j=-k+nL}^{-1}\bigcap_{i=0}^{\lfloor\frac{k-nL}{n}\rfloor}\left[\mathcal{A}\backslash\boxed{\text{I}}\right]_{j+in}\right)\\ &\leq (k-nL)(1+\psi_\mu(n))^{\lfloor\frac{k-nL}{n}\rfloor}\mu\left(\left[\mathcal{A}\backslash\boxed{\text{I}}\right]\right)^{\lfloor\frac{k-nL}{n}\rfloor+1}\end{aligned}$$

the second line being obtained by considering one symbol out of every n. Applying (1.3) with $M = n$, $L = n^2\sqrt{n}$, and $k = K_n = \sqrt{T_{n+1} - T_n}$, since $\psi_\mu(n) \to 0$, we have $\mu(\Gamma^{T'_n}_{l,\geq K_n}) \underset{n\to\infty}{\longrightarrow} 0$ and the rate of convergence is exponential. □

Density of auxiliary states

By auxiliary state, we mean any element of $\mathcal{A}\backslash\mathcal{B}$, that is to say $\boxed{\text{I}}$, $\boxed{\text{W}}$ and any element of $\mathcal{A}$ which is not of the form $(b, \#, \#, \#, \#, \#)$.

Proposition 1.2.5. For t large enough, an acceptable segment is formatted and contains only initialised processes.

Proof. In a segment of length k, Fact 5 ensures that the segment is formatted if $t \geq k(1 + \log k)$. All remaining uninitialised processes may take up to k more steps to be erased.

When $T_n \leq t < T_{n+1}$, for an acceptable segment of length k, we have $k(2 + \log k) \leq K_n(2 + \log(K_n)) = o(T_n)$ by Proposition 1.2.2. Taking n large enough, we conclude. □

Proposition 1.2.6. Let $\mu \in \mathcal{M}^{\text{full}}_{\sigma-\text{erg}}(\mathcal{A}^{\mathbb{Z}})$ and $u \in \mathcal{B}^{[0,\ell]}$ for some fixed ℓ. For a given length k such that $n + 1 \leq k \leq K_n$, we have:

- If $t \in [T_n + k, T_{n+1}]$,

$$\left|\mu\left(F^{-t}([u]) \mid \Gamma^{T_n}_{\ell,k}\right) - \widehat{\delta_{w_n}}([u])\right| = O\left(\frac{1}{\sqrt{n}}\right);$$

- If $t \in [T_n, T_n + k]$,

$$\left| \mu\left(F^{-t}([u]) \mid \Gamma^{T_n}_{\ell,k}\right) - \left(\frac{k-(t-T_n)}{k} \widehat{\delta_{w_{n-1}}}([u]) + \frac{t-T_n}{k} \widehat{\delta_{w_n}}([u]) \right) \right| = O\left(\frac{1}{\sqrt{n}}\right).$$

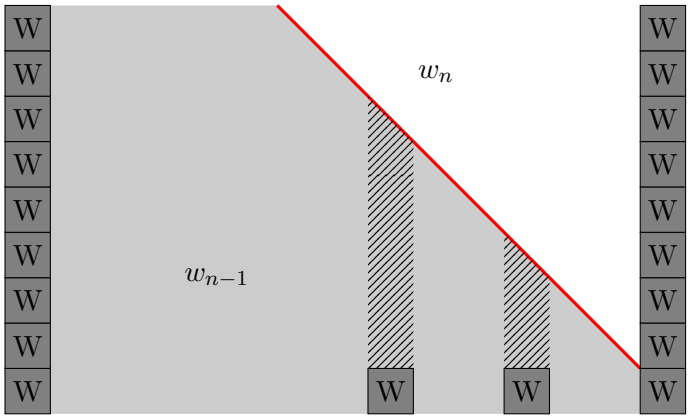


Figure 1.11: Illustration of Proposition 1.2.6. The output is not correctly written in dashed areas because of the destruction of a wall.

Proof. We write $\Gamma^{T_n}_{[i,i+k]} = \left\{ x \in \mathcal{A}^{\mathbb{Z}} \mid [i, i+k] \text{ is a segment at time } T_n \right\}$, so that

$$\Gamma^{T_n}_{\ell,k} = \bigsqcup_{i=-k+\ell+1}^{0} \Gamma^{T_n}_{[i,i+k+1]} = \bigsqcup_{i=0}^{k-\ell-1} \sigma^i\left(\Gamma^{T_n}_{[-1,k]}\right) \qquad \text{(disjoint union)}.$$

Suppose $x \in \Gamma^{T_n}_{[-1,k]}$. Since such a segment is acceptable, it is formatted as long as n is large enough, and any uninitialised counter or wall has been destroyed. Since $|w_n| = O(\sqrt{n})$ (smaller than the computing space), the copying process uses $O(\sqrt{n})$ auxiliary cells.

First point: The tail of the copying process progresses at speed one, so at time $T_n + k$ the copy of the word is finished (since $T_n + k \leq T_{n+1}$), and the segment contains only by copies of w_n except for the time counter, computation and merging counter area ($O(\sqrt{n})$ cells) and a merging signal (one cell).

Therefore for all $x \in \Gamma^{T_n}_{[-1,k]}$, one has $\left|\text{Freq}(u, F^t(x)_{[0,k-1]}) - \widehat{\delta_{w_n}}([u])\right| = \frac{O(\sqrt{n})}{k} = O\left(\frac{1}{\sqrt{n}}\right)$, taking into account the last copy of w_n in the segment which can be incomplete ($|w_n| \leq \sqrt{n}$), and since $k \geq n$. Thus we have

$$\left| \frac{1}{k} \sum_{i=0}^{k-1} \mu\left(F^{-t}([u]_i) \mid \Gamma^{T_n}_{[-1,k]}\right) - \widehat{\delta_{w_n}}([u]) \right| = O\left(\frac{1}{\sqrt{n}}\right).$$

Since μ is σ-invariant, $\mu\left(F^{-t}([u]_i) \mid \Gamma^{T_n}_{[-1,k]}\right) = \mu\left(F^{-t}([u]_0) \mid \Gamma^{T_n}_{[-1-i,k-i]}\right)$. So:

$$\begin{aligned}\mu\left(F^{-t}([u]_0) \mid \Gamma^{T_n}_{\ell,k}\right) &= \sum_{i=-k+\ell}^{-1} \mu\left(F^{-t}([u]_0) \mid \Gamma^{T_n}_{[-1-i,k-i]}\right) \cdot \mu\left(\Gamma^{T_n}_{[-1-i,k-i]} \mid \Gamma^{T_n}_{\ell,k}\right) \\ &= \frac{1}{k-\ell} \sum_{i=1}^{k-\ell} \mu\left(F^{-t}([u]_0) \mid \Gamma^{T_n}_{[i-1,i+k]}\right)\end{aligned}$$

by σ-invariance and disjoint union of $\Gamma^{T_n}_{\ell,k}$. The result follows.

Second point: When $t \in [T_n, T_n + k]$, the copy is still taking place, with $t - T_n$ cells containing copies of w_n and the rest containing copies of w_{n-1}, except for $O(\sqrt{n})$ various auxiliary states, and possibly defects when a wall has been destroyed at time T_n (there are at most $\frac{k}{n}$ of them). Therefore

$$\left|\mathrm{Freq}\left(u, F^t(x)_{[0,k-1]}\right) - \left(\frac{k-(t-T_n)}{k}\widehat{\delta_{w_{n-1}}}([u]) + \frac{t-T_n}{k}\widehat{\delta_{w_n}}([u])\right)\right| = \frac{1}{k}O(\sqrt{n})\cdot\frac{k}{n} = O\left(\frac{1}{\sqrt{n}}\right),$$

since $k \geq n$. Using the same reasoning as the first point, we conclude. □

Proof of Theorem 1.2.1 - first point

Let $\mu \in \mathcal{M}^{\mathrm{full}}_{\psi-\mathrm{mix}}(\mathcal{A}^{\mathbb{Z}})$ and $u \in \mathcal{B}^{[0,\ell]}$. By Propositions 1.2.3 and 1.2.4, $\mu\left(\bigcup_{k=n+1}^{K_n} \Gamma^{T'_n}_{\ell,k}\right) \underset{n\to\infty}{\longrightarrow} 1$ exponentially fast, and $\Gamma^t_{\ell,k} = \Gamma^{T'_n}_{\ell,k}$ for $t \in [T'_n, T'_{n+1} - 1]$. Therefore:

$$\exists C > 0, \max_{T_n \leq t < T_{n+1}} \left| F^t_*\mu([u]) - \sum_{k=n+1}^{K_n} \mu\left(F^{-t}([u]) | \Gamma^t_{\ell,k}\right) \mu\left(\Gamma^t_{\ell,k}\right) \right| = O\left(e^{-Cn}\right).$$

Take n large enough that $T_n = T'_n$. By Proposition 1.2.6,

$$\begin{aligned}\max_{T'_n \leq t < T'_{n+1}} \Bigg| F^t_*\mu([u]) - \sum_{k=n+1}^{K_n} \mu(\Gamma^{T_n}_{\ell,k}) \Bigg(&\max\left(0, \frac{k-(t-T_n)}{k}\right) \widehat{\delta_{w_{n-1}}}([u]) \\ &+ \min\left(1, \frac{t-T_n}{k}\right) \widehat{\delta_{w_n}}([u])\Bigg)\Bigg| = O\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

Let f_n be the piecewise linear function defined by:

$$\begin{array}{rccl} f_n: & [T_n, T_{n+1}] & \longrightarrow & [0,1] \\ & t & \longmapsto & \displaystyle\sum_{k=n+1}^{K_n} \min\left(1, \frac{t-T_n}{k}\right) \mu\left(\Gamma^{T_n}_{\ell,k}\right) + \frac{t-T_n}{T_{n+1}-T_n} \mu\left(\Gamma^{T_n}_{\ell,>K_n}\right). \end{array}$$

The second term is chosen so that $f_n(T_n) = 0$ and $f_n(T_{n+1}) = 1$, but it converges to 0 exponentially fast and thus does not affect the equation by more than $O\left(\frac{1}{\sqrt{n}}\right)$. Therefore:

$$\max_{T_n \leq t < T_{n+1}} \left| F^t_*\mu([u]) - \left(f_n(t)\widehat{\delta_{w_n}}([u]) + (1-f_n(t))\widehat{\delta_{w_{n-1}}}([u])\right)\right| = O\left(\frac{1}{\sqrt{n}}\right).$$

$$\max_{T_n \le t < T_{n+1}} d_{\mathcal{M}}\left(F_*^t\mu, \left[\widehat{\delta_{w_{n-1}}}, \widehat{\delta_{w_n}}\right]\right) = O\left(\frac{1}{\sqrt{n}}\right),$$

so $\mathcal{V}(F,\mu) \subset \mathcal{V}((w_n)_{n\in\mathbb{N}})$. Since f_n is $\frac{1}{n}$-Lipschitz on $[T_n, T_{n+1}]$, any $\nu \in \left[\widehat{\delta_{w_{n-1}}}, \widehat{\delta_{w_n}}\right]$ is at distance at most $\frac{1}{n}$ of an element of the form $\left(f_n(t)\widehat{\delta_{w_n}} + (1-f_n(t))\widehat{\delta_{w_{n-1}}}\right)$ for $T_n \le t < T_{n+1}$.

We conclude that $\mathcal{V}(F,\mu) = \mathcal{V}((w_n)_{n\in\mathbb{N}})$.

Rate of convergence For clarity, assume that w_n is computable in space $O(\sqrt{n})$ by repeating elements if necessary.

By Proposition 1.2.2 we have $T_n = \Theta(\lfloor\sqrt{n}\rfloor q^{\lfloor\sqrt{n}\rfloor})$ so, writing $n(t)$ the current value of n at time t, we have $n(t) = \Theta(\log(t)^2)$ and $O\left(\frac{1}{\sqrt{n(t)}}\right) = O\left(\frac{1}{\log t}\right)$.

We find that the rate of convergence is:

$$d_{\mathcal{M}}\left(F_*^t\mu, \mathcal{V}\left((w_n)_{n\in\mathbb{N}}\right)\right) \le d_{\mathcal{M}}\left(F_*^t\mu, \left[\widehat{\delta_{w_{n(t)-1}}}, \widehat{\delta_{w_{n(t)}}}\right]\right) + \sup_{\nu\in\left[\widehat{\delta_{w_{n(t)-1}}}, \widehat{\delta_{w_{n(t)}}}\right]} d_{\mathcal{M}}\left(\nu, \mathcal{V}\left((w_n)_{n\in\mathbb{N}}\right)\right)$$

$$= O\left(\frac{1}{\log(t)}\right) + \sup\left\{d_{\mathcal{M}}\left(\nu, \mathcal{V}\left((w_n)_{n\in\mathbb{N}}\right)\right) : \nu \in \bigcup_{n\ge n(t)} \left[\widehat{\delta_{w_n}}, \widehat{\delta_{w_{n+1}}}\right]\right\},$$

by the last proof.

Proof of Theorem 1.2.1 - second point

Assume that $\mathcal{V}((w_i)_{i\in\mathbb{N}}) = \{\nu\}$ and let F be the cellular automaton associated with this sequence as described above, and consider $\mu \in \mathcal{M}^{\text{full}}_{\sigma-\text{erg}}(\mathcal{A}^{\mathbb{Z}})$. Since μ is not assumed to be ψ-mixing, Proposition 1.2.4 does not apply, and there is no guarantee most segments are acceptable. However large segments are still rare; more precisely, $\mu(\Gamma^t_{0,\ge k}) \underset{k\to\infty}{\longrightarrow} 0$ for all t since all sets $\Gamma^t_{0,k}$ are disjoint.

CLAIM 1: $F_*^t\mu([\mathcal{A}\backslash\mathcal{B}]) \underset{t\to\infty}{\longrightarrow} 0$, *i.e., the density of auxiliary states tends to 0.*

Proof. Suppose we are in an initial segment of length k. Detached time counters, Turing machines and merging counters initially present are destroyed in less than k steps. Similarly, left merging signals and copy auxiliary states initially present progress at speed -1, so they are destroyed before time k. Any uninitialised wall is destroyed after $k(1+\log k)$ steps at most, and any counter attached to it are destroyed after less than k more steps. For all those states, the probability of apparition after time $k(2+\log k)$ is less than $\mu(\Gamma^0_{0,\ge k}) \underset{k\to\infty}{\longrightarrow} 0$.

At time T'_n, all segments are longer than n, so the density of initialised walls and initialised auxiliary states inside each segment is $O\left(\frac{\sqrt{n}}{n}\right)$.

Only uninitialised formatting counters and right merging signals remain. Inside each segment, call **non-formatted area** the interval between the initialised formatting counter of the left wall and the rightmost cell containing one of those two states. At each step, this area decreases by one cell to its left but may grow by one cell to its right as a counter or signal progresses. Notice that merging with other segments cannot increase this area since segments of length n at time T_n are formatted (see Figure 1.12).

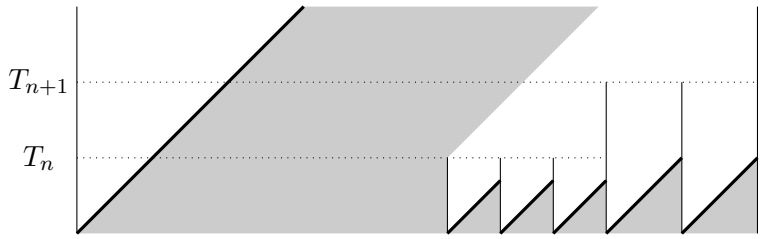


Figure 1.12: Illustration of the last part of the proof of Claim 1. Slanted lines are formatting counters and grey areas are potentially non-formatted.

Therefore, a segment at time T_n can contain a non-formatted area longer than $\sqrt{n}$ only if it is issued from a segment longer than $\sqrt{n}$ initially. Other segments have a non-formatted area smaller than $\sqrt{n}$ for a length larger than n. By σ-invariance,

$$\mu(\{x \in \mathcal{A}^{\mathbb{Z}} \mid x_0 \text{ is in a non-formatted area}\}) \leq \frac{\sqrt{n}}{n} + \mu\left(\Gamma^0_{0,\geq\sqrt{n}}\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, for $a \in \mathcal{A}\backslash\mathcal{B}$, we have $F^t_*\mu([a]) \underset{t\to\infty}{\to} 0$. □

CLAIM 2: *For any $n \in \mathbb{N}$, $d_{\mathcal{M}}\left(F^t_*\mu, Conv\left((\widehat{\delta_{w_i}})_{i\geq n}\right)\right) \underset{t\to\infty}{\longrightarrow} 0$, where $Conv(X)$ is the convex hull of the set X.*

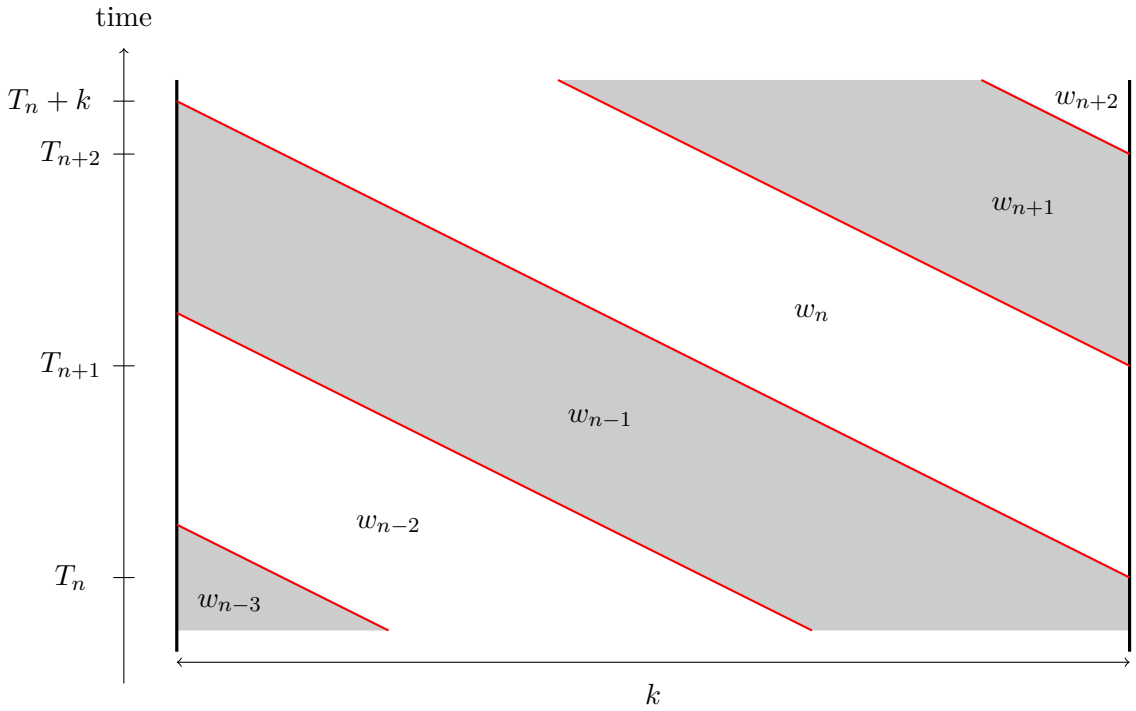


Figure 1.13: Illustration of Claim 2. When $t > T_n + k$, a segment of length k is a succession of stripes containing $w_n, w_{n+1}, \ldots$ plus a negligible part of auxiliary states and defects.

Proof. Consider a segment of length k at time T_n. At time $T_n + k$ the copying process for w_n is finished, but since the segment is not necessarily acceptable, other copying processes may have started in the meanwhile. Therefore, the segment contains:

- auxiliary states, with negligible frequency;
- strips containing repeated copies of w_n, then $w_{n+1}, w_{n+2}\ldots$ separated by ongoing copy processes (the frequency of auxiliary copy states being negligible).

See Figure 1.13. Since $\mu(\Gamma^{T_n}_{\ell,\geq k}) \underset{k\to\infty}{\longrightarrow} 0$, we have $d_{\mathcal{M}}\left(F_*^{T_n+t}\mu, Conv((\widehat{\delta_{w_i}})_{i\geq n})\right) \underset{t\to\infty}{\longrightarrow} 0$. □

The second point of the Theorem 1.2.1 follows easily from Claim 2.

Remark. It does not follow from the last claim that the sequence $(F_*^t\mu)$ is close to any of the $\widehat{\delta_{w_i}}$ at any point, which is the reason why the result holds only for a single measure. Controlling the length of the segments as needed in the proof of the first point requires ψ-mixing.

Section 1.3
Removing the auxiliary states

Before stating consequences of Theorem 1.2.1, we consider in this section the case where using auxiliary states are not allowed; that is, given a uniformly computable sequence of words on $\mathcal{B}$, we build a cellular automaton $\mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ that perform the same task as in the previous section.

A direct extension is impossible, for reasons that are detailed in the next section. However, if the limit measure does not have full support, the previous results can be extended by using a word not charged by the measure to encode the auxiliary states in a sense.

Theorem 1.3.1 (Realisation of a computable polygonal path of measures with no auxiliary states)**.**
Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly computable sequence of words of $\mathcal{B}^*$, where $\mathcal{B}$ is a finite alphabet, and assume there exists a word u that does not appear as subword in any of the w_n. Then there is a cellular automaton $F : \mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ such that for any measure $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{B}^{\mathbb{Z}})$, $\mathcal{V}(F,\mu) = \mathcal{V}((w_n)_{n\in\mathbb{N}})$.

However, because of the destructive nature of the formatting counter in the modified construction, the proof in Section 1.2.5 cannot be adapted and we cannot weaken the hypothesis to $\mu \in \mathcal{M}^{\text{full}}_{\sigma-\text{erg}}(\mathcal{B}^{\mathbb{Z}})$ when $\mathcal{K}$ is a singleton.

Let $\mathcal{A} \supset \mathcal{B}$ be the alphabet and F the CA associated to the sequence $(w_n)_{n\in\mathbb{N}}$ by Theorem 1.2.1. Our aim is to provide an encoding of any configuration of $\mathcal{A}^{\mathbb{Z}}$ in $\mathcal{B}^{\mathbb{Z}}$ and a cellular automaton F' that behaves similarly to F after encoding.

Denote $U_d \subset \mathcal{B}^d$ the set of words of length d with prefix u, that do not contain u as subword (except at the first letter), and that do not end with a prefix of u. $|U_d| \underset{d\to\infty}{\longrightarrow} \infty$, so for d large enough, we can find an injection $\varphi : \mathcal{A}\backslash\mathcal{B} \to U_d$ (encoding the auxiliary states), and we extend it by putting $\varphi = Id$ on $\mathcal{B}$. For a finite word, we define $\varphi(u_1 \dots u_n) = \varphi(u_1) \dots \varphi(u_n)$, and this can be naturally extended further to configurations $\Phi : \mathcal{A}^{\mathbb{Z}} \mapsto \mathcal{B}^{\mathbb{Z}}$ by considering that $\varphi(a_0)$ starts on the column zero. Notice that this encoding is not σ-invariant.

Let $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ be the set of configurations such that the word u does not appear on the main layer ($\mathbf{T}$ is a subshift of finite type). Since u marks unambiguously the beginning of a word of $\varphi(\mathcal{A}\backslash\mathcal{B})$, the restriction $\Phi : \mathbf{T} \to \mathcal{B}^{\mathbb{Z}}$ is injective.

We can define locally a decoding $\Psi : \Phi(\mathbf{T}) \to \mathbf{T}$ such that $\Psi \circ \Phi = \text{Id}$, by looking d cells to the right for occurrences of u. If u appears, we are an **output cell**, that is, the image by φ of a single letter $b \in \mathcal{B}$ (corresponding to $(b, \#, \#, \#, \#, \#)$ for $b \in \mathcal{B}$ in the previous construction); otherwise, we belong in an **auxiliary cluster**, the image by φ of a letter $A\backslash B$ that occupy d cells while containing one letter of output. See Figure 1.14 for an example.

Intuitively, we want to build a cellular automaton that behaves similarly as the automaton defined in Theorem 1.2.1, where elements $(b, \#, \#, \#, \#, \#)$ are represented by output cells

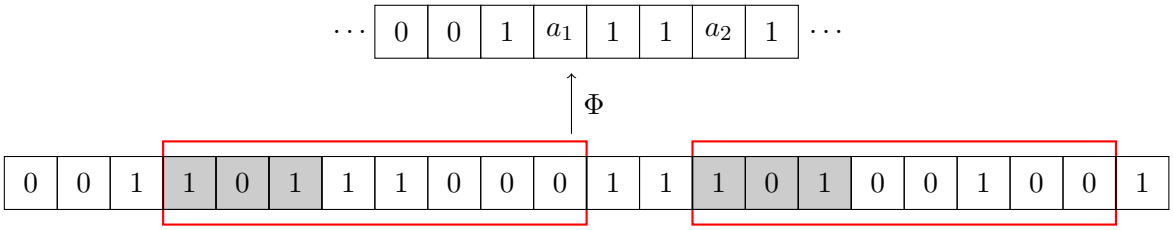


Figure 1.14: Encoding of the auxiliary states with $u = 101$ and $d = 3$. In this case $U_d \subset 101 \cdot \mathcal{A}^3 \cdot 00$.

and all other elements by auxiliary clusters. However, Φ and Ψ are not σ-invariant, so $\Phi \circ F \circ \Psi$ is not a cellular automaton. Instead, we build manually a cellular automaton on $\mathcal{B}^{\mathbb{Z}}$ that behaves in roughly the same way as $\Phi \circ F \circ \Psi$.

Provided the neighbourhood is larger than $[-4d, 4d]$, each cell can "read" the cluster in which it belongs, and the three clusters to its right and left. At time 0, if a word u is not the prefix of a word of U_d, it is replaced by a word b^d and can never be created again. To avoid creating an auxiliary cluster by mistake, we fix to this purpose a letter $b \in \mathcal{B}$ such that $b^d \notin U_d$. Similarly, auxiliary clusters that are destroyed for any reason leave behind them output b cells.

Remark. For clarity, in all diagrams of this section, we suppose that $\mathcal{B} = \{0, 1\}$, $d = 3$ (it would be much larger in a real implementation) and we represent auxiliary clusters as blocks with layers, instead of words from $\mathcal{B}^d$. Also we fix $b = 0$ in the definition above.

The different parts of the construction are modified in the following way.

- $\boxed{\text{I}}$ and $\boxed{\text{W}}$ clusters, time counters, and Turing machines have the same behaviour as in the previous construction. However, since the counters take more space, it is necessary to erase $3d$ cells to the left and right of each $\boxed{\text{I}}$ cluster at time 0.

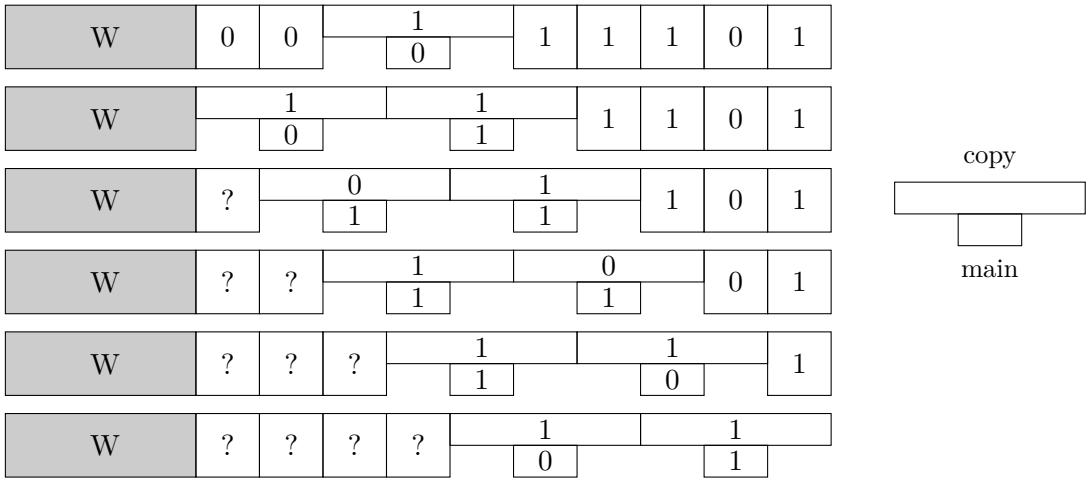


Figure 1.15: End of the copying process described in Figure 1.8, copying the word 1101.

- The tail of copying processes progresses to the left at speed one, and behaves normally as long as it does not meet another auxiliary state (see Figure 1.15). When the process has finished the copy, it is destroyed and leaves b cells behind.
- Formatting counters progress to the right at speed d. This is too fast to keep information on the output layer, so the counter leaves behind output cells b defined above. Any other signal it meets (e.g. copying process or length-measuring signal) is similarly erased.

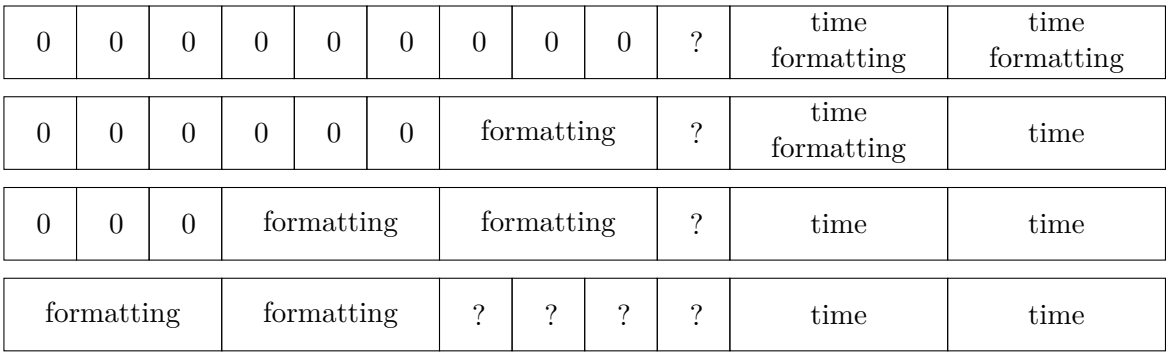


Figure 1.16: A formatting counter gets offset when entering the time counter area. Notice the auxiliary clusters being replaced by output cells containing $b = 0$.

- When close to a time counter, it may happen that the formatting counter cannot progress by d cells exactly (see Figure 1.16). In this case, it is offset by less than d cells, and

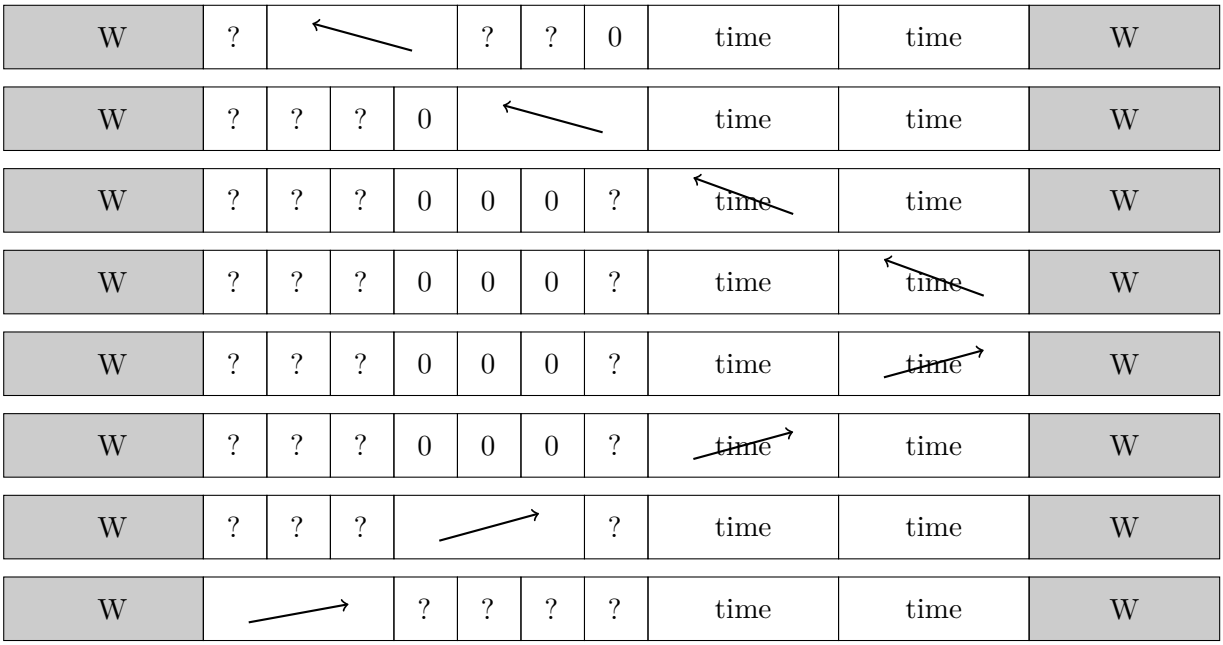


Figure 1.17: Determination of length. Here $d = 3$, $t_0 = 8$ and $o = 1$, for a measured length of 13.

formatting clusters separated by small offsets in this way are still considered to be the same counter for the rule of the automaton. The subsequent comparison process is unchanged.

- Merging signals which determine length of segments also progress at speed d. To avoid possible interactions with copying processes (similarly to the case of formatting counters), the determination of length starts only after the copy is finished. Thus a merging signal is only offset when entering the time counter area. After bouncing off the right wall, it returns to the left wall where its offset can be measured. If it takes t_0 steps to return with an offset of α, then the segment has length $\frac{t_0}{2} \cdot d + \alpha$ (see Figure 1.17). This value is compared to n and the rest of the process is not modified.

In this way, Propositions 1.2.4 and 1.2.5 can be extended. We can check that at time t, with $T_n \leq t < T_{n+1}$, the copy process followed by the process of determination of length for segments of size $n+1$ still take less than $T_{n+1} - T_n$ steps. Furthermore, the frequency of auxiliary states is multiplied by a fixed constant d. Hence the proof in Section 1.2.5 can be adapted, and the theorem follows.

Section 1.4
Problems solved with those constructions

In this section, we use the construction developed in Sections 1.2 and 1.3 to solve various problems, starting with the characterisation of reachable limit measures and μ-limit measure sets. We consider the case when auxiliary states are not allowed, the case of Cesàro mean convergence, consequences of these results for the decidability of asymptotic properties of cellular automata, and various other extensions.

1.4.1 Characterisation of reachable μ-limit measures sets

Previous results

The first effort towards understanding the computational content of the asymptotic behaviour of cellular automata is due to Hurd [Hur87], who provided examples of ω-limit sets whose language is not recursively enumerable. This work was continued in articles such as [Maa95] and more recently [BCV14], with no full characterisation being reached.

Concerning μ-limit sets, Boyer, Poupet and Theyssier were the first to use similar constructions to build computationally complex μ-limit sets [BPT06]. More recently, Boyer, Delacourt and Sablik proved the following:

Theorem 1.4.1 ([BDS10])**.**
Let $\mathcal{B}$ be an alphabet and (Σ_i) be a uniformly computable sequence of generable subshifts of $\mathcal{B}^{\mathbb{Z}}$, that is, assume that we have a Turing machine taking as input an integer i that writes letters from $\mathcal{B}$ on a bi-infinite tape, never stopping or erasing a letter, and such that the limit configuration x_i^∞ satisfies

$$\forall u \in \mathcal{A}^*, \mathrm{Freq}(u, x_i^\infty) \Leftrightarrow u \in \mathcal{L}(\Sigma_i).$$

Then there exists an alphabet $\mathcal{A} \supset \mathcal{B}$ and a cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that

$$\Lambda_\lambda(F) = \overline{\bigcup_i \Sigma_i},$$

where λ is the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$.

The notion of generable subshift is not unlike the notion of simulable measure, and includes such examples as transitive sofic subshifts and substitutive subshifts associated to a primitive substitution (see [FM10] for definitions). However, this result does not give a full characterisation.

Our construction is inspired by these articles, and an ongoing work by the same authors aims at reaching a full computability characterisation of μ-limit sets, using some techniques developed in the present thesis and new ideas [BDP$^+$].

The connected case

Reciprocals of the computable obstructions described in Section 1.1 follow directly from Theorem 1.2.1.

Corollary 1.4.2 (Sufficient conditions for being a single limit measure)**.**
Let $\nu \in \mathcal{M}_\sigma^{\text{s-comp}}(\mathcal{B}^{\mathbb{Z}})$ be a limit-computable measure. There is an alphabet $\mathcal{A} \supset \mathcal{B}$ and a cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that for any $\mu \in \mathcal{M}_{\sigma-\text{erg}}^{\text{full}}(\mathcal{A}^{\mathbb{Z}})$, one has $F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu$.

Proof. Combine Proposition 1.1.1 with the second point of Theorem 1.2.1. □

Corollary 1.4.3 (Sufficient conditions for being a μ-limit measures set)**.** Let $\mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$ be a compact, Π_2-computable and connected (Π_2-CCC) subset of $\mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$. There is an alphabet $\mathcal{A} \supset \mathcal{B}$ and a cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that for any $\mu \in \mathcal{M}_{\psi-\text{mix}}^{\text{full}}(\mathcal{A}^{\mathbb{Z}})$, one has $\mathcal{V}(F, \mu) = \mathcal{K}$.

These are in particular a full characterisation of limit measures and connected μ-limit measures sets that are reachable from some computable initial measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.

Proof. Combine Proposition 1.1.6 with the first point of Theorem 1.2.1. □

Open question. Can the rate of convergence be improved, or can we prove that this is the best possible rate?

Since Theorem 1.3.1 is a counterpart to the second point of Theorem 1.2.1 without auxiliary states, it is natural to give a similar counterpart to Corollary 1.4.3. Corollary 1.4.2 does not have a counterpart since its proof uses the second point of Theorem 1.2.1, which does not extend to the construction with no auxiliary states.

Definition 1.4.1. A word $u \in \mathcal{A}^*$ is said to be **charged** by a set $\mathcal{K} \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ if there exists $\nu \in \mathcal{K}$ such that $\nu([u]) > 0$.

Corollary 1.4.4 (Sufficient conditions for being a μ-limit measures set – no auxiliary states)**.**
Let $\mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$ be a non-empty Π_2-CCC subset of $\mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$ that does not charge a word $u \in \mathcal{B}^*$. Then there is a cellular automaton $F : \mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ such that for any measure $\mu \in \mathcal{M}_{\psi-\text{mix}}^{\text{full}}(\mathcal{B}^{\mathbb{Z}})$, $\mathcal{V}(F, \mu) = \mathcal{K}$. In particular, any limit-computable measure which does not have full support can be obtained this way.

Proof. Since $\mathcal{K}$ does not charge u, we can assume without loss of generality that no word in the uniformly computable sequence $(w_n)_{n\in\mathbb{N}}$ associated to $\mathcal{K}$ by Proposition 1.1.1 contains u as subword. Thus Theorem 1.3.1 applies. □

This leaves open in particular the case of limit measures with full support. A direct extension would be impossible, since if $F_*^t\mu$ converges to a limit measure with full support, the automaton F must be surjective. In particular, a surjective automaton leaves the uniform measure invariant (see Proposition 3.1.1).

Open question. Which sets of measures can be reached by surjective cellular automata?

