PERMUTATION STATISTICS RELATED TO A CLASS OF NONCOMMUTATIVE SYMMETRIC FUNCTIONS AND GENERALIZATIONS OF THE GENOCCHI NUMBERS

FLORENT HIVERT, JEAN-CHRISTOPHE NOVELLI, LENNY TEVLIN, AND JEAN-YVES THIBON

Abstract. We prove conjectures of the third author [L. Tevlin, Proc. FPSAC'07, Tianjin] on two new bases of noncommutative symmetric functions: the transition matrices from the ribbon basis have nonnegative integral coefficients. This is done by means of two composition-valued statistics on permutations and packed words, which generalize the combinatorics of Genocchi numbers.

1. Introduction

In the theory of noncommutative symmetric functions [4], the self dual commutative Hopf algebra Sym of ordinary symmetric functions is replaced by a pair of mutually dual Hopf algebras ($\mathbf{Sym}, QSym$), respectively called Noncommutative Symmetric Functions, and Quasi-symmetric functions. The usual bases of Sym are usually lifted only on one side (with the notable exception of Schur functions, which admit natural analogs on both sides). In particular, monomial symmetric functions m_μ split into the quasi-monomial functions M_I on the quasi-symmetric side, and their dual basis h_μ is lifted on the noncommutative side, in the form of the homogeneous products S^I.

In [14], the third author has proposed a construction of noncommutative monomial and forgotten symmetric functions, and conjectured positivity properties of certain transition matrices involving the new bases. The purpose of the present article is to prove these conjectures, by providing combinatorial interpretations.

These interpretations rely on new permutations statistics, which generalize the combinatorics related to the Genocchi numbers.

Notations. We shall depart from the notation of [14] and write Ψ_I instead of M^I, and L_I instead of L^I. Other notations are as in [4]. See [3] for background on quasideterminants.

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2. Background

2.1. **Noncommutative symmetric functions.** Recall that the algebra **Sym** of noncommutative symmetric functions is a graded free associative algebra over a sequence S_n of indeterminates, with $\deg S_n = n$. Among other sequences of generators, the *noncommutative power sums of the first kind* Ψ_n are defined by an oriented analog of Newton's recursion, which may be solved in terms of quasideterminants [4, 3]. The following definition [14] refines formulas (39) and (40) of [4], and defines an analog of the monomial basis which extends the Ψ_n.

Definition 2.1. *The* ***noncommutative monomial symmetric function*** *corresponding to a composition $I = (i_1, \dots, i_r)$ is defined as a quasideterminant of an r by r matrix:*

$$r\Psi_I \equiv r\Psi_{(i_1,\dots,i_r)} = (-1)^{r-1} \begin{vmatrix} \Psi_{i_r} & 1 & 0 & \dots & 0 & 0 \\ \Psi_{i_{n-1}+i_r} & \Psi_{i_{n-1}} & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\dots+i_r} & \dots & \dots & \dots & \Psi_{i_2} & n-1 \\ \boxed{\Psi_{i_1+\dots+i_r}} & \dots & \dots & \dots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix} \tag{1}$$

where r is the length of I. In particular,

$$\Psi_{(n)} = \Psi_n, \text{ and } \Psi_{1^r} = \Lambda_r \tag{2}$$

where Λ_r is an elementary symmetric function.

The quasideterminants may be recursively evaluated by means of the following generalized Newton relations:

$$\begin{aligned} r\Psi_{i_1,\dots,i_r} &= \Psi_{i_1}\Psi_{i_2,\dots,i_r} - \Psi_{i_1+i_2}\Psi_{i_3,\dots,i_r} + \dots \\ &+ (-1)^{s-1}\Psi_{i_1+\cdots+i_s}\Psi_{i_{s+1},\dots,i_r} + \cdots + (-1)^r\Psi_{i_1+\cdots+i_r}. \end{aligned} \tag{3}$$

From a noncommutative analog of the quasi-monomial basis M_I, one can define an analog of Gessel's fundamental basis F_I by

$$L_I = \sum_{J \succeq I} \Psi_J. \tag{4}$$

Define the coefficients G_{IJ} by the expansion

$$R_I = \sum_J G_{IJ} L_J. \tag{5}$$

It has been conjectured in [14] that these numbers are nonnegative integers. Our aim is to prove this fact by means of a combinatorial interpretation.

2.2. **Free quasi-symmetric functions.** Let us fix an infinite ordered alphabet $A = \{a_1 < \cdots < a_n < \dots\}$. The *standardized word* $\mathrm{Std}(w)$ of a word $w \in A^*$ is the permutation obtained by iteratively scanning w from left to right, and labelling $1, 2, \dots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on.