On an unrelated note, the dynamics on the μ-limit measures set in our construction is always the identity. In the next section, we will enforce richer dynamics in order to reach non-connected μ-limit measure sets. We still do not get a good understanding of the possible dynamics of cellular automata on their μ-limit measures sets, which could lead to results similar to those of Di Lena and Margara on ω-limit sets [LM09].

Towards the non-connected case

In Corollary 1.4.3 the μ-limit measures set is assumed to be connected. Indeed, in the construction of Theorem 1.2.1, each word w_n is copied progressively on each segment, so that we reach the closure of an infinite polygonal path which is connected. However, non-connected μ-limit measures sets also have some topological obstructions. For example, if $\mathcal{V}(F,\mu)$ is finite, we have the following proposition.

Proposition 1.4.5. Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ such that $\mathcal{V}(F,\mu)$ is finite. Then F_* induces a cycle on $\mathcal{V}(F,\mu)$.

Proof. Let $d = \min\{d_{\mathcal{M}}(\nu,\nu') : \nu,\nu' \in \mathcal{V}(F,\mu) \text{ with } \nu \neq \nu'\} > 0$ and consider $\nu \in \mathcal{V}(F,\mu)$. It is possible to extract a sequence $(n_i)_{i\in\mathbb{N}}$ such that $d_{\mathcal{M}}(F_*^{n_i}\mu,\nu) < \frac{d}{3}$ and $d_{\mathcal{M}}(F_*^{n_i+1}\mu,\nu) > \frac{2d}{3}$. Since $d_{\mathcal{M}}(F_*^n\mu,\mathcal{V}(F,\mu)) \underset{n\to\infty}{\longrightarrow} 0$, we have $d_{\mathcal{M}}(F_*^{n_i}\mu,\nu) \underset{i\to\infty}{\longrightarrow} 0$. By continuity of F_*, $d_{\mathcal{M}}(F_*^{n_i+1}\mu,F_*\nu) \underset{i\to\infty}{\longrightarrow} 0$.

One deduces that for all $\nu \in \mathcal{V}(F,\mu)$ there exists $\nu' \in \mathcal{V}(F,\mu)$ such that $F_*\nu = \nu'$. So there is $k \in \mathbb{N}$ such that $\mathcal{V}(F,\mu) = \{\nu_0,\dots,\nu_{k-1}\}$ and $F_*\nu_i = \nu_{i+1}$ where the addition is modulo k. □

We exhibit some examples of more sophisticated behaviours based on the construction in Theorem 1.2.1. The first one is a family of cellular automata where $\mathcal{V}(F,\mu)$ is a finite set of connected components, which is a partial reciprocal of Proposition 1.4.5. The second one is a family of cellular automata where $\mathcal{V}(F,\mu)$ has an infinite number of connected components. However these are not total characterisations of the possible μ-limit measures sets.

Example (Finite set of connected components)**.** Suppose $\mathcal{K} = \{\nu_0,\dots,\nu_{k-1}\} \subset \mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$ is a finite set of σ-invariant limit-computable measures such that $G\nu_i = \nu_{i+1}$ for some periodic cellular automaton $G : \mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ ($G^p = Id$ for some $p \in \mathbb{N}$). Then there is an alphabet $\mathcal{A} \supset \mathcal{B}$ and a cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that $\mathcal{V}(F,\mu) = \mathcal{K}$ for $\mu \in \mathcal{M}^{\text{full}}_{\sigma-\text{erg}}(\mathcal{A}^{\mathbb{Z}})$. Indeed, let F be the cellular automaton satisfying $F_*^t\mu \to \nu_0$ obtained by Theorem 1.2.1, and consider the cellular automaton that applies G on the main layer and applies the local rule of F once every k steps.

The same idea holds if $\mathcal{K}$ is a finite union of Π_2-CCC sets which are mapped by a periodic cellular automaton $G : \mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$.

This is of course a very restricted case and any kind of general characterisation is still an open problem.

Example (Infinite set of connected components)**.** We give a sketch of a modification of the construction of Theorem 1.2.1 to obtain examples of cellular automata where $\mathcal{V}(F,\mu)$ has an infinite number of connected components. This is the first such example to our knowledge. The construction uses the firing squad cellular automaton introduced by Mazoyer [Maz96] $F_{\mathtt{FS}} : \mathcal{B}_{\mathtt{FS}} \to \mathcal{B}_{\mathtt{FS}}$, which has the following properties:

- the alphabet contains 4 states $\left\{ F \text{ (dashed)} , \text{ (solid)} , \text{ (dashed)} , O \text{ (solid)} \right\} \subset \mathcal{B}_{\mathtt{FS}}$;
- if $x_{[0,n]} =$ O (dashed) $\square^{n-1}$ (solid) then $F^{2n}_{\mathtt{FS}}(x)_{[0,n]} =$ F (dashed) $^{n+1}$;
- the state F (dashed) does not appear in $(F^t_{\mathtt{FS}}(x)_j)_{(t,j)\in[0,n]\times[0,2n-1]}$.

Consider a uniformly computable family $(\mathcal{K}_i)_{i\in\mathbb{N}}$ of disjoint Π_2-CCC subsets of $\mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$. There is a uniformly computable sequence of words $(w_n)_{n\in\mathbb{N}}$ of $\mathcal{B}^*$ such that $\mathcal{V}((w_n)_{n\in\mathbb{N}} = \bigcup_{i\in\mathbb{N}} \mathcal{K}_i$. Define $w'_n = w_n \times$ (dashed) $^{|w_n|} \in (\mathcal{B}\times\mathcal{B}_{\mathtt{FS}})^*$ and consider the cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ given by Theorem 1.2.1 which produces $\mathcal{V}((w'_n)_{n\in\mathbb{N}})$, with $\mathcal{A} \supset \mathcal{B}\times\mathcal{B}_{\mathtt{FS}}$. We modify F to obtain $\widetilde{F}$ in the following way.

- at time T_n, when the copy of w_n is initiated, we initialise a counter on another layer to count the length k of the segment;
- at time $t = T_{n+1} - 2k$, the state O (solid) appears on the left border of each segment (this is a computable number and the time counter keeps track of current time);
- All F (dashed) symbols are immediately transformed into (dashed) symbols.

This requires the segments to be shorter than $T_{n+1} - T_n$ cells, but the probability that $[0,l]$ belongs to such a segment tends to 1 as time tends to infinity (Proposition 1.2.4). Furthermore, the state F (dashed) appears only at times $(T_n)_{n\in\mathbb{N}}$. Therefore, in those segments, $\widetilde{F_*\mu}$ approximates the measure $\widehat{\delta_{w_n}} \times \widehat{\delta \text{ F (dashed)}}$ at time T_{n+1} and the measure $\widehat{\delta_{w_n}} \times \widehat{\delta \text{ (dashed)}}$ at time $T_{n+1}+1$.

For an initial measure $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$, one has $\mathcal{V}(\widetilde{F},\mu) = (\bigcup_i \mathcal{K}_i) \times \widehat{\delta \text{ F (dashed)}} \cup \mathcal{K}'$ with $\mathcal{K}' \subset \mathcal{M}_\sigma\left(\mathcal{B}\times\left(\mathcal{B}_{\mathtt{FS}} \setminus \{ F \text{ (dashed)} \}\right)^{\mathbb{Z}}\right)$. In particular, $\mathcal{V}(\widetilde{F},\mu)$ has an infinite number of connected components.

Open question. Is it possible to characterise all compact subsets of $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ that can be reached as μ-limit measures set of some cellular automaton when μ is computable?

1.4.2 Convergence in Cesàro mean

In this section, by adapting the enumeration $(w_n)_{n\in\mathbb{N}}$, we obtain similar results on $\mathcal{V}'(F,\mu)$, the set of limit points for the Cesàro mean sequence.

Corollary 1.4.6 (Sufficient conditions for being a μ-limit measures set in Cesàro mean)**.** Let $\mathcal{B}$ be a finite alphabet and $\mathcal{K}' \subset \mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$ a Π_2-CCC set. There exist an alphabet

$\mathcal{A} \supset \mathcal{B}$, and a cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that for any $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$, one has $\mathcal{V}'(F, \mu) = \mathcal{K}'$.

$\mathcal{V}'(F, \mu)$ is connected by Proposition 0.1.9, and if we suppose that the initial measure μ is computable, we obtain a full characterisation of reachable subsets $\mathcal{K}'$.

This corollary is a consequence of the following stronger result, where we have control over both $\mathcal{V}(F, \mu)$ and $\mathcal{V}'(F, \mu)$.

Corollary 1.4.7. Let $\mathcal{B}$ be a finite alphabet and $\mathcal{K}' \subset \mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$ two Π_2-CCC sets. There exist an alphabet $\mathcal{A} \supset \mathcal{B}$ and a cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that for any $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$, one has:

- $\mathcal{V}(F, \mu) = \mathcal{K}$;
- $\mathcal{V}'(F, \mu) = \mathcal{K}'$.

By Proposition 0.1.9, $\mathcal{V}'(F, \mu)$ is included in the convex hull of $\mathcal{V}(F, \mu)$. Here we need the stronger hypothesis that it is included in $\mathcal{V}(F, \mu)$. Therefore, if we suppose the initial measure is computable, this is a characterisation of pairs of connected subsets $(\mathcal{K}, \mathcal{K}')$ such that $\mathcal{K}' \subset \mathcal{K}$ that can be reached in this way.

Proof. We use notations from the proof of Proposition 1.1.6. Notably $(w_n)_{n\in\mathbb{N}}$ and $(w'_n)_{n\in\mathbb{N}}$ are the uniformly computable sequences of words associated to $\mathcal{K}$ and $\mathcal{K}'$, respectively, and $\mathbf{V}_k$ and $\mathbf{V}^t_k$ are defined with regard to $\mathcal{K}$.

We define a new sequence of words $(w''_n)_{n\in\mathbb{N}}$ in the following manner, using a similar method as Proposition 1.1.6. For $n \in \mathbb{N}$, let $i_n \leq n$ be the maximal value such that one can find a path $w_n = u_0, u_1, \dots, u_l = w'_n, u_{l+1}, \dots, u_{l'} = w_{n+1}$ with $u_1, \dots u_{l-1}, u_{l+1}, \dots, u_{l'} \in Vb^t_{i_n}$ and $d_{\mathcal{M}}(u_k, u_{k+1}) \leq 4b(i_n)$ for all k.

Let $P_n : \{0, \dots, p_n\} \to \mathbf{V}^t_{i_n}$ be such a path. Since $\mathbf{V}^t_{i_n} \subset \mathcal{A}^{\leq i_n+1}$, this path is of length $p_n \leq 2|A|^{i_n+1} \leq 2|A|^{|w_n|+1} < 2|A|^{n+1}$.

For $i \in [|\mathcal{A}|^{n^2}, |\mathcal{A}|^{(n+1)^2}]$, we define:

- if $i < |\mathcal{A}|^{n^2} + p_n$, $w''_i = P_n(i - |\mathcal{A}|^{n^2})$;
- otherwise, $w''_i = w'_n$.

and let F be the CA associated to $(w''_n)_{n\in\mathbb{N}}$ by Theorem 1.2.1. Since all elements of $(w_n)_{n\in\mathbb{N}}$ appear, we can prove as in Proposition 1.1.6 that $\mathcal{V}(F, \mu) = \mathcal{V}((w''_n)_{n\in\mathbb{N}}) = \mathcal{K}$.

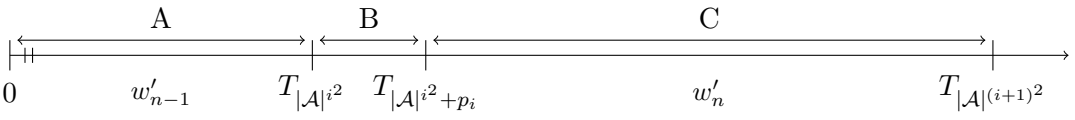


Figure 1.18: Intuitively, we prove $A + B \ll C$, then $B \ll A$.

We have
$$\frac{|\mathcal{A}|^{n^2}+p_n}{|\mathcal{A}|^{(n+1)^2}-(|\mathcal{A}|^{n^2}+p_n)} < \frac{|\mathcal{A}|^{n^2+1}}{|\mathcal{A}|^{(n+1)^2}-|\mathcal{A}|^{n^2+1}} \underset{n\to\infty}{\longrightarrow} 0.$$
In other words, the subset $[0,|\mathcal{A}|^{n^2}+p_n]$ is (asymptotically) of negligible density in $[0,|\mathcal{A}|^{(n+1)^2}]$. Since $T_{i+1}-T_i = q^{\lfloor\sqrt{i}\rfloor}$ (where q is defined in Section 1.2.3) is an increasing sequence, the subset $[0,T_{|\mathcal{A}|^{n^2}+p_n}]$ is of negligible density in $[0,T_{|\mathcal{A}|^{(n+1)^2}}]$. This means that, putting $t_{n+1}=T_{|\mathcal{A}|^{(n+1)^2}}, d(\varphi^F_{t_{n+1}}(\mu),\widehat{\delta_{w'_{n+1}}}) \underset{n\to\infty}{\longrightarrow} 0$.

Furthermore, notice that for $x,y\in\mathbb{R}_+$, when $y\leq\sqrt{x}$, we have $\lfloor\sqrt{x+y}\rfloor\leq\lfloor\sqrt{x}\rfloor+1$ and $\lfloor\sqrt{x-y}\rfloor\geq\lfloor\sqrt{x}\rfloor-1$. Thus :
$$T_{|\mathcal{A}|^{n^2}+p_n}-T_{|\mathcal{A}|^{n^2}} < q^{|\mathcal{A}|^{\frac{n^2}{2}}+1}\cdot 2|\mathcal{A}|^{n+1}.$$
$$T_{|\mathcal{A}|^{n^2}+p_n} > T_{|\mathcal{A}|^{n^2}}-T_{|\mathcal{A}|^{n^2}-|\mathcal{A}|^{\frac{n^2}{2}}} > q^{|\mathcal{A}|^{\frac{n^2}{2}}-1}\cdot|\mathcal{A}|^{\frac{n^2}{2}},$$
and therefore
$$\frac{T_{|\mathcal{A}|^{n^2}+p_n}-T_{|\mathcal{A}|^{n^2}}}{T_{|\mathcal{A}|^{n^2}+p_n}} \underset{n\to\infty}{\longrightarrow} 0.$$
This means that, when $t'_n=T_{|\mathcal{A}|^{n^2}+p_n}, d(\varphi^F_{t'_n}(\mu),\widehat{\delta_{w'_n}}) \underset{n\to\infty}{\longrightarrow} 0$.

To sum up, we have two sequences of times $t_0<t'_0<\cdots<t_n<t'_n<\dots$ such that, for all $n\in\mathbb{N}$, the Cesàro mean sequence $(\varphi^F_t(\mu))_{t\in\mathbb{N}}$ is (asymptotically) close to $\widehat{\delta_{w'_n}}$ between times t_n and t'_n, and is close to $\widehat{\delta_{w'_{n+1}}}$ at time t_{n+1}. Furthermore, between times t'_n and t_{n+1}, $\varphi^F_t(\mu)$ is by definition a convex combination of $\varphi^F_{t'_n}(\mu)$ and $\widehat{\delta_{w'_{n+1}}}$, and thus it is close to the segment $[\widehat{\delta_{w'_n}},\widehat{\delta_{w'_{n+1}}}]$. We conclude that asymptotically, the sequence is close to $\mathcal{V}((w'_n)_{n\in\mathbb{N}})$, and thus its set of limit points is $\mathcal{K}'$. □

This result has a counterpart with no auxiliary states, using Theorem 1.3.1.

Corollary 1.4.8 (Sufficient conditions for being a μ-limit measures set in Cesàro mean – no auxiliary states)**.**
Let $\mathcal{K}'\subset\mathcal{K}\subset\mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$ be two non-empty Π_2-CCC sets that both do not charge the same word $u\in\mathcal{B}^*$. Then there exists a cellular automaton $F:\mathcal{B}^{\mathbb{Z}}\to\mathcal{B}^{\mathbb{Z}}$ such that for any $\mu\in\mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$,

- $\mathcal{V}(F,\mu)=\mathcal{K}$;
- $\mathcal{V}'(F,\mu)=\mathcal{K}'$.

As we remarked in the previous subsection, the existence of a non-charged word u is a necessary hypothesis.

Open question. Is it possible to extend Corollary 1.4.7 and 1.4.8 when $\mathcal{K}'\not\subset\mathcal{K}$?

Using Example 1.4.1 we can only provide some examples where $\mathcal{V}(F,\mu) \cap \mathcal{V}'(F,\mu) = \emptyset$, typically where $\mathcal{V}(F,\mu)$ is a finite set and $\mathcal{V}'(F,\mu)$ is the singleton containing its barycentre.

1.4.3 Decidability consequences

Using this construction, we show that for any nontrivial property, there is no algorithm that, given as an input an alphabet and a cellular automaton on this alphabet, can decide whether its μ-limit measures set satisfy this property. This is analogous to the classical result by Rice on Turing machines.

In the context of cellular automata, a similar result on ω-limit sets was found by Kari [Kar94], and Delacourt proved that this was also true for μ-limit sets:

Theorem 1.4.9 (Rice theorem on μ-limit sets, [Del11])**.**
Let P be a property on subshifts nontrivial on μ-limit sets (i.e. not always or never true). Then it is undecidable, given an alphabet $\mathcal{A}$ and a CA $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, whether $\Lambda_\lambda(F)$ satisfies P, where λ is the uniform measure on $\mathcal{A}^{\mathbb{Z}}$.

In both results, the alphabet is considered as an input of the problem, and this will also be true for the following results; we consider the case where the alphabet is fixed only in Corollary 1.4.13.

Corollary 1.4.10 (Rice theorem on μ-limit measures sets)**.**
Let P be a nontrivial property on non-empty Π_2-CCC sets of $\mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$ (i.e. not always or never true). Then it is undecidable, given an alphabet $\mathcal{A}$ and a CA $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, whether $\mathcal{V}(F,\mu)$ satisfies P for $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$.

To see that it is a generalisation of Theorem 1.4.9, consider that for every reachable μ-limit set Λ, there is a Π_2-computable compact set of measures $\mathcal{K}$ such that $\overline{\bigcup_{\nu\in\mathcal{K}} \operatorname{supp}(\nu)} = \Lambda$, and this set can be supposed connected since taking the convex hull of a set of measures does not add any configuration to the union of their support.

Proof. We proceed by reduction to the halting problem. Since P is nontrivial, let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Π_2-CCC sets that satisfies and does not satisfy P, respectively. By Proposition 1.1.6, there exists two computable sequences of words $(w_n)_{n\in\mathbb{N}}, (w'_n)_{n\in\mathbb{N}} \in (\mathcal{A}^*)^{\mathbb{N}}$ such that $\mathcal{K}_1 = \mathcal{V}((w_n)_{n\in\mathbb{N}}), \mathcal{K}_2 = \mathcal{V}((w'_n)_{n\in\mathbb{N}})$.

Now let $\mathcal{TM}$ be a Turing machine. Define the sequence $(w''_n)_{n\in\mathbb{N}}$ in the following way.

- If $\mathcal{TM}$ halts on the empty input in less than n steps, $w''_n = w_n$.
- Otherwise, $w''_n = w'_n$.

This sequence is uniformly computable by simulating n steps of the Turing machine and computing the corresponding sequence. Therefore, we can use the previous construction to build a CA F such that $\mathcal{V}(F,\mu) = \mathcal{V}((w''_n)_{n\in\mathbb{N}})$. If $\mathcal{TM}$ halts on the empty input, then $w''_n = w_n$ for n large enough; otherwise, $w''_n = w'_n$ for n large enough. Thus, $\mathcal{V}(F,\mu)$ satisfies P if and only if $\mathcal{TM}$ halts. □

The same reasoning holds for a single limit and the Cesàro mean sequence.

Corollary 1.4.11 (Rice theorem on single limit measures)**.**
Let P be a nontrivial property on $\mathcal{M}_\sigma^{\text{s-comp}}(\mathcal{B}^{\mathbb{Z}})$. Then it is undecidable, given an alphabet $\mathcal{A}$ and a CA $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, whether $F_*^t\mu \to \nu$ where ν satisfies P for $\mu \in \mathcal{M}_{\sigma-\text{erg}}^{\text{full}}(\mathcal{B}^{\mathbb{Z}})$.

Corollary 1.4.12 (Rice theorem on Cesàro mean μ-limit measures sets)**.**
Let P be a nontrivial property on non-empty Π_2-CCC sets of $\mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$. Then it is undecidable, given an alphabet $\mathcal{A}$ and a CA $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, whether $\mathcal{V}'(F, \mu)$ satisfies P for $\mu \in \mathcal{M}_{\psi-\text{mix}}^{\text{full}}(\mathcal{B}^{\mathbb{Z}})$.

Corollary 1.4.13 (Rice theorem on μ-limit measures sets – no auxiliary states)**.**
Let $\mathcal{B}$ be an alphabet, $\mu \in \mathcal{M}_{\psi-\text{mix}}^{\text{full}}(\mathcal{B}^{\mathbb{Z}})$, $u \in \mathcal{B}^*$, and P be a nontrivial property on non-empty Π_2-CCC sets that do not charge u. Then it is undecidable, given a CA $F : \mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$, whether $\mathcal{V}(F, \mu)$ satisfies P.

This result extends to single measures and Cesàro mean μ-limit measures set, in a similar way as Corollaries 1.4.11 and 1.4.12.

Actually, if λ is the uniform Bernoulli measure, the problem of whether $\mathcal{V}(F, \lambda) = \{\lambda\}$ is equivalent to the surjectivity of F, which is decidable [AP72]. More generally, if we fix an alphabet, the question of which nontrivial properties on limit measures and μ-limit measures sets are decidable remains open.

Section 1.5

Computation on the space of measures

The construction developed in Section 1.2 has several limitations. In particular, it "ignores" any information contained in the initial measure, erasing it with the formatting process, and has the same typical asymptotic behaviour regardless of the initial measure. In this section, we modify the construction to perform computation on the space of probability measures, that is, we want the μ-limit measures set to be a function of the initial measure; this requires to keep some information in the construction. When the initial measure is not computable, we can use this information as a "source" of noncomputability to reach μ-limit measures sets that would be unreachable otherwise.

Computation with access to an oracle

The obstructions shown in Section 1.1 can be generalised to obstructions on $\mu \mapsto \mathcal{V}(F, \mu)$, including cases where the initial measure is not necessarily computable, by considering computability with access to an oracle $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.

Definition 1.5.1 (Turing machine with oracle)**.**
A **Turing machine with oracle** in $\mathcal{M} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ has the same behaviour as a classical Turing machine, except that an oracle $\mu \in \mathcal{M}$ is fixed prior to computation. The machine can query the oracle at any time during the computation by writing $u \in \mathcal{A}^*$ and $n \in \mathbb{N}$ on an special additional **oracle tape** and entering a special **oracle state**. At this step, the content of the oracle tape is considered as the oracle input and, after one step, the contents of the oracle tape are replaced by an approximation of $\mu([u])$ up to an error 2^{-n} and the computation resumes.

We extend the definitions of Section 1.1 to define notions of computability when the Turing machines have access to an oracle.

Definition 1.5.2 (Function computable with oracles on countable sets)**.**
Let $\mathcal{M} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and X, Y two countable sets. A function $f : \mathcal{M} \times X \to Y$ is **computable with oracles in** $\mathcal{M}$ if there exists a Turing machine with oracle in $\mathcal{M}$ which takes as input $x \in X$ and returns $y = f(\mu, x) \in Y$, up to reasonable encoding.

Definition 1.5.3 (Function computable with oracles on uncountable sets)**.**
Let $\mathcal{M} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.
A function $\varphi : \mathcal{M} \longrightarrow \mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$ is **computable with oracles in** $\mathcal{M}$ if there exists a computable function with oracles in $\mathcal{M}$ $f : \mathcal{M} \times \mathbb{N} \longrightarrow \mathcal{B}^*$ such that $|\varphi(\mu) - \widehat{\delta_{f(\mu,n)}}| \leq 2^{-n}$. This is an extension of the previous definition where the image is not countable, hence the abuse of notation.
A sequence of functions $(f_n : \mathcal{M} \times \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \longrightarrow \mathbb{R})_{n\in\mathbb{N}}$ is **uniformly computable with oracles in** $\mathcal{M}$ if:

- there exists $a : \mathcal{M} \times \mathbb{N} \times \mathbb{N} \times \mathcal{A}^* \longrightarrow \mathbb{Q}$ computable with oracles in $\mathcal{M}$ such that $\left| f_n(\mu, \widehat{\delta_w}) - a(\mu, n, m, w) \right| \leq \frac{1}{m}$ for all $\mu \in \mathcal{M}$, $w \in \mathcal{A}^*$ and $n, m \in \mathbb{N}$;

- there exists $b : \mathcal{M}\times\mathbb{N} \longrightarrow \mathbb{Q}$ computable with oracles in $\mathcal{M}$ such that $d_{\mathcal{M}}(\nu,\nu') < b(\mu,m)$ implies $|f_n(\mu,\nu) - f_n(\mu,\nu')| \leq \frac{1}{m}$ for all $\mu \in \mathcal{M}$ and $n, m \in \mathbb{N}$.

A function $f : \mathcal{M} \times \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \longrightarrow \mathbb{R}$ is **Σ_n-computable with oracles in $\mathcal{M}$** (resp. Π_n) if there exists a uniformly computable sequence of functions with oracles in $\mathcal{M}$ ($f_{i_1,\dots,i_n} : \mathcal{M} \times \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \to \mathbb{R})_{i_1,\dots,i_n\in\mathbb{N}}$ such that:

$$f = \sup_{i_1\in\mathbb{N}} \inf_{i_2\in\mathbb{N}} \sup_{i_3\in\mathbb{N}} \cdots f_{i_1,\dots,i_n} \qquad \left(\text{resp. } f = \inf_{i_1\in\mathbb{N}} \sup_{i_2\in\mathbb{N}} \inf_{i_3\in\mathbb{N}} \cdots f_{i_1,\dots,i_n}\right).$$

f is **$\mathbf{\Delta_n}$-computable with oracles in $\mathcal{M}$** if it is both Σ_n-computable and Π_n-computable with oracles in $\mathcal{M}$.

Let $\mathfrak{K}$ be the set of compact subsets of $\mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$. Defining the computability of a function $\Psi : \mathcal{M} \longrightarrow \mathfrak{K}$ can be done in various ways, similarly as in Proposition 1.1.4. For example, Ψ is **Π_n-computable** if the distance function $\mu,\nu \mapsto d_{\Psi(\mu)}(\nu)$ is Σ_n-computable with oracles in $\mathcal{M}$.

The proofs of Sections 1.1.2 and 1.1.3 can be easily adapted in this framework.

Proposition 1.5.1 (Computability obstruction for a general initial measure)**.**
For any cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$,

- $\mu \longmapsto F_*\mu$ is computable with oracles in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ (equivalent to Proposition 1.1.3);
- $\mu \longmapsto \mathcal{V}(F,\mu)$ and $\mu \longmapsto \mathcal{V}'(F,\mu)$ are Π_2-computable with oracles in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ (equivalent to Proposition 1.1.5).

Furthermore, if $\Psi : \mathcal{M} \longrightarrow \mathfrak{K}$ is a Π_2-computable function with oracles in $\mathcal{M}$ and if every element of $\Psi(\mathcal{M})$ is connected, then there exists a computable function $f : \mathcal{M} \times \mathbb{N} \longrightarrow \mathcal{A}^*$ with oracles in $\mathcal{M}$ such that $\Psi(\mu) = \mathcal{V}((f(\mu,n))_{n\in\mathbb{N}})$, where $\mathcal{V}((f(\mu,n))_{n\in\mathbb{N}})$ is the closure of the limit points of the polygonal path (equivalent to Proposition 1.1.6).

Towards a reciprocal

In this section, we give a partial reciprocal to Proposition 1.5.1. To use the initial measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$) as an oracle, we need to keep some information from the initial configuration. We adapt the original construction in the following way:

Each segment keeps a sample of the initial configuration, using the frequency of patterns inside this sample as an oracle in the computation. We need to ensure that the frequency of a pattern $u \in \mathcal{A}^k$ in this sample is close to $\mu([u])$ with a high probability. For this, we use Theorem III.1.7 of [Shi96] applied on a measure $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$ that ensures we have an exponential rate of convergence for every length. Formally, for any $k, m, n \in \mathbb{N}$, $c > 0$:

$$\mu\left(\left\{x \in \mathcal{A}^{\mathbb{Z}} : \max_{u\in\mathcal{A}^k}\{|\mu([u]) - \text{Freq}(u, x_{[0,n]})|\} \geq \varepsilon\right\}\right) \leq (k+m)\psi(m)^{\frac{n}{k}} \left(\frac{n}{k}+1\right)^{\text{Card}(A)^k} 2^{-\frac{nc\varepsilon^2}{4k}}.$$

However, in our construction, we are unable to keep all information from the initial configuration since the formatting process destroys information in the segment. In all the following,

we will only keep information about the density of $\boxed{\text{I}}$ symbols, and the reached μ-limit set of measures depends on this parameter only. The same method could be adapted to keep information about longer words, only considering the positions of $\boxed{\text{I}}$ symbols.

Theorem 1.5.2. Let $\Psi : \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\{0,1\}^{\mathbb{Z}}) \to \mathfrak{K}$ be a Π_2-computable function where $\mathfrak{K}$ is a set of compact connected subsets of $\mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$. Assume that if $\mu, \mu' \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\{0,1\}^{\mathbb{Z}})$ are such that $\mu([1]) = \mu'([1])$, we have $\Psi(\mu) = \Psi(\mu')$.

Then there exists an alphabet $\mathcal{A} \supset \mathcal{B}$ and a cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that for all $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$, we have $\mathcal{V}(F, \mu) = \Psi(\pi\mu)$ where π is the 1-block map defined by $\pi(x)_i = 1$ when $x_i = \boxed{\text{I}}$, and $\pi(x)_i = 0$ otherwise.

Notice that since only one density is considered, it would be equivalent in this case to consider a Π_2-computable function with oracles in $\mathbb{R}$ $\mathbb{R} \to \mathfrak{K}$. We kept this statement more technical to be consistent with the general case.

Proof. Let $f : \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\{0,1\}^{\mathbb{Z}}) \times \mathbb{N} \longrightarrow \mathcal{A}^*$ be a computable function with oracles in $\mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\{0,1\}^{\mathbb{Z}})$ such that $\Psi(\mu) = \mathcal{V}((f(\mu, n))_{n\in\mathbb{N}})$ and consider the associated Turing machine with oracle.

Let F be the cellular automaton defined in Theorem 1.2.1 that simulated the Turing machine corresponding to $((f(\mu, n))_{n\in\mathbb{N}})$. Of course we need to specify the behaviour of the automata when the machine performs an oracle query.

We add a new layer $\mathcal{A}_{\texttt{oracle}}$ in which each segment at time t stores the frequency of the state $\boxed{\text{I}}$ in this segment at time 0. To do that, we modify the construction in the following way:

- We subdivide the layer $\mathcal{A}_{\texttt{oracle}}$ in two parts, on which each wall $\boxed{\text{W}}$ keeps on its left:
  - the first counter for the number of $\boxed{\text{I}}$ symbols that have been destroyed in its left segment;
  - the second counter for the length of this segment, 0 if the segment is not formatted.
- Another counter accompanies each formatting counter, measuring the length of the segment as it progresses.
- The second counter is initialised as 0. When the time counter attached to this wall makes a comparison with an initialised formatting counter (the comparison returns the result "="), the second counter stores the length of the segment. It may take the value 0 again if it merges with a non-formatted segment (see Figure 1.19).
- When a wall is destroyed by a merging process, it sends to its right an **oracle signal** at speed 1 containing the information stored in its oracle counters. Such a signal should not cross a formatting counter, so it is slowed down if necessary.
- When a wall's counters are (c_1, c_2) and a signal (c'_1, c'_2) comes from its left, there are three cases:
  - If $c_2 = 0$, the left segment cannot be formatted; the signal cannot come from an initialised wall and can be safely ignored. The counters does not change.
  - If $c_2 \neq 0$, the left segment has been formatted and all false signals erased. Thus the information comes from an initialised wall. The new number of $\boxed{\text{I}}$ symbols is $c''_1 = c_1 + c'_1 + 1$ to take the merging into account.

∗ If $c_2' = 0$, the segment just merged with a non-formatted segment and $c_2'' = 0$;
∗ otherwise $c_2'' = c_2 + c_2'$.

The counters take the values (c_1'', c_2'').

See Figure 1.19. We remark that if the length of the segment is k, the information can be coded in space $\log(k)$, and it is possible to actualise the values before another signal can come from the left.

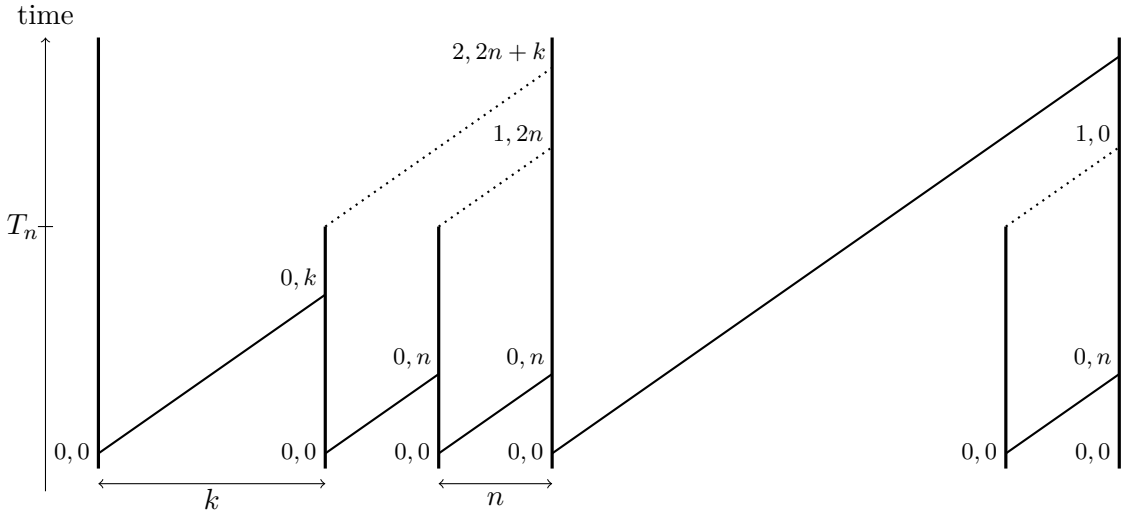


Figure 1.19: Each wall has its counter displayed when its value changes. Slanted thick lines are formatting counters, dotted lines are signals transmitting information.

- If two symbols $\boxed{\text{I}}$ are too close in the initial configuration, they are destroyed by the bootstrapping process (see Section 1.2.2). If a $\boxed{\text{I}}$ is in a group of $\boxed{\text{I}}$ separated by two cells or less, the rightmost $\boxed{\text{I}}$ sends a formatting counter and the leftmost one starts a time counter. Thus a group of $\boxed{\text{I}}$ separated by two cells or less behave as a single symbol for initialisation purposes. Each $\boxed{\text{I}}$ symbol except the leftmost one is transformed immediately into an oracle signal $(1, d)$, where d is the distance to the nearest $\boxed{\text{I}}$ to its left. The other cells present initially are erased.
- The Turing machine simulation described in Section 1.2.3 can be adapted to simulate a Turing machine with oracle. When there is an oracle query for the value of $\mu([\![\boxed{\text{I}}]\!])$ with precision 2^{-i} at time $t \in [T_n, T_{n+1}]$, there are two possibilities:
  - if $n^{-\frac{1}{6}} \leq 2^{-i}$, the Turing machine uses the information stored in the oracle layer to return the frequency of $\boxed{\text{I}}$ on the segment at time 0, and this corresponds to an approximation of $\mu([\![\boxed{\text{I}}]\!])$ with sufficient precision;
  - if $n^{-\frac{1}{6}} > 2^{-i}$, the computation stops, and the last word successfully computed is output. The same thing happens until a time when enough information is available.

Let us check that $\mathcal{V}(F, \mu) = \Psi(\pi_*\mu)$ for $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$. It is clear that the density of auxiliary states tends to 0, so if the sample approximates correctly $\mu([\![\boxed{\text{I}}]\!])$, the sequence of words $(w_n)_{n\in\mathbb{N}}$ produced by the cellular automaton correspond to $(f(\mu, n))_{n\in\mathbb{N}}$ up to some

repetition. Thus we only need to prove that the probability that a cell belongs to a segment whose sample corresponds to a "bad" approximation tends to 0 when t tends to ∞. Recall that $\Gamma^{T_n}_{[i,j]} = \{x \in \mathcal{A}^{\mathbb{Z}} \mid [i,j] \text{ is a segment at time } T_n\}$.

$$\begin{aligned} B_n &= \mu\left(\left\{x \in \mathcal{A}^{\mathbb{Z}} : x_0 \text{ belongs in a segment with a "bad" sample at time } T_n\right\}\right) \\ &= \sum_{i<0,j>0} \mu\left(\left\{x \in \Gamma^{T_n}_{[i,j]} : |\mu([\boxed{\text{I}}]) - Freq(\boxed{\text{I}}, x_{[i,j]})| > n^{-\frac{1}{6}}\right\}\right) \\ &= \sum_{k>0} k \cdot \mu\left(\left\{x \in \Gamma^{T_n}_{[0,k]} : |\mu([\boxed{\text{I}}]) - Freq(\boxed{\text{I}}, x_{[0,k]})| > n^{-\frac{1}{6}}\right\}\right), \end{aligned}$$

by σ-invariance. By restricting ourselves to $n \leq k \leq K_n$, and for any $m \in \mathbb{N}$ large enough that $\psi_\mu(m) < 1$:

$$\begin{aligned} B_n &\leq \mu\left(\Gamma^{T_n}_{0,\geq K_n}\right) + \sum_{k=n}^{K_n} k \cdot \mu\left(\left\{x \in \mathcal{A}^{\mathbb{Z}} : |\mu([\boxed{\text{I}}]) - Freq(\boxed{\text{I}}, x_{[0,k]})| > n^{-\frac{1}{6}}\right\}\right) \\ &\leq \mu\left(\Gamma^{T_n}_{0,\geq K_n}\right) + K_n^2(1+m)\psi_\mu(m)^n \left(n+1\right)^{\mathrm{Card}(A)} 2^{-\frac{c}{4}n^{\frac{2}{3}}} \\ &\underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

The result follows. □

This result may seem surprising since the same cellular automaton has very different asymptotic behaviours depending on the initial measure.

Open question. Is it possible to improve Theorem 1.5.2 and characterise functions $\Psi : \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\{0,1\}^{\mathbb{Z}}) \to \mathfrak{K}$, where $\mathfrak{K}$ is the set of compact subsets of $\mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$, that are realisable as the action of a cellular automaton F in the sense that for all μ, $\mathcal{V}(F,\mu) = \Psi(\mu)$?

Particles in cellular automata

2

Section 2.0
Introduction

In this chapter, we take an approach that is closer to Wolfram's original approach. Instead of constructing "artificial" cellular automata that exhibit a desired typical asymptotic behaviour, we prove that some simple cellular automata found "in the wild" have a typical asymptotic behaviour that corresponds to the visual intuition we get by iterating them on a random configuration.