With a permutation σ, we associate the polynomial

$$\mathbf{F}_\sigma := \sum_{\mathrm{Std}(w)=\sigma^{-1}} w\,. \tag{6}$$

These polynomials span a subalgebra of $\mathbb{K}\langle A\rangle$, called **FQSym** for Free Quasi-Symmetric functions [2]. Note that the field $\mathbb{K}$ is assumed to be of characteristic zero. Their product rule is given by

$$\mathbf{F}_{\sigma'}\mathbf{F}_{\sigma''} = \sum_{\sigma\in\sigma'\Cup\sigma''} \mathbf{F}_\sigma\,, \tag{7}$$

where the *shifted shuffle* $\sigma' \Cup \sigma''$ of two packed words is defined as

$$\sigma' \Cup \sigma'' = \sigma' \sqcup\!\sqcup \sigma''[|\sigma|], \tag{8}$$

the k-shift $w[k]$ of a word w being obtained by replacing each letter w_i by $w_i + k$, and $\sqcup\!\sqcup$ is the usual shuffle product on words defined recursively by

$$(au) \sqcup\!\sqcup (bv) = a \cdot (u \sqcup\!\sqcup (bv)) + b \cdot ((au) \sqcup\!\sqcup v), \tag{9}$$

with $u \sqcup\!\sqcup \epsilon = \epsilon \sqcup\!\sqcup u = u$ if ϵ is the empty word.

We shall make use of the basis $\mathbf{G}_\sigma$ of **FQSym**, dual to $\mathbf{F}_\sigma$, defined by $\mathbf{G}_\sigma := \mathbf{F}_{\sigma^{-1}}$.

2.3. **Word quasi-symmetric functions.** The *packed word* $u = \mathrm{pack}(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < b_2 < \ldots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. A word u is said to be *packed* if $\mathrm{pack}(u) = u$. We denote by PW the set of packed words. With such a word, we associate the polynomial

$$\mathbf{M}_u := \sum_{\mathrm{pack}(w)=u} w\,. \tag{10}$$

These polynomials span a subalgebra of $\mathbb{K}\langle A\rangle$, called **WQSym** for Word Quasi-Symmetric functions [5, 11] (and called **NCQSym** in [1]), the invariants of the non-commutative quasi-symmetrizing action. Their product rule is given by

$$\mathbf{M}_{u'}\mathbf{M}_{u''} = \sum_{u\in u' *_W u''} \mathbf{M}_u\,, \tag{11}$$

where the *convolution* $u' *_W u''$ of two packed words is defined as

$$u' *_W u'' = \sum_{\substack{v,w;u=v\cdot w\,\in\,\mathrm{PW}\\ \mathrm{pack}(v)=u',\mathrm{pack}(w)=u''}} u\,. \tag{12}$$

3. A statistic on permutations generalizing Genocchi numbers

Genocchi numbers (sequence A001469 of [13]) are known to count a large variety of combinatorial objects, among which numerous sets of permutations. Our statistic derives directly from the most classical of those sets: it is the number of permutations of $\mathfrak{S}_{2n}$ such that each even integer is followed by a smaller integer and each odd integer is either followed by a greater one, or at the last position of the permutation.

Let us define the *Genocchi descent set* (G-descent set for short) of a permutation $\sigma \in \mathfrak{S}_n$ as

$$\mathrm{GDes}(\sigma) := \{i \in [2, n] | \sigma_j = i \Longrightarrow \sigma_{j+1} < \sigma_j\}. \tag{13}$$

In other words, $\mathrm{GDes}(\sigma)$ is the set of *values* of the descents of σ, different from the usual set $\mathrm{Des}(\sigma)$ which records the *positions* of the descents of σ. Astonishingly enough, this G-statistic behaves very differently from the classical descent statistic. From the G-descent set, we define the *Genocchi composition of descents* (or G-composition, for short) $\mathrm{GC}(\sigma)$ of a permutation, as the integer composition I of n whose descent set is $\{d-1 | d \in \mathrm{GDes}(\sigma)\}$.

The following tables represent the G-composition of all permutations of $\mathfrak{S}_2$, $\mathfrak{S}_3$, and $\mathfrak{S}_4$.