For many simple cellular automata, self-organisation takes a particular form: from a random configuration, after a short transitional regime, regions consisting in a simple repeated pattern emerge and grow in size, while the boundaries between them persist under the action of the cellular automaton and can be followed from an instant to the next. Therefore their movement (time evolution) can be defined inductively, and in this case we call these boundaries **particles**. In the simplest cases, these particles evolve at constant speed and are annihilated on collision; however, they can sometimes present a periodic behaviour or even perform a random walk, and the collisions may give birth to new particles following some more or less complicated rules.

This type of behaviour was first observed empirically in elementary cellular automata #18, #122, #126, #146, and #182 [Gra83, Gra84], then #54, #62, #184 [BNR91], etc. These automata attracted attention mainly because their dynamics seemed neither too simple nor too chaotic, giving hope to understand their underlying structure better. In Figure 2.1, we show many such automata iterated on the uniform configuration.

Roughly speaking, studying particles in cellular automata requires two steps:

- Identifying and describing the particles for a CA, usually as finite words;
- Describing the particle dynamics and understand its effect on the properties of the CA.

Historically, this study was often performed on individual or small groups of similar-looking CA, and the first step was done in a case-by-case manner. See for example [Fis90b, Fis90a] for the 3-state cyclic automaton, [BF95, BF05] for Rule $\#184$ and other automata with the same dynamics, [Gra84, EN92] for Rule $\#18$... Other works such as [Elo94] skip the first step and study particle dynamics in an abstract manner, deducing dynamical properties of automata by making assumptions on the dynamics of their particles and providing some example of such particles.

The first general formalism of particles in cellular automata was introduced by Pivato: regions are characterised by a subshift Σ and particles are defects in a configuration of Σ. In particular, he developed some invariants to characterise the persistence of a defect [Piv07a, Piv07c] and he described the different dynamics of propagation of a defect [Piv07b].

Our focus is on the second step, and more precisely we are interested in how the dynamics of the particles affect the typical asymptotic behaviour. In other words, we assume the existence of a set of particles with good properties (found by Pivato's methods or otherwise), and

we try to explain the self-organisation that is observed experimentally. In particular, most of these behaviours cannot be observed through the ω-limit set alone (Section 0.1.3 or [KM00]), which means our notion of typical asymptotic behaviour is relevant.

Considering only restricted cases for the dynamics of the particles seems reasonable since almost any nontrivial property of the μ-limit set is undecidable [Del11], and the same is true about limit measures (Section 1.4 or [HdMS13]). To determine the μ-limit set in some cases, Kůrka suggests an approach based on particle weight function which assigns weights to certain words [Kůr03]. However, this method does not cover any case when a defect can remain in the μ-limit set. Hence we aim at a more general approach, in terms of particle dynamics as well as initial measures.

One of our main motivations for this study is the class of captive automata introduced by Guillaume Theyssier in [The04]. A captive cellular automaton is defined by a local rule f on some neighbourhood $\mathcal{N}$ satisfying $f((x_i)_{i\in\mathcal{N}}) \in \{x_i : i \in \mathcal{N}\}$; in other words, a colour cannot appear unless it is already present in the neighbourhood. In op.cit., the author proved that these automata share interesting algebraic properties, and also noticed an interesting phenomenon by drawing a captive cellular automaton at random (fixed alphabet and neighbourhood): most captive automata exhibited the type of behaviour we described above. We give a sample of this phenomenon in Figure 2.1 with two captive CA an one ordinary CA drawn randomly. Any kind of general result concerning self-organisation of captive cellular automata remains a challenging open problem.

Another motivation for the study of particle-like objects in cellular automata stems not from empirical observations, but from computational considerations. Indeed, when performing algorithmic tasks with cellular automata, the most natural approach is to use particles (often called signals in this context) as a means of communication. The construction in Chapter 2 of the present thesis is a good example, and other examples include the density classification problem [GKL87, dSM92, Fat13], the firing range squad problem [Maz96], the problem of designing an intrinsically universal cellular automaton [OR09]... We argue that this is not only a human bias. When studying Rule #110, Cook found that particles were the key to understand the universality of the automaton [Coo04]; similarly, in [HC97, HCM98], the authors found an underlying particle system in the Rule #54 automaton to understand its physical behaviour and computational content. Following a different approach, Das, Mitchell and Crutchfield [DMC94] used genetic algorithms to "breed" a cellular automaton capable of simple algorithmic tasks, and the resulting automaton was found to use signals in a similar manner.

Therefore there seems to be a deep link between particles and computation in cellular automata, and our results may help prove that some automata computing with particles behave in the expected way. For example, in Section 2.1.5, we use our results to prove some asymptotic properties of Fatès' candidate to perform the algorithmic task of density classification [Fat13].

In Section 2.1, we show that when particles have good collision properties (coalescence), only particles moving in one particular direction can remain in the μ-limit set (μ being an initial σ-ergodic measure). We introduce our own formalism of particle system in Section 2.1.1 to describe the dynamics of the particles, and Section 2.1.2 is dedicated to the proof itself. We introduce a simplified version of Pivato's formalism in Section 2.1.3, since this is by far the simplest way to find such a particle system in most examples.

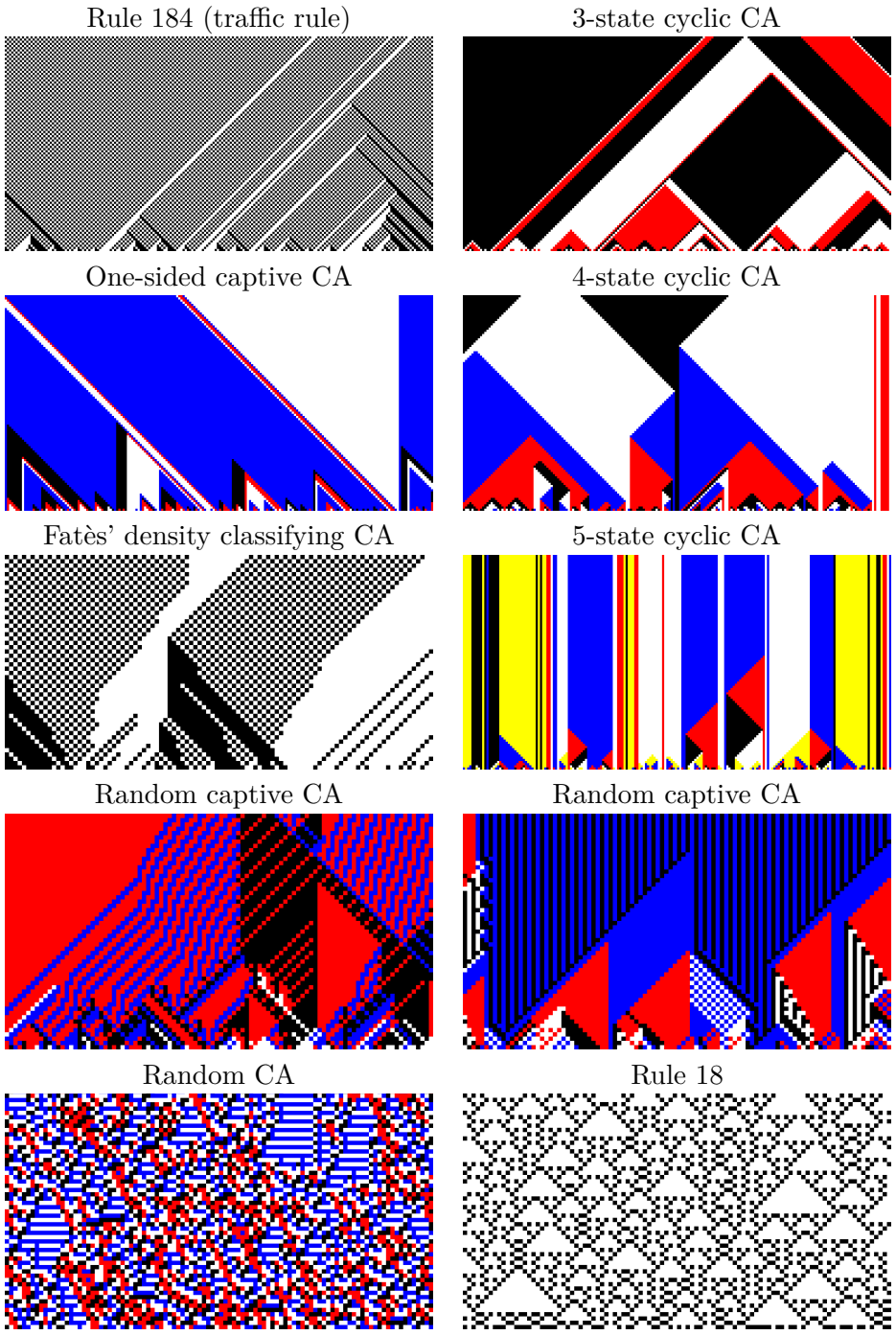


Figure 2.1: Space-time diagrams of some cellular automata with particles, starting from a configuration drawn uniformly at random.

We spend Section 2.1.4 on various examples of automata that fall under these hypotheses, thus explaining the behaviours observed in the first 3 lines in Figure 2.1: remaining particles (if any) all have the same speed. However, our formalism is not general enough to include

defects in a sofic subshift that can have a particle-like behaviour, such as in Rule $\#18$ (see the bottom right picture in Figure 2.1 and [EN92]).

In Section 2.2, we improve the previous qualitative results with a quantitative approach, considering the time evolution of some parameters when the automata have simple dynamics. This research direction was inspired by [KFD11], where the authors consider the waiting time before a particle crosses the central column. Using the same approach as in [BF95, KM00], we show that the behaviour of these automata can be described by a random walk process (Section 2.2.1), and we approximate this process by a Brownian motion using scale invariance (Section 2.2.3).

This approach can be used for various natural parameters such as the density of particles at time t (Section 2.2.4) or the rate of convergence to the limit measure (Section 2.2.5). This generalises some of the results from [KFD11] and [BF05], particularly in terms of conditions on the initial measures, since they were only known for initial Bernoulli measures. We apply these results to some automata exhibiting these dynamics in Section 2.2.6.

Section 2.3 is devoted to the study of the limit measures of the 3-state cyclic automaton C_3, that we were unable to describe using the results of the previous section. Experimental simulations suggested a rather surprising behaviour when the automaton is iterated on a nonuniform Bernoulli measure $\mu = \mathrm{Ber}_{(\lambda_1,\lambda_2,\lambda_3)}$:

$$C_{3*}^t \mu \underset{t\to\infty}{\longrightarrow} \lambda_2 \widehat{\delta_0} + \lambda_0 \widehat{\delta_1} + \lambda_1 \widehat{\delta_2}.$$

Intuitively, when looking at a finite window for t large enough, we observe almost surely a monochromatic region, and the probability to see each colour $i \in \{0,1,2\}$ is equal to the initial frequency of $i-1 \mod 3$.

Using the same approach as in the previous section, but adding more information to the random walk and managing to keep this information in the scaling and approximation process, we prove this typical asymptotic behaviour.

Section 2.1 is an improved version of [HdMS11], with Section 2.1.5 being a new result. Sections 2.2.1 to 2.2.3 were published in [HdMS12], with some examples from Section 2.2.6. Section 2.3 is a yet unpublished collaboration between Antony Quas and ourselves.

Section 2.1

Particle-based organisation: qualitative results

In this section, we take a qualitative approach to self-organisation: that is, we assume some properties on the dynamics of the particles of some cellular automata and try to deduce properties of the μ-limit measures set of the cellular automaton, with no regard to how fast this organisation takes place. In Section 2.1.1, we introduce the formalism of a particle system to describe the particle dynamics. In Section 2.1.2, we state our main result, which can be simplified as follows: if the particles evolve at constant speed and have good collision properties, then only one of them can survive in the μ-limit set. In Section 2.1.3, we introduce the most relevant parts of Pivato's formalism in the restricted case that is consistent with our own formalism; this is useful to find actual particle systems in our examples in a way that correspond to the intuition, as we see in Section 2.1.4. Finally, we explore in Section 2.1.5 the extension of our result to probabilistic cellular automata.

2.1.1 Particles

Definition 2.1.1 (Particle system)**.**

Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton. A **particle system** for F is a tuple $(\mathcal{P}, \pi, \phi)$, where:

- $\mathcal{P}$ is a finite set whose elements are called **particles**;
- $\pi : \mathcal{A}^{\mathbb{Z}} \mapsto (\mathcal{P} \cup \{0\})^{\mathbb{Z}}$ is a factor;
- $\phi : \mathcal{A}^{\mathbb{Z}} \times \mathbb{Z} \mapsto 2^{\mathbb{Z}}$ (subsets of $\mathbb{Z}$) is a function called **update function**,

such that the update function satisfies the following properties for all $x \in \mathcal{A}^{\mathbb{Z}}$ and $k \in \mathbb{Z}$, denoting $\mathrm{Part}_{\mathcal{P},\pi}(x) = \{k \in \mathbb{Z} \ : \ \pi(x)_k \in \mathcal{P}\}$, and omitting $\mathcal{P}$ and π when the particle system is fixed by the context:

Locality There is a constant $r > 0$ (its **radius**) such that $\phi(x, k) \subset [k - r, k + r]$.

The particles cannot "jump" arbitrarily far. By constant we mean it does not depend on x and k.

Surjectivity $\mathrm{Part}(F(x)) = \phi(x, \mathbb{Z})$.

A particle at time $t + 1$ cannot appear from nowhere; it must be the image of some particle at time t.

Particle control $\forall k \in \mathbb{Z}$, $\begin{array}{l} k \in \mathrm{Part}(x) \Rightarrow \forall k' \in \phi(x, k), k' \in \mathrm{Part}(F(x)); \\ k \notin \mathrm{Part}(x) \Rightarrow \phi(x, k) = \emptyset. \end{array}$

If a particle is sent somewhere, it remains a particle; conversely, a particle cannot come from a non-particle.

Disjunction $k < k' \Rightarrow \phi(x,k) = \phi(x,k')$ or $\max \phi(x,k) < \min \phi(x,k')$.

Two different particles crossing is considered an interaction, in which case their common image is the resulting set of particles. This assumption excludes half-progression half-interaction cases where two particles share a part of their image.

Intuitively, the update function associate to each particle at time t (given as a coordinate in a configuration) its set of images at time $t+1$ under the action of the cellular automaton. This image can be one particle if the particle simply persists, but also $\emptyset$ if it disappears or many particles. Particles that interact share the same image.

The four hypotheses ensure that the update function accurately describe the time evolution of the particles.

Notice that since the factor and update function are locally defined, all these conditions can be checked in an automatic manner by simple enumeration of patterns up to a certain length.

In the context of a fixed particle system for F, we use shorthands for the composition of the update function, defined inductively:

$$\phi^t(x,k) = \bigcup_{k' \in \phi(x,k)} \phi^{t-1}(F(x),k') \text{ and } \phi^{-1} \circ \phi(x,k) = \{k' \in \mathbb{Z} \mid \phi(x,k') = \phi(x,k)\}.$$

If $\phi(x,k)$ is a singleton, we use "$\phi(x,k)$" instead of "the only member of $\phi(x,k)$" as an abuse of notation.

We postpone the discussion on how to actually find a particle system in a given cellular automata to Section 2.1.3. We now look for assumptions on the dynamics of the particles that let us deduce that some particles disappear asymptotically. Simulations suggest that this is the case when the particles are forced to collide, and that these collisions are destructive in the sense that the total number of particles decreases; thus we introduce the notion of coalescence.

Definition 2.1.2 (Coalescence)**.**
Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton, and $(\mathcal{P}, \pi, \phi)$ a particle system for F. This particle system is **coalescent** if, for every $x \in \mathcal{A}^{\mathbb{Z}}$ and $k \in \mathrm{Part}(x)$, the particle has one of two possible behaviours:

Progression $|\phi(x,k)| = |\phi^{-1}(\phi(x,k))| = 1$, and $\pi(x)_k = \pi(F(x))_{\phi(x,k)}$

(the particle persists and its type does not change), or

Destructive interaction $|\phi(x,k)| < |\phi^{-1}(\phi(x,k))|$

(particles collide and strictly fewer particles are created).

Progressing and interacting particles of a configuration $x \in \mathcal{A}^{\mathbb{Z}}$ are denoted $\mathrm{Prog}_{\mathcal{P},\pi,\phi}(x)$ and $\mathrm{Inter}_{\mathcal{P},\pi,\phi}(x)$, respectively, and $\mathcal{P}, \pi$ and ϕ are omitted when the particle system is clear from the context. $k \in \mathrm{Prog}_{\mathcal{P},\pi,\phi}(x)$ is the case when we use "$\phi(x,k)$" to mean "the only member of the singleton $\phi(x,k)$".

Notice that, regardless of coalescence, we have because of locality $|\phi(x,k)| + |\phi^{-1}(\phi(x,k))| \leq 2r+2$, where r is the radius of the update function. The $+2$ is due to the fact that, if two coordinates are at distance r, the shortest segment containing those points has length $r+1$. See Figure 2.2 for a visual proof.

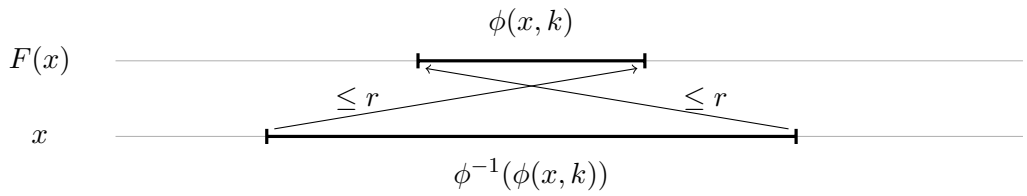


Figure 2.2: Visual proof that $|\phi(x,k)| + |\phi^{-1}(\phi(x,k))| \leq 2r+2$.

Even though the main result makes no reference to the speed of a particle, introducing this notion lets us state a corollary that is easier to use as well as corresponding more clearly to the intuition.

Definition 2.1.3 (Speed)**.**

Let F be a cellular automata and $(\mathcal{P}, \pi, \phi)$ be a particle system for F.

A particle $p \in \mathcal{P}$ has **speed $v \in \mathbb{Z}$** if for any configuration $x \in \mathcal{A}^{\mathbb{Z}}$ and $k \in \mathbb{Z}$ such that $\pi(x)_k = p$, we have one of the following:

Eventual interaction $\exists t, \phi^t(x,k) \in \mathrm{Inter}(F^t(x))$;

Progression at speed v $\forall t, \phi^t(x,k) \in \mathrm{Prog}(F^t(x))$ and $\phi^t(x,k) - k \underset{t\to\infty}{\sim} vt$.

2.1.2 A particle-based self-organisation result

We recall Definition 0.1.4: $\mathrm{Freq}(u,x)$ stands for the frequency with which the pattern u appears in the configuration x, and similarly for $\mathrm{Freq}(S,x)$ where S is a set of patterns.

We introduce the following notations for all the subsequent proofs. For $n \in \mathbb{N}$, let $\mathbf{B}_n$ be the set $[-n,n] \subset \mathbb{Z}$. Suppose that F is a cellular automaton $\mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. In the context of a fixed particle system $(\mathcal{P}, \pi, \phi)$, we introduce the densities of particles in a configuration $x \in \mathcal{A}^{\mathbb{Z}}$:

$$\text{For } p \in \mathcal{P},\ \mathcal{D}_p(x) = \mathrm{Freq}(p, \pi(x)) \quad \text{and} \quad \mathcal{D}(x) = \mathrm{Freq}(\mathcal{P}, \pi(x));$$

$$\mathcal{D}_{\mathrm{Prog}}(x) = \limsup_{t\to\infty} \frac{1}{2t+1} |\mathrm{Prog}(x) \cap \mathbf{B}_t| \quad \text{and similarly for } \mathcal{D}_{\mathrm{Inter}}(x),$$

the last two definitions applying only if the particle system is coalescent.

For $\mu \in \mathcal{M}_{\sigma-\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}})$, the $\limsup$ can be replaced by a simple limit in the definition of frequency for μ-almost all configurations. This implies for example that $\mathcal{D}(x) = \sum_{p\in\mathcal{P}} \mathcal{D}_p(x)$ for μ-almost all x.

First of all, the following proposition clarifies how controlling the frequency of interactions gives us information about the evolution of the density of the different kinds of particles.

Proposition 2.1.1 (Evolution of densities)**.**

Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton, $\mu \in \mathcal{M}_{\sigma-\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}})$, and $(\mathcal{P}, \pi, \phi)$ a coalescent particle system for F. Then, for μ-almost all $x \in \mathcal{A}^{\mathbb{Z}}$:

(i) $\mathcal{D}(F(x)) \leq \mathcal{D}(x) - \frac{1}{r+1}\mathcal{D}_{\mathrm{Inter}}(x)$;

(ii) $\forall p \in \mathcal{P}, \mathcal{D}_p(F(x)) \leq \mathcal{D}_p(x) + \mathcal{D}_{\mathrm{Inter}}(x)$.

where r is the radius of the update function ϕ.

Proof. (i) By surjectivity of the update function, we have $\mathrm{Part}(F(x)) = \bigcup_{k\in\mathrm{Part}(x)} \phi(x,k)$. Furthermore, by locality,

$$\forall x \in \mathcal{A}^{\mathbb{Z}},\ \forall n \in \mathbb{N},\ \mathrm{Part}(F(x)) \cap \mathbf{B}_n \subseteq \bigcup_{k\in\mathrm{Part}(x)\cap\mathbf{B}_{n+r}} \phi(x,k)$$
$$\subseteq \bigcup_{k\in\mathrm{Prog}(x)\cap\mathbf{B}_{n+r}} \phi(x,k) \quad \sqcup \bigcup_{k\in\mathrm{Inter}(x)\cap\mathbf{B}_{n+r}} \phi(x,k).$$

The second line being obtained by coalescence: since $\mathrm{Part}(x) = \mathrm{Prog}(x) \sqcup \mathrm{Inter}(x)$, particles in $F(x)$ are either images of progressing particles or of interacting particles. By disjunction:

$$\forall x \in \mathcal{A}^{\mathbb{Z}},\ \left|\bigcup_{k\in\mathrm{Prog}(x)\cap\mathbf{B}_{n+r}} \phi(x,k)\right| = |\mathrm{Prog}(x) \cap \mathbf{B}_{n+r}|$$

$$\text{and} \quad \forall x \in \mathcal{A}^{\mathbb{Z}},\ \left|\bigcup_{k\in\mathrm{Inter}(x)\cap\mathbf{B}_{n+r}} \phi(x,k)\right| \leq \frac{r}{r+1}\left|\phi^{-1}\left(\bigcup_{k\in\mathrm{Inter}(x)\cap\mathbf{B}_{n+r}} \phi(x,k)\right)\right|$$
$$\leq \frac{r}{r+1}\left|\mathrm{Inter}(x) \cap \mathbf{B}_{n+2r}\right|.$$

This first equality is because progressing particles are "one-to-one". The ratio $\frac{r}{r+1}$ is due to the condition of coalescence plus the remark that $|\phi(x,k)| + |\phi^{-1}(\phi(x,k))| \leq 2r+2$. The last inequality is by locality.

$$\forall x \in \mathcal{A}^{\mathbb{Z}},\ |\mathrm{Part}(F(x)) \cap \mathbf{B}_n| \leq |\mathrm{Prog}(x) \cap \mathbf{B}_{n+r}| + \frac{r}{r+1}|\mathrm{Inter}(x) \cap \mathbf{B}_{n+2r}|$$

Then, passing to the limit:

$$\forall_\mu x \in \mathcal{A}^{\mathbb{Z}},\ \mathcal{D}(F(x)) \leq \mathcal{D}_{\mathrm{Prog}}(x) + \frac{r}{r+1}\mathcal{D}_{\mathrm{Inter}}(x) = \mathcal{D}(x) - \frac{1}{r+1}\mathcal{D}_{\mathrm{Inter}}(x).$$

(ii) Similarly, for any particle $p \in \mathcal{P}$, one has for all $x \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{N}$:

$$\{k \in \mathbf{B}_n \mid \pi(F(x))_k = p\} \subseteq \bigcup_{k\in\mathrm{Part}(x)\cap\mathbf{B}_{n+r}} \phi(x,k) \quad \text{(locality)}.$$

For $k \in \mathrm{Prog}(x)$, if $\pi(F(x))_{\phi(x,k)} = p$, then by definition of coalescence $\pi(x)_k = p$. For μ-almost all x, using $\mathrm{Part}(x) = \mathrm{Prog}(x) \sqcup \mathrm{Inter}(x)$, we conclude that $\mathcal{D}_p(F(x)) \leq \mathcal{D}_p(x) + \mathcal{D}_{inter}(x)$ by passing to the limit. □

We state our main result. A simple version (Corollary 2.1.3) states that in a coalescent particle system with a σ-ergodic initial measure, if all particles can be assigned a speed, then only particles with one fixed speed can remain asymptotically. The more general result is designed to handle more difficult cases such as particles performing random walks, as we can see on the last example of Section 2.1.4.

Definition 2.1.4 (Clashing)**.**

Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton, $(\mathcal{P}, \pi, \phi)$ a coalescent particle system for F, and $\mathcal{P}_1$ and $\mathcal{P}_2$ two subsets of $\mathcal{P}$. We say that $\mathcal{P}_1$ **clashes with** $\mathcal{P}_2$ μ-almost surely if, for every $n \in \mathbb{N}^*$ and μ-almost all $x \in \mathcal{A}^{\mathbb{Z}}$,

$$\pi(x)_0 \in \mathcal{P}_1 \text{ and } \pi(x)_n \in \mathcal{P}_2 \Longrightarrow \exists t \in \mathbb{N}, \phi^t(x, 0) \in \mathrm{Inter}(F^t(x)) \text{ or } \phi^t(x, n) \in \mathrm{Inter}(F^t(x))$$

The intuition behind clashing particles in the following: if two clashing particles are present, then they end up interacting (almost surely) and thus decreasing the global frequency of particles. This is why they cannot both persist asymptotically. Note that clashing is oriented left to right: particles with speed +1 clash with particles of speed -1, but the converse is not true.

Theorem 2.1.2 (Main result)**.**
Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton, μ an initial σ-ergodic measure and $(\mathcal{P}, \pi, \phi)$ a coalescent particle system for F where $\mathcal{P}$ can be partitioned into sets $\mathcal{P}_1 \dots \mathcal{P}_n$ such that, for every $i < j$, $\mathcal{P}_i$ clashes with $\mathcal{P}_j$ μ-almost surely.

Then all particles appearing in the μ-limit set belong to the same subset, i.e. there exists a i such that

$$\forall p \in \mathcal{P},\ p \in \mathcal{L}(\pi(\Lambda_\mu(F))) \Rightarrow p \in \mathcal{P}_i.$$

If furthermore there exists a j such that $\mathcal{P}_j$ clashes with itself μ-almost surely, then this set of particles does not appear in the μ-limit set, i.e.

$$\forall p \in \mathcal{P},\ p \in \mathcal{L}(\pi(\Lambda_\mu(F))) \Rightarrow p \notin \mathcal{P}_j.$$

Corollary 2.1.3 (Main result - version with speedy particles)**.**
Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton, μ an initial σ-ergodic measure and $(\mathcal{P}, \pi, \phi)$ a coalescent particle system for F.

If each particle $p \in \mathcal{P}$ has speed $v_p \in \mathbb{R}$, then there is a speed $v \in \mathbb{R}$ such that:

$$\forall p \in \mathcal{P}, p \in \mathcal{L}(\pi(\Lambda_\mu(F))) \Rightarrow v_p = v.$$

Proof of Theorem 2.1.2. Let $i = 1$, $j = 2$ for clarity and assume there are two particles $p_1 \in \mathcal{P}_1, p_2 \in \mathcal{P}_2$ appearing in $\mathcal{L}(\pi(\Lambda_\mu(F)))$. By definition, this means that $\pi_* F_*^t \mu([p_i]) \underset{t\to\infty}{\nrightarrow} 0$ for $i \in \{1, 2\}$.

For all $x \in \mathcal{A}^{\mathbb{Z}}$, $(\mathcal{D}(F^t(x)))_{t\in\mathbb{N}}$ is a decreasing sequence of positive reals. For all $t \in \mathbb{N}$, by Birkhoff's ergodic theorem (Corollary 0.1.5) applied to $\pi_* F_*^t \mu$, we have for μ-almost all x $\mathcal{D}(F^t(x)) = \pi_* F_*^t \mu([\mathcal{P}])$. Therefore, for μ-almost all x, $(\mathcal{D}(F^t(x)))_{t\in\mathbb{N}} = (\pi_* F_*^t \mu([\mathcal{P}]))_{t\in\mathbb{N}}$ and therefore $\pi_* F_*^t \mu([\mathcal{P}]) \underset{t\to\infty}{\longrightarrow} d_\infty \geq 0$.

For $x \in \mathcal{A}^{\mathbb{Z}}$, denote $\mathcal{D}_{\mathcal{P}_i}(x) = \mathrm{Freq}(\mathcal{P}_i, \pi(x))$. By the first point of Proposition 2.1.1, we can see that for μ-almost all $x \in \mathcal{A}^{\mathbb{Z}}$,

$$\sum_{t\in\mathbb{N}} \mathcal{D}_{\mathrm{Inter}}(F^t(x)) \leq (r+1) \left(\sum_{t\in\mathbb{N}} \mathcal{D}(F^{t+1}(x)) - \mathcal{D}(F^t(x)) \right) \leq (r+1)(\mathcal{D}(x) - d_\infty) < +\infty.$$

Again by Birkhoff's theorem, $(\mathcal{D}_{\mathcal{P}_i}(F^t(x)))_{t\in\mathbb{N}} = (\pi_* F_*^t \mu([\mathcal{P}_i]))_{t\in\mathbb{N}}$ for μ-almost all x. By the second point of Proposition 2.1.1,

$$\text{For } i \in \{1,2\},\ \sup_{n\in\mathbb{N}} |\mathcal{D}_{\mathcal{P}_i}(F^{t+n}(x)) - \mathcal{D}_{\mathcal{P}_i}(F^t(x))| \leq \sum_{n=0}^{\infty} \mathcal{D}_{\text{Inter}}(F^{t+n}(x)) \to 0.$$

Thus $\pi_* F_*^t \mu([\mathcal{P}_i])$ is a Cauchy sequence and admits a limit $d_i \neq 0$.

Since clashing particles are present with positive frequency, they generate interactions that decrease the global density of particles. We will reach a contradiction with the fact that the global density tends to a limit.

Fix $\varepsilon < \frac{d_1 \cdot d_2}{2r+5}$ and T large enough such that for $t \geq T$, $\pi_* F_*^t \mu([\mathcal{P}]) - d_\infty < \varepsilon$ and $|\pi_* F_*^t \mu([\mathcal{P}_i]) - d_i| < \varepsilon$ for $i \in \{1,2\}$. By Birkhoff's ergodic theorem (Corollary 0.1.6) applied on $\pi_* F_*^T \mu$, we have:

$$\frac{1}{K} \sum_{k=0}^{K} \pi_* F_*^T \mu\left([p_1]_0 \cap [p_2]_k\right) \underset{K\to\infty}{\longrightarrow} \pi_* F_*^T \mu([p_1]) \cdot \pi_* F_*^T \mu([p_2]),$$

and $\pi_* F_*^T \mu([p_1]) \cdot \pi_* F_*^T \mu([p_2]) \geq d_1 \cdot d_2 - (d_1 + d_2 - \varepsilon)\varepsilon \geq d_1 \cdot d_2 - 2\varepsilon$. In particular, one can find a k such that $\pi_* F_*^T \mu([p_1]_0 \cap [p_2]_k) > d_1 \cdot d_2 - 3\varepsilon$. By Birkhoff's theorem, this means that words of $V_k = p_1(\mathcal{P} \cup \{0\})^{k-1} p_2 \subset (\mathcal{P} \cup \{0\})^*$ have frequency at least $d_1 \cdot d_2 - 3\varepsilon$ in $F^T(x)$, for μ-almost all $x \in \mathcal{A}^{\mathbb{Z}}$.

Since $\mathcal{P}_1$ and $\mathcal{P}_2$ clash μ-almost surely, any occurrence of V_k yields an interaction:

$$\begin{aligned}
\forall_\mu x \in \mathcal{A}^{\mathbb{Z}},\ \mathcal{D}(F^T(x)) - d_\infty &\geq \frac{1}{r+1} \sum_{t=T}^{\infty} \mathcal{D}_{\text{Inter}}(F^t(x)) && \text{Proposition 2.1.1(i)} \\
&\geq \frac{1}{2r+2} \text{Freq}(V_k, F^T(x)) && \mathcal{P}_1 \text{ and } \mathcal{P}_2 \text{ clash } \mu\text{-almost surely} \\
&\geq \frac{1}{2r+2}(d_1 \cdot d_2 - 3\varepsilon) > \varepsilon,
\end{aligned}$$

which is a contradiction with the definition of ε. □

Proof of Corollary 2.1.3. Consider the set of speeds $\{v_p \ :\ p \in \mathcal{P}\}$ and order it as $v_1 > v_2 > \cdots > v_n$. Now partition the set of particles into $(\mathcal{P}_{v_i})_{0\leq i\leq n}$ where $\mathcal{P}_{v_i}$ is the set of particles with speed v_i, and apply the theorem.

We only need to show that for any $i < j$, $\mathcal{P}_{v_i}$ clashes with $\mathcal{P}_{v_j}$ μ-almost surely. Let $p_i \in \mathcal{P}_{v_i}$ and $p_j \in \mathcal{P}_{v_j}$, and $x \in \mathcal{A}^{\mathbb{Z}}$ such that $\pi(x)_0 = p_i$ and $\pi(x)_n = p_j$ for some $n \in \mathbb{N}^*$. If both particles satisfy the second property in the definition of speed (Progression at speed v), then for some t large enough we have $\phi^t(x,0) > \phi^t(x,n)$, which is forbidden by coalescence since two particles in progression cannot cross. Thus at some time t we have either $\phi^t(x,0) \in \text{Inter}(F^t(x))$ or $\phi^t(x,n) \in \text{Inter}(F^t(x))$. □

2.1.3 Defects

Before giving a series of examples where this result can be used to describe the typical asymptotic behaviour of a cellular automaton, we present the formalism introduced by Pivato in [Piv07a, Piv07c] that defines particles as defects with respect to a F-invariant subshift Σ. Indeed, this formalism gives us an easier way to find the particle systems in our examples.

Intuitively, the F-invariant subshift describes the homogeneous regions that persist under the action of F in the space-time diagram, and defects are the borders between these regions. This allows us to define $\mathcal{P}$ and π in a way that corresponds to the intuition, even though it gives no information on the dynamics (update function ϕ).

General definitions

For a cellular automaton F, consider Σ a F-invariant subshift. The **defect field** of $x \in \mathcal{A}^{\mathbb{Z}}$ with respect to Σ is defined as:

$$\mathcal{F}_x^{\Sigma} : \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{N} \cup \{\infty\} \\ k & \mapsto & \max\left\{n \in \mathbb{N} : x_{k+[-\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil]} \in \mathcal{L}_n(\Sigma)\right\} \end{array},$$

where the result is possibly 0 or ∞ if the set is empty or infinite. Intuitively, this function returns the size of the largest word admissible for Σ centred on a cell. A **defect** in a configuration x relative to Σ is a local minimum of $\mathcal{F}_x^{\Sigma}$. Then the interval $[k, l]$ between two defects forms a homogeneous region in the sense that $x_{[k+1,l]} \in \mathcal{L}(\Sigma)$.

However, it is not true that we can always make a correspondence between defects and a finite set of words (forbidden patterns), so as to obtain a finite set of particles and a factor. This is the case only when the set of forbidden patterns is finite, that is, when Σ is a SFT. In this case, a defect corresponds to the centre of the occurrence of a forbidden word. This is a limitation of our result.

The examples given in Figure 2.1 suggest that defects can usually be classified using one of these approaches:

- Regions correspond to different subshifts and defects behave according to their surrounding regions (**interfaces** - e.g. cyclic automaton);
- Regions correspond to the same periodic subshift and defects correspond to a "phase change" (**dislocations** - e.g. rule 184 automaton).

Interfaces

Let Σ be a SFT and assume Σ can be decomposed as a disjoint union $\Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ of F-invariant σ-transitive SFTs (the **domains**). The intuition is that between two defects, each region belongs to the language of only one of the domains, and we can classify defects according to which domain the regions surrounding them on the left and the right correspond to. Since each domain is F-invariant, this classification is conserved under the action of F for non-interacting defects.

Formally, since the different domains $(\Sigma_k)_{k\in[1,n]}$ are disjoint SFTs, there is a length $\alpha > 0$ such that $(\mathcal{L}_\alpha(\Sigma_k))_{k\in[1,n]}$ are disjoint (if two subshifts share arbitrarily long words, they share

a configuration by closure). In particular, if $u \in \mathcal{L}_\alpha(\Sigma)$, then there is a unique k such that $u \in \mathcal{L}(\Sigma_k)$: we say that u **belongs to the domain** k. Thus, for a given configuration, we can assign a choice of a domain to each homogeneous region between two consecutive defects, and this choice is unique if this region is larger than α.

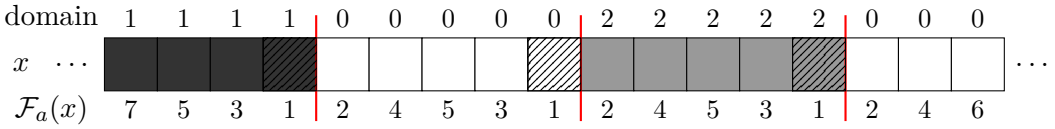


Figure 2.3: Interfaces between monochromatic domains, marked by slanted patterns. Red lines show the visual intuition of a domain change.

We call these defects **interface defects** and we can classify them according to the domain of the surrounding regions. Let $\mathcal{P} = \{p_{ij} \ : \ (i,j) \in [1,n]^2\}$ be the set of particles. Define the factor $\pi : \mathcal{A}^{\mathbb{Z}} \to (\mathcal{P} \cup \{0\})^{\mathbb{Z}}$ of order $\max(r, 2\alpha)$, where r is the radius of Σ, in the following way. For $x \in \mathcal{A}^{\mathbb{Z}}$ and $k \in \mathbb{Z}$:

- if $x_{k+[-\lfloor \frac{r}{2} \rfloor, \lceil \frac{r}{2} \rceil]} \in \mathcal{L}(\Sigma)$, $\pi(x)_k = 0$;
- else, let $u_1 = x_{[k-m,k]}$ where $m = \max\{n \leq \alpha \ : \ x_{[k-n,k]} \in \mathcal{L}(\Sigma)\}$
  $u_2 = x_{[k+1,k+m]}$ where $m = \max\{n \leq \alpha \ : \ x_{[k+1,k+m]} \in \mathcal{L}(\Sigma)\}$
  d_i a domain to which u_i belongs $(i \in \{1,2\})$

  and put $\pi(x)_k = p_{d_1 d_2}$.

Notice that the domain choice is unique when domains are larger than α cells; otherwise, there may be a choice (arbitrary, or fixed beforehand).

Dislocations

Contrary to interface defects that mark a change between languages of different SFT, dislocation defects mark a "change of phase" inside a single SFT. We call these defects **dislocation defects**.

Let Σ be a σ-transitive SFT of order $r > 1$. We say that Σ is P**-periodic** if there exists a partition $V_1, \dots, V_P$ of $\mathcal{L}_{r-1}(\Sigma)$ such that

$$a_1 \cdots a_r \in \mathcal{L}_r(\Sigma) \quad \Leftrightarrow \quad \exists i \in \mathbb{Z}/P\mathbb{Z},\ a_1 \cdots a_{r-1} \in V_i \text{ and } a_2 \cdots a_r \in V_{i+1}.$$

The **period** of Σ is the maximal $P \in \mathbb{N}$ such that Σ is P-periodic. For example, the orbit of a finite word $u \in \mathcal{A}^*$, defined as $\{\sigma^k(^\infty u^\infty) : k \in \mathbb{Z}\}$ is a periodic SFT of period less than $|u|$.

We thus associate to each $x \in \Sigma$ its **phase** $\varphi(x) \in \mathbb{Z}/P\mathbb{Z}$ such that $x_{[0,r-2]} \in V_{\varphi(x)}$. Obviously, $\varphi(\sigma^k(x)) = \varphi(x) + k \mod p$. For $x \in \mathcal{A}^{\mathbb{Z}}$, we say that an homogeneous region $[a,b]$ (i.e. a region such that $x_{[a,b]} \in \Sigma$) is **in phase** k if $\exists y \in \Sigma, \varphi(y) = k, x_{[a,b]} = y_{[a,b]}$. If $b - a > r - 2$, the phase of a region is unique and means $x_{[a,a+r-2]} \in V_{k+a \mod p}$.

As we can see in Figure 2.4, the finite word corresponding to a defect (here 00 or 11) does not depend only on the phase of the surrounding region but also on the position of the defect. More precisely, since $\varphi(\sigma(x)) = \varphi(x) + 1$, a defect in position j with a region in phase f to

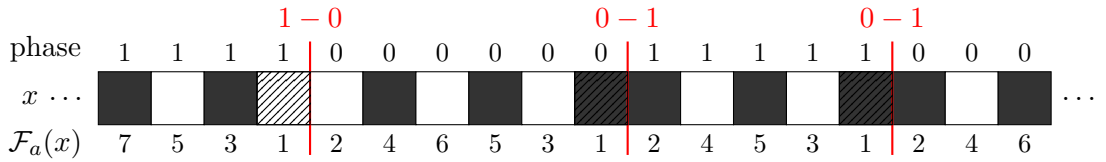


Figure 2.4: Dislocations in the checkerboard subshift ($P = 2$), marked by slanted patterns. Red lines show the visual intuition of a change of phase, with the surrounding local phases.

its left and a defect in position 0 with a region in phase $f + j \mod P$ to its left "observe" the same finite word to their left.

Therefore, we define for each defect its **local phases** $\varphi([i, j])+j \mod P$ (left) and $\varphi([j, k])+j \mod P$ (right), where j is the position of the defect and $[i, j]$ and $[j, k]$ are the surrounding homogeneous regions.

Now we classify the defects according to the local phase of the surrounding regions. Let $\mathcal{P} = \{p_{ij} : (i, j) \in \mathbb{Z}/P\mathbb{Z}^2\}$ be the set of particles. Since defects correspond to the centre of occurrences of forbidden words and the phase of a region can be locally distinguished, the factor $\pi : \mathcal{A}^{\mathbb{Z}} \to (\mathcal{P} \cup \{0\})$ of order $2r - 2$ is defined exactly as in the interface case. The choice of local phase is unique if the region is larger than $r - 1$ cells.

In the general case, those two formalisms can be mixed by fixing a decomposition $\Sigma = \bigsqcup_{i \in \mathcal{A}} \Sigma_i$ where some of the Σ_i have nonzero periods. We can classify defects according to the domains and local phase of the surrounding regions in a similar manner. Except for arbitrary choices for small regions, obtaining the set of particles and the factor from the SFT decomposition can be done in an entirely automatic way.

2.1.4 Examples

Rule 184

We consider the "traffic" automaton (rule #184). This automaton has been very well studied, especially in the case of initial Bernoulli measures [BF95, BF05]. Even for more general probability measures, the results we present here are not new [KM00]; we present it as a simple case to better understand the formalism.

Proposition 2.1.4. Let F_{184} be the traffic automaton and $\mu \in \mathcal{M}_{\sigma-\mathrm{erg}}$. Then:

$$\mu([00]) > \mu([11]) \Rightarrow 11 \notin \Lambda_\mu(F_{184});$$
$$\mu([00]) < \mu([11]) \Rightarrow 00 \notin \Lambda_\mu(F_{184});$$
$$\mu([00]) = \mu([11]) \Rightarrow \Lambda_\mu(F_{184}) = \{^\infty 01^\infty, ^\infty 10^\infty\}.$$

Proof. We consider the checkerboard SFT $\Sigma = \{^\infty(01)^\infty, ^\infty(10)^\infty\}$, which is 2-periodic and F_{184}-invariant. Using the dislocation formalism, we define the phases $\varphi(^\infty(01)^\infty) = 0$ and $\varphi(^\infty(10)^\infty) = 1$, obtaining a set of particles defined by their local phases $\{p_{01}, p_{10}\}$.

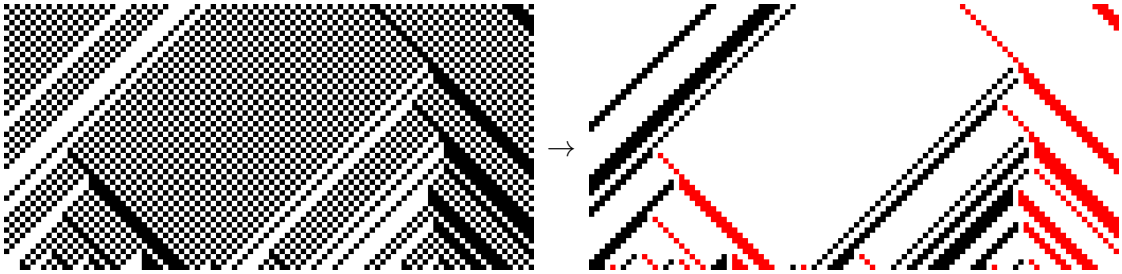


Figure 2.5: Particle system for the traffic automaton.

The corresponding factor of order $r = 2$ is defined by the local rule:

$$\begin{array}{ccc} 00 & \to & p_{01} \\ 11 & \to & p_{10} \\ \text{otherwise} & \to & 0 \end{array}.$$

Indeed, consider $x \in \mathcal{A}^{\mathbb{Z}}$ with a defect $x_{01} = 00$. The phase of the 0 in position 0 is 0 and the phase of the 0 in position 1 is 1, so this corresponds to a particle p_{01}. Changing the position of the defect would not change the particle since the local phase would be modified accordingly.

The update function is defined in the intuitive manner: with p_{01} evolving at speed -1 and p_{10} at speed $+1$ and both particles being sent to $\emptyset$ in case of collision.

$$\forall x \in \mathcal{A}^{\mathbb{Z}}, \forall k \in \mathbb{Z}, \phi(x,k) = \begin{cases} \{k+1\} & \text{if } \pi(x)_k = p_{10} \text{ and } \pi(x)_{k+1} \neq p_{01} \text{ and } \pi(x)_{k+2} \neq p_{01} \\ \{k-1\} & \text{if } \pi(x)_k = p_{01} \text{ and } \pi(x)_{k-1} \neq p_{10} \text{ and } \pi(x)_{k-2} \neq p_{10} \\ \emptyset & \text{otherwise (and in particular if } \pi(x)_k = 0) \end{cases}$$

We now check that the particle system satisfies all necessary conditions. To do that, one should verify that the update function is defined properly, that is, show that for all $x \in \mathcal{A}^{\mathbb{Z}}$ and $k \in \mathbb{Z}$ we have:

$$\pi(F(x))_{k+1} = p_{10} \Leftrightarrow \pi(x)_k = p_{10} \text{ and } \pi(x)_{k+1} \neq p_{01} \text{ and } \pi(x)_{k+2} \neq p_{01},$$

and similarly for p_{01}. This is tedious due to the high number of cases but can be easily automated by enumerating all patterns of length 4. The different conditions follow from this claim:

Locality Obvious by definition of ϕ.

Surjectivity The ($\Rightarrow$) direction of the claim implies that $\pi(F(x))_{k+1} = p_{10} \Rightarrow \pi(x)_k = p_{10}$ and $\{k+1\} = \phi(x,k)$. The other cases are similar.

Particle control The first condition is simply the ($\Leftarrow$) direction of the claim. The second condition is by definition of ϕ.

Disjunction For $k < k'$, to have $\phi(x,k) > \phi(x,k')$, the only way would be to have $\pi(x)_{k'} = p_{01}$, $\pi(x)_k = p_{10}$ and $k' = k+1$. In that case, by definition, $\phi(x,k) = \phi(x,k') = \emptyset$.

Coalescence and speeds Obvious by definition of ϕ.

Therefore we can apply Corollary 2.1.3 and only one type of particle remains in $\Lambda_\mu(F_{184})$.

Furthermore, since the collisions are of the form $p_{01} + p_{10} \to \emptyset$, it is clear that for all $x \in \mathcal{A}^{\mathbb{Z}}$, $\mathcal{D}_{p_{01}}(F_{184}(x)) - \mathcal{D}_{p_{01}}(x) = \mathcal{D}_{p_{10}}(F_{184}(x)) - \mathcal{D}_{p_{10}}(x)$. Therefore, which particle remains is decided according to whether $\mu([00]) > \mu([11])$ or the opposite, both particles disappearing in case of equality. □

n-state cyclic automaton

The n-state cyclic automaton C_n is a particular captive cellular automaton defined on the alphabet $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$ by the local rule

$$c_n(x_{i-1}, x_i, x_{i+1}) = \begin{cases} x_i + 1 & \text{if } x_{i-1} = x_i + 1 \text{ or } x_{i+1} = x_i + 1; \\ x_i & \text{otherwise.} \end{cases}$$

See Figure 2.1 for an example of space-time diagram.

This automaton was introduced by [Fis90b]. In this paper, the author shows that for all Bernoulli measure μ, the set $[i]_0$ (for $i \in \mathcal{A}$) is a μ-attractor iff $n \geq 5$. Simulations starting from a random configuration suggest the following: for $n = 3$ or 4, monochromatic regions keep increasing in size; for $n \geq 5$, we observe the convergence to a fixed point where small regions are delimited by vertical lines. We use the main result to explain this observation.

Proposition 2.1.5. Define:

$$\begin{aligned} u_+ &= \{ab \in \mathcal{A}^2 : (b-a) \mod n = +1\}; \\ u_- &= \{ab \in \mathcal{A}^2 : (b-a) \mod n = -1\}; \\ u_0 &= \{ab \in \mathcal{A}^2 : (b-a) \mod n \neq \pm 1\}. \end{aligned}$$

Then, for any measure $\mu \in \mathcal{M}_{\sigma-\mathrm{erg}}((\mathbb{Z}/nZ)^{\mathbb{Z}})$, only one of those three sets may intersect the language of $\Lambda_\mu(C_n)$.

If furthermore μ is a Bernoulli measure, then the persisting set can only be u_0.

Proof. We consider the interface defects relatively to the decomposition $\Sigma = \bigsqcup_{i \in \mathcal{A}} \Sigma_i$, where $\Sigma_i = \{^\infty i^\infty\}$. Σ is a C_n-invariant SFT of order $r = 2$, and defects are exactly transitions between colours. Thus we define $\mathcal{P} = \{p_{ab} \ : \ ab \in \mathcal{A}^2\}$. One cell is enough to distinguish between domains ($\alpha = 1$) and we obtain a factor π of order 2 defined by the local rule:

$$\begin{array}{ccc} \mathcal{A}^2 & \to & \mathcal{P} \cup \{0\} \\ a \cdot a & \mapsto & 0 \\ a \cdot b & \mapsto & p_{ab} \end{array} \qquad \text{for all } a, b \in \mathcal{A}.$$

Simulations suggest that p_{ab} evolves at constant speed $+1$ if $ab \in u_+$, -1 if $ab \in u_-$ and 0 if $ab \in u_0$. Particles progress at their assigned speed unless they meet another particle, in which case they collide and disappear. We group together the particles of same speed, writing $p_+ = \{p_{ab} \ : \ ab \in u_+\}$ and p_- and p_0 similarly. Formally, for $x \in \mathcal{A}^{\mathbb{Z}}$ and $k \in \mathbb{Z}$ the update function is defined as:

$$\phi(x,k) = \begin{cases} \{k+1\} & \text{if } \pi(x)_k \in p_+ \text{ and } \pi(x)_{k+1} \notin p_0 \cup p_- \text{ and } \pi(x)_{k+2} \notin p_-; \\ \{k-1\} & \text{if } \pi(x)_k \in p_- \text{ and } \pi(x)_{k-1} \notin p_0 \cup p_+ \text{ and } \pi(x)_{k-2} \notin p_+; \\ \{k\} & \text{if } \pi(x)_k \in p_0 \text{ and } \pi(x)_{k+1} \notin p_- \text{ and } \pi(x)_{k-1} \notin p_+; \\ \emptyset & \text{otherwise (and in particular if } \pi(x)_k = 0). \end{cases}$$

As previously, checking that this particle system satisfies all conditions necessary to apply Corollary 2.1.3 is tedious but can be automated, since it consists mostly in checking that the update function actually describes the dynamics of the particles on all words of length 6. Since $[p_+] = \pi([u_+])$ and so on, we obtain the result.

If μ is a Bernoulli measure: Consider the "mirror" application $\gamma((a_k)_{k\in\mathbb{Z}}) = (a_{-k})_{k\in\mathbb{Z}}$. γ is continuous, and thus measurable. We have $\mu(\gamma([u])) = \mu([u^{-1}]) = \mu([u])$, where $(u_1 \cdots u_n)^{-1} = u_n \cdots u_1$. But $\pi(x)_k \in p_+ \Leftrightarrow \pi(\gamma(x))_{-k} \in p_-$, and conversely; since $F \circ \gamma = \gamma \circ F$, all measures $F^t_*\mu$ are γ-invariant, and thus no particle in p_+ or p_- can persist in $\mathcal{L}(\pi(\Lambda_\mu(F)))$ (since otherwise, the symmetrical particle would persist too). □

For small values of n or particular initial measures, this proposition can be refined in the following manner:

$n = 3$ p_0 is empty. Given the combinatorics of collisions, where a particle in p_+ can only disappear by colliding with a particle in p_-, we see that particles in p_+ persist if and only if $\pi_*\mu([p_+]) > \pi_*\mu([p_-])$, and symmetrically. In the equality case (in particular, for any Bernoulli measure), no defect can persist in the μ-limit set, which means that $\Lambda_\mu(F)$ is a set of monochromatic configurations.

$n = 4$ If μ is a Bernoulli measure, the result of [Fis90b] shows that $[i]_0$ cannot be a μ-attractor for any i. In other words, for μ-almost all x, $F^t(x)$ does not converge, which means that particles in p_+ or p_- cross the central column infinitely often (even though their probability to appear tends to 0). This could not happen if particles in p_0 were persisting in $\pi(\Lambda_\mu(F))$, and thus $\Lambda_\mu(F)$ is a set of monochromatic configurations.

$n \geq 5$ If μ is a nondegenerate Bernoulli measure, the result of [Fis90b] shows that $[i]_0$ is a μ-attractor for all i. This means that some particles in p_0 persist in $\pi(\Lambda_\mu(F))$, and any configuration of $\Lambda_\mu(F)$ contains only homogeneous regions separated by vertical lines.

One-sided captive cellular automata

We consider the family of captive cellular automata $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ of neighbourhood $\{0,1\}$, which means that the local rule $f : \mathcal{A}^{\{0,1\}} \to \mathcal{A}$ satisfies $f(a_0a_1) \in \{a_0, a_1\}$. See Figure 2.1 for an example of space-time diagram.

Proposition 2.1.6. Let F be a one-sided captive automaton and $\mu \in \mathcal{M}_{\sigma-\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}})$. Define:

$$u_+ = \{ab \in \mathcal{A}^2 \ : \ a \neq b, \ f(a,b) = a\}$$
$$u_- = \{ab \in \mathcal{A}^2 \ : \ a \neq b, \ f(a,b) = b\}$$

Then either $u_+ \notin \mathcal{L}(\Lambda_\mu(F))$ or $u_- \notin \mathcal{L}(\Lambda_\mu(F))$.

If moreover, for all $a, b \in \mathcal{A}$, the local rule satisfies $f(ab) = f(ba)$ and μ is a Bernoulli measure, then $\Lambda_\mu(F) \subseteq \{^\infty a^\infty \ : \ a \in \mathcal{A}\}$ (no particle remains).

Proof. We consider the interface defects relative to the decomposition $\Sigma = \bigsqcup_{i\in\mathcal{A}} \Sigma_i$ where $\Sigma_i = \{^\infty i^\infty\}$ and obtain the same particles $\mathcal{P}$ and factor π as the n-state cyclic automata. p_{ab} evolve at speed -1 if $f(a,b) = b$ and 0 if $f(a,b) = a$, and we define p_{-1} and p_0 accordingly. The update function is defined as follows:

$$\forall x \in \mathcal{A}^{\mathbb{Z}}, \ \forall k \in \mathbb{Z}, \ \phi(x,k) = \begin{cases} \{k\} & \text{if } \pi(x)_k \in p_0 \text{ and } \pi(x)_{k+1} \notin p_{-1} \\ \{k-1\} & \text{if } \pi(x)_k \in p_{-1} \text{ and } \pi(x)_{k-1} \notin p_0 \\ \emptyset & \text{otherwise} \end{cases}$$

As in the two previous examples, we can check that the update function describes the particles dynamics on all words of length 3, and deduce the properties of locality, growth, surjectivity, coalescence and speed from there. We then apply the main result.

If μ is a Bernoulli measure: Then μ is invariant under the mirror application γ and $F \circ \gamma = \gamma \circ F$ by hypothesis. As in the previous example, we conclude that no particle can persist in $\Lambda_\mu(F)$. □

An automaton performing random walks

Let F be defined on the alphabet $\mathcal{A} = (\mathbb{Z}/2\mathbb{Z})^2$ on the neighbourhood $\{-2, \dots, 2\}$ by the local rule f defined as follows:

$$f : (a_{-2}, b_{-2}), \dots, (a_2, b_2) \mapsto (a_{-2}+a_2, c) \text{ where } c = \begin{array}{l} 1 \text{ if } (a_{-1}, b_{-1}) = (0,1) \text{ or } (a_0, b_0) = (1,1); \\ 0 \text{ otherwise.} \end{array}$$

Intuitively, the first layer performs addition mod 2 at distance 2, while the ones on the second layer behave as particles, moving right if the first layer contains a 1 and not moving if it contains a 0. Two colliding particles simply merge.

Proposition 2.1.7. Let $\nu \in \mathcal{M}_{\sigma-\mathrm{erg}}((\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}})$ and $\mu = \lambda \times \nu$, where λ is the uniform measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$.

Then $F_*^t \mu \underset{t\to\infty}{\longrightarrow} \lambda \times \widehat{\delta_0}$.

Proof. Pivato's formalism is not necessary here. Consider the set of particles $\mathcal{P} = \{1\}$ and

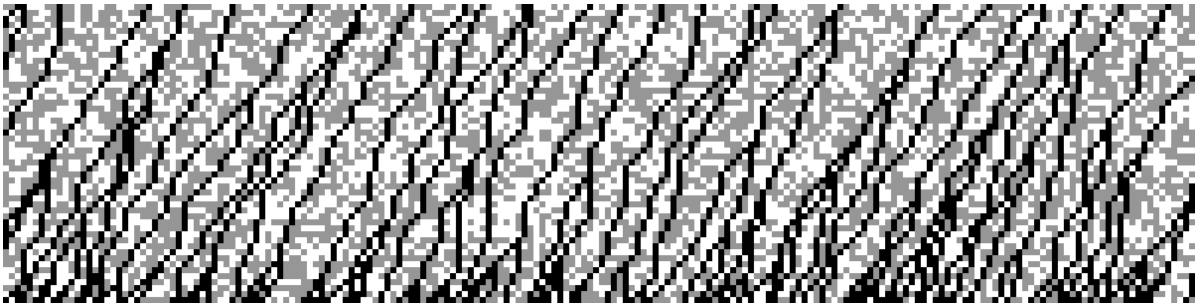


Figure 2.6: Automaton performing random walks iterated on the uniform measure. ■ is a particle, while the second layer is represented by □ (0) or ■ (1).

the factor π that is the projection on the second layer. The update function is defined as:

$$\forall x \in \mathcal{A}^{\mathbb{Z}},\ \forall k \in \mathbb{Z},\ \phi(x,k) = \begin{cases} \{k+1\} & \text{if } x_k = (1,1); \\ \{k\} & \text{if } x_k = (0,1); \\ \emptyset & \text{otherwise.} \end{cases}$$

Intuitively, each particle performs a random walk with independent steps and no bias. Thus Corollary 2.1.3 is not sufficient to conclude, and we need to use the general result of Theorem 2.1.2 by proving that $\{1\}$ clashes with itself.

Let $k \in \mathbb{N}$. We prove that, when x is chosen according to μ_k the conditional measure of μ relative to the event $\pi(x)_0 = \pi(x)_k = 1$, $\phi^t(x,k) - \phi^t(x,0)$ performs an unbiased and independent random walk with a "death condition" on 0 (particle collision).

Writing $(a_n^t, b_n^t) = F^t(x)_n$, we have $a_0^t = \sum_{n=0}^t \binom{n}{t} a^0_{-2t+4n} \mod 2$ by straightforward induction. Consider the evolution of $\phi^t(x,k) - \phi^t(x,0)$ at each step:

$$\begin{aligned} \delta_t(x) &= (\phi^{t+1}(x,k) - \phi^t(x,k)) - (\phi^{t+1}(x,0) - \phi^t(x,0)) \\ &= a^t_{\phi^t(x,k)} - a^t_{\phi^t(x,0)} \\ &= \left(\sum_{n=0}^{t} \binom{n}{t} a^0_{\phi^t(x,k)-2t+4n} \mod 2 \right) - \left(\sum_{n=0}^{t} \binom{n}{t} a^0_{\phi^t(x,0)-2t+4n} \mod 2 \right). \end{aligned}$$

Notice that:

- $a^0_{\phi^t(x,k)+2t}$ has coefficient 1 in the left-hand term and 0 in the right-hand term;
- $a^0_{\phi^t(x,0)-2t}$ has coefficient 0 in the left-hand term and 1 in the right-hand term.

Because the particle cannot move by more than one cell per step, these variables did not appear in the expression of any previous $\delta_{t'}(x), t' < t$. Because the initial measure is uniform, all variables a_n^0 are chosen independently and fairly between 0 and 1. Since both terms are sums of variables taking values in $\mathbb{Z}/2\mathbb{Z}$, this is enough to show that the terms are independent of each other, independent from all previous $\delta_{t'}(x), t' < t$, and are fairly

distributed between 0 and 1. Therefore $\delta_t(x)$ takes value 0 with probability $\frac{1}{2}$, -1 with probability $\frac{1}{4}$ and $+1$ with probability $\frac{1}{4}$, independently from all previous $\delta_{t'}$.

Therefore $\phi^t(x,k)-\phi^t(x,0)$ performs an unbiased and independent random walk, which implies that $\mu_k(\{x \ : \ \forall t, \phi^t(x,k) > \phi^t(x,0)\}) = 0$ (standard result in one-dimensional random walks). This shows that the particles almost surely end up being in interaction (by definition of coalescence), and therefore $\{1\}$ clashes with itself μ-almost surely. Applying the theorem, we find that no particle can remain in $\Lambda_\mu(F)$.

Therefore, if we write π_i the factor projecting on the i-th coordinate, $\pi_{2*}F_*^t\mu \to \widehat{\delta_0}$. Since the addition mod 2 automaton is surjective, it leaves the uniform measure invariant: see Theorem 3.1.1. Therefore $\pi_{1*}F_*^t\mu = \lambda$, and we conclude that $F_*^t\mu \to \lambda \times \widehat{\delta_0}$. □

2.1.5 Probabilistic cellular automata

This approach can be adapted to non-deterministic cellular automata, and in particular probabilistic cellular automata. We use here a generalised version of the standard definition.

Definition 2.1.5.
Let $\mathcal{A}$ be a finite alphabet and $\mathcal{N} \subset \mathbb{Z}$. We define the application that applies a bi-infinite sequence of local rules to a configuration componentwise:

$$\Phi_{\mathcal{N}} : \begin{array}{ll} (\mathcal{A}^{\mathcal{A}^{\mathcal{N}}})^{\mathbb{Z}} \times \mathcal{A}^{\mathbb{Z}} & \to \mathcal{A}^{\mathbb{Z}} \\ ((f_i)_{i\in\mathbb{Z}}, (x_i)_{i\in\mathbb{Z}}) & \mapsto (f_i((x_{i+r})_{r\in\mathcal{N}})_{i\in\mathbb{Z}}. \end{array}$$

Definition 2.1.6 (Generalised probabilistic cellular automaton)**.**
A **generalised probabilistic cellular automaton** $\tilde{F}$ on the alphabet $\mathcal{A}$ with neighbourhood $\mathcal{N}$ is defined by a measure on bi-infinite sequence of local rules $\nu \in \mathcal{M}_\sigma((\mathcal{A}^{\mathcal{A}^{\mathcal{N}}})^{\mathbb{Z}})$.
For a configuration $x \in \mathcal{A}^{\mathbb{Z}}$, $\tilde{F} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is then defined as:

$$\text{For any borelian } U, \ \tilde{F}(x)(U) = \int_{(\mathcal{A}^{\mathcal{A}^{\mathcal{N}}})^{\mathbb{Z}}} 1_U(\Phi_{\mathcal{N}}(f,x))\mathrm{d}\nu(f).$$

A deterministic cellular automaton F defined by a local rule f corresponds in this formalism to a Dirac $\nu = \widehat{\delta_f}$ (in which case the image measure is a Dirac on the image configuration), and usual probabilistic cellular automata correspond to the case where ν is a Bernoulli measure; in other words, the local rule that applies at each coordinate is drawn independently among a finite set of local rules $\mathcal{A}^{\mathcal{N}} \to \mathcal{A}$.

Definition 2.1.7 (Action on the space of measures)**.**
A generalised probabilistic cellular automaton defined by a measure $\nu \in \mathcal{M}_\sigma((\mathcal{A}^{\mathcal{A}^{\mathcal{N}}})^{\mathbb{Z}})$ extends naturally to an action $\tilde{F}_* : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \to \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ by defining

$$\tilde{F}_*\mu(U) = \int_{\mathcal{A}^{\mathbb{Z}}} \int_{(\mathcal{A}^{\mathcal{A}^{\mathcal{N}}})^{\mathbb{Z}}} 1_U(\Phi_{\mathcal{N}}(f,x))\mathrm{d}\nu(f)\mathrm{d}\mu(x).$$

The μ-limit measures set of $\tilde{F}$, $\mathcal{V}(\tilde{F},\mu)$, is the set of adherence values of the sequence $(\tilde{F}_*{}^t\mu)_{t\in\mathbb{N}}$, and the μ-limit set can be defined as

$$\Lambda_\mu(\tilde{F}) = \overline{\bigcup_{\eta\in\mathcal{V}(\tilde{F},\mu)} \text{supp } \eta}$$

The definitions of a particle system extend directly, except that the update function also depends on the choice of the local rules as well as on the configuration. Therefore we write $\phi(x, n, (f_i))$ instead of $\phi(x, n)$, where $x \in \mathcal{A}^{\mathbb{Z}}, n \in \mathbb{Z}$ and $(f_i) \in (\mathcal{A}^{\mathcal{A}^{\mathcal{N}}})^{\mathbb{Z}}$, and the composition notation is simplified as follows (inductively):

$$\phi^t\left(x, n, (f^k)_{0\leq k<t}\right) = \bigcup_{m\in\phi(x,n,f^1)} \phi^{t-1}\left(\Phi_{\mathcal{N}}(f^{t-1}, x), m, f^{t-1}\right),$$

where each $f^t \in (\mathcal{A}^{\mathcal{A}^{\mathcal{N}}})^{\mathbb{Z}}$ is a bi-infinite sequence of local rules.

A particle system is said to be coalescent ν-almost surely if the coalescence conditions hold for all $x \in \mathcal{A}^{\mathbb{Z}}$ and ν-almost every $f \in (\mathcal{A}^{\mathcal{A}^{\mathcal{N}}})^{\mathbb{Z}}$, and a particle $p \in \mathcal{P}$ has speed v ν^∞-almost surely if the speed conditions hold for ν^∞-almost every sequence $(f^t)_{t\in\mathbb{N}}$, where ν^∞ is the product measure (i.e. each f^t is drawn independently according to ν). The clashing conditions are extended similarly.

Theorem 2.1.8 (Main result - probabilistic automata)**.**
Let $\tilde{F} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be a probabilistic cellular automaton defined by $\nu \in \mathcal{M}_\sigma((\mathcal{A}^{\mathcal{A}^{\mathcal{N}}})^{\mathbb{Z}})$, μ an initial σ-ergodic measure and $(\mathcal{P}, \pi, \phi)$ a ν^∞-almost surely coalescent particle system for $\tilde{F}$ where $\mathcal{P}$ can be partitioned into sets $\mathcal{P}_1 \dots \mathcal{P}_n$ such that, for any $i < j$, $\mathcal{P}_i$ clashes with $\mathcal{P}_j$ μ, ν^∞-almost surely.

Then all particles appearing in the μ-limit set belong to the same subset, i.e. there exists a i such that

$$\forall p \in \mathcal{P},\ p \in \mathcal{L}(\pi(\Lambda_\mu(F))) \Rightarrow p \in \mathcal{P}_i.$$

If furthermore there exists a j such that $\mathcal{P}_j$ clashes with itself μ, ν^∞-almost surely, then this set of particles does not appear in the μ-limit set, i.e.

$$\forall p \in \mathcal{P},\ p \in \mathcal{L}(\pi(\Lambda_\mu(F))) \Rightarrow p \notin \mathcal{P}_j.$$

Corollary 2.1.9 (Main result with speedy particles - probabilistic automata)**.**
Let $\tilde{F} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a probabilistic cellular automaton defined by $\nu \in \mathcal{M}_\sigma((\mathcal{A}^{\mathcal{A}^{\mathcal{N}}})^{\mathbb{Z}})$, μ an initial σ-ergodic measure and $(\mathcal{P}, \pi, \phi)$ a ν^∞-almost surely coalescent particle system for $\tilde{F}$. Assume that each particle $p \in \mathcal{P}$ has speed $v_p \in \mathbb{R}$ ν^∞-almost surely, then there is a speed $v \in \mathbb{R}$ such that:

$$\forall p \in \mathcal{P},\ p \in \mathcal{L}(\pi(\Lambda_\mu(F))) \Rightarrow v_p = v.$$

The proof of these statements are exactly the same as the proofs of Theorem 2.1.2 and Corollary 2.1.3, except that every statement in the proof holds ν^∞-almost surely.

Example: Fatès' density classifying candidate

For any real $p \in [0, 1]$, consider the probabilistic automaton $\tilde{F}$ defined on the neighbourhood $\mathcal{N} = \{-1, 0, 1\}$ by local rules drawn independently between the traffic rule (rule #184, probability p) and the majority rule (rule #232, probability $1 - p$). This corresponds to the case where ν is a Bernoulli measure.

This automaton was introduced by Fatès in [Fat13] as a candidate to solve the density classification problem. Even though the following result does not answer this question, it is new to our knowledge.

Proposition 2.1.10.
Let $\mu \in \mathcal{M}_{\sigma-\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}})$ and p be a real in $[0,1]$.
Then $\Lambda_\mu(\tilde{F}) \subset \{^\infty 0^\infty, ^\infty 1^\infty, ^\infty(01)^\infty, ^\infty(10)^\infty\}$. As a consequence, any limit measure of $(\tilde{F}_*^t \mu)_{t\in\mathbb{N}}$ is a convex combination of $\widehat{\delta_0}, \widehat{\delta_1}$ and $\widehat{\delta_{01}}$.

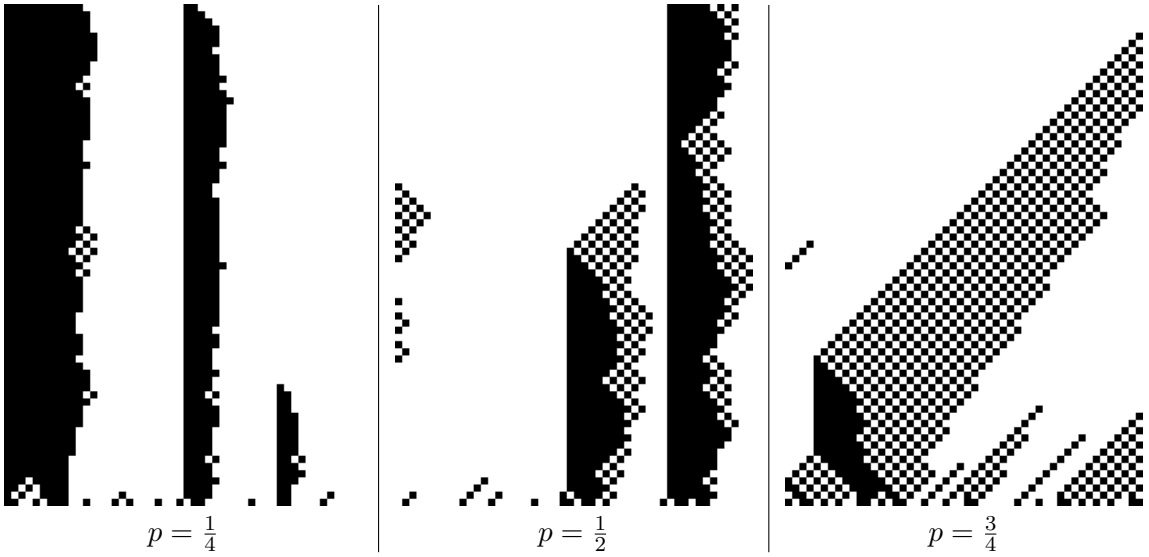


Figure 2.7: Dynamics of the traffic-majority automaton iterated on the initial measure $\mathrm{Ber}(\frac{3}{5}, \frac{2}{5})$. Density classification is more efficient with p close to 1.

Proof. The cases $p = 0, 1$ correspond to deterministic automata and can be treated easily.

The visual intuition suggests to consider interface defects according to the decomposition $\Sigma_0 \sqcup \Sigma_1 \sqcup \Sigma_2$, where $\Sigma_0 = \{^\infty 0^\infty\}$, $\Sigma_1 = \{^\infty 1^\infty\}$ (monochromatic subshifts) and $\Sigma_2 = \{^\infty(01)^\infty, ^\infty(10)^\infty\}$ (checkerboard subshift), since those SFTs are invariant under the action of both rules. The set of particles would be $\mathcal{P} = \{p_{i,j} : i \neq j \in \{0,1,2\}\}$.

However, as Figure 2.8 shows, the particle p_{10} can "explode" and give birth to two particles p_{12} and p_{20}, contradicting the condition of coalescence. To solve this problem, we tweak the particle system by replacing each particle p_{10} by one particle p_{12} and one particle p_{20}.

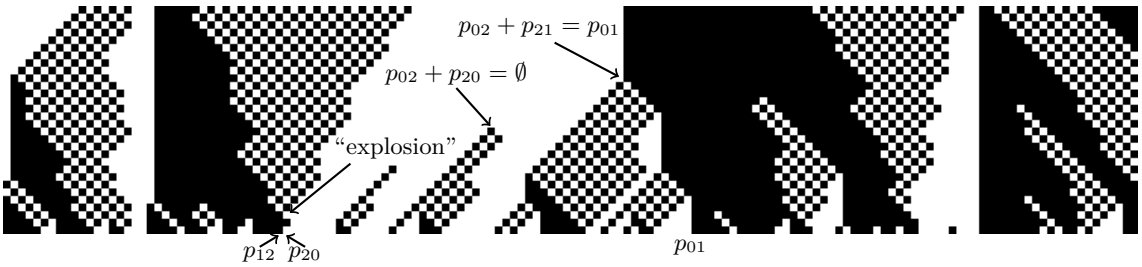


Figure 2.8: Fatès' traffic-majority probabilistic automaton, with $p = \frac{3}{4}$.

The corresponding factor π is defined on the neighbourhood $\{0, \dots, 3\}$ by the local rule:

0011	$\mapsto$	p_{01}	_110	$\mapsto$	p_{12}
0010	$\mapsto$	p_{02}	_100	$\mapsto$	p_{20}
1011	$\mapsto$	p_{21}	otherwise	$\mapsto$	0

where the wildcards _ can take both values.

In the absence of interactions, the update function $\phi(x, k, f)$ can be defined in the following manner.

Regardless of the rule that is applied, p_{01}, p_{02} and p_{21} move at a constant speed 0, $+1$ and -1 respectively. A particle p_{12} move at speed -1 if rule #184 is applied at its position and at speed $+1$ otherwise (independent random walk with bias $1-2p$), except if a particle p_{20} prevents its movement to the right, in which case it does not move. The particle p_{20} behaves symmetrically. Furthermore all interactions are of the form $p_{ij} + p_{ji} \to \emptyset$ or $p_{ij} + p_{jk} \to p_{ik}$ (when $(i, j, k) \neq (1, 2, 0)$, by the last remark). A formal definition of the update function would be tedious, but it is entirely described by these remarks. The various conditions of locality, disjunction, particle control, surjectivity and coalescence are proved similarly to the previous examples.

Assume $p \geq \frac{1}{2}$. We show that no particle can remain asymptotically by applying the main result on the sets $(\mathcal{P}_i)_{0 \leq i \leq 4}$: $\{p_{02}\}$, $\{p_{20}\}$, $\{p_{01}\}$, $\{p_{12}\}$ and $\{p_{21}\}$. We need only to show the clashes relative to the second and fourth sets since all other clashes are consequences of the speed of these particles.

Let $k \in \mathbb{N}$ and x be such that $\pi(x)_0 = p_{02}$ and $\pi(x)_k \in \{p_{12}, p_{20}\}$. Since p_{02} progresses at speed 1, the distance $\phi^t(x, k) - \phi^t(x, 0)$ cannot increase, and it decreases by at least one with probability p (respectively $1 - p$). It is clear that the particles end up in interaction ν^∞-almost surely. Showing that p_{12} and p_{20} clash with p_{21} is symmetric.

Let x be such that $\pi(x)_0 = p_{20}$ and $\pi(x)_k = p_{01}$. As long as there are no interactions, the distance $\phi^t(x, k) - \phi^t(x, 0) = -\phi^t(x, 0)$ performs an independent random walk of bias $2p - 1$, where a increasing step is sometimes replaced by a constant step. Such a random walk reaches 0 ν^∞-almost surely, which shows that the particles end up in interaction. The clashes between p_{01} and p_{12}, and between $\{p_{20}\}$ and $\{p_{12}\}$, are proved in a similar manner. The same proof holds for $p \leq \frac{1}{2}$ by exchanging the roles of p_{20} and p_{12}.