(14)

2	11
12	21

3	21	12	111
123	132	213	321
	231		
	312		

(15)

4	31	22	211	13	121	112	1111
1234	1243	1324	1432	2134	2143	3214	4321
	1342	2314	2431		3421		
	1423	3124	3142		4213		
	2341		3241				
	2413		4132				
	3412		4231				
	4123		4312				

More combinatorial properties of these numbers, including a hook-length formula will be given in [12].

4. A **Sym** quotient of **FQSym**

Let $\sim$ be the equivalence relation defined by $\sigma \sim \tau$ iff $\mathrm{GC}(\sigma) = \mathrm{GC}(\tau)$. Let $\mathcal{J}$ be the subspace of **FQSym** spanned by the differences

$$\{\mathbf{F}_\sigma - \mathbf{F}_\tau | \sigma \sim \tau\}. \tag{16}$$

Theorem 4.1. *$\mathcal{J}$ is a two-sided ideal of* **FQSym***, and the quotient* $\mathbf{T} = \mathbf{FQSym}/\mathcal{J}$ *is isomorphic to* **Sym** *as an algebra.*

Moreover, let T_I be the image in $\mathbf{T}$ of the $\mathbf{F}_\sigma$ such that $\mathrm{GC}(\sigma) = I$. Then

$$T_I T_J = \sum_K C_{I,J}^K T_K, \tag{17}$$

where $C_{I,J}^K$ is computed as follows. Let K' and K'' be the compositions such that $|K'| = |I|$ and either $K = K' \cdot K''$, or $K = K' \triangleright K''$. If K' is not coarser than I or *if K'' is not finer than J, then $C_{I,J}^K$ is 0. Otherwise,*

$$C_{I,J}^K = \binom{|I| + l(J) - l(I)}{l(K) - l(I)} \tag{18}$$

Proof – We have to prove that the set (with multiplicities) of G-compositions of the shifted shuffle of two permutations depends only on the G-compositions of the permutations.

Let $\sigma \in \mathfrak{S}_m$ and $\tau \in \mathfrak{S}_n$ and let I and J be their respective G-compositions. Let $K = (k_1, \dots, k_r)$ be a composition of $m+n$ and let us compute the number of permutations μ in $\sigma \Cup \tau$ such that $\mathrm{GC}(\mu) = K$. We shall need the unique compositions K' and K'' such that $K = K' \cdot K''$ or $K = K' \triangleright K''$, with $|K'| = |I|$.

Let us now consider which letters can follow the letters x from 1 to m in μ. We have four cases:

(1a) x is a G-descent of μ and is not a G-descent of σ,
(2a) x is a G-descent of μ and is a G-descent of σ,
(3a) x is not a G-descent of μ and is not a G-descent of σ,
(4a) x is not a G-descent of μ and is a G-descent of σ.

The first case implies that K cannot be the G-composition of a word in the shifted shuffle of σ and μ. Let us now restrict to compositions K such that K' is coarser than I. The second case implies that x has to be followed in μ by a letter coming from σ. The third one implies nothing about x. The fourth one implies that x has to be followed in μ by a letter coming from τ.

Let $f(\sigma)$ be the number of occurrences of the fourth case. Let $g(\sigma)$ be the number of occurrences of the third and fourth cases, plus one.

Let us now consider which letters can follow the letters x from $m+1$ to $m+n$ in μ. We have again four cases:

(1b) x is a G-descent of μ and is not a G-descent of $\tau[m]$,
(2b) x is a G-descent of μ and is a G-descent of $\tau[m]$,
(3b) x is not a G-descent of μ and is not a G-descent of $\tau[m]$,
(4b) x is not a G-descent of μ and is a G-descent of $\tau[m]$.

The first case implies that x has to be followed in μ by a letter coming from σ. The second case implies nothing about x. The third one implies that x has to be followed in μ by a letter coming from τ. The fourth one implies that K cannot be the G-composition of a word in the shifted shuffle of σ and μ. We now restrict to compositions K such that K'' is finer than J.

This preliminary analysis proves that the number of permutations μ with G-composition K is equal to the number of ways of separating the letters of τ in any number of blocks with given necessary separations (case 1b) and necessary non-separations (case 3b) and put those blocks in the middle of blocks of letters of σ, themselves separated into this number of blocks with given necessary separations (case 4a), and necessary non-separations (case 2a). The *number* of such blocks for each pair of permutations depends only on the lengths of their G-compositions, and *a fortiori* only on their G-compositions, so that our equivalence relation on permutations indeed induces a quotient algebra of **FQSym**.

Let us now determine the structure constants of this algebra. The previous remark can be reformulated as follows. We have two cases depending on whether $K = K'\cdot K''$ or $K = K' \triangleright K''$.