Applying Theorem 2.1.8, we conclude that only one particle p_{ij} can remain in the μ-limit set. However, if we consider $V_k = \{x \in \Lambda_\mu(F) : \pi(x)_k = p_{ij}\}$, we notice that

configurations in V_k are of the form $y \cdot z$, where $y \in \mathcal{A}^{]-\infty,k]}$ is admissible for Σ_i and $z \in \mathcal{A}^{[k+1,+\infty[}$ is admissible for Σ_j; in particular, they contain only one particle, and the $(V_k)_{k\in\mathbb{Z}}$ are disjoint. By σ-invariance, for any measure $\eta \in \mathcal{V}(\tilde{F}, \mu)$, $\eta(V_k)$ is independent from k and $\eta\left(\bigcup_k V_k\right) = \sum_k \eta(V_k) \leq 1$. Consequently, $\eta(V_k) = 0$, which means $V_k \notin \mathrm{supp}(\eta)$, and we conclude that no particle remain in the μ-limit set. In other words, $\Lambda_\mu(F) \subset \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$. □

Section 2.2

Particle-based organisation: quantitative results

For some cellular automata with simple defect dynamics, the previous results can be refined with a quantitative approach: that is, to determine the asymptotic distribution of random variables related to the particles. In [KFD11], Kůrka, Formenti and Dennunzio considered $T_n(x)$, the entry time after time n on the initial configuration x, which is the waiting time before a particle appears in a given position after time n. They restricted their study to a gliders automaton, which is a cellular automaton on 3 states: a background state and two particles evolving at speeds 0 and -1 that annihilate on contact. Thus, we have one entry time for each type of particle ($T_n^+(x)$ and $T_n^-(x)$). When the initial configuration is drawn according to the Bernoulli measure of parameters $(\frac{1}{2}, 0, \frac{1}{2})$, which means that each cell contains, independently, a particle of each type with probability $\frac{1}{2}$, they proved that:

$$\forall \alpha \in \mathbb{R}^+,\ \mu\left(\frac{T_n^-(x)}{n} \leq \alpha\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi} \arctan \sqrt{\alpha}.$$

They also called to develop formal tools in order to be able to handle more complex automata, starting with the $(-1, 1)$ symmetric case.

In Section 2.2.2, we extend this result to allow arbitrary values for the particle speeds v_- and v_+, and relax the conditions on the initial measure to some α-mixing conditions. Then, when $v_- < 0$ and $v_+ \geq 0$, we have:

$$\forall x \in \mathbb{R}^+,\ \mu\left(\frac{T_n^-(x)}{n} \leq \alpha\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi} \arctan\left(\sqrt{\frac{-v_-\alpha}{v_+ - v_- + v_+\alpha}}\right),$$

and symmetrically if we exchange $+$ and $-$. The proof relies on the fact that the behaviour of gliders automata can be characterised by some random walk process; this idea was introduced by Belitsky and Ferrari in [BF95] and was already used in [KM00] and [KFD11]. In our case, a particle appearing in a position corresponds to a minimum between two concurrent random walks. Under α-mixing conditions, we rescale this process and approximate it with a Brownian motion. Thus we obtain the explicit asymptotic distribution of entry times.

This method, consisting in associating a random walk to each gliders automata and studying this random walk using scale invariance, is not limited to this particular conjecture concerning entry times. Indeed, we see in the next two sections that it can be used to study the asymptotic behaviour of two other, arguably more natural, parameters: the particle density at time t and the rate of convergence to the limit measure. However, we obtain only an upper bound instead of an explicit asymptotic distribution. There is no doubt this method can be adapted to other parameters in a similar way.

Furthermore, these results can be extended to other automata with similar behaviour, such as those in Figure 2.1, by factorising them onto a gliders automaton. This point is discussed in Section 2.2.6.

2.2.1 Gliders automata and random walks

Remember that, as an abuse of notation, we write $[a, b]$ instead of $\{a, \dots, b\} \subset \mathbb{Z}$ when the context is clear.

Definition 2.2.1 (Gliders automata)**.**
Let $v_- < v_+ \in \mathbb{Z}$. The (v_-, v_+)**-gliders automaton** (or GA) G is the cellular automaton of neighbourhood $[-v_+, -v_-]$ defined on the alphabet $\mathcal{A} = \{-1, 0, +1\}$ by the local rule:

$$f(x_{-v_+} \dots x_{-v_-}) = \begin{cases} +1 & \text{if } x_{-v_+} = +1 \text{ and } \forall N \leq -v_-, \sum_{n=-v_++1}^{N} x_n \geq 0 \\ -1 & \text{if } x_{-v_-} = -1 \text{ and } \forall N \geq -v_+, \sum_{n=N}^{-v_--1} x_n \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In all the following, $\mathcal{A} = \{-1, 0, +1\}$ and the diagrams are represented with the convention $\square = 0, \blacksquare = +1, \blacksquare = -1$.

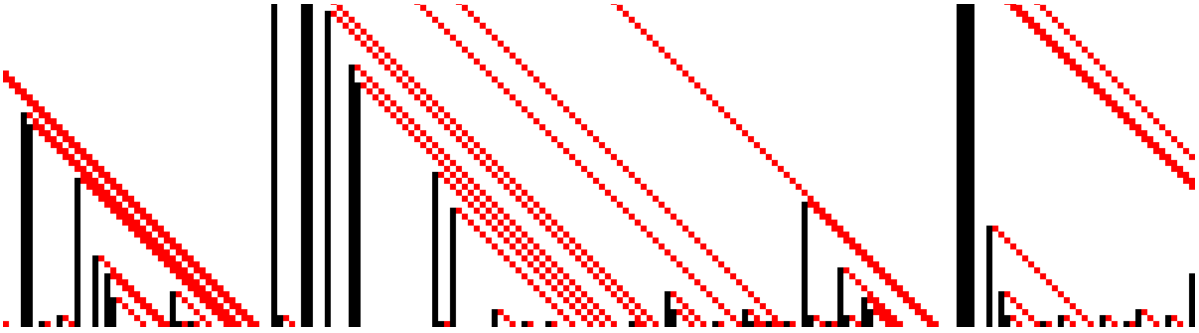


Figure 2.9: Space-time diagram of the $(-1, 0)$-gliders automaton on a random initial configuration.

Our results apply on automata with simple defects dynamics, namely, automata admitting a particle system with $\mathcal{P} = \{\pm 1\}$ and whose update function corresponds to a gliders automaton. We first prove our results for gliders automata before generalising them in Section 2.2.6. Let us introduce some tools that turn the study of the dynamics of a gliders automaton into the study of some random walk.

Definition 2.2.2 (Random walk associated with a configuration)**.**
Let $x \in \{-1, 0, 1\}^{\mathbb{Z}}$. Define the partial sums S_x by:

$$S_x(0) = 0 \quad \text{and} \quad \forall k \in \mathbb{Z},\ S_x(k+1) - S_x(k) = x_k.$$

We extend S_x to $\mathbb{R}$ by putting $S_x(t) = (\lceil t \rceil - t) S_x(\lfloor t \rfloor) + (t - \lfloor t \rfloor) S_x(\lceil t \rceil)$ for $t \in \mathbb{R}$. We also introduce the rescaled process $S_x^k : t \mapsto \frac{S_x(kt)}{\sqrt{k}}$.

This random walk is simpler to study than the space-time diagram of the gliders automaton, and actually contains the same amount of information, as shown by the following technical lemmas.

Definition 2.2.3.
Let $f : \mathbb{R} \to \mathbb{R}$ and $U \subset \mathbb{R}$. We define $\underset{U}{\operatorname{argmin}}\ f$ by:

$$\forall t \in U,\ t = \underset{[t_0,t_1]}{\operatorname{argmin}}\ f \iff \forall t' \in U \backslash \{t\}, f(t) < f(t').$$

In other words, t realises the strict minimum of f on U; this point is not always defined.

Lemma 1. Let G be the (v_-, v_+)-gliders automaton. For all $j \in \mathbb{Z}$ and $n \geq 1$,

$$j = \underset{[j,\ j+n]}{\operatorname{argmin}}\ S_{G(x)} \iff j - v_+ = \underset{[j-v_+,\ j+n-v_-]}{\operatorname{argmin}}\ S_x,$$
$$j = \underset{[j-n,\ j]}{\operatorname{argmin}}\ S_{G(x)} \iff j - v_- = \underset{[j-n-v_+,\ j-v_-]}{\operatorname{argmin}}\ S_x.$$

Proof. We prove those equivalences by induction on n. At each step, we prove only the first equivalence, the other one being symmetric.

Base case.

$$\begin{aligned} S_{G(x)}(j) < S_{G(x)}(j+1) &\Leftrightarrow G(x)_j = +1 \\ &\Leftrightarrow x_{j-v_+} = +1 \text{ and } \forall N \leq -v_-, \sum_{t=-v_++1}^{N} x_{j+t} \geq 0 \\ &\Leftrightarrow S_x(j - v_+) < \min_{[j+1-v_+,\ j+1-v_-]} S_x. \end{aligned}$$

Induction. Assume both equivalences hold for some $n \geq 1$.

Suppose $j = \underset{[j,\ j+n+1]}{\operatorname{argmin}}\ S_{G(x)}$. In particular $j = \underset{[j,\ j+n]}{\operatorname{argmin}}\ S_{G(x)}$, and by induction hypothesis $j - v_+ = \underset{[j-v_+,\ j+n-v_-]}{\operatorname{argmin}}\ S_x$. We distinguish two cases:

- if $S_x(j+n-v_-+1) > S_x(j-v_+)$, then $j - v_+ = \underset{[j-v_+,\ j+n-v_-+1]}{\operatorname{argmin}}\ S_x$ and we conclude;
- otherwise, this means that $S_x(j+n-v_-+1) = S_x(j-v_+)$ (the walk can decrease by at most one at each step), and thus

$$j + n - v_- + 1 = \underset{[j-v_++1,\ j+n-v_-+1]}{\operatorname{argmin}}\ S_x.$$

By induction hypothesis,

$$j + n + 1 = \underset{[j+1,\ j+n+1]}{\operatorname{argmin}}\ S_{G(x)},$$

and in particular $S_{G(x)}(j+n+1) < S_{G(x)}(j+1)$. Therefore $S_{G(x)}(j+n+1) \leq S_{G(x)}(j)$, a contradiction with the first assumption.

The converse is proved in a similar manner.

□

Lemma 2. Let G be the (v_-, v_+)-gliders automaton. For all $j \in \mathbb{Z}$ and $k \geq 0$,

$$G^t(x)_j = -1 \iff j - v_- t + 1 = \operatorname*{argmin}_{[j-v_+t,\ j-v_-t+1]} S_x$$
$$G^t(x)_j = +1 \iff j - v_+ t = \operatorname*{argmin}_{[j-v_+t,\ j-v_-t+1]} S_x$$

This is illustrated in Figure 2.10.

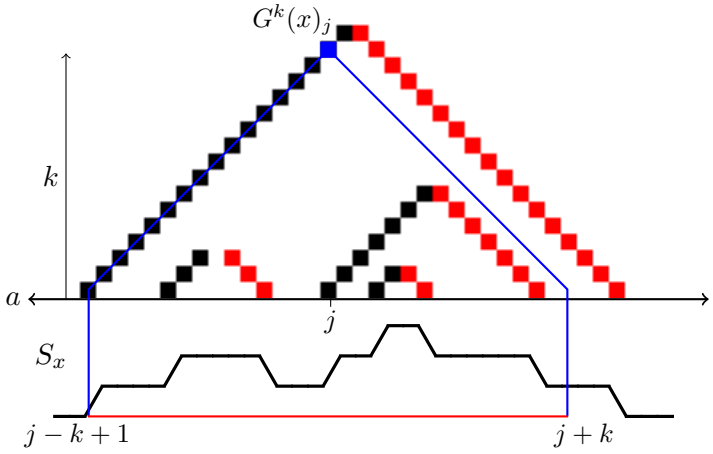


Figure 2.10: Illustration of Lemma 2. A strict minimum is reached on $j-k+1$.

Proof. By induction on t, proving only the first equivalence at each step:

Base case $(t=0)$**.** By definition of S_x, $S_x(j+1) < S_x(j) \Leftrightarrow x_j = -1$.

Induction. Assume that both equivalences hold for a given itme t. By applying the induction hypothesis on $G(x)$, $G^{t+1}(x)_j = -1 \Leftrightarrow j - v_- t + 1 = \operatorname*{argmin}_{[j-v_+t,\ j-v_-t+1]} S_{G(x)}$ and we conclude by applying Lemma 1.

□

2.2.2 Entry times

The main result of Section 2.1 implies that, for any σ-ergodic initial measure μ, $\Lambda_G(\mu)$ contains at most one kind of particle, which one depending on whether $\mu([+1]) > \mu([-1])$ or the

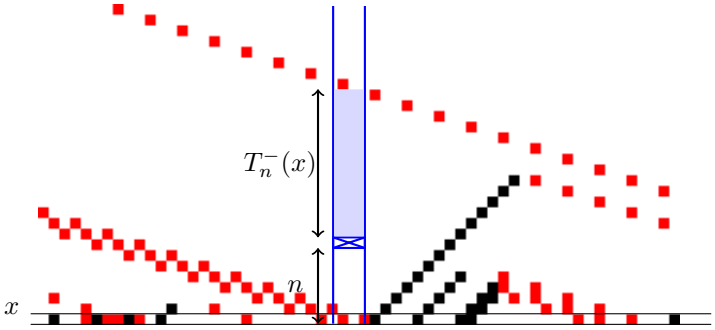


Figure 2.11: An entry time for the (-3,1)-gliders automaton.

opposite. When $\mu([+1]) = \mu([-1])$, $\Lambda_G(\mu)$ only contains the particleless configuration $^\infty 0^\infty$. In other words, $G_*^t\mu \to \widehat{\delta_0}$, which means that the probability of seeing a particle in any fixed finite window tends to 0 as $t \to \infty$.

Definition 2.2.4 (Entry times)**.**
Let $v_- < 0 \leq v_+ \in \mathbb{Z}$, G the (v_-, v_+)-GA and $x \in \{-1, 0, 1\}^{\mathbb{Z}}$. We define:

$$T_n^-(x) = \min\{k \in \mathbb{N} \ : \ \exists i \in [0, |v_-| - 1],\ G^{k+n}(x)_i = -1\},$$

with $T_n^-(x) = \infty$ if this set is empty. This is the **entry time** of x into the set $\{b \in \{-1, 0, 1\}^{\mathbb{Z}} \ : \exists i \in [0, |v_-|-1], b_i = -1\}$ after time n at position 0. We define $T_n^+(x)$ in a symmetrical manner.

The size of the considered window is such that any particle "passing through" the column 0 appears in this window exactly once (See Figure 2.11). Of course entry times for particles of speed 0 make no sense. From now on, we only consider T^- for simplicity, all the results being valid for T^+.

As a consequence of Birkhoff's ergodic theorem, when $\mu([-1]) > \mu([+1])$, -1 particles persist μ-almost surely and their density converges to a positive number. Therefore:

- $\mu(T_n^+(x) = \infty) \underset{n\to\infty}{\longrightarrow} 1$;
- $\forall \alpha > 0, \mu\left(\frac{T_n^-(x)}{n} \leq \alpha\right) \underset{n\to\infty}{\longrightarrow} 1$,

and symmetrically. This is why we only consider the case $\mu([-1]) = \mu([+1])$. Kůrka et al. proved the following result:

Theorem 2.2.1 ([KFD11])**.**
For the $(-1, 0)$-GA ("Asymmetric gliders") with an initial measure $\mu = \text{Ber}\left(\frac{1}{2}, 0, \frac{1}{2}\right)$:

$$\forall \alpha > 0,\ \mu\left(\frac{T_n^-(x)}{n} \leq \alpha\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi} \arctan\sqrt{\alpha}.$$

In the same article, they conjectured that this result could be extended to any initial Bernoulli measure of parameters $(p, 1-2p, p)$ for $0 \leq p \leq \frac{1}{2}$ by replacing the right-hand term by $\frac{2}{\pi} \arctan\sqrt{2p\alpha}$. We will prove that this conjecture is actually incorrect.

To state our result, we introduce two particular subclasses of $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. We recall the definition of the α-mixing coefficients of a measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$:

$$\alpha_\mu(n) = \sup\{|\mu(A \cap B) - \mu(A)\mu(B)| : A \in \mathfrak{B}_{]-\infty,0]}, B \in \mathfrak{B}_{[n,+\infty[}\}.$$

Define:

- $\mathrm{Ber}_=$ the set of Bernoulli measures on $\{-1,0,+1\}^{\mathbb{Z}}$ and parameters $(p, 1-2p, p)$ for some $0 < p \leq \frac{1}{2}$;
- $\mathcal{M}ix$ the set of measures $\mu \in \mathcal{M}_\sigma(\{-1,0,+1\}^{\mathbb{Z}})$ satisfying:
  - $\int_{\mathcal{A}^{\mathbb{Z}}} x_0 \, \mathrm{d}\mu(x) = 0$;
  - $\sum_{k=0}^{\infty} \int_{\mathcal{A}^{\mathbb{Z}}} x_0 \cdot x_k \, \mathrm{d}\mu(x)$ converges absolutely to a real $\sigma_\mu^2 > 0$ (asymptotic variance);
  - $\exists \varepsilon > 0, \sum_{n \geq 0} \alpha_\mu(n)^{\frac{1}{4}-\varepsilon} < \infty$.

In particular, $\mathrm{Ber}_= \subset \mathcal{M}ix$.

Theorem 2.2.2 (Main result)**.**
For any (v_-, v_+)-GA with $v_- < 0$ and $v_+ \geq 0$ and any initial measure $\mu \in \mathcal{M}ix$,

$$\forall \alpha > 0, \ \mu\left(\frac{T_n^-(x)}{n} \leq \alpha\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi} \arctan\left(\sqrt{\frac{-v_-\alpha}{v_+ - v_- + v_+\alpha}}\right).$$

Notice that this limit is independent from μ (as long as $\mu \in \mathcal{M}ix$), disproving the conjecture when $\mu \in \mathrm{Ber}_=$.

2.2.3 Brownian motion and proof of the main result

The third hypothesis for $\mathcal{M}ix$ is chosen so that the large-scale behaviour of the partial sums $S_x(t)$ can be approximated by a Brownian motion. This invariance principle is the core of our proofs. The first and second conditions ensures that the Brownian motion obtained this way have no bias and nonzero variance, respectively.

Definition 2.2.5 (Brownian motion)**.**
A **Brownian motion** (or **Wiener process**) B of mean 0 and variance σ^2 is a continuous time stochastic process taking values in $\mathbb{R}$ such that:

- $B(0) = 0$,
- $t \mapsto B(t)$ is almost surely continuous,
- $B(t_2) - B(t_1)$ follow the normal law of mean 0 and variance $(t_2 - t_1)\sigma^2$;
- For $t_1 < t_2 \leq t_1' < t_2'$, increments $B(t_2) - B(t_1)$ and $B(t_2') - B(t_1')$ are independent.

See [MP10] for a general introduction to Brownian motion.

Proposition 2.2.3 (Rescaling property)**.**
Let B be a Brownian motion. Then, for any $k > 0$, $t \mapsto \frac{1}{\sqrt{k}} B(kt)$ is a Brownian motion with same mean and variance.

We now state some invariance principles, which consists in appoximating rescaled random walks by Brownian motion. We use a strong version, which guarantees an almost sure convergence by considering a copy of the process in a richer probability space.

Theorem 2.2.4 ([ZC96], Corollary 9.3.1)**.**
Let $X = (X_i)_{i\in\mathbb{N}}$ a family of random variables taking values in $\{-1, 0, 1\}$. We denote $\alpha_X(n)$ its α-mixing coefficients defined as:

$$\alpha_X(n) = \sup\{|P(A \cap B) - P(A)P(B)| : t \in \mathbb{N}, A \in X_{[0,t]}, B \in X_{[t+n,+\infty[}\},$$

where $X_{[a,b]}$ is the sigma-algebra generated by $(X_a, \dots, X_b)$.

Assume that:

(1) $\forall i, \mathbb{E}(X_i) = 0$;

(2) $\frac{1}{t}\mathbb{E}\left(\sum_{i,j=1}^{\lfloor t\rfloor} X_i \cdot X_j\right)$ converges absolutely to some positive real σ^2;

(3) $\exists\varepsilon > 0, \sum_{n=1}^{\infty} \alpha_X(n)^{\frac{1}{4}+\varepsilon}$.

Then we can define two processes $X' = (X'_i)_{i\in\mathbb{N}}$ and B on a richer probability space $(\Omega, \mathbb{P})$ such that:

1. X and X' have the same distribution;
2. B is a Brownian motion of mean 0, variance σ^2;
3. for any $\varepsilon > 0$,
$$\left|\sum_{i=1}^{\lfloor t\rfloor} X_i - B(t)\right| = O\left(t^{\frac{1}{4}+\varepsilon}\right) \quad \mathbb{P}\text{-almost surely.}$$

Corollary 2.2.5. Let $\mu \in \mathcal{M}ix$. For any fixed constants $q < r \in \mathbb{R}$, we can define a process $X' = (X'_i)_{i\in\mathbb{Z}}$ and a family of processes $(t \mapsto B_n(t))_{n\in\mathbb{N}}$ on a richer probability space $(\Omega, \mathbb{P})$ such that:

1. X' has distribution μ;
2. every B_n is a Brownian motion of mean 0 and variance $\sigma_\mu^2 > 0$;
3. for any $\varepsilon > 0$, denoting $S_{X'}$ the piecewise linear function defined by $S_{X'}(0) = 0$ and $S_{X'}(k+1) - S_{X'}(k) = X'_k$ for all $k \in \mathbb{Z}$,
$$\forall n \in \mathbb{N}, \sup_{t\in[q,r]} \left|\frac{S_{X'}(nt)}{\sqrt{n}} - B_n(t)\right| = O\left(n^{-\frac{1}{4}+\varepsilon}\right) \quad \mathbb{P}\text{-almost surely.}$$

Proof. We apply Theorem 2.2.4 on $(x_i)_{i\in\mathbb{N}}$, where x has distribution μ. Because μ is σ-invariant, this is a stationary process. The first and third conditions are satisfied by definition of $\mathcal{M}ix$. For the second condition,

$$\frac{1}{n}\mathbb{E}(S_X(n)^2) = \frac{1}{n}\sum_{0\le i,j\le n} \mathbb{E}(x_i \cdot x_j) \underset{n\to\infty}{\longrightarrow} \sigma_\mu^2$$

by stationarity. We obtain two processes $X^1 = (X_i^1)_{i\in\mathbb{N}}$ and B^1 on a richer probability

space $(\Omega, \mathbb{P})$ such that X^1 has the same distribution as x, B^1 is a Brownian motion of mean 0, variance σ_μ^2, and:

$$\forall \varepsilon > 0, \left| \sum_{i=1}^{\lfloor t \rfloor} X_i^1 - B^1(t) \right| = \underset{t \to +\infty}{O} \left(t^{\frac{1}{4}+\varepsilon} \right) \quad \mathbb{P}\text{-almost surely.}$$

Since the variables X_i^1 take value in $\{-1, 0, 1\}$, we have for any t $\left| \sum_{i=1}^{\lfloor t \rfloor} X_i^1 - S_{X^1}(t) \right| < 1$ (a staircase and piecewise linear function having the same values on $\mathbb{N}$). Therefore:

$$\forall \varepsilon > 0, \left| S_{X^1}(t) - B^1(t) \right| = \underset{t \to +\infty}{O} \left(t^{\frac{1}{4}+\varepsilon} \right) \quad \mathbb{P}\text{-almost surely.}$$

$$\forall \varepsilon > 0, \forall n \in \mathbb{N}, \ \frac{1}{\sqrt{n}} \left| S_{X^1}(tn) - B^1(tn) \right| = \underset{n \to \infty}{O} \left(n^{-\frac{1}{4}+\epsilon} \right) \cdot \underset{t \to \infty}{O} \left(|t|^{\frac{1}{4}+\epsilon} \right) \quad \mathbb{P}\text{-almost surely.}$$

For any $r \in \mathbb{R}_+^2$, taking the sup for $t \in [0, r]$, we obtain:

$$\forall \varepsilon > 0, \forall n \in \mathbb{N}, \ \sup_{t \in [0,r]} \left| \frac{S_{X^1}(tn)}{\sqrt{n}} - \frac{B^1(tn)}{\sqrt{n}} \right| = \underset{n \to \infty}{O} \left(n^{-\frac{1}{4}+\epsilon} \right) \quad \mathbb{P}\text{-almost surely.}$$

By rescaling property $B_n^1 : t \mapsto \frac{B^1(tn)}{\sqrt{n}}$ is a Brownian motion of same mean and variance as B^1.

To extend the result to negative values, we apply the theorem again to $(x_{-i-1})_{i \in \mathbb{N}}$, obtaining a process X^2 and a Brownian motion B^2 satisfying the same asymptotic bound on $t \to -\infty$. Joining both parts, we can see that the process $X' = \dots X_{-2}^2, X_{-1}^2, X_0^1, X_1^1 \dots$ have distribution μ and $B_n : t \mapsto B_n^1(t)$ if $t \geq 0$, $B_n^2(t)$ if $t < 0$ is a Brownian motion. □

For a survey of invariance principles under different assumptions, see [MR12].

Using this last result, we prove the main theorem.

Proof of Theorem 2.2.2.

For any $x \in \{-1, 0, 1\}^{\mathbb{Z}}$, Lemma 2 applied on the column 0 gives:

$$\begin{aligned} T_n^-(x) &= \min \left\{ k \in \mathbb{N} \mid \exists j \in [0, -v_-[\ , S_x(-v_-(n+k)+j+1) < \min_{[-v_+(n+k)+j, \ -v_-(n+k)+j]} S_x \right\} \\ &= \min \left\{ k \in \mathbb{N} \mid \exists j \in [0, -v_-[\ , S_x(-v_-(n+k)+j+1) < \min_{[-v_+(n+k)+j, \ -v_- n]} S_x \right\} \end{aligned}$$

Note that if this condition is reached on $k \in \mathbb{N}$, since S_x is piecewise linear, it is attained for t as soon as $t > k - 1$ and reciprocally. Thus:

$$T_n^-(x) = \inf \left\{ t \geq 0 \mid \exists j \in [0, -v_-[\ , S_x(-v_-(n+t)+j+2) < \min_{[-v_+(n+t)+j+1, -v_- n]} S_x \right\}$$

Replacing j by 0 in this expression adds to the infimum a value comprised between 0 and $\frac{-v_- - 1}{-v_-}$ (remember $v_- < 0$). Since the infimum is necessarily an integer, we compensate by taking the integer part:

$$\begin{aligned} T_n^-(x) &= \left\lfloor \inf\left\{ t \geq 0 \mid S_x(-v_-(n+t)+2) < \min_{[-v_+(n+t)+1, -v_- n]} S_x \right\} \right\rfloor \\ &= \left\lfloor \inf\left\{ t \geq 0 \mid S_x^n\left(-v_-\left(1+\frac{t}{n}\right)+\frac{2}{n}\right) < \min_{[-v_+(1+\frac{t}{n})+\frac{1}{n}, -v_-]} S_x^n \right\} \right\rfloor \\ &= \left\lfloor n \cdot \inf\left\{ t \geq 0 \mid S_x^n\left(-v_-(1+t)+\frac{2}{n}\right) < \min_{[-v_+(1+t)+\frac{1}{n}, -v_-]} S_x^n \right\} \right\rfloor \end{aligned}$$

Dividing by n, since S_x^n is $\sqrt{n}$-Lipschitz and $t - \frac{1}{n} \leq \frac{\lfloor nt \rfloor}{n} \leq t$ for all $t, n \in \mathbb{R} \times \mathbb{N}$:

$$\begin{aligned} &\mu\left(\min_{[-v_-, -v_-(1+\alpha)]} S_x^n + \frac{4}{\sqrt{n}} < \min_{[-v_+(1+\alpha), -v_-]} S_x^n \right) \leq \mu\left(\frac{T_n^-(x)}{n} \leq \alpha \right) \\ &\mu\left(\frac{T_n^-(x)}{n} \leq \alpha \right) \leq \mu\left(\min_{[-v_-, -v_-(1+\alpha)]} S_x^n - \frac{3}{\sqrt{n}} < \min_{[-v_+(1+\alpha), -v_-]} S_x^n \right) \end{aligned} \tag{1}$$

Using Corollary 2.2.5, we build a process X' and a family of processes $(B_n)_{n\in\mathbb{N}}$ on a richer probability space $(\Omega, \mathbb{P})$ such that X' is distributed according to μ and the B_n are Brownian motions.

$$\forall n \in \mathbb{N}, \sup_{[-v_+(1+\alpha), -v_-(1+\alpha)]} \left| \frac{S_{X'}(nt)}{\sqrt{n}} - B_n(t) \right| = O\left(n^{-\frac{1}{4}+\varepsilon}\right) \quad \mathbb{P}\text{-almost surely.}$$

By symmetry, $B_n^l(t) = B_n(-v_- - t) - B_n(-v_-)$ and $B_n^r(t) = B_n(-v_- + t) - B_n(-v_-)$ are two independent Brownian motions on $[0,\ v_- - v_+(1+\alpha)]$ and $[0,\ -v_-\alpha]$, respectively. Consequently, for any $\varepsilon > 0$ and n large enough:

$$\begin{aligned} \mu\left(\min_{[-v_-, -v_-(1+\alpha)]} S_x^n - \varepsilon < \min_{[-v_+(1+\alpha), -v_-]} S_x^n \right) &= \mathbb{P}\left(\min_{[-v_-, -v_-(1+\alpha)]} S_{X'}^n - \varepsilon < \min_{[-v_+(1+\alpha), -v_-]} S_{X'}^n \right) \\ &\leq \mathbb{P}\left(\min_{[-v_-, -v_-(1+\alpha)]} B_n - 2\varepsilon < \min_{[-v_+(1+\alpha), -v_-]} B_n \right) \\ &\leq \mathbb{P}\left(\min_{[0, -v_-\alpha]} B_n^l - 2\varepsilon < \min_{[0, -v_- + v_+(1+\alpha)]} B_n^r \right) \end{aligned} \tag{2}$$

and a symmetrical lower bound for the first term of (1). We evaluate this last term.

For any Brownian motion B and $b > 0$, we have by rescaling $\mathbb{P}\left(\min_{[0,b]} B \geq m \right) = \mathbb{P}\left(\min_{[0,1]} B \geq \frac{m}{\sqrt{b}} \right)$. Furthermore, since B_n^l and B_n^r are independent, so are $\min_{[0,1]} B_x^l$ and

$\min_{[0,1]} B_x^r$. Denote μ_m the law of the minimum of a Brownian motion on $[0,1]$, which is defined by the density function:

$$\begin{array}{rcll} \mathbb{R} & \to & \mathbb{R} & \\ t & \mapsto & \frac{e^{-t^2}}{2} & \text{if } t \leq 0, \qquad \text{(see [MP10]).} \\ & & 0 & \text{otherwise.} \end{array}$$

This means that for any $y, z > 0$:

$$\begin{aligned} \mathbb{P}\left(\min_{[0,y]} B_n^l < \min_{[0,z]} B_n^r\right) &= \int_{-\infty}^{0}\int_{-\infty}^{0} 1_{\{\sqrt{y}\cdot m_1 \leq \sqrt{z}\cdot m_2\}} d\mu_m(m_2) d\mu_m(m_1) \\ &\underset{(i)}{=} \frac{4}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\frac{\sqrt{z}\cdot m_2}{\sqrt{y}}} e^{\frac{-m_1^2}{2}} e^{\frac{-m_2^2}{2}} dm_1 dm_2 \\ &\underset{(ii)}{=} \frac{2}{\pi}\int_{\pi}^{\pi+\arctan\left(\sqrt{\frac{y}{z}}\right)}\int_{0}^{+\infty} r e^{\frac{-r^2}{2}} dr d\theta \\ &= \frac{2}{\pi}\arctan\left(\sqrt{\frac{y}{z}}\right) \end{aligned} \tag{3}$$

(i) by using the law of the minimum of a Brownian motion, (ii) by passing in polar variables. For $\varepsilon > 0$, a similar calculation gives:

$$\begin{aligned} \left|\mathbb{P}\left(\min_{[0,y]} B_n^l - 2\varepsilon < \min_{[0,z]} B_n^r\right) - \mathbb{P}\left(\min_{[0,y]} B_n^l < \min_{[0,z]} B_n^r\right)\right| &\leq \frac{4}{2\pi}\int_{-\infty}^{0}\int_{\frac{\sqrt{z}\cdot m_2}{\sqrt{y}}}^{\frac{\sqrt{z}\cdot m_2 + 2\varepsilon}{\sqrt{y}}} e^{\frac{-m_1^2}{2}} e^{\frac{-m_2^2}{2}} dm_1 dm_2 \\ &\leq \frac{8\varepsilon}{2\pi\sqrt{y}}\int_{-\infty}^{0} e^{\frac{-y m_2^2}{2z}} e^{\frac{-m_2^2}{2}} dm_2 \\ &\underset{\varepsilon\to 0}{\longrightarrow} 0 \end{aligned} \tag{4}$$

To sum up, the right-hand term in (2) converges to $\frac{2}{\pi}\arctan\left(\sqrt{\frac{-v_-\alpha}{v_+ - v_- + v_+\alpha}}\right)$ as $\varepsilon \to 0$. The first term in (1) can be bounded from below by the same method. Since ε can be taken as small as possible by taking n large enough, the theorem follows. □

2.2.4 Particle density

Definition 2.2.6 (Particle density in a configuration)**.**
The -1 **particle density** in $x \in \{-1,0,1\}^{\mathbb{Z}}$ is defined as $d_-(x) = \text{Freq}(-1, x)$. $d_+(x)$ is defined in a symmetrical manner.

In all the following, any result on d_- also holds for d_+ by symmetry.

Theorem 2.2.6 (Decrease rate of the particle density)**.**

Let G be a (v_-, v_+)-GA with initial measure $\mu \in \mathcal{M}ix$. Then:

$$\forall_\mu x \in \{-1,0,1\}^{\mathbb{Z}},\ \forall \varepsilon > 0,\ d_-(G^t(x)) = O\left(t^{-\frac{1}{4}+\varepsilon}\right)$$

If furthermore $\mu \in \mathrm{Ber}_=$:

$$\forall_\mu x \in \{-1,0,1\}^{\mathbb{Z}},\ d_-(G^t(x)) \sim t^{-\frac{1}{2}}$$

Proof. When $\mu \in \mathcal{M}ix$, it is in particular σ-ergodic, and so are its images $G^t_*\mu$. By Birkhoff's ergodic theorem, one has $d_-(G^t(x)) = G^t_*\mu([-1]) = \mu(G^t(x)_0 = -1)$ for μ-almost all $x \in \{-1,0,1\}^{\mathbb{Z}}$.

We first prove the theorem when G is the $(-1,0)$-gliders automaton. By Lemma 2,

$$\mu(G^t(x)_0 = -1) = \mu\left(S_x(t+1) < \min_{[0,t]} S_x\right).$$

Equivalent ($\mu \in \mathrm{Ber}_=$): By symmetry,

$$\mu\left(S_x(t+1) < \min_{[0,t]} S_x\right) = \mu\left(S_x(0) < \min_{[1,t+1]} S_x\right),$$

which is the probability that the random walk starting from 0 remains strictly positive during t steps, also known as its probability of survival. According to [Red01], when the random walk is symmetric and the steps are independent, we have the equivalent $\mu(G^t(x)_0 = -1) \sim \frac{1}{\sqrt{t}}$.

Upper bound:

$$\mu\left(S_x(t+1) < \min_{[0,t]} S_x\right) \leq \mu\left(S_x^{t+1}(1) = \min_{[0,1]} S_x^{t+1}\right).$$

Using Corollary 2.2.5, we have:

$$\begin{aligned}
\mu\left(S_x^{t+1}(1) = \min_{[0,1]} S_x^{t+1}\right) &= \mathbb{P}\left(S_{X'}^{t+1}(1) = \min_{[0,1]} S_{X'}^{t+1}\right) \\
&\leq \mathbb{P}\left(B_{t+1}(1) \leq \min_{[0,1]} B_{t+1} + C_{t+1}\right) \\
&\leq \mathbb{P}\left(B_{t+1}(0) \leq \min_{[0,1]} B_{t+1} + C_{t+1}\right),
\end{aligned}$$

where $C_{t+1} = \sup_{[0,1]} \left|S_{X'}^{t+1} - B_{t+1}\right| = O\left(t^{-\frac{1}{4}+\varepsilon}\right)$ $\mathbb{P}$-almost surely, and where the third line is obtained by symmetry of the Brownian motion.

Furthermore $\mathbb{P}\left(\min_{[0,1]} B_{t+1} > -C_{t+1}\right) = \int_{-C_{t+1}}^{0} e^{-x^2/2}dx \leq C_{t+1} = O\left(t^{-\frac{1}{4}+\varepsilon}\right)$.

General case (any $v_- < v_+$): Let G' be the (v_-, v_+)-GA. Then

$$G' = \sigma^{v_+} \circ G^{v_+ - v_-}.$$

To conclude, it is enough to see that the particle density is σ-invariant and decreasing under the action of G. □

2.2.5 Rate of convergence

Theorem 2.2.7 (Rate of convergence to the limit measure)**.**
Let G be the (v_-, v_+)-GA with initial measure $\mu \in \mathcal{M}ix$. Then:

$$\forall \varepsilon > 0,\ d_{\mathcal{M}}(G_*^t\mu, \widehat{\delta_0}) = O\left(t^{-1/4+\varepsilon}\right)$$

If furthermore $\mu \in \mathrm{Ber}_=$:

$$d_{\mathcal{M}}(G_*^t\mu, \widehat{\delta_0}) = \Omega\left(t^{-1/2}\right)$$

Proof. We first prove the theorem when G is the $(-1, 0)$-gliders automaton. By defining $0^\ell \in \mathcal{A}^\ell$ the word containing only zeroes, the distance can be rewritten:

$$\forall t \in \mathbb{N}, d_{\mathcal{M}}(G_*^t\mu, \widehat{\delta_0}) = \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} G_*^t\mu\left(\mathcal{A}^{\mathbb{Z}} \backslash [0^\ell]\right).$$

Lower bound ($\mu \in \mathrm{Ber}_=$): $d_{\mathcal{M}}(G_*^t\mu, \widehat{\delta_0}) > G_*^t\mu\left(\mathcal{A}^{\mathbb{Z}}\backslash[0]\right)$. We conclude with Theorem 2.2.6.

Upper bound: We give an upper bound for $G_*^t\mu(\mathcal{A}^{\mathbb{Z}}\backslash[0^\ell]) = \mu(\exists 0 \leq d \leq \ell, G^t(x)_d = \pm 1)$ for $\ell \in \mathbb{N}$ and $t \in \mathbb{N}$. By Lemma 2,

$$\forall d \in \mathbb{Z},\ G^t(x)_d = +1 \Leftrightarrow S_x(d) < \min_{[d+1, d+t]} S_x.$$

Therefore:

$$\begin{aligned} G_*^t\mu\left(\bigcap_{d=0}^{\ell}[+1]_d\right) &\leq \mu\left(\min_{[0,\ell]} S_x < \min_{[\ell+1,t]} S_x\right) \\ &\leq \mu\left(\min_{[0,t]} S_x \geq -\ell\right) \\ &\leq \mu\left(\min_{[0,1]} S_x^t \geq -\frac{\ell}{\sqrt{t}}\right) \end{aligned}$$

By Corollary 2.2.5, using the same notations as in the previous proofs:

$$\begin{aligned} G_*^t\mu\left(\exists 0 \leq d \leq \ell, x_d = +1\right) &\leq \mathbb{P}\left(\min_{[0,1]} S_{X'}^t \geq -\frac{\ell}{\sqrt{t}}\right) \\ &\leq \mathbb{P}\left(\min_{[0,1]} B_t \geq -\frac{\ell}{\sqrt{t}} - C_t\right) \qquad \text{where } C_t = O\left(t^{-\frac{1}{4}+\varepsilon}\right) \\ &= O\left(t^{-\frac{1}{4}+\varepsilon}\right) \end{aligned}$$

for any $\varepsilon > 0$, following the same calculations as in Section 2.2.4. The case of -1 particles is symmetrical, and we conclude.

General case: Apply the same method as in the previous section, considering that $d_{\mathcal{M}}$ and all considered measures are σ-invariant and that any CA is Lipschitz w.r.t $d_{\mathcal{M}}$. □

2.2.6 Extension to other cellular automata

Definition 2.2.7. Let F_1, F_2 be two CAs on $\mathcal{A}^{\mathbb{Z}}$ and $\mathcal{B}^{\mathbb{Z}}$, respectively. We say that F_1 **factorises onto** F_2 if there exists a **factor** $\pi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ such that $\pi \circ F_1 = F_2 \circ \pi$.

In other words, F_1 admits a particle system $(\mathcal{P}, \pi, \phi)$ with $\mathcal{B} = \mathcal{P} \cup \{0\}$ and where ϕ is a cellular automaton on $\mathcal{B}^{\mathbb{Z}}$.

In this section, we extend the Theorems 2.2.2 and 2.2.6 to automata that factorise onto a gliders automaton, and discuss conditions for the extension of Theorem 2.2.7. In Section 2.1.3, we exhibited a general method to find such a factor using experimental intuition when such a factor is not obvious.

In other words, using the formalism from Section 2.1, we extend the theorems to automata that admit a particle system $(\mathcal{P}, \pi, \phi)$, where $\mathcal{P} = \{-1, +1\}$ and ϕ acts as a gliders automaton.

In order to extend the theorem to such CAs, starting from an initial measure μ, we must first ensure that $\pi_*\mu \in \mathcal{M}ix$. We show that the third condition in the definition of $\mathcal{M}ix$ is invariant under factor.

Proposition 2.2.8. Let $\pi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ be a factor, $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and $k > 0$ any real such that $\sum_{n\geq 0} \alpha_\mu(n)^k < \infty$. Then, $\sum_{n\geq 0} \alpha_{\pi_*\mu}(n)^k < \infty$.

Proof. We keep the notations from the definition of $\alpha_\mu(n)$. π is defined by a local rule with neighbourhood $\mathcal{N} \subset [-r, r]$ for some $r > 0$. Then, $\pi^{-1}\mathfrak{B}^0_{-\infty} \subset \mathfrak{B}^r_{-\infty}$ and $\pi^{-1}\mathfrak{B}^\infty_n \subset \mathfrak{B}^{+\infty}_{n-r}$. By σ-invariance, we have for all n $\alpha_{\pi_*\mu}(n) < \alpha_\mu(n - 2r)$, and the result follows. □

Hence, if $\mu \in \mathcal{M}ix$, we only have to prove that $\pi_*\mu$ weighs evenly the sets of particles -1 and $+1$, and that the corresponding asymptotic variance is not zero. Under these assumptions, we can extend some of the previous results with the forbidden patterns playing the role of the particles.

Corollary 2.2.9. Let $F : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ be a CA and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$. Suppose that F factorises onto a (v_-, v_+)-GA via a factor π such that $\pi_*\mu \in \mathcal{M}ix$.

Then Theorem 2.2.2 and the first point of Theorem 2.2.6 hold if we replace "$x_k = \pm 1$" by "$\pi(x)_k = \pm 1$".

Examples: (In all the following, we use the convention □ $= 0$, ■ $= 1$, ■ $= 2$, ■ $= 3$.)

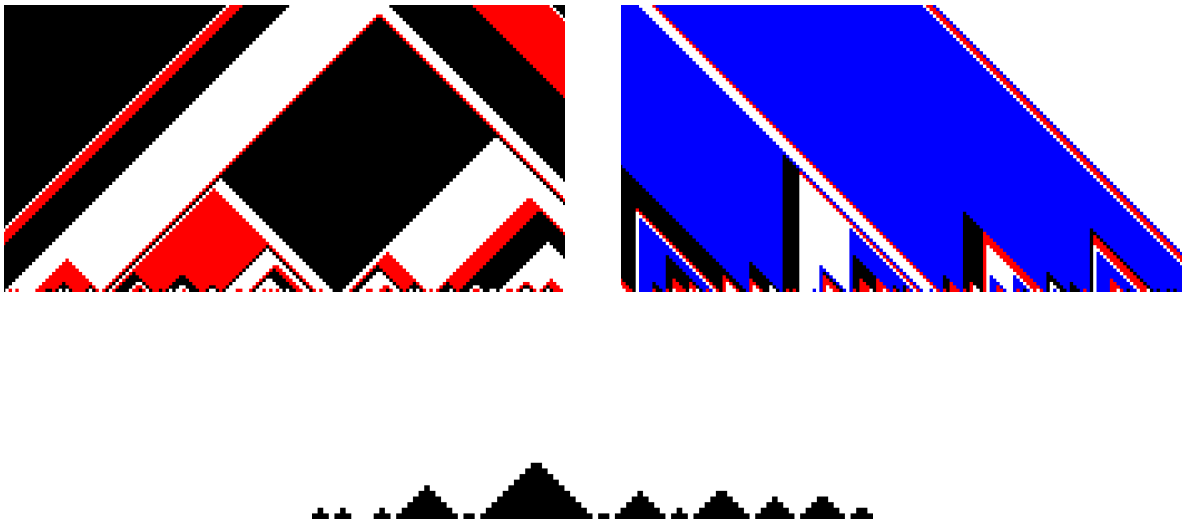


Figure 2.12: The 3-state cyclic CA, a one-sided captive CA and the product CA.

Traffic automaton: Let $\mathcal{A} = \{0, 1\}$ and F_{184} be the elementary CA corresponding to rule #184. F_{184} factorises on the $(-1, +1)$-gliders automaton, using the factor found in Section 2.1.4:

$$\begin{array}{ccc} 00 & \mapsto & +1 \\ 11 & \mapsto & -1 \\ \text{otherwise} & \to & 0 \end{array}$$

This factor is represented in Figure 2.5. If μ is a measure such that $\pi_*\mu \in \mathcal{M}ix$, then Theorem 2.2.2 and the first point of Theorem 2.2.6 hold.

For example, this is true for the 2-step Markov measure defined by the matrix $\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$ and the eigenvector $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ with $p > 0$. A particular case is the Bernoulli measure of pa-

rameters $(\frac{1}{2}, \frac{1}{2})$. Theorem 2.2.7 can also be extended by considering $d_{\mathcal{M}}(F^t_{184*}\mu, \widehat{\delta_{01}})$, since this distance can be bounded knowing the density of particles.

3-state cyclic automaton: Let $\mathcal{A} = \mathbb{Z}/3\mathbb{Z}$ and C_3 be the 3-state cyclic automaton. We consider the factor π defined in Section 2.1.4:

$$\begin{array}{lcll} ab & \mapsto & +1 & \text{if } a = b+1 \mod 3 \\ ab & \mapsto & -1 & \text{if } a = b-1 \mod 3 \\ ab & \mapsto & 0 & \text{if } a = b \end{array}$$

If μ is such that $\pi_*\mu \in \mathcal{M}ix$, then Theorem 2.2.6 applies. This is true in particular when μ is any 2-step Markov measure defined by a matrix $(p_{ij})_{1\leq i,j\leq 3}$ satisfying $p_{01}+p_{12}+p_{20} = p_{10}+p_{21}+p_{02}$, all of these values being nonzero, with $(\mu_i)_{1\leq i\leq 3}$ its only eigenvector. This includes any nondegenerate Bernoulli measure. However, even when the limit measure is known (e.g. starting from the uniform measure), Theorem 2.2.7 does not apply.

One-sided captive automata: Let F be any one-sided captive cellular automaton defined by a local rule f. As explained in Section 2.1.4, F factorises onto the $(-1, 0)$-gliders automaton with a factor defined by:

$$\begin{array}{lcll} ab & \mapsto & +1 & \text{if } a \neq b, f(a,b) = a \\ ab & \mapsto & -1 & \text{if } a \neq b, f(a,b) = b \\ ab & \mapsto & 0 & \text{if } a = b \end{array}$$

For an initial measure μ, if $\pi_*\mu \in \mathcal{M}ix$, then Theorem 2.2.2 and the first point of Theorem 2.2.6 apply.

Notice that this class of automata contains the identity ($\forall a, b \in \mathcal{A}, f(a,b) = b$) and the shift σ ($\forall a, b \in \mathcal{A}, f(a,b) = a$). However, since we have in each case $\pi^{-1}(+1) = \emptyset$ or $\pi^{-1}(-1) = \emptyset$, it is impossible to find an initial measure that weighs evenly each kind of particle, and so $\pi_*\mu$ cannot belong in $\mathcal{M}ix$. The limit measure, however, depends on the exact rule, and Theorem 2.2.7 does not apply.

Counter-example:

Product automaton: Let $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$ and F_{128} be the CA of neighbourhood $\{-1, 0, 1\}$ defined by the local rule $f(x_{-1}, x_0, x_1) = x_{-1} \cdot x_0 \cdot x_1$. Using the formalism from Section 2.1.3, we can see that F_{128} factorises onto the $(-1, 1)$-GA by the factor

$$\pi : \left\{ \begin{array}{lcl} 01 & \to & +1 \\ 10 & \to & -1 \\ \text{otherwise} & \to & 0 \end{array} \right.$$

If μ is any Bernoulli measure, then $\pi_*\mu$ satisfies all conditions of $\mathcal{M}ix$ except that $\sigma_\mu = 0$; indeed, we can check that for $\pi_*\mu$-almost all configurations, the particles $+1$ and -1 alternate. Hence, only one particle can cross any given column after time 0, and therefore $\forall \alpha > 0,\ \mu\left(\frac{T_n^-(x)}{n} \leq \alpha\right) \underset{n\to\infty}{\longrightarrow} 0$. Furthermore, any particle survives up to time t only if it is the border of a initial cluster of black cells larger than $2t$ cells, which happens with a probability $\mu([1])^{2t}$ exponentially decreasing in t.

Even though we showed that the asymptotic distributions of entry times are known for some class of cellular automata and a large class of measures, this covers only very specific dynamics. It is not known how these results extend for more than 2 particles and/or other kind of particle interaction. In particular, there is no obvious stochastic process characterising the behaviour of such automata that would play the role of S_x in our proofs.

Section 2.3

A case study: the 3-state cyclic automaton

As we saw in the previous section, information about the behaviour of the particles is not always sufficient to determine the limit measure(s). In this section, we perform an in-depth study of the typical asymptotic behaviour of an automaton with simple defect dynamics (namely, the 3-state cyclic automaton defined earlier), but whose limit measure cannot in most cases be described with the methods of the previous sections. Without changing our general approach, we refine our tools to describe more precisely the small-scale relationship between the random walk process and the space-time diagram of the automaton in this particular case.

In this section, C_3 denotes the one-sided 3-state cyclic automaton on the neighbourhood $\{0,1\}$ and alphabet $\mathcal{A} = \mathbb{Z}/3\mathbb{Z}$ by the local rule:

$$f(u_0,u_1) = \begin{cases} u_0+1 & \text{if } u_1 = u_0+1 \mod 3, \\ u_0 & \text{otherwise.} \end{cases}$$

The two-sided version of this CA was introduced by Fisch ([Fis90b],[Fis90a]) as a deterministic model to emulate an interacting particle system with stochastic dynamics. We consider the one-sided version here to simplify proofs, but all our results hold for the two-sided version.

As described in Section 2.1.4, this automaton factorises on the (-1,0)-gliders automaton by the factor $\pi : 10, 02, 21 \mapsto +1,\ 01, 20, 12 \mapsto -1$, otherwise $\mapsto 0$.

We showed using Corollary 2.1.3 that, starting from any ergodic measure μ that satisfies $\pi_*\mu([+1]) = \pi_*\mu([-1])$, no particle remains in the μ-limit set. In other words, any accumulation point of $(C_{3*}^t\mu)_{t\in\mathbb{N}}$ is a convex combination of $\widehat{\delta_0}$, $\widehat{\delta_1}$ and $\widehat{\delta_2}$. In general, this argument is not sufficient to determine exactly the limit measure. Actually, when μ is a Bernoulli measure, we show the following result:

Theorem 2.3.1 (Main result)**.**
Let μ be a Bernoulli measure on $(\mathbb{Z}/3\mathbb{Z})^{\mathbb{Z}}$ with nonzero parameters $(\lambda_0,\lambda_1,\lambda_2)$. Then:

$$C_{3*}^t\mu \underset{t\to\infty}{\longrightarrow} \lambda_2\widehat{\delta_0} + \lambda_0\widehat{\delta_1} + \lambda_1\widehat{\delta_2}.$$

This is quite surprising since it implies that the asymptotic frequency of each colour i is equal to the initial frequency of its "prey" $i-1$.

We dedicate the rest of the section to this proof. Since we already know that any limit point of $(C_{3*}^t\mu)_{t\in\mathbb{N}}$ is a convex combination of $\widehat{\delta_0}$, $\widehat{\delta_1}$ and $\widehat{\delta_2}$, it remains to show that for each i, $\mu(C_3^t(x)_0 = i) \to \lambda_{i-1}$.

As in Definition 2.2.2 for any $x \in \mathcal{A}^{\mathbb{Z}}$, we define the random walk S_x by:

$$S_x(0) = 0 \quad \text{and} \quad \forall k \in \mathbb{Z},\ S_x(k+1) - S_x(k) = \begin{cases} -1 & \text{if } x_{k+1} - x_k = 1 \mod 3 \\ 0 & \text{if } x_{k+1} - x_k = 0 \mod 3 \\ +1 & \text{if } x_{k+1} - x_k = -1 \mod 3 \end{cases}$$

We also introduce the piecewise linear and rescaled process S_x^k. See Section 2.2 for details.

Since μ is a Bernoulli measure, S_x is a Markov chain on the state space $\mathbb{Z}$, with the particular property that $S_x(t+1) - S_x(t)$ only depends on $S_x(t) \mod 3$.

Lemma 3.

$$\forall x \in \mathcal{A}^{\mathbb{Z}},\ \forall t \in \mathbb{N},\ C_3^t(x)_0 - x_0 \equiv \min_{[0,t]} S_x \mod 3$$

Proof. By induction, the minimum decreasing by one only when a particle crosses the 0 column. □

Definition 2.3.1 (Colour, bases, dives)**.** Let $x \in \mathcal{A}^{\mathbb{Z}}$.

The **colour** at time t is $x_0 + S_x(t) \mod 3$.

t is a **base** if $t = \operatorname{argmin}_{[0,t]} S_x$.

The **dive** starting from time t is the interval $[t, t+t']$ where $t' = \min_{T \in \mathbb{N}}\{S_x(T) < S_x(t)\}$.

Sketch of the proof: By Lemma 3, the colour in the central column at time t corresponds to the colour of the last minimum (last **base**). An interval between two bases is a dive, and a dive cannot contain a base. We define the **type** of a dive as the constant value of $C_3^t(x)_0$ on that dive, which is the colour of the last base.

Therefore we need to understand the relation between the length of a dive and the colour of the initial point. Since the large-scale behaviour of the walk is similar to a Brownian motion, it is known that the length of a dive has infinite expectation, so that we cannot simply compare the expectations. To understand this relationship, we partition each dive into sub-dives starting from points of colour 0, and determined the expected number of sub-dives depending on the colour of the initial point (Lemma 4). Since the time spent outside of these dives is negligible (Lemma 5), it does not affect the asymptotic frequency of each color (Lemma 6).

Let us begin the proof. We partition time in consecutive dives in the following way:

Definition 2.3.2 (Dive partition)**.**

We define inductively two families $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ of time intervals:

- Let $t_0 = \min\{t \in \mathbb{N} \mid x_0 + S_x(t) = 0[3]\}$;
- X_0 is the dive starting from t_0. Denote its end point t_0';
- $Y_n = [t_n' + 1, t_{n+1} - 1]$, where $t_{n+1} = \min\{t \in \mathbb{N} \mid x_0 + S_x(t_n' + t) = 0[3]\}$;
- X_n is the dive starting from t_n. Denote its end point t_n'.

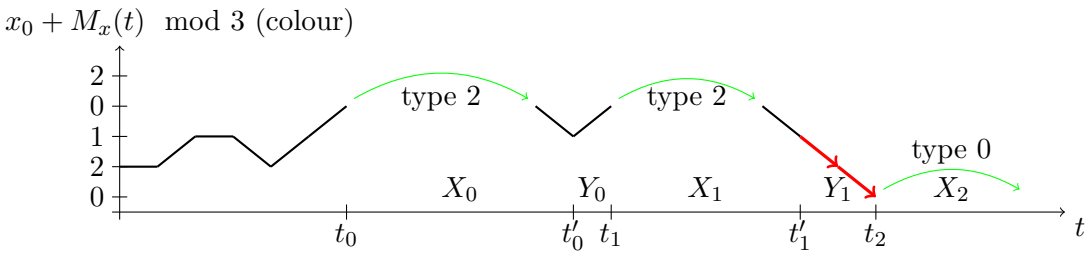


Figure 2.13: The first three dives of a random walk. Red edges correspond to a change of base.

Furthermore define $n(t) = \max\{n \mid t_n \leq t\}$.

The times t_i are stopping times, that is, they are μ-almost surely finite and for all i and t, the event $t_i = t$ only depends on the value of the variables $\{x_k \ : \ k < t\}$. As a consequence (strong Markov property), for all i, the values of $S_x(t)$ for $t \geq t_i$ given $S_x(t_i)$ are independent from $\{x_k \ : \ k \leq t_i\}$. The same statements apply to times t'_i.

Lemma 4. The stochastic process $(t(X_i))_{i\in\mathbb{N}}$, which represents the type of each successive dive, is a Markov chain of transition matrix:

$$\begin{pmatrix} \lambda_2 & \lambda_2 & \lambda_2 \\ \frac{\lambda_0}{\lambda_0+\lambda_2} & \frac{\lambda_0}{\lambda_0+\lambda_2} & 0 \\ \frac{\lambda_1\lambda_2}{\lambda_0+\lambda_2} & \frac{\lambda_1\lambda_2}{\lambda_0+\lambda_2} & 1-\lambda_2 \end{pmatrix}.$$

Proof. Suppose that $t = t'_i$, that is, we just finished the dive X_i. It follows that $x_0 + S_x(t'_i) = 1 \mod 3$. The values of $S_x(t)$ for $t > t'_i$ given $S_x(t'_i)$ are independent from $\{x_k \ : \ k \leq t'_i\}$. We remarked earlier that the time evolution of S_x depended only on the colour of the initial point, which is here 1. This shows that the relative evolution of the walk $S_x(t'_i + t) - S_x(t'_i)$ is independent from $S_x(t'_i)$ and $\{x_k \ : \ k \leq t'_i\}$.

By definition of the dive X_i, the value of the last minimum before t'_i is one of $\{S_x(t'_i) + 1, S_x(t'_i), S_x(t'_i) - 1\}$ (if it was lower, X_i would be included in a larger dive of type 0, which is a contradiction). More precisely, these three values correspond to the cases where the type of X_i is $2, 1$ and 0, respectively. The type of the next dive, $t(X_{i+1})$, depends only on the last minimum before time t'_i and the future time evolution $S_x(t'_i + t) - S_x(t'_i)$ for $t > 0$. As we just saw, knowing the last minimum before time t'_i is the same as knowing $t(X_i)$. Since $S_x(t'_i + t) - S_x(t'_i)$ is independent from all past variables, this shows that for all $(k_j)_{0\leq j\leq i+1} \in (\mathbb{Z}/3\mathbb{Z})^{i+2}$:

$$\mu\Big(t(X_{i+1}) = k_{i+1} \mid t(X_i) = k_i, \ldots, t(X_0) = k_0\Big) = \mu\Big(t(X_{i+1}) = k_{i+1} \mid t(X_i) = k_i\Big).$$

In other words, $(t(X_i))_{i\in\mathbb{N}}$ is a Markov chain.

Now let $i \in \mathbb{N}$ and assume for clarity that $t(X_i) = 1$. After time t'_i, S_x can remain constant for some time, but the probability that it takes a step up before any step down

is $\frac{\lambda_0}{\lambda_0+\lambda_2}$. In this case the dive X_{i+1} starts and it is again of type 1.

In the other case, the walk reaches a new base of colour 2. From there, consider the two following events:

(i) S_x reaches a new base of colour 0, i.e. it takes a step down from its current position;

(ii) S_x starts a dive of type 2, i.e. it takes two steps up from its current position.

We determine the probability that event (i) happens before event (ii). We can ignore constant steps and only consider steps up and down by renormalising. Then any path for which event (i) happens first is an alternating up-down path ending with a down step to reach the new base of colour 0. Hence :

$$\begin{aligned}\mu\left(\text{event (i) happens before event (ii)}\right) &= \frac{\lambda_0}{\lambda_0+\lambda_1}\cdot\sum_{n\in\mathbb{N}}\frac{\lambda_1}{\lambda_0+\lambda_1}\cdot\frac{\lambda_2}{\lambda_0+\lambda_2}\\ &= \frac{\lambda_0}{\lambda_0+\lambda_1}\cdot\frac{1}{1-\frac{\lambda_1\lambda_2}{(\lambda_0+\lambda_1)(\lambda_0+\lambda_2)}}\\ &= \frac{\lambda_0}{\lambda_0+\lambda_1}\cdot\frac{(\lambda_0+\lambda_2)(\lambda_0+\lambda_1)}{\lambda_0}\\ &= \lambda_0+\lambda_2\end{aligned}$$

and in this case the dive X_{i+1} is of type 0. To sum up,

- $\mu\big(X_{i+1}=1 \mid X_i=1\big)=\frac{\lambda_0}{\lambda_0+\lambda_2}$;
- $\mu\big(X_{i+1}=0 \mid X_i=1\big)=\left(1-\frac{\lambda_0}{\lambda_0+\lambda_2}\right)(\lambda_0+\lambda_2)=\lambda_2$;
- $\mu\big(X_{i+1}=2 \mid X_i=1\big)=\left(1-\frac{\lambda_0}{\lambda_0+\lambda_2}\right)\lambda_1=\frac{\lambda_1\lambda_2}{\lambda_0+\lambda_2}$.

The other columns of the matrix are obtained in a similar manner. □

Thus $(t(X_i))_{i\in\mathbb{N}}$ is an irreducible and aperiodic Markov chain of eigenvector $\begin{pmatrix}\lambda_0\\ \lambda_2\\ \lambda_1\end{pmatrix}$, which means that $\mu(t(X_i)=j)\underset{i\to\infty}{\longrightarrow}\lambda_{j-1}$. To see that the matrix is aperiodic, notice that for each state j, $\mu\big(t(X_{i+1})=j \mid t(X_i)=j\big)>0$.

Now let us consider the lengths $\ell(X_i)=t'_i-t_i$ and $\ell(Y_i)=t_{i+1}-t'_i$ for $i\in\mathbb{N}$. Since t_i and t'_i are stopping times, we deduce by the strong Markov property that all lengths $(\ell(X_i))_{i\in\mathbb{N}}$ and $(\ell(Y_i))_{i\in\mathbb{N}}$ are independent random variables.

Furthermore, $\ell(X_i)$ only depends on $\{S_x(t)\ :\ t\geq t_i\}$. By the strong Markov property, this means that $\ell(X_i)$ given $S_x(t_i)$ is independent from $\{x_k\ :\ k\leq t_i\}$. By we remarked earlier than the relative time evolution $S_x(t+k)-S_x(t)$ for $k>0$ only depended on $S_x(t) \mod 3$, and this is sufficient to know the length of the dive. Since the dives X_i all start by definition from a point of colour 0, their length $\ell(X_i)$ all follow the same law $\ell(X)$. By a similar argument, the lengths $\ell(Y_i)$ all follow the same law $\ell(Y)$.

An interval Y_i corresponds to a path that never leaves a "strip" of width 2 which have a fixed positive probability to end at each instant, and thus $\mathbb{E}[l(Y)]<\infty$. An interval X_i corresponds

to a first-passage process in a symmetrical random walk, which implies $\mathbb{E}[l(X)] = \infty$ (see [Red01]).

Now we investigate the relationship between $t(X_i), \ell(X_i)$ and $\ell(Y_i)$. On the one hand, $t(X_i)$ only depends on the last minimum before time t_i and therefore is independent from $\{x_k : k \geq t_i\}$. On the other hand, we saw earlier that $\ell(X_i)$ was independent from $\{x_k : k \leq t_i\}$ by strong Markov property. Therefore $\ell(X_i)$ is independent from all previous $(t(X_k))_{k<i}$ and $(\ell(X_k))_{k<i}$, and $t(X_i)$ is independent from all previous $(\ell(X_k))_{k<i}$. However $\ell(Y_i)$ is dependent on $t(X_i)$ and $t(X_{i+1})$, but we prove that these lengths can be neglected.

Define $Y(t) = t_0 + \sum_{i=0}^{n(t)} \ell(Y_i)$.

Lemma 5. For almost all x, $Y(t) = O\left(\sqrt{t}\right)$.

Proof. Let us show that for almost all x, $n(t) = O(\sqrt{t})$. For a fixed constant $C > 0$,

$$\begin{aligned}\mu\left(n(t) > C\sqrt{t}\right) &\leq \mu\left(\sum_{i=0}^{C\sqrt{t}} \ell(X_i) < t\right) \\ &\leq \mu\left(\ell(X_0) < t \cap \ell(X_1) < t \cap \cdots \cap \ell(X_{C\sqrt{t}}) < t\right) \\ &\leq \mu\left(\ell(X_0) < t\right)^{C\sqrt{t}},\end{aligned}$$

the third step being by independence of the $(\ell(X_i))_{i\in\mathbb{N}}$. Let us give a bound on $\mu\left(\ell(X_0) < t\right)$.

Let $k_0, \ldots k_n \ldots$ be the consecutive points such that $x_0 + S_x(k_i) = 0$[3]. Define the walk S'_x as $S'_x(i) = \frac{1}{3}(x_0 + S_x(k_i))$. That is, S'_x follows S_x but jumps from a point of colour 0 to the next. Since the $(k_i)_{i\in\mathbb{N}}$ are stopping times and their colour is fixed, S'_x is now a standard independent, unbiased random walk.

By definition, $k_0 = t_0$. If $S_x(t_0 + 3) = S_x(t_0) + 3$, i.e. S_x goes up three steps consecutively, then $S'_x(1) = \frac{1}{3}S_x(t_0) + 1$. If from there on S'_x stays above 1 for t steps, then S_x stays above 1 for (at least) t steps, and a fortiori $\ell(X_0) \geq t$.

This first event happens with probability $\lambda_0 \cdot \lambda_1 \cdot \lambda_2$. The second event can be restated in the following way: the survival time of the walk S'_x starting from 1 is greater that t. According to [Red01], this happens with probability $\frac{C'}{\sqrt{t}}$ for some fixed constant C'. Now, fixing $D = \lambda_0 \cdot \lambda_1 \cdot \lambda_2 \cdot C'$, we have for any constant C:

$$\mu\left(n(t) > C\sqrt{t}\right) < \underbrace{\left(1 - \frac{D}{\sqrt{t}}\right)^{C\sqrt{t}}}_{\underset{t\to\infty}{\longrightarrow} e^{-D/C}}$$

and therefore $\mu\left(n(t) = O(\sqrt{t})\right) = 1$.

As a consequence, since the Y_i have bounded expectation, $Y(t) = O(\sqrt{t})$ μ-almost surely. □

Lemma 6. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(\alpha) = o(\alpha)$. Then:

$$\mu\Big(C_3^t(x)_0 \quad \text{is constant on} \quad t \in [\alpha - f(\alpha), \alpha]\Big) \underset{\alpha\to\infty}{\longrightarrow} 1.$$

Proof. For α large enough, we have:

$$\begin{aligned}\mu\Big(C_3^t(x)_0 \text{ is constant on } [\alpha - f(\alpha), \alpha]\Big) &\geq \mu\left(\min_{[0,\alpha-f(\alpha)]} S_x \leq \min_{[\alpha-f(\alpha),\alpha]} S_x\right)\\ &\geq \mu\left(\min_{[0,1-\frac{f(\alpha)}{\alpha}]} S_x^\alpha \leq \min_{[1-\frac{f(\alpha)}{\alpha},1]} S_x^\alpha\right)\end{aligned}$$

By Theorem 2.2.5, we build a process X' distributed according to μ and a family of Brownian motions $(B_n)_{n\in\mathbb{N}}$ such that $\sup_{[0,1]} |S_x^n(t) - B_n(t)| \underset{n\to\infty}{\to} 0$. For any $\varepsilon > 0$, $\mu\left(\min_{[0,1-\varepsilon]} S_{X'}^\alpha \leq \min_{[1-\varepsilon,1]} S_{X'}^\alpha\right) \underset{\alpha\to\infty}{\longrightarrow} \mu\left(\min_{[0,1-\varepsilon]} B_\alpha \leq \min_{[1-\varepsilon,1]} B_\alpha\right)$. By Theorem 5.28 in [MP10], the location of the last minimum of a Brownian motion on $[0, 1]$ is arcsine-distributed. Therefore, for α large enough:

$$\mu\Big(C_3^t(x)_0 \text{ is constant on } [\alpha - f(\alpha), \alpha]\Big) \geq \frac{2}{\pi}\arcsin\sqrt{1-\varepsilon} - \varepsilon$$

and since this number tends to 1 as $\varepsilon \to 0$, we conclude. □

Proof of Theorem 2.3.1. Define $n'(t) = n(t - Y(t))$. In other words, $n'(t) = \max_{n\in\mathbb{N}} \sum_{i=0}^{n} \ell(X_i) \leq t$. Since all lengths $(\ell(X_i))_{i\in\mathbb{N}}$ are independent from the types $(t(X_i))_{i\in\mathbb{N}}$, the same can be said for $n'(t)$.

Combining Lemmas 6 and 5, we can see that

$$\mu\Big(C_3^t(x)_0 \text{ is constant on } [\alpha - Y(\alpha), \alpha]\Big) \underset{\alpha\to\infty}{\longrightarrow} 1.$$

If $t \in X_i$, we have $C_3^t(x)_0 = t(X_i) = t(X_{n(t)})$. This is still true if $t \in Y_{n(t)}$ except if the process has already reached a new base during this time interval $Y_{n(t)}$. By the previous remark, the probability that this happens tends to 0 as $t \to \infty$.

Therefore, μ-almost surely, for any $i \in \{0,1,2\}$:

$$\begin{aligned}\mu\left(C_3^t(x)_0 = i\right) &\underset{t\to\infty}{\sim} \mu\left(C_3^{t-Y(t)}(x)_0 = i\right)\\ &\underset{t\to\infty}{\sim} \sum_{j=0}^{\infty} \mu\left(n'(t) = j \cap t(X_j) = i\right)\\ &\underset{t\to\infty}{\sim} \sum_{j=0}^{\infty} \mu\left(n'(t) = j\right) \cdot \mu\left(t(X_j) = i\right)\end{aligned}$$

For any $\varepsilon > 0$, we can find N large enough so that $|\mu\left(t(X_N) = i\right) - \lambda_{i-1}| < \varepsilon$. Furthermore, $\mu(n'(t) > N) \underset{t\to\infty}{\longrightarrow} 1$. Consequently:

$$\begin{aligned}\mu\left(C_3^t(x)_0 = i\right) &\leq \mu\left(n'(t) < N\right) + \sum_{j=N}^{\infty} \mu\left(n'(t) = j\right) \cdot \mu\left(t(X_j) = i\right)\\ &\leq \varepsilon + (\lambda_{i-1} + \varepsilon) \qquad \text{for } t \text{ large enough.}\end{aligned}$$

Since this is true for all i and ε, and $\sum_{i=0,1,2} \mu\left(C_3^t(x)_0 = i\right) = 1$, we conclude:

$$\forall i \in \{0,1,2\},\ \mu\left(C_3^t(x)_0 = i\right) \underset{t\to\infty}{\longrightarrow} \lambda_{i-1}.$$

□

Randomisation and rigidity

3

Section 3.0
Introduction

Some cellular automata exhibit a typical asymptotic behaviour that is in a sense the opposite of the self-organisation that we considered in the previous chapter: instead of "order emerging from disorder", that is, simple patterns emerging from an initial configuration with no particular property (e.g. drawn uniformly), we observe visually "disorder emerging from order", that is, convergence towards a state where all patterns are present with the same probability even if the initial measure is heavily biased towards some patterns. In measure-theoretical terms, it means that the distribution at time t converges to the uniform measure. This phenomenon is called **randomisation**.

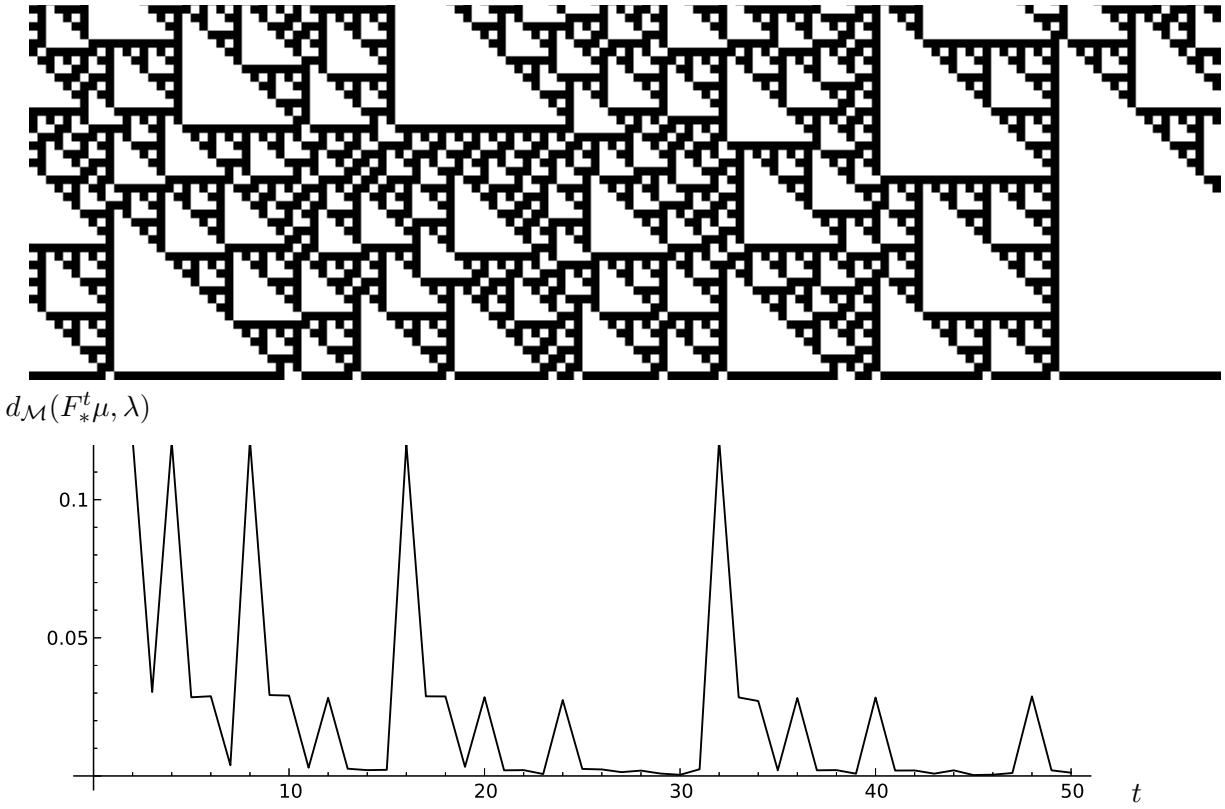


Figure 3.1: Automaton defined by $F_{102}(x)_0 = x_0 + x_1 \mod 2$, starting from the initial Bernoulli measure with parameters $(\frac{1}{10}, \frac{9}{10})$. The graph gives experimental values of the distance to the uniform measure as a function of time.

This kind of behaviour was first observed in elementary CA such as Rule $\#102$, which corresponds to the cellular automaton performing addition mod 2 on the neighbourhood $\{0, 1\}$. In this case, the phenomenon at hand is slightly more subtle: for any initial measure μ, the

distribution at time t becomes closer to the uniform measure as long as t avoids a subsequence of density 0 [PY02]. Notice in Figure 3.1 how white cells are predominant near the powers of two, and the distribution seems more uniform far from them. This phenomenon is called **randomisation in density**, but in most papers the weakest property of **randomisation in Cesàro mean** is considered instead.

Definition 3.0.1 (Randomisation)**.**
Let F be a cellular automaton $\mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, λ the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$ and $\mathcal{M} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ a **class of initial measures**.

F **randomises** $\mathcal{M}$ if:

$$\forall \mu \in \mathcal{M}, F_*^t \mu \underset{t\to\infty}{\longrightarrow} \lambda.$$

F **randomises** $\mathcal{M}$ **in density** if for any $\mu \in \mathcal{M}$ this convergence holds on a subsequence of density one.

F **randomises** $\mathcal{M}$ **in Cesàro mean** if:

$$\forall \mu \in \mathcal{M}, \ \frac{1}{T}\sum_{t=0}^{T} F_*^t \mu \underset{T\to\infty}{\longrightarrow} \lambda.$$

For example, $\mathcal{M}$ can be $\mathrm{Ber}(\mathcal{A}^{\mathbb{Z}})$ the set of nondegenerate Bernoulli measures on $\mathcal{A}^{\mathbb{Z}}$, or even a larger class under σ-ergodicity or mixing conditions. For a fixed class $\mathcal{M}$, randomisation implies randomisation in density which implies randomisation in Cesàro mean. Notice however that a notion of "randomisation for every initial measure" (taking $\mathcal{M} = \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$) is unreasonable, because for example the uniform measure cannot be reached from a measure supported by a periodic orbit. Still, even for $\mathcal{M} = \mathrm{Ber}(\mathcal{A}^{\mathbb{Z}})$, there is no known randomising cellular automaton (for simple convergence).

This property shares many characteristics with **density classification** on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ as introduced in [Pac88]. Instead of converging towards the uniform measure, the automaton must converge towards $\widehat{\delta_0}$ or $\widehat{\delta_1}$ depending on which of 0 or 1 appears more frequently in the initial measure (the equality case is not specified). Both properties are linked to the problem of robust or reliable computation, that is, computation in presence of noise, which was introduced in [vN56] in the context of Boolean circuits. If density classification corresponds to the ability to keep a bit of memory (monochromatic configuration) in presence of noise [GKL87], also called **fault tolerance**, randomisation corresponds to the ability to correct to imperfection of a source of randomness. Another similarity is that no cellular automaton is known to classify density in dimension one, but experimental candidates are known. This justifies an experimental approach to find preliminary evidence for randomisation, similarly as [Fat13] for density classification.