In the first case, the number of permutations τ with G-composition I is

$$\sum_{k=1}^{\max(n,m)} \binom{m-l(I)}{k-1-(l(I)-l(K'))}\binom{n}{l(K'')+1-k}. \tag{19}$$

This sum of binomial coefficients is easily simplified, and one gets

$$\binom{m-l(I)+l(J)}{l(K')+l(K'')-l(I)}. \tag{20}$$

In the second case, the number of permutations τ with G-composition I is

$$\sum_{k=1}^{\max(n,m)} \binom{m-l(I)}{k-1-(l(I)-l(K'))}\binom{n}{l(K'')-k}. \tag{21}$$

Similarly, this sum of binomial coefficients reduces to

$$\binom{m-l(I)+l(J)}{l(K')+l(K'')-1-l(I)}, \tag{22}$$

so that C_{IJ}^K is indeed given by (18) in any case.

One can notice that these coefficients coincide, in the special case $I = (n)$ with those of the product $L_I L_J$ (see Proposition 4.6 of [14]), so that, since the L_n are algebraic generators of **Sym**, the T_n are algebraic generators of **T**. Moreover, since **Sym** is free over the sequence $L_n = S_n$, the algebra **T** is free over the T_n. ∎

Example 4.2. Let $I = (2, 2, 1)$, $J = (1, 3)$, and $K = (4, 2, 1, 1, 1)$. We can choose $\sigma = 32514$ and $\tau = 2134$.

We then have $K' = (4, 1)$ and $K'' = (1, 1, 1, 1)$. The coefficient of K in $T_I T_J$ is $\binom{5+2-3}{5-3} = 6$ and, indeed, there are six permutations in the shifted shuffle $32514 \Cup 2134$ with G-composition K:

(23) 372685194, 376825194, 376829514, 736825194, 736829514, 768392514.

Those six permutations are obtained as follows: σ has one necessary separation between 3 and 2 and one necessary non-separation between 1 and 5, and nothing after 4. The permutation $\tau[5]$ has two necessary separations, between 8 and 9, and after 9, and one necessary non-separation between 6 and 8. Then one inserts the blocks of $\tau[5]$ in σ, satisfying the separation/non-separation constraints and gets the six permutations.

Note 4.3. Note that this quotient is not a Hopf quotient, since as one can easily check, $\mathcal{J}$ is not a coideal. For example, $231 \sim 312$ but

$$\overline{\Delta}(\mathbf{F}_{231}) = \mathbf{F}_{12} \otimes \mathbf{F}_1 + \mathbf{F}_1 \otimes \mathbf{F}_{21}, \text{ and } \overline{\Delta}(\mathbf{F}_{312}) = \mathbf{F}_{21} \otimes \mathbf{F}_1 + \mathbf{F}_1 \otimes \mathbf{F}_{12}. \tag{24}$$

5. Change of bases in **Sym**

Thanks to the previous result, we have a map going from **Sym** to itself in a very unusual way: start with the injection of **Sym** into **FQSym***, and compose it with the self duality isomorphism of **FQSym**, which reads

$$R_I := \sum_{\mathrm{D}(\sigma)=I} \mathbf{G}_\sigma = \sum_{\mathrm{D}(\tau^{-1})=I} \mathbf{F}_\tau, \tag{25}$$

where D is the composition whose descent set is equal to the descent set of σ, and then go from **FQSym** to **Sym** by the G-quotient homomorphism.

Let ϕ be the composition of those maps and let R'_I be the image of R_I by ϕ:

$$\begin{array}{rccc} \phi: & \mathbf{Sym} & \to & \mathbf{T} \\ & R_I & \mapsto & R'_I. \end{array} \tag{26}$$

By definition of ϕ, we have

$$R'_I := \sum_{\mathrm{D}(\sigma^{-1})=I} \overline{\mathbf{F}_\sigma} = \sum_{\substack{\mathrm{D}(\sigma^{-1})=I \\ \mathrm{GC}(\sigma)=J}} T_J, \tag{27}$$

where $\overline{\mathbf{F}}_\sigma$ is the image of $\mathbf{F}_\sigma$ by the G-quotient homomorphism. Then, since $L_n = R_n$ and $R'_n = \overline{\mathbf{F}_{12\ldots n}} = T_n$, we have $\phi(L_n) = T_n$ for all n, so that, thanks to the product formulas of L_n and T_n, $\phi(L_I) = T_I$ for all compositions I.