It is clear that a randomising automaton (simple or in Cesàro mean) must be surjective (Theorem 3.1.1). The first positive results concerned automata that are group homomorphisms, since the group structure gives access to many powerful tools. For example, the first historical result is due to Lind ([Lin84]), who proved that the cellular automaton performing addition mod 2 on the neighbourhood $\{0, 1\}$ randomised in Cesàro mean the class of nondegenerate Bernoulli measures. However, he also showed that this result did not hold for simple randomisation, as we detailed above. Numerous results extended the set of initial

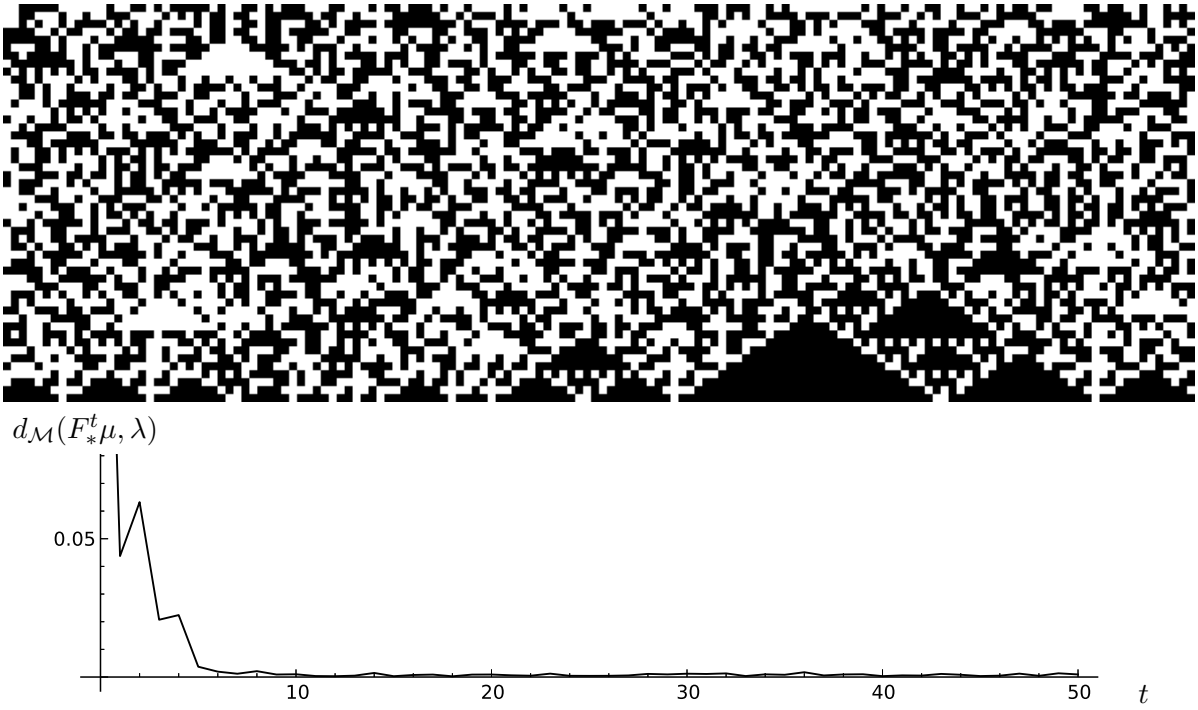


Figure 3.2: Automaton defined by $F(x)_0 = x_{-1} + x_0 \cdot x_1 + x_2 \mod 2$, starting from the initial Bernoulli measure with parameters $(\frac{1}{10}, \frac{9}{10})$. One of the candidates for randomisation when $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$. The graph gives experimental values of the distance to the uniform measure as a function of time.

measures and the set of cellular automata for randomisation in Cesàro mean in various ways [MM98, FMMN99, PY02, PY03]. All these results are summed up in Section 3.1.1.

Nevertheless, none of these articles found sufficient conditions or even a candidate for simple randomisation. It seems that a group structure, which was used in virtually all proofs of randomisation in density and Cesàro mean, is actually detrimental for simple randomisation. For example, in the case where the alphabet is $\mathbb{Z}/n\mathbb{Z}$, Proposition 3.1.2 suggests that such automata cannot be good candidates. We took different approaches to add a perturbation to an algebraic automaton such as composing its local rule with a permutation; we obtained good candidates experimentally, but with all usual algebraic tools unavailable, we were unable to provide a proof.

With this experimental approach, we conjectured that such properties as bipermutativity or positive expansivity are key to the randomising process. In Section 3.1.2, we make a brute-force exploration of the space of bipermutative automata for small alphabets and neighbourhoods and give experimental statistics that back up this conjecture. This approach is not practical for positive expansivity since no algorithm for testing this property is known [Luk10].

A related topic is the study of **rigidity**, that is, how measures invariant under the action of certain cellular automata among a class $\mathcal{M} \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ must be the uniform measure. $\mathcal{M}$ is usually the class of measures that satisfy some conditions such as σ-ergodicity, full support, nonzero entropy, σ-mixing, etc. Of course a single limit measure (pointwise or in Cesàro

mean) is always invariant under the action of the automaton, so a rigidity result could be a first step towards a randomisation result. This problematic was developed around Furstenberg conjecture [Fur67] relative to measures on the unit circle invariant by multiplication by 2 and 3. Concerning cellular automata, some results concerning restricted classes of automata [Miy79, Miy94, HMM03, Piv05, Sab07] were an inspiration to find randomisation candidates.

In Section 3.2.1, we give an alternative and simpler proof of a rigidity result on the addition mod 2 automaton from [Miy94]. In Section 3.2.2, we consider a family of bipermutative CA on the alphabet $\mathbb{Z}/n\mathbb{Z}$ obtained by composing the local rule of an affine automaton by a permutation (in particular, they belong to a class we conjectured to be randomising). Using the same tools as in [HMM03], we obtained a rigidity result under σ-ergodicity and entropy conditions. In particular, when n is prime, this result implies that all invariant measures satisfying those conditions are the uniform measure.

This chapter is the result of an ongoing collaboration between Irène Marcovici, Alejandro Maass and ourselves.

Section 3.1
Randomisation

3.1.1 State of the art and negative results

In this section, we state some necessary conditions for randomisation, as well as some partial positive results concerning randomisation in Cesàro mean. In particular, we show that the cellular automata that have been studied the most for randomisation in Cesàro mean are not good candidates for randomisation in simple convergence. Instead, we conjecture that another class is more suited to this purpose. This claim is backed up by experimental evidence in the next section.

Proposition 3.1.1 (Hedlund [Hed69])**.**
$F_*\lambda = \lambda$ if, and only if, F is a surjective automaton.

In particular, if F randomises any nonempty class (directly or in Cesàro mean), then F is surjective. Indeed, if $F_*^t\mu \underset{t\to\infty}{\longrightarrow} \lambda$, it follows that $F_*\lambda = \lambda$. The same is true for Cesàro mean convergence.

Proof of Proposition 3.1.1.
($\Rightarrow$) For any $u \in \mathcal{A}^*$, $\lambda(F^{-1}([u])) = \lambda([u]) \neq 0$, so $F^{-1}([u]) \neq \emptyset$. For $x \in \mathcal{A}^{\mathbb{Z}}$, $F^{-1}(x) = \bigcap_n F^{-1}([x_{[-n,n]}]_{-n}) \neq \emptyset$ by closure.
($\Leftarrow$) Without loss of generality, assume the neighbourhood of F is $[0, r]$ for some $r > 0$. The preimage of any cylinder $[u]$ is just an union of cylinders $[v]$ with $|v| = |u| + r$, and conversely the image on any cylinder $[v]$ is included in a cylinder $[u]$ with $|v| = |u| + r$. Therefore:

$$\forall n \in \mathbb{N}, \sum_{u\in\mathcal{A}^n} \mathrm{Card}\left\{v \in \mathcal{A}^{n+r} \ : \ F([v]) \subset [u]\right\} = |\mathcal{A}|^{n+r}$$

We aim at proving that every term of this sum is equal. Suppose that there exists $u \in \mathcal{A}^*$ such that $\mathrm{Card}\{v \in \mathcal{A}^{|u|+r} \ : \ F([v]) \subset [u]\} < |\mathcal{A}|^r$. Now, for any $i > 0$, consider the set $W_i = \{w \in \mathcal{A}^{i|u|+(i-1)r)} \ : \ \forall j \leq i, u \sqsubset_{j(|u|+r)} w\}$. It is clear that $|W_i| = |\mathcal{A}|^{(i-1)r}$.

However, for any $w' \in \mathcal{A}^{i\cdot(|u|+r)}$, $F([w']) \subset W_i$ implies $F([w'_{[j(|u|+r),(j+1)(|u|+r)]}]) \subset [u]$ for any $j \leq i$. Therefore there are at most $(|\mathcal{A}|^r - 1)^i$ such w'. For i large enough, $(|\mathcal{A}|^r-1)^i < |W_i| = |\mathcal{A}|^{(i-1)r}$. To sum up, we have a set of words of length $n = i|u|+(i-1)r$, and each of these words have a preimage that is a set of words of length $n + r$; however, for cardinality reasons, one of these preimages must be empty, which is in contradiction with surjectivity.

Therefore, for all $u \in \mathcal{A}^*$, we have $\mathrm{Card}\{v \in \mathcal{A}^{|u|+r} \ F([v]) \subset [u]\} = |\mathcal{A}|^r$, and

$$F_*\lambda([u]) = \sum_{\substack{v\in\mathcal{A}^{|u|+r}\\ F([v])\subset[u]}} \lambda([v]) = |\mathcal{A}|^r \cdot \frac{1}{|\mathcal{A}|^{|u|+r}} = \frac{1}{|\mathcal{A}|^{|u|}},$$

which means that $F_*\lambda = \lambda$. □

Most positive results in the literature concern cases where the set $\mathcal{A}^{\mathbb{Z}}$ is endowed with a group structure $(\mathcal{A}^{\mathbb{Z}}, +)$ and the cellular automata is a morphism for this structure. This is the case in particular if $(\mathcal{A}, +)$ is itself a finite group, in which case $\mathcal{A}^{\mathbb{U}}$ (for a finite $\mathbb{U}$) and $\mathcal{A}^{\mathbb{Z}}$ are groups for the componentwise group operation (product groups). Unless explicitly mentioned, we only consider abelian groups.

Definition 3.1.1 (Linear and affine automata)**.**
A cellular automaton F is **algebraic** if there exists an abelian group structure $(\mathcal{A}^{\mathbb{Z}}, +)$ such that F is an endomorphism $\mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$.

If $\mathcal{A}^{\mathbb{Z}}$ has a product group structure, it is equivalent to define F by a local rule f acting on the neighbourhood $\mathcal{N}$ such that $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$ is an homomorphism. In this case, we say F is **linear**.

F is **affine** if it can be written as $F(x) = G(x) + {}^{\infty}a^{\infty}$, where $a \in \mathcal{A}$, G is a linear automaton, and the operation is componentwise.

F **factorises on** a (nontrivial) affine automaton if there is an alphabet $\mathcal{B}$, with $|\mathcal{B}| > 2$, a factor $\pi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ and an affine automaton $G : \mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}^{\mathbb{Z}} & \xrightarrow{F} & \mathcal{A}^{\mathbb{Z}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{B}^{\mathbb{Z}} & \xrightarrow{G} & \mathcal{B}^{\mathbb{Z}} \end{array}$$

The term "linear" comes from the case $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$, where an automaton is linear if and only if its local rule $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$ can be written $f((a_i)_{i\in\mathcal{N}}) = \sum_{i\in\mathcal{N}} x_i a_i$ for some choice of $(x_i)_{i\in\mathcal{N}} \in (\mathbb{Z}/n\mathbb{Z})^{\mathcal{N}}$. We also use the term **nontrivial** to exclude the identity and the powers of the shift.

Definition 3.1.2 (Bipermutative automata)**.**
Let F be a cellular automaton defined by a local rule f acting on the neighbourhood $[a, b]$, for $a, b \in \mathbb{Z}$. F is **bipermutative** if, for all $\omega \in \mathcal{A}^{b-a-1}$, the following conditions are satisfied:

- $a \mapsto f(\omega \cdot a)$ is bijective (**right-permutative**);
- $a \mapsto f(a \cdot \omega)$ is bijective (**left-permutative**).

F is **permutative** if it is right or left-permutative.

We present some positive results relative to the randomisation problem, giving in each case the class of automata and the class of initial measures. By one site Cesàro convergence, we mean that only the asymptotic distribution of $F^t(x)_0$ is described. By affine + $(*)$, we mean the class of affine cellular automata on $\mathbb{Z}/n\mathbb{Z}$ such that for every prime divisor p of n, at least two coefficients are relatively prime to p. For most recent results, two main methods were used: stochastic processes and renewal theory (lines 3 and 5), and harmonic analysis (line

4). In each case, hypotheses on the initial class of measures are quite technical and related to these methods, and we refer to each original article for full definitions. However, all these sets of measures include nondegenerate Bernoulli and full-support Markov measures.

Class of automata	Class of initial measures	Type of convergence	Reference
$F(x)_0 = x_0 + x_1 \mod 2$	nondegenerate Bernoulli	Cesàro mean	[Lin84]
subclass of right-permutative $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$	nondegenerate Bernoulli	one site Cesàro	[MM98]
$F(x)_0 = \alpha x_0 + \beta x_1$ $\lvert\mathcal{A}\rvert$ prime power	complete connections summable decay	Cesàro mean	[FMMN99]
$\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$ affine + $(*)$	harmonically mixing	Cesàro mean	[PY02, PY03]
$\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$ affine + $(*)$	complete connections summable decay	Cesàro mean	[HMM03]

Notice that all these results were concerned with Cesàro mean convergence. Indeed, we have the following result:

Proposition 3.1.2. For $n \in \mathbb{N}$, let $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$ and endow $\mathcal{A}^{\mathbb{Z}}$ with the product group. Then an affine cellular automaton cannot randomise the set $\mathrm{Ber}(\mathcal{A}^{\mathbb{Z}})$ of nondegenerate Bernoulli measures on $\mathcal{A}^{\mathbb{Z}}$.

This is a new result even though the proof is a generalisation of a result of Lind [Lin84] in the case $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$ and $F(x)_0 = x_0 + x_1$, and the main ideas were present in [PY02, PY03] in the case $|\mathcal{N}| = 2$. We first introduce the convolution product.

Definition 3.1.3 (Convolution product)**.**
Let $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. $\mu \otimes \nu$, the **convolution product of μ and ν**, is defined as:

$$\text{For any borelian } U,\ \mu \otimes \nu(U) = \int_{\mathcal{A}}^{\mathbb{Z}} \int_{\mathcal{A}}^{\mathbb{Z}} 1_U(x+y) \mathrm{d}\mu(x) \mathrm{d}\nu(y).$$

The convolution product is associative.

To clarify the structure of this proof, we give a sketch of the simple case $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$ and $F(x)_0 = x_0 + x_1$. By induction, we have $F^t(x) = \sum_{k=0}^{t} \binom{t}{k} x_k$. When $t = 2^n$, one can check (still by induction) that $F^{2^n}(x) = x_0 + x_{2^n}$. From that, we can deduce that for any initial Bernoulli measure μ, $F_*^{2^n}\mu \to \mu \otimes \mu$, and most Bernoulli measures do not satisfy $\mu \otimes \mu = \lambda$.

This phenomenon taking place around powers of two explains the behaviour observed in Figure 3.1, and this result shows that a similar phenomenon occurs for larger alphabets and neighbourhoods.

Proof of Proposition 3.1.2. Let F be an affine cellular automaton on alphabet $(\mathbb{Z}/n\mathbb{Z}, +)$. Since endomorphisms of $\mathbb{Z}/n\mathbb{Z}$ are all of the form $x \mapsto x \cdot y$ for some $y \in \mathbb{Z}/n\mathbb{Z}$, the local rule f of F can be written as $f((x_i)_{i\in\mathcal{N}}) = \sum_{i\in\mathcal{N}} \alpha_i \cdot x_i + \beta$ for some neighbourhood $\mathcal{N}$ and some $(\alpha_i) \in (\mathbb{Z}/n\mathbb{Z})^{\mathcal{N}}, \beta \in \mathbb{Z}/n\mathbb{Z}$.

Decomposing n is prime factors, we get an integer k, a sequence of distinct primes $(p_i)_{i\leq n}$ and a sequence of positive integers $(q_i)_{i\leq n}$ such that $n = \prod_{i\leq n} p_i^{q_i}$, and by the Chinese remainder theorem:

$$\mathbb{Z}/n\mathbb{Z} \simeq \prod_{i\leq n} \mathbb{Z}/p_i^{q_i}\mathbb{Z}.$$

Therefore:

$$\mathrm{Hom}((\mathbb{Z}/nZ)^{\mathcal{N}}, \mathbb{Z}/nZ) \simeq \prod_{i\leq n} \mathrm{Hom}((\mathbb{Z}/p_i^{q_i}\mathbb{Z})^{\mathcal{N}}, \mathbb{Z}/p_i^{q_i}\mathbb{Z}).$$

In other words, a linear automaton $(\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}} \to (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}}$ can be decomposed in a unique way as a product of linear automata $(\mathbb{Z}/p_i^{q_i}\mathbb{Z})^{\mathbb{Z}} \to (\mathbb{Z}/p_i^{q_i}\mathbb{Z})^{\mathbb{Z}}$, and the same is true for affine automata. Since $\mathcal{M}_\sigma((\mathbb{Z}/p_i^{q_i}\mathbb{Z})^{\mathbb{Z}}) \subset \mathcal{M}_\sigma((\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}})$ up to the inclusion function that sends a Bernoulli to a Bernoulli, it is enough to show that at least one of these automata does not randomise $\mathrm{Ber}(\mathbb{Z}/p_i^{q_i}\mathbb{Z})$.

Therefore we assume without loss of generality that $\mathcal{A} = \mathbb{Z}/p^q\mathbb{Z}$ for some prime p and some $q > 0$, and also that $\mathcal{N} = [0, d]$ for some $d > 0$ by composing F with a power of σ if necessary. We keep the notation $f((x_i)_{0\leq i\leq d}) = \sum_{i=0}^{d} \alpha_i \cdot x_i + \beta$ for some $(\alpha_i) \in (\mathbb{Z}/p^q\mathbb{Z})^{[0,d]}$, $\beta \in \mathbb{Z}/p^q\mathbb{Z}$.

Define the polynomial $P_F \in \mathbb{Z}/p^q\mathbb{Z}[X]$ as $P_F(X) = \sum_{i=0}^{d} \alpha_i X^i$. We prove by straightforward induction that $F^n(x)_0 = \sum_{i=0}^{dn} \alpha_n^i x_i + \beta_n$, where α_n^i is the coefficient of $P_F(X)^n$ corresponding to X^i and β_n is some constant.

Furthermore, by the multinomial formula, we have:

$$P_F(X)^n = \sum_{\ell=0}^{nd} \left[\sum_{\substack{k_1+2k_2+\cdots+dk_d=\ell \\ k_0+k_1+\cdots+k_d=n}} \binom{n}{k_0, \ldots, k_d} \prod_{i=0}^{d} \alpha_i^{k_i} \right] X^\ell. \tag{3.1}$$

where $\binom{n}{k_1,\ldots,k_d}$ is the **multinomial coefficient** defined as $\frac{n!}{k_1!\cdots k_n!}$. Now we show that, when $n = p^k$ for any $k \geq q$, $P(X)^{p^k}$ admits less than $d \cdot p^q + 1$ nonzero coefficients (modulo p^q). We use the following generalisation of Kummer's classical result about congruences of binomial coefficients:

Theorem 3.1.3 (Fray, [Fra67])**.**

Let p be a prime number, $k_0, \ldots, k_d \in \mathbb{N}^{d+1}$ and $n = k_0 + \cdots + k_d$. The number:

$$\max \left\{ k : p^k \middle| \binom{n}{k_0, \ldots, k_d} \right\}$$

is equal to the number of carries that appear when adding $k_1 + \cdots + k_d = n$ in base p.

Corollary 3.1.4. Let p be a prime number, $k \geq q > 0$ two integers and $k_0, \dots, k_d \in \mathbb{N}^{d+1}$ satisfying $\sum_i k_i = p^k$. We have

$$p^q \mid \binom{p^k}{k_0, \dots, k_d} \text{ unless all } k_i \text{ are divisible by } p^{k-q}.$$

This corollary follows directly from the theorem by noting that, since p^k is written $100\cdots00$ in base p, the only way to get less than q carries in the addition $k_0 + \cdots + k_d = p^k$ is that the last $k-q$ digits of each k_i are all zeroes. See Figure 3.3 for an illustration.

^{1}	^{1}	1	0	0	0
+	1	0	0	0	0
+	1	2	0	0	0
1	0	0	0	0	0

Figure 3.3: Illustration of the corollary with $p = 3$, $k = 5$. To get $q = 2$ carries or less, the $3 = k - q$ last digits of each k_i must be 0.

In particular, in Equation 3.1 with $n = p^k$, the coefficient of X^ℓ is nonzero only if ℓ is divisible by p^{k-q}, which makes for at most $d \cdot p^q + 1$ nonzero coefficients. In other words,

$$\forall k \geq q, P_F(X)^{p^k} = \sum_{i=0}^{dp^q} \alpha_{p^k}^{ip^{k-q}} X^{ip^{k-q}}.$$

The coefficients $\left(\alpha_{p^k}^{ip^{k-q}}\right)_{0 \leq i \leq dp^q} \in (\mathbb{Z}/p^q\mathbb{Z})^{dp^q+1}$ and β_{p^k} can only take a finite number of values. Therefore we can find a sequence $(k_n)_{n\in\mathbb{N}}$ where each $\alpha_{p^{k_n}}^{ip^{k_n-q}}$ is equal to a constant a_i and the $\beta_{p^{k_n}}$ is equal to a constant b. In other words,

$$\forall n \in \mathbb{N}, P_F(X)^{p^{k_n}} = \sum_{i=0}^{dp^q} a_i X^{ip^{k_n-q}}$$

$$\forall n \in \mathbb{N}, \forall x \in \mathcal{A}^{\mathbb{Z}}, F^{p^{k_n}}(x)_0 = \sum_{i=0}^{dp^q} a_i x_{ip^{k_n-q}} + b$$

We now show that $F_*^{p^{k_n}}\mu \underset{t\to\infty}{\longrightarrow} \bigotimes_{i=0}^{dp^q} a_i^*\mu \otimes \widehat{\delta_b}$, where $a_i^*\mu$ is the image measure of μ under multiplication by a_i. We show that this convergence holds for every cylinder $[u]$, which is sufficient since cylinders form a base of the topology. For $u \in \mathcal{A}^l$ and any $n \in \mathbb{N}$, and

denoting $b^\ell = b \cdots b \cdots b \in \mathcal{A}^\ell$,

$$\begin{aligned} F_*^{p^{k_n}}\mu([u]) &= \sum_{\substack{v_0,\dots,v_{dp^q}\in\mathcal{A}^\ell \\ a_0v_0dots+a_{dp^q}v_{dp^q}+b^\ell=u}} \mu\left([v_0]_0 \cap [v_1]_{p^{k_n-q}} \cap \cdots \cap [v_{dp^q}]_{dp^{k_n}}\right) \\ &= \sum_{\substack{v_0,\dots,v_{dp^q}\in\mathcal{A}^\ell \\ a_0v_0+\cdots+a_{dp^q}v_{dp^q}+b^\ell=u}} \mu([v_0]_0)\cdot\mu([v_1]_{p^{k_n-q}})\cdots\mu([v_{dp^q}]_{dp^{k_n}}) + \underset{n\to\infty}{O}(1) \\ &= \bigotimes_{i=0}^{dp^q} a_i^*\mu([u]) + \underset{n\to\infty}{O}(1) \end{aligned}$$

where the second line is obtained by dp^q applications of shift-mixing, since all cylinders depend on variables located $p^{k_n-q} - l \to \infty$ cells apart. Therefore $F_*^{p^{k_n}}\mu$ converges in weak-$*$ convergence to $\bigotimes_{i=0}^{dp^q} a_i^*\mu([u]) \otimes \widehat{\delta_b}$. It suffices then to find a nondegenerate Bernoulli measure such that $\bigotimes_{i=0}^{dp^q} a_i^*\mu([u]) \otimes \widehat{\delta_b} \neq \lambda$, for example by considering $\mu = \mathrm{B}er\left(1-\varepsilon, \frac{\varepsilon}{p^q-1}, \dots, \frac{\varepsilon}{p^q-1}\right)$ for ϵ small enough that $(1-\varepsilon)^{dp^q} > \frac{1}{p^q}$. □

One should not conclude that randomisation is impossible for any affine automata on any Bernoulli initial measure. For example, take $\mathcal{A} = \mathbb{Z}/5\mathbb{Z}$, F the linear automaton defined by $F(x)_0 = x_0 + 2\cdot x_1 \mod 5$ and $\mu = \mathrm{Ber}\left(\frac{1+\frac{1}{\sqrt{5}}}{2}, 0, \frac{1-\frac{1}{\sqrt{5}}}{4}, \frac{1-\frac{1}{\sqrt{5}}}{4}, 0\right)$. Then one can check that $F_*\mu = a$, which implies of course that $F_*^t\mu \to a$. This should be considered as coincidental, as opposed to a "structural" randomisation on a large class.

As a side remark, this proof gives an insight as to the elements of $\mathcal{V}(F,\mu)$, for example in the simple case $F(x)_0 = x_0 + x_1$ with $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$ and an initial σ-mixing measure μ. For any n and k, $\mathcal{V}(F,\mu)$ contains all measures of the form $\bigotimes_{i=1}^{2^n} F_*^k\mu$, and also measures $\nu_{n,k}$ such that $F_*^k\nu_{n,k} = \bigotimes_{i=1}^{2^n}\mu$. By closure, $\mathcal{V}(F,\mu)$ also contains the uniform measure. We believe that these cases cover all measures of $\mathcal{V}(F,\mu)$, but this is not proven and the $\nu_{n,k}$ are not well understood.

Roughly speaking, we just saw that the algebraic structure brings some "randomising power", but also too much regularity in the space-time diagram which prevents the automaton from randomising in pointwise convergence. Visually, we obtained good results by adding a perturbation consisting in composing the local rule of a linear automaton by a permutation. We conjecture that the "randomising power" does not come from algebraicity but from more general properties, while the regularity of algebraic automata is actually detrimental to the randomising process.

Definition 3.1.4 (Positively expansive automaton)**.**
A cellular automaton F is **positively expansive** if it admits an **expansivity constant** $\varepsilon > 0$ such that:

$$\forall x \neq y \in \mathcal{A}^{\mathbb{Z}}, \exists t \in \mathbb{N}, d(F^t(x), F^t(y)) \geq \varepsilon.$$

Conjecture 3.1.5. A nontrivial bipermutative automaton (not a power of σ) randomises in Cesàro mean. Furthermore, it randomises directly if, and only if, it does not factorise on an affine automaton.

A stronger version of this conjecture would be replace the bipermutativity hypothesis by positive expansiveness. However, the decidability of positive expansiveness is still an open problem in the one-dimensional case (see for example [Luk10]), hence an exhaustive experimental test of this conjecture seems unreasonable. However we give in next section an example of a positively expansive, non-bipermutative automaton that seems to randomise experimentally.

3.1.2 Experimental evidence

In this section, we give some experimental evidence for the randomisation process.

Proposition 3.1.6. If $|\mathcal{A}| + |\mathcal{N}| \leq 5$, then any bipermutative cellular automaton defined on alphabet $\mathcal{A}$ and neighbourhood $|\mathcal{N}|$ is affine.

Proof. Taking $\mathcal{N}$ of the form $[0, r]$ without loss of generality, we enumerated with the help of a computer all pairs $(|\mathcal{A}|, |\mathcal{N}|)$ with $|\mathcal{A}| + |\mathcal{N}| \leq 5$ and all bipermutative local rules for each pair. We then enumerated all possible group structures on $\mathcal{A}$, and all affine automata for each group structure. Both sets were equal. □

Proposition 3.1.2 is a strong hint that affine automata are not good randomisation candidates. In the following, we consider the case $|\mathcal{A}| = 2$ and $|\mathcal{N}| = 4$, which are the values for which the set of cellular automata to test is the smallest.

We further reduce the set to explore by removing cellular automata that behave in essentially the same way. Let $\mathfrak{S}_{\mathcal{A}}$ be the set of permutations of the set $\mathcal{A}$.

Definition 3.1.5 (Mirror and permutation operators)**.**
The **mirror operator** is defined as:

$$\forall a \in \mathcal{A}^{\mathbb{Z}}, \forall i \in \mathbb{Z},\ b(a)_i = a_{-i}.$$

Let $\tau \in \mathfrak{S}_{\mathcal{A}}$. The **permutation operator** associated to τ is defined as:

$$\forall a \in \mathcal{A}^{\mathbb{Z}}, \forall i \in \mathbb{Z},\ T_\tau(a)_i = \tau(a_i).$$

Definition 3.1.6 (Equivalence)**.**
We say that two cellular automata F and G are **mirror-equivalent**, and we write $F \underset{mir}{\sim} G$, if $F \circ b = b \circ G$.
We say that two cellular automata F and G are **permutation-equivalent**, and we write $F \underset{perm}{\sim} G$, if there is a permutation $\tau \in \mathfrak{S}_{\mathcal{A}}$ such that $F \circ T_\tau = T_\tau \circ G$.
We simply say that F and G are **equivalent** if $F \sim G$, where $\sim$ is the symmetric and transitive cloture of $\underset{mir}{\sim} \cup \underset{perm}{\sim}$.

Proposition 3.1.7. This equivalence relation preserves affinity and bipermutativity. If the initial class of measures is invariant under the mirror and permutation operators, then it also preserves the randomising character of cellular automata.

Proof. This is a direct consequence of the definition and of the fact that the uniform Bernoulli measure is invariant under these operators. □

Thus, we perform our tests only for one representant per equivalence class. For the same reason, we assume the neighbourhood of the tested automata is $[0, r]$ for some $r > 0$.

Experimental procedure

To test experimentally the randomising properties of cellular automata, we fix an arbitrary nonuniform, nondegenerate, computable Bernoulli measure μ and we approximate the value of $d_{\mathcal{M}}(F^t_*\mu, \lambda)$ for t ranging from 0 to 50, where λ is the uniform measure. The value of 50 is arbitrary but is experimentally large enough to make a difference between apparently randomising and nonrandomising behaviours.

The choice of the particular initial measure may impact the phenomenon, in particular with "coincidental randomisation" as described earlier. However, we did not have enough resources to test many initial measures with enough repetitions to get significant results (each test took between one and two days). We repeated the test with a few different initial measures and fewer repetitions, obtaining the same apparent (though not significant) results regardless of the initial measure.

Our experimental procedure is as follows:

First fix a maximal pattern length ℓ_{max}. We have

$$\left| d_{\mathcal{M}}(F^t_*\mu, \lambda) - \sum_{l=0}^{\ell_{max}} \frac{1}{2^l} \max_{u \in \mathcal{A}^l} \left| F^t_*\mu([u]) - \frac{1}{|\mathcal{A}|^l} \right| \right| \leq \frac{1}{2^{\ell_{max}}}$$

We choose ℓ_{max} so as to control the first error term $\frac{1}{2^{\ell_{max}}}$.

Draw independently n words $(x^i_{[0,\ell_{max}+50r-1]})_{0 \leq i \leq n}$ according to μ, meaning that the word u is drawn with probability $\mu([u])$. Then compute $F^t(x^i)_{[0,\ell_{max}-1]}$ for $t \in [0, 50]$ and each i. For every $t \in [0, 50]$, $l \leq \ell_{max}$ and $u \in \mathcal{A}^l$, $(1_{[u]}(F^t(x_i)_{[0,l-1]}))_{i \leq n}$ is a sequence of independent Bernoulli variables of parameter $F^t_*\mu([u])$. Therefore $d^{n,t}_u = \sum_{i=0}^{n} 1_{[u]}(F^t(x_i)_{[0,l-1]})$ follows the corresponding binomial law of parameters $(n, F^t_*\mu([u]))$, and by the central limit theorem,

$$\frac{1}{\sqrt{n}} \frac{d^n_u - nF^t_*\mu([u])}{F^t_*\mu([u])(1 - F^t_*\mu([u]))} \quad \underset{\text{law}}{\longrightarrow} \quad \mathcal{N}(0, 1),$$

where $\mathcal{N}(0, 1)$ is the standard normal distribution centered on 0 of variance 1. More precisely, if $D^{n,t}_u$ and Φ are the cumulative distribution functions of the above distribution and a standard normal distribution, respectively, the Berry-Esseen theorem states that:

$$\sup_{x \in \mathbb{R}} |D^{n,t}_u(x) - \Phi(x)| \leq \frac{1}{\sqrt{n}} \frac{C(1 - 2F^t_*\mu([u]))}{F^t_*\mu([u])^2(1 - F^t_*\mu([u]))^2}$$

where $C < 0.4748$ [She11]. Denote $\varepsilon^{n,t}_u$ this error term, which depends on our estimate of $F^t_*\mu([u])$, but it is typically very small when the estimate is close to $\frac{1}{2}$.

Intuitively, for n large enough, $\frac{d^n_u}{n}$ is almost distributed like a Gaussian centered on $F^t_*\mu([u])$, of variance $\frac{1}{\sqrt{n}} \frac{1}{F^t_*\mu([u])(1-F^t_*\mu([u]))} \leq \frac{1}{2\sqrt{n}}$. If we draw a variable according to this Gaussian, it

would fall in the interval $F_*^t\mu([u]) \pm \frac{1}{2\sqrt{n}}\Phi^{-1}\left(\frac{1+p}{2}\right)$ with probability at least p. That means that, to obtain a confidence level p on the value of $F_*^t\mu([u])$, we get a confidence interval:

$$\left|\frac{d_u^n}{n} - F_*^t\mu([u])\right| \leq \frac{1}{2\sqrt{n}} \cdot \Phi^{-1}\left(\frac{1+p}{2} + \varepsilon_u^{n,t}\right) \quad \text{with confidence level } p.$$

Now notice that

$$\mathbb{P}\left(\max_{u\in\mathcal{A}^l}\left|F_*^t\mu([u]) - \frac{d_u^{n,t}}{n}\right| \geq C\right) \leq |\mathcal{A}|^l \cdot \mathbb{P}\left(\left|F_*^t\mu([v]) - \frac{d_v^n}{n}\right| \geq C\right)$$

for any constant $C > 0$ and any arbitrary $v \in \mathcal{A}^l$. Thus, to get a confidence level p on the maximum of $|\mathcal{A}|^l$ positive terms, it is enough to get a confidence level $1 - \frac{1-p}{|\mathcal{A}|^l}$ on each individual term. Therefore:

$$\max_{u\in\mathcal{A}^l}|F_*^t\mu([u]) - d_u^{n,t}| \leq \frac{1}{2\sqrt{n}} \cdot \Phi^{-1}\left(1 - \frac{1-p}{2|\mathcal{A}|^l} + \max_{u\in\mathcal{A}^l}\varepsilon_u^{n,t}\right) \quad \text{with confidence level } p.$$

To sum up,

$$\left|d_{\mathcal{M}}(F_*^t\mu,\lambda) - \sum_{l=0}^{\ell_{max}}\frac{1}{2^l}\max_{u\in\mathcal{A}^l}\left|\frac{d_u^{n,t}}{n} - \frac{1}{|\mathcal{A}|^l}\right|\right| \leq \frac{1}{2\sqrt{n}} \cdot \sum_{l\leq\ell_{max}}\Phi^{-1}\left(1 - \frac{1-p}{2|\mathcal{A}|^l} + \max_{u\in\mathcal{A}^l}\varepsilon_u^{n,t}\right) + \frac{1}{2^{\ell_{max}}}$$

In all the following, we use $p = 0.99$, $\ell_{max} = 7$ and we draw $n = 400,000$ configurations. The radius of the confidence interval varies with time since $\varepsilon_u^{n,t}$ depends on the estimate of each $F_*^t\mu([u])$. Nevertheless, since this term is experimentally negligible, we give only an upper bound.

Results for $\mathcal{A} = 2, |\mathcal{N}| = 4$

The results of our tests are shown in Figures 3.7 (pointwise convergence) and 3.8 (convergence in Cesàro mean). For each equivalence class of bipermutative cellular automata (obtained by exhaustive enumeration and testing), we indicate whether it corresponds to linear cellular automata and we underline one representant (using the same numbering system as the elementary cellular automata, see Definition 0.1.15) on which the test is performed. On the right side is drawn our estimation of $d_{\mathcal{M}}(F_*^t\mu,\lambda)$ for t ranging from 0 to 50, where μ is the Bernoulli measure of parameters $\frac{1}{4}$ and $\frac{3}{4}$. The plotted value has a confidence level at $p = 0.99$ up to an error at most 0.0102.

The results seem to support the conjecture, keeping in mind that when $|\mathcal{A}| = 2$, all automata factorising on a linear automata are linear themselves. To better understand the phenomenon, we also represent one space-time diagram for the representants of the first two classes on Figure 3.4. It should be apparent that the peaks in the first graph correspond to times when the density of 0 and 1 are not uniform (more zeroes than one).

Example (Non-abelian linear automaton)**.**
We consider the simplest cellular automaton with a noncommutative group structure: take $\mathcal{A} = \mathfrak{S}_3$, neighbourhood $\mathcal{N} = \{0,1\}$ and local rule $f(a_0,a_1) = a_0 \circ a_1$. We approximate $d_{\mathcal{M}}(F_*^t\mu,\lambda)$ for t ranging from 0 to 50, where μ is the Bernoulli measure of parameters $(\frac{1}{32},\frac{1}{32},\frac{1}{16},\frac{1}{8},\frac{1}{4},\frac{1}{2})$

Linear automata Bipermutative, non-affine automata

Figure 3.4: Space-time diagrams of rules 21930 (left) and 22185 (right). Arrows correspond to the peaks in the first graph of the table.

(arbitrary). The plotted value have a confidence level at $p = 0.99$ up to an error at most 0.0102. Even though the following graph show peaks located at powers of two (16 then 32), the confidence interval is way too broad to reach a conclusion (notice the scale is about 10 times smaller than other graphs). We did not have enough computing power and time to reach a definitive conclusion.

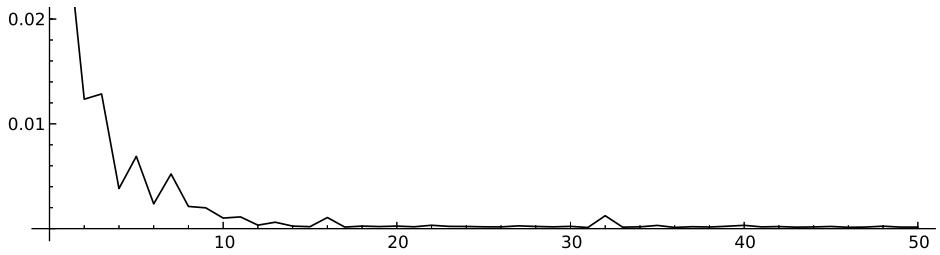


Figure 3.5: $d_{\mathcal{M}}(F_*^t\mu, \lambda)$ as a function of t where F is non-abelian linear.

Example (Positively expansive, non bipermutative, randomising automaton)**.**
We use the notion of two-sided permutation cellular automata defined in [JNY13], where the authors prove the expansiveness of a class of automata not very different from bipermutative automata. We consider here the simplest member from this class: a cellular automaton F defined on the alphabet $\{0, 1\}$, the neighbourhood $\{0, 1, 2, 3\}$, and the local rule

$$f(u_0, u_1, u_2, u_3) = \begin{cases} u_0 + u_2 & \text{if } u_1 = 0; \\ u_0 + u_3 & \text{if } u_1 = 1. \end{cases}$$

It is clear that this automaton is not bipermutative (and thus cannot factorise on a linear automaton), and it is positively expansive as proved in [JNY13], Theorem 4.7. Experimental results (Figure 3.6) seem to suggest that this automaton randomises. Similarly to the other experiments, this diagram shows an estimation of $d_{\mathcal{M}}(F_*^t\mu, \lambda)$ for t ranging from 0 to 50, where μ is the Bernoulli measure of parameters $\frac{1}{4}$ and $\frac{3}{4}$. The plotted value have a confidence level at $p = 0.99$ up to an error at most 0.0102.

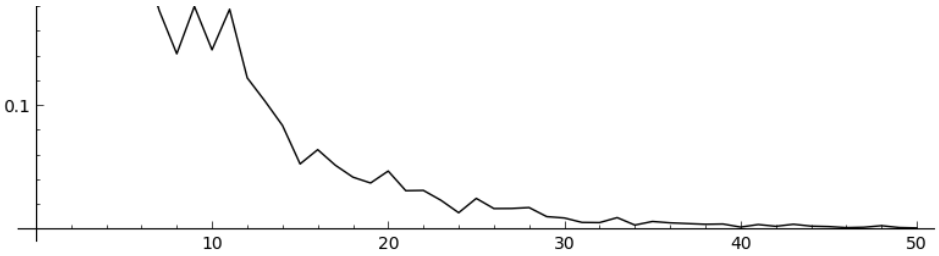


Figure 3.6: $d_{\mathcal{M}}(F_*^t\mu, \lambda)$ as a function of t where F is positively expansive.

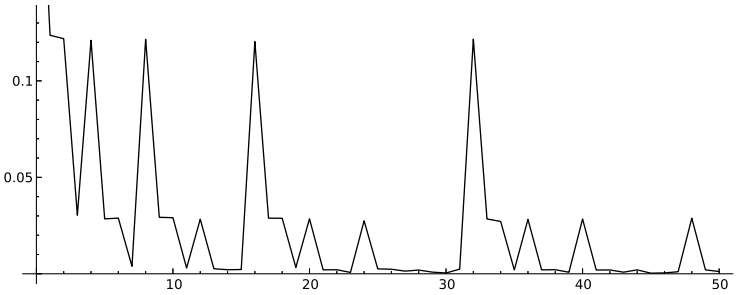
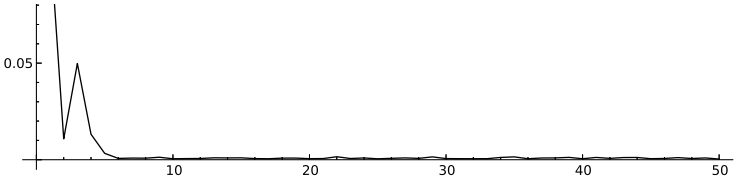
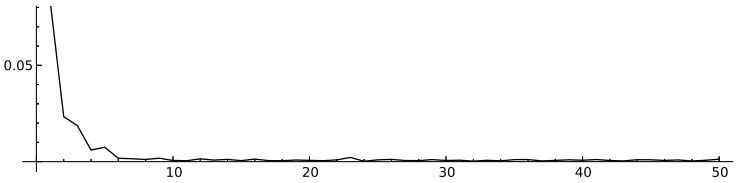
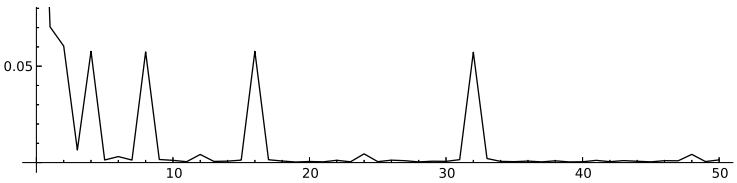
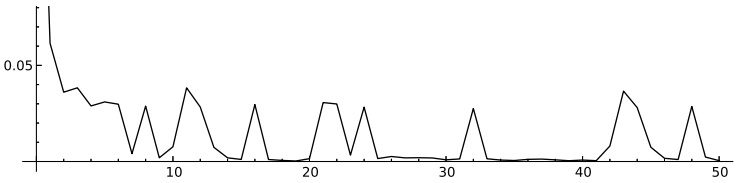
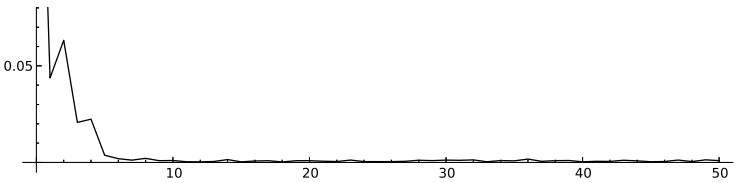
equivalence class	linearity $f(a,b,c,d) =$	$d_{\mathcal{M}}(F_*^t\mu, \lambda)$ as a function of t where $\mu = Ber(\frac{1}{4}, \frac{3}{4})$ and $t \in \{0, \ldots, 50\}$.
<u>21930</u> 43605	linear $a + d$	
<u>22185</u> 27285	non affine $a + (1 - b)\times$ $(1 - c) + d$	
<u>22950</u> 26010 39525 42585	non affine $a + b\times$ $(1 - c) + d$	
<u>23205</u> 26265 39270 42330	linear $a + b + d$	
<u>27030</u> 38505	linear $a + b + c + d$	
<u>38250</u> 43350	non affine $a + b \times c + d$	

Figure 3.7: Experimental results for pointwise convergence, $\mathcal{A} = 2, |\mathcal{N}| = 4$.

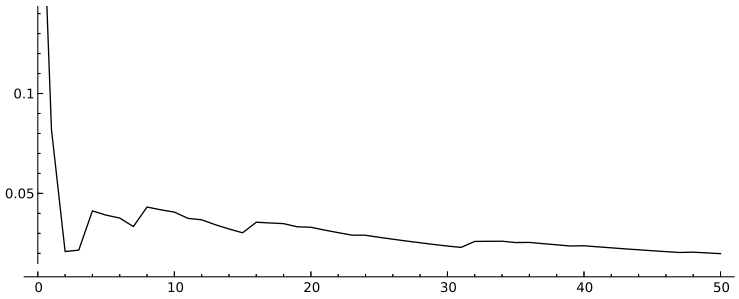
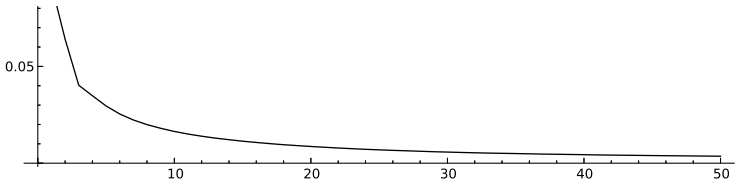
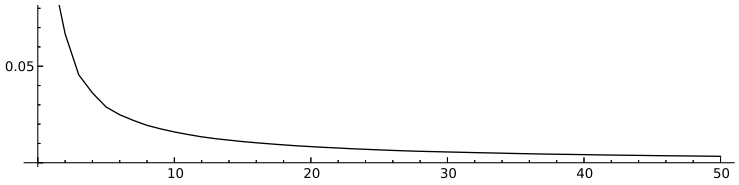
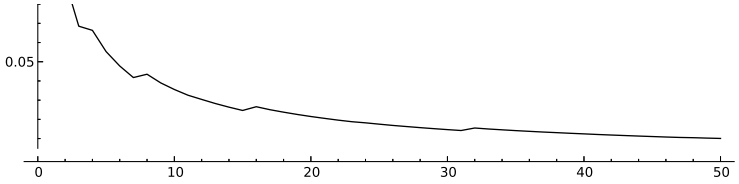
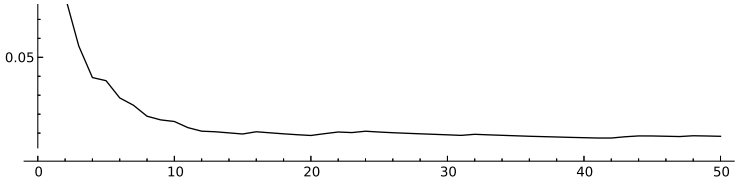
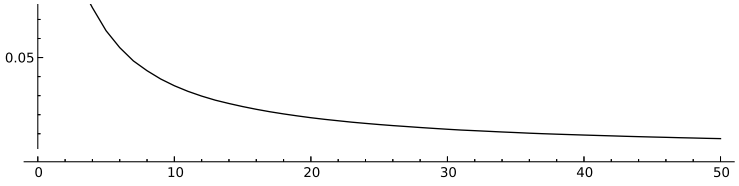
equivalence class	linearity $f(a,b,c,d) =$	$\sum_{t\leq T} d_{\mathcal{M}}(F_*^t\mu,\lambda)$ as a function of T where $\mu = Ber(\frac{1}{4},\frac{3}{4})$ and $T \in \{0,\dots,50\}$.
<u>21930</u> 43605	linear $a+d$	
<u>22185</u> 27285	non affine $a+(1-b)\times$ $(1-c)+d$	
<u>22950</u> 26010 39525 42585	non affine $a+b\times$ $(1-c)+d$	
<u>23205</u> 26265 39270 42330	linear $a+b+d$	
<u>27030</u> 38505	linear $a+b+c+d$	
<u>38250</u> 43350	non affine $a+b\times c+d$	

Figure 3.8: Experimental results for Cesàro mean convergence, $\mathcal{A} = 2, |\mathcal{N}| = 4$.

Section 3.2
Rigidity

Surjective CA admit the uniform Bernoulli measure as invariant measure, but they may have many other invariant measures. For some cellular automata, we can obtain **rigidity** results, which consists in proving that any F-invariant measure satisfying some properties (excluding for example measures supported by periodic orbits) must be the uniform measure. Since a single limit measure (in pointwise or Cesàro mean convergence) is invariant, this could be a first step towards a randomisation proof.

Rigidity methods for probability measures stem from Furstenberg's conjecture [Fur67], that states that any nonatomic probability measure on the unit circle invariant under multiplication by 2 and 3 (mod 1) must be uniform. This conjecture received a partial positive answer [Rud90] when the considered measure has nonzero entropy, but the general case is still open.

3.2.1 Mixing criterion

Let us consider the cellular automaton defined on the alphabet $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$ by $F_p(x)_k = x_k + x_{k+1} \mod p$. The uniform Bernoulli measure λ is an invariant measure of F_p, but F_p has many other invariant measures, such as $\widehat{\delta_0}$ and other measures supported by a periodic orbit. We prove below a first rigidity result for that CA. Let us mention that Miyamoto has obtained similar results when $p = 2$ using harmonic analysis [Miy79, Miy94].

Proposition 3.2.1 (Miyamoto, [Miy94])**.**
Let $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$ and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be a F_2-invariant measure that is a convex combination of σ-mixing measures. Then μ is a convex combination of $\widehat{\delta_0}$ and λ.

Proposition 3.2.2 (Marcovici)**.**
Let $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$ and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be a σ-mixing, F_p-invariant measure with full support on $\mathcal{A}^{\mathbb{Z}}$. Then μ is the uniform measure λ.

Proof. For $n \in \mathbb{N}$, define the CA $G_n = F_p^{p^n}$. Since $x \mapsto x^p$ is a morphism, we have by straightforward induction $G_n(x)_0 = x_0 + x_{p^n}$ (**scaling property**).

Let μ be a σ-mixing measure with full support on $\mathcal{A}^{\mathbb{Z}}$ that is invariant under the action of F_p. Then, for any $n \in \mathbb{N}$, μ is an invariant measure of G_n. Let us fix some $\ell \geq 1$ and consider the cylinders of length ℓ. Let $w \in \mathcal{A}^\ell$ be such that $\mu([w]) = \min_{u \in \mathcal{A}^\ell} \mu([u])$, and assume that there exists some $w' \in \mathcal{A}^\ell$ such that $\mu([w']) > \mu([w])$. We set $\varepsilon = \mu([w']) - \mu([w]) > 0$.

Since μ is σ-mixing:

$$\forall \alpha > 0, \exists n \in \mathbb{N}, \forall u, v \in \mathcal{A}^\ell, \ \left| \mu\left([u] \cap \sigma^{-p^n}[v]\right) - \mu([u])\mu([v]) \right| < \frac{\alpha}{2^\ell}.$$

The measure μ is invariant under G_n, so:

$$\forall u \in \mathcal{A}^\ell,\ \mu([u]) = \mu(G^{-n}([u])) = \sum_{v\in\mathcal{A}^\ell} \mu([v] \cap \sigma^{-p^n}([u-v])),$$

where the linear operations on words are made bit by bit modulo p.

In particular, we obtain :

$$\begin{aligned}
\mu([w]) &= \sum_{v\in\mathcal{A}^\ell} \mu([v] \cap \sigma^{-p^n}[w-v]) \\
&\geq \sum_{v\in\mathcal{A}^\ell} \mu([v])\mu([w-v]) - \alpha \\
&\geq \sum_{\substack{v\in\mathcal{A}^\ell \\ v\neq w-w'}} \mu([v])\mu([w-v]) + \mu([w-w'])\mu([w']) - \alpha \\
&\geq \sum_{\substack{v\in\mathcal{A}^\ell \\ v\neq w-w'}} \mu([v])\mu([w]) + \mu([w-w'])(\mu([w]) + \varepsilon) - \alpha \\
&\geq \mu([w]) + \varepsilon\mu([w-w']) - \alpha.
\end{aligned}$$

Any choice of $\alpha < \varepsilon\mu([w-w'])$ gives a contradiction. Thus, the only F_p-invariant measure of full support that is σ-mixing is the uniform measure. □

3.2.2 Entropy criterion

In this section, we use a criterion based on σ-ergodicity and entropy that ensures that invariant measures of A family of bipermutative CA must be the uniform measure λ. Since F commutes with the shift σ, the couple (F, σ) defines a $\mathbb{Z}^2$ action on $\mathcal{A}^\mathbb{Z}$.

Let $\mathcal{P}$ be a finite partition of $\mathcal{A}^\mathbb{Z}$, that is, a covering of $\mathcal{A}^\mathbb{Z}$ by a finite number of disjoint sets. We denote by $\mathcal{P}_\ell$ the partition corresponding to $\{[u]_{-\ell} \ :\ u \in \mathcal{A}^{2\ell+1}\}$. The refinement of two partitions $\mathcal{P}$ and $\mathcal{P}'$ is the partition defined by:

$$\mathcal{P} \vee \mathcal{P}' = \{A \cap B \ : A \in \mathcal{P} \text{ and } B \in \mathcal{P}'\}.$$

Definition 3.2.1 (Entropy)**.**

The **entropy of the partition** $\mathcal{P}$ is defined by:

$$H_\mu(\mathcal{P}) = -\sum_{A\in\mathcal{P}} \mu(A)\log(\mu(A)).$$

The **entropy of a cellular automaton** $F : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ is defined by:

$$h_\mu(F) = \lim_{\ell\to\infty} \lim_{N\to\infty} \frac{1}{N} H_\mu\left(\bigvee_{n=0}^{N-1} F^{-n}(\mathcal{P}_\ell)\right).$$

This limit exists by subadditivity. We refer for example to [Wal82] for a complete introduction to entropy.

Note that the entropy of the shift σ can also be written:

$$h_\mu(\sigma) = \lim_{k\to\infty} \sum_{u\in\mathcal{A}^k} \mu([u]) \log \mu([u]).$$

In the following theorems, for different cellular automata, we characterise F-invariant measures that satisfy the following conditions $(*)$:

(i) $\mu \in \mathcal{M}_{\sigma-\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}})$,

(ii) $h_\mu(F) > 0$,

Ergodicity with respect to σ is a strong assumption, but the assumption of ergodicity for the action (F, σ) is not sufficient to guarantee the results.

Theorem 3.2.3 (Host, Maass, Martinez [HMM03])**.**
Let $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$ with p prime, and let F be the linear CA defined by $F(x)_0 = ax_0 + bx_1 + c$ for some $a, b \in \mathbb{Z}/p\mathbb{Z}^*, c \in \mathbb{Z}/p\mathbb{Z}$.

If μ is F-invariant and satisfies $(*)$, then $\mu = \lambda$.

We generalise the tools of [HMM03] to prove a similar rigidity result when the local function is "perturbed" by composition with a permutation.

Theorem 3.2.4.
Let $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$, and let F be a bipermutative CA defined by $F(x)_0 = \rho(ax_0 + bx_1 + c)$ for some $a, b \in \mathbb{Z}_n^*, c \in \mathbb{Z}_n$, and $\rho \in \mathfrak{S}(\mathcal{A})$.

If μ is F-invariant and satisfies $(*)$, then $h_\mu(F) = \log k$, where k divides n. In particular, if n is a prime number, then $h_\mu(F) = \log n$ and $\mu = \lambda$.

First let us introduce some preliminary results.

We denote by $\mathfrak{B}$ the Borel $\sigma-$algebra of $\mathcal{A}^{\mathbb{Z}}$. We set $\mathfrak{B}_1 = F^{-1}(\mathfrak{B})$, and given a measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$, we define μ_x as its conditional measure with respect to $\mathfrak{B}_1$ at point x. That is, for $A \in \mathfrak{B}$, one has $\mu_x(A) = \mathbb{E}(\mathbf{1}_A|\mathfrak{B}_1)(x)$, and $\mu(\cdot) = \int_{\mathcal{A}^{\mathbb{Z}}} \mu_x(\cdot)\mathrm{d}\mu(x)$.

For $x \in \mathcal{A}^{\mathbb{Z}}$, we set $\mathcal{F}(x) = \{y \in \mathcal{A}^{\mathbb{Z}} : F(y) = F(x)\}$ the **fiber** of x.

Lemma 7. Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a cellular automaton and let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. We have:

1. $\sigma_*\mu_x = \mu_{\sigma(x)}$,
2. the support of the measure μ_x is $\mathrm{supp}(\mu_x) = \mathcal{F}(x)$.

Let us now assume that F is a bipermutative CA of neighbourhood $\mathcal{N} = \{0, 1\}$. That means that, for any $x \in \mathcal{A}^{\mathbb{Z}}$ and for any $a \in \mathcal{A}$, there exists a unique element $y \in \mathcal{F}(x)$ such that $y_0 = a$. For $\omega \in \mathfrak{S}(\mathcal{A})$, we can thus define the operator:

$$\begin{array}{rccl} T_\omega : & \mathcal{A}^{\mathbb{Z}} & \longrightarrow & \mathcal{A}^{\mathbb{Z}} \\ & x & \longmapsto & y \quad \text{such that } y \in \mathcal{F}(x) \text{ and } y_0 = \omega(x_0). \end{array}$$

For $\omega \in \mathfrak{S}(\mathcal{A})$, we also define:

$$\phi_\omega(x) = T_{\omega *}\mu_x(\{x\}) = \mu_x(T_\omega^{-1}(x)) = \mu_x(\{T_{\omega^{-1}}(x)\}).$$

In particular, we have $\phi_{\mathrm{Id}}(x) = \mu_x(\{x\})$ and $\phi_\omega(x) = \phi_{\mathrm{Id}}(T_{\omega^{-1}}(x))$. Finally, we set:

$$E_\omega = \{x \in \mathcal{A}^{\mathbb{Z}} : \phi_\omega(x) > 0\}.$$

Proposition 3.2.5 (Entropy formula for bipermutative CA, Section 4.3 of [HMM03])**.**
Let F be a bipermutative CA with $\mathcal{N} = \{0, 1\}$ and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. Then:

$$h_\mu(F) = \int_{\mathcal{A}^{\mathbb{Z}}} -\ln \phi_{\mathrm{Id}}(x) d\mu(x).$$

Proposition 3.2.6. Let μ be a (σ, F)-invariant measure, ergodic for σ and of positive entropy for F. The following properties are satisfied:

1. $\phi_{\mathrm{Id}} \circ \sigma = \phi_{\mathrm{Id}} \quad \mu$-a.e.;
2. There is a constant c such that $\phi_{\mathrm{Id}} \circ F = \phi_{\mathrm{Id}} = c \quad \mu$-a.e.;
3. $\mu(E_{\mathrm{Id}}) = 1$;
4. For $\omega \in \mathfrak{S}(\mathcal{A})$, $T_{\omega^{-1}*}\mathbf{1}_{E_\omega *}\mu$ is absolutely continuous with respect to μ, that is: if $\mu(A) = 0$, then $\mu(T_\omega(A) \cap E_\omega) = 0$;
5. $\phi_\omega = \phi_{\mathrm{Id}} \quad \mu$-a.e. in E_ω.

Proof.
1. By Lemma 7, $\sigma_* \mu_x = \mu_{\sigma(x)}$. Consequently, $\phi_{\mathrm{Id}}(\sigma(x)) = \mu_{\sigma(x)}(\{\sigma(x)\}) = \mu_x(\{x\})$.
2. By the first point, for any real r, the set $\phi_{\mathrm{Id}}^{-1}(\{r\})$ is σ-invariant. By σ-ergodicity of μ, it means the function ϕ_{Id} is equal μ-a.e. to some constant c. Since μ is F-invariant, it follows that $\phi_{\mathrm{Id}}(F(x)) = c$ for μ-a.e. x. We thus obtain $\phi_{\mathrm{Id}}(F(x)) = \phi_{\mathrm{Id}}(x) = c$ for μ-a.e. x.
3. By the first point and the σ-ergodicity of μ, we have $\mu(E_{\mathrm{Id}}) = 0$ or $\mu(E_{\mathrm{Id}}) = 1$. Since $h_\mu(F) > 0$ by hypothesis, it follows from Proposition 3.2.5 that $\mu(E_{\mathrm{Id}}) = 1$.
4. We have $\mu(A) = \int_{\mathcal{A}^{\mathbb{Z}}} \mu_x(A) d\mu(x)$, so that if $\mu(A) = 0$, then $\mu_x(A) = 0$ μ-a.e. In particular, for μ-a.e. $x \in T_\omega(A)$, $0 = \mu_x(A) \geq \mu_x(T_\omega^{-1}(x)) = \phi_\omega(x)$, thus $x \notin E_\omega$.
5. By the second point, $\phi_{\mathrm{Id}}(F(x)) = \phi_{\mathrm{Id}}(x)$ for μ-a.e. x. Using the fourth point, we obtain that for μ-almost every $x \in E_\omega$, $\phi_{\mathrm{Id}}(F(T_{\omega^{-1}}(x))) = \phi_{\mathrm{Id}}(T_{\omega^{-1}}(x))$. Since by definition $F(T_{\omega^{-1}}(x)) = F(x)$, it comes $\phi_{\mathrm{Id}}(x) = \phi_{\mathrm{Id}}(T_{\omega^{-1}}(x)) = \phi_\omega(x)$ for μ-a.e. $x \in E_\omega$.

□

Proposition 3.2.7. Let $\omega \in \mathfrak{S}(\mathcal{A})$. If there exists $d \in \mathbb{N}$ such that $T_\omega \circ \sigma^d = \sigma^d \circ T_\omega$, then for any $\omega' \in \mathfrak{S}(\mathcal{A})$, we have: $\phi_\omega = \phi_{\omega' \circ \omega}$ μ-a.e. in $E_{\omega'}$.

Proof. For any $d > 0$, σ^d-invariant measures form a closed, convex set and their extremal points are σ^d-ergodic measures [Wal82]. Therefore every σ^d-invariant measure can be written as a convex combination of σ^d-ergodic components.

Let ν be such a σ^d-ergodic component of μ. The measure $F_*\nu$ is σ^d-invariant and ergodic for σ^d, and it is absolutely continuous with respect to $F_*\mu = \mu$. Thus, $F_*\nu$ is an ergodic component of μ for σ^d, and it is equal to $\sigma^j\nu$ for some $j \in \{0, \dots, d-1\}$, so that $F_*^d\nu = \sigma^{jd}\nu = \nu$.

The function ϕ_ω is σ^d-invariant, since $\phi_\omega(\sigma^d(x)) = \phi_{\mathrm{Id}}(T_{\omega^{-1}}(\sigma^d(x))) = \phi_{\mathrm{Id}}(\sigma^d(T_{\omega^{-1}}(x)))$, and by the first point of Proposition 3.2.6, $\phi_{\mathrm{Id}}(\sigma^d(T_{\omega^{-1}}(x))) = \phi_{\mathrm{Id}}(T_{\omega^{-1}}(x)) = \phi_\omega(x)$ μ-a.e. Thus, for each ergodic component ν of μ (for σ^d), ϕ_ω is equal ν-a.e. to some constant $c_{\nu,\omega}$. And since $F_*^d\nu = \nu$, we obtain that $\phi_\omega(F^d(x)) = \phi_\omega(x) = c_{\nu,\omega}$ ν-a.e. This is true for each ergodic component of μ. Consequently, $\phi_\omega(F^d(x)) = \phi_\omega(x)$ μ-a.e.

Using the fourth point, we obtain that for μ-a.e. $x \in E_{\omega'}$, $\phi_\omega(F^d(T_{\omega'^{-1}}(x))) = \phi_\omega(T_{\omega'^{-1}}(x))$. Since $F^d(T_{\omega'^{-1}}(x)) = F^d(x)$, it follows that $\phi_\omega(F^d(T_{\omega'^{-1}}(x))) = \phi_\omega(F^d(x)) = \phi_\omega(x)$ μ-a.e. Finally, $\phi_\omega(T_{\omega'^{-1}}(x)) = \phi_\omega(x)$, that is, $\phi_\omega(x) = \phi_{\omega'\circ\omega}(x)$ for μ-a.e. $x \in E_{\omega'}$. □

Proof of Theorem 3.2.4. For $k \in \mathbb{Z}/n\mathbb{Z}$, let $\omega_k \in \mathfrak{S}(\mathcal{A})$ be the permutation defined by $\omega_k(j) = j + k$. For simplicity, we replace the notations $T_{\omega_k}, \phi_{\omega_k}, E_{\omega_k}$ by T_k, ϕ_k, E_k respectively.

Set $v = b^{-1}a$ (by hypothesis, F is bipermutative and a, b are invertible in $\mathbb{Z}/n\mathbb{Z}$) and let d be such that $v^{2d} = 1$. Observe that $T_k \circ \sigma^{2d} = \sigma^{2d} \circ T_k$, since two elements of the same fiber can be represented as follows.

$$\begin{array}{ccccccccc} \dots & x_0 & & x_1 & & x_2 & & x_3 & \dots & x_{2d} & \dots \\ \dots & x_0 + k & & x_1 - kv & & x_2 + kv^2 & & x_3 - kv^3 & \dots & x_{2d} + k & \dots \end{array}$$

Let μ be a (σ, F)-invariant measure, ergodic for σ and of positive entropy for F. We know by Proposition 3.2.6, third point, that $\mu(E_0) = 1$, and that there exists a constant c such that $\phi_k(x) = c$ μ-a.e. in E_k.

By Proposition 3.2.7, for any $i, k \in \mathcal{A}$, $\phi_k = \phi_{i+k}$ μ-a.e. in E_i.

Let us notice that by definition, $\sum_{j=0}^{n-1} \phi_j(x) = \mu_x(\mathcal{F}(x)) = 1$. Let j be the smallest element of $\{1, \dots, n\}$ such that $\mu(E_j) > 0$. There exists such a j, since otherwise, we would have $c = 1$ and by Proposition 3.2.5 $h_\mu(F) = 0$. Then, in E_j, we have μ-a.s. $c = \phi_0 = \phi_j = \phi_{2j} = \phi_{3j} = \dots$ Moreover, for values i that are not in the subgroup of $\mathbb{Z}/n\mathbb{Z}$ generated by j, we have μ-a.s. $\phi_i = 0$, since otherwise, we would get a contradiction with the definition of j. Consequently, $c = \gcd(j, n)/n$, and by Proposition 3.2.5, $h_\mu(F) = -\log c$. If n is prime, then the only possibility is that $\gcd(j, n) = 1$ and $h_\mu(F) = \log n$, so that $\mu = \lambda$, meaning that μ is the uniform measure. □

Conclusion and perspectives 4

In this thesis, we considered various questions centered around limit measures of cellular automata. The interest in limit measures comes from two main reasons: they are a good description of the typical asymptotic behaviour of a cellular automata iterated on a random configuration, and as such are a good tool to describe self-organisation phenomena; and they are a natural notion of output when considering that cellular automata are performing computation on sources of randomness.

The main result of this thesis is, in Chapter 1, the characterisation of all measures that can be reached as limit measures by a cellular automaton starting from a simple measure μ by computability conditions. The main tools were computability analysis on spaces of probability measures, and a sophisticated construction involving Turing machine simulations to compute any probability measure in cellular automata with minimal hypotheses. This result completely characterises probability measures that can be algorithmically simulated by cellular automata and partly explains the variety of behaviours observed in simulations. The remaining gaps are mostly the question of which full support measures can be reached in this way without increasing the alphabet, and in which ways it is possible to have the simulated (output) measure depend on the initial (input) measure).

In Chapter 2, we studied various questions related to the typical asymptotic behaviour of a family of cellular automata comporting finite words behaving like interacting particles. We explored consequences of the particle dynamics on the limit measure, especially when they exhibits good collision properties, and on the asymptotic time evolution of some parameters related to the distribution of the particles. This required to model the time evolution of the particles by a well-known random walk process, which gave us access to standard tools: in particular, the invariance principle to approximate the process by a Brownian motion. Refining this approach allowed us to handle one more difficult example, but generalising this work to more general particle dynamics (and thus wider classes of cellular automata) will require further work.

In Chapter 3, we looked for a cellular automaton that converges to the uniform Bernoulli measure starting from a wide class of initial measure (randomisation phenomenon). We conjectured that a link exists between some combinatorial and dynamical properties and this phenomenon, which was backed up by experimental statistics. However, proving that even one cellular automaton exhibits such a behaviour is still an open problem. We provided a first step of a tentative proof through the study of conditions that force an invariant measure under the action of the cellular automaton to be the uniform Bernoulli measure.

Now we would like to mention some open questions that are of a particular interest to us and that we are or have been studying, but for which we did not get obtain results that warranted a mention in this thesis.

Characterising the limit measures in dimension $d > 1$

In Chapter 1, we characterised measures and sets of measures that can be reached asymptotically (as μ-limit measures set) by one-dimensional cellular automata iterated on some simple initial measure. The reason this result was limited to dimension one is the highly *ad hoc* nature of the construction, which relies on many properties of dimension one such as the fact that each segment has exactly two neighbours.

Open question. What sets of measures $\mathcal{V} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}^d})$ can be obtained as μ-limit measures set of some cellular automata $F : \mathcal{B}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d}$ with $\mathcal{B} \supset \mathcal{A}$ for some simple initial measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}^d})$?

Martin Delacourt suggested a construction based on a similar approach of partitioning space in disjoint areas and performing computation independently on each area. However, the shape of the areas and the auxiliary processes are very different, and some technical difficulties have not been solved yet. This is an ongoing collaboration, but we are confident that a similar result holds for higher dimensions.

Computational power and dynamics of surjective CA

In Section 1.3, we provided a construction with no auxiliary states that could reach any semi-computable measure without full support at the limit. Doing the same in the full support case would require the corresponding cellular automaton to be surjective; therefore, to fill the gap left by Corollary 1.4.4, we need to understand the computational power of surjective automata in the context of simulation probability measures.

Since surjective cellular automata leave the uniform measure invariant, a direct extension of Theorem 1.3.1 – and consequently Corollary 1.4.4 – is impossible. Nevertheless, we cannot conclude that surjective automata have no computational power at all: in Section 3.1, and particularly Proposition 3.1.2, we saw that not all surjective automata converge to the uniform measure when starting from a nonuniform measure. Therefore one can ask the following question:

Open question. Let us fix an initial measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. What measures or sets of measures can be reached at the limit by iterating a surjective cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ on μ ?

As we explained above, the answer depends on the choice of μ, and in particular the question is only relevant if $\mu \neq \lambda$, in contrast with the non-surjective case. We do not expect that all semi-computable measures can be reached in this way due to the highly constrained dynamics of surjective automata. This could be a very good example of how restricting the dynamics of the model can influence its computational power.

4-state cyclic automaton

Using results from [Fis90b] and Section 2.1, we can show that the 4-state cyclic automaton iterated on the uniform measure does not have any particle remaining in the μ-limit set. However, the proof of Fisch solves the main difficulty and is highly technical. We are unable to give a simpler proof of this result within our framework, and to generalise it to other Bernoulli measures as we did with the 3-state cyclic automaton.

Open question. If μ is a nondegenerate Bernoulli measure, do we have:

$$F^t\mu \to \nu \text{ with } \nu \in Conv(\widehat{\delta_0}, \widehat{\delta_1}, \widehat{\delta_2}, \widehat{\delta_3})?$$

By analogy with our results in Section 2.3, it seems even more challenging to determine exactly the weight given to each Dirac in the limit measure(s) depending on the parameters

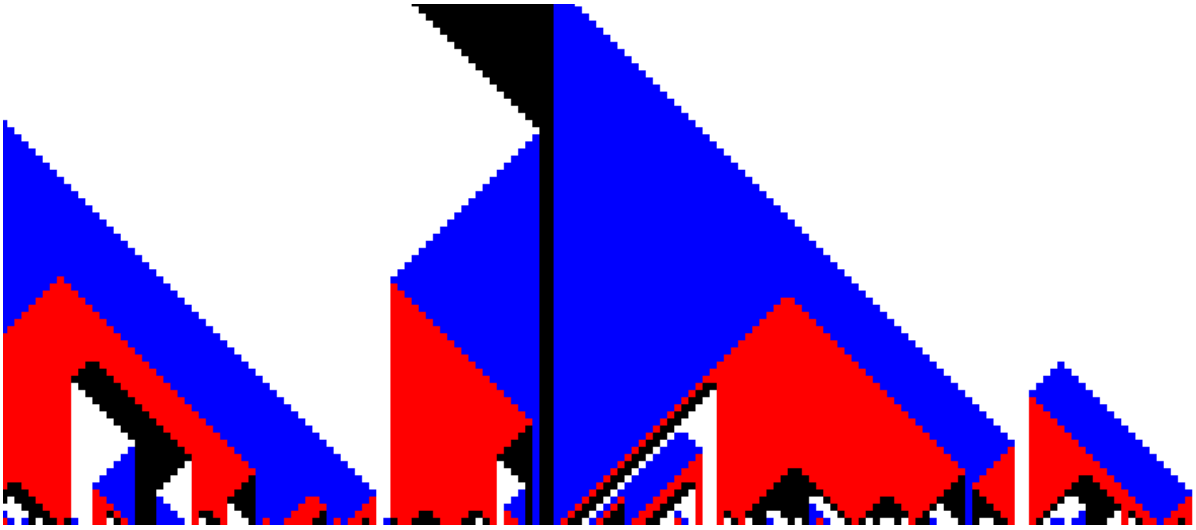


Figure 4.1: The 4-state cyclic automaton.

of the initial Bernoulli measure. In particular, it is difficult to find a simple stochastic process that describes the time evolution of the CA, playing the same role as the Brownian motion in Section 2.3. Experimental statistics did not let a clear relationship appear as in Theorem 2.3.1.

Density classification problem

Definition 4.0.2 (Density classification problem)**.**
Let F be a cellular automaton on alphabet $\{0,1\}$ and $M \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ a **class of initial measures**.
F performs **density classification** on M if:

$$\forall \mu \in M, \begin{array}{l} \mu([0]) > \mu([1]) \Rightarrow F_*^t \mu \underset{t\to\infty}{\longrightarrow} \widehat{\delta_0} \\ \mu([1]) > \mu([0]) \Rightarrow F_*^t \mu \underset{t\to\infty}{\longrightarrow} \widehat{\delta_1} \end{array},$$

the behaviour not being specified when $\mu([0]) = \mu([1])$.

Open question. Can we find a cellular automaton performing density classification for a large class of initial measures?

If the randomisation problem can be seen as noise correction in a perfect source of randomness, this problem can be seen as noise correction to keep one bit of information. This problem was introduced in [Pac88], with the main candidate being the Gács-Kurdiumov-Levy cellular automaton [GKL87, dSM92], shown in Figure 4.2.

Recently, many researchers took an experimental approach to find new rules that seemed to perform best on finite configurations (see [Fat13] for a survey), including probabilistic cellular automata. Recent progress [BFMM12] solved the problem in dimension 2 and higher; however, it is still open in dimension one even for small classes.

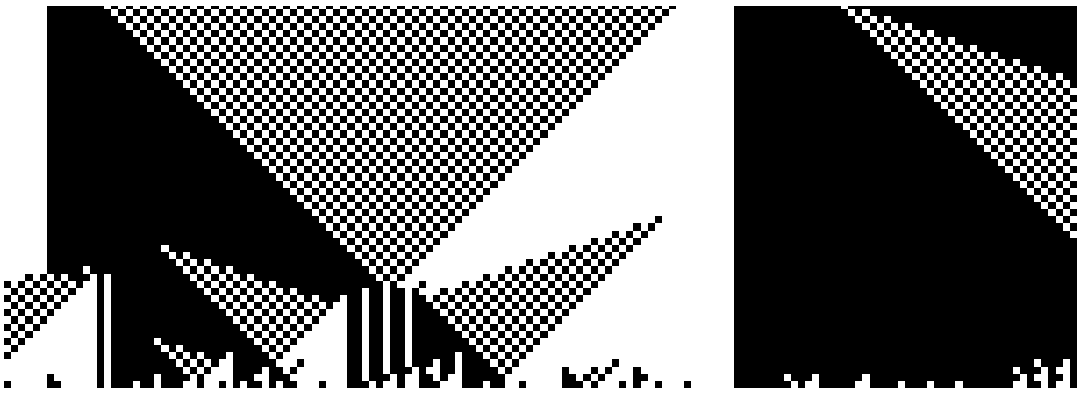


Figure 4.2: The Gács-Kurdiumov-Levy cellular automaton, a candidate for density classification, iterated on an initial measure $\mu = \mathrm{Ber}(\frac{1}{4}, \frac{3}{4})$.

In Section 1.5, we constructed a cellular automaton that converged to a limit measure that was a function of a parameter, the density of a special state in the initial measure. In addition, we saw in Section 1.3 that it was possible under some circumstances to get rid of the auxiliary states in this kind of construction. Combining these approaches, and barring some technical difficulties, we may be able to construct a density classifying CA.

Notice that, by opposition to the current candidates which are relatively simple cellular automata, such a solution would be highly sophisticated with an extremely large neighbourhood and very slow convergence.

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