Since the T_n are algebraic generators of $\mathbf{T}$, the algebra morphism ϕ is an isomorphism of algebras, so that, applying ϕ^{-1} to Equation (27), one gets

Theorem 5.1. *Let I be a composition of n. Then*

$$R_I = \sum_{J \vDash n} G_{IJ} L_J, \tag{28}$$

where G_{IJ} is the number of permutations satisfying $\mathrm{D}(\sigma^{-1}) = I$ and $\mathrm{GC}(\sigma) = J$. In particular, the G_{IJ} are nonnegative integers.

Examples of the transition matrices are given in Section 7.1, together with the same matrices filled with the corresponding permutations. The Genocchi numbers appear as the sums of the values in the rows indexed by compositions of the form (2^n) or $(2^n 1)$.

Combining this last result with Equation (4), one then gets

Corollary 5.2. *Let I be a composition of n. Then*

$$R_I = \sum_{J \vDash n} K_{IJ} \Psi_J, \tag{29}$$

where K_{IJ} are nonnegative integers.

One can easily describe those integers in terms of permutations. They can be described in a much more natural way in terms of packed words as one shall see in the following section.

Note 5.3. Theorems 17 and 5.1 prove Conjecture 4.1 of [14]. Theorem 5.1 proves Conjecture 5.4 of [14]. Corollary 5.2 proves Conjecture 5.3 of [14]. Corresponding statements for the "forgotten" basis are obtained by applying the canonical involution ω.

6. Permutations replaced by packed words

The previous section was devoted to the study of the transition matrices from R to L, we now apply a similar analysis to the transition matrices from R to Ψ. As already mentioned, the latter can be described in terms of the former, since there is a very simple transition matrix from L to Ψ: it is the matrix of the refinement order on compositions. Nevertheless, as the sum of the entries of the transition matrix $M_n(R, L)$ is $n!$, the sum of the entries of a transition matrix $M_n(R, \Psi)$ is the nth ordered Bell number (sequence A000670 of [13]) counting, for example, set compositions (ordered set partitions), or packed words.

This suggests the existence of two statistics on packed words giving back the entries of the transition matrices, exactly as in the (R, L) case. The algebraic context is essentially the same as before if one replaces the algebra **FQSym** by the algebra **WQSym** (see [5, 11]).

The proof of the connection between the two statistics and the matrices $M(R, \Psi)$ follows the same guidelines as the previous proof. We first define a composition-valued statistic on packed words, then prove that this statistic defines a quotient of **WQSym**, isomorphic to **Sym** as an algebra. Then, comparing the structure constants of the natural base with those of the Ψ_I, we prove that they are mapped to each other by a simple isomorphism, hence giving the coefficients of the matrix $M(R, \Psi)$.

6.1. A statistic on packed words.

Let w be a packed word. The *Word composition* (W-composition) of w is the composition whose descent set is given by the positions of the last occurrences of each letter in w. For example,

$$\mathrm{WC}(1543421323) = (2, 3, 2, 2, 1). \tag{30}$$

Indeed, the descent set is $\{2, 5, 7, 9, 10\}$ since the last 5 is in position 2, the last 4 is in position 5, the last 1 is in position 7, the last 3 is in position 9, the last 2 is in position 10.

The following tables represent the W-compositions of all packed words in PW_2 and PW_3. One can recover from the matrix $\mathfrak{M}'_4$ in Section 7.2 the W-compositions in PW_4: it is the composition indexing their row.

(31)

2	11
11	12
	21

3	21	12	111
111	112	122	123
	121	211	132
	212		213
	221		231
			312
			321

6.2. **A Sym quotient of WQSym.** Let $\sim$ be the equivalence relation on packed words defined by $u \sim v$ iff $\mathrm{WC}(u) = \mathrm{WC}(v)$. Let $\mathcal{J}'$ be the subspace of **WQSym** spanned by the differences

$$\{\mathbf{M}_u - \mathbf{M}_v \,|\, u \sim v\}. \tag{32}$$

Theorem 6.1. *$\mathcal{J}'$ is a two-sided ideal of* **WQSym**, *and the quotient* $\mathbf{T}'$ *defined by* $\mathbf{T}' = \mathbf{WQSym}/\mathcal{J}'$ *is isomorphic to* **Sym** *as an algebra.*

Moreover, let U_I be the image of $\mathbf{M}_u$ in $\mathbf{T}'$. Then

$$U_I U_J := \sum_K D_{I,J}^K U_K, \tag{33}$$

where $D_{I,J}^K$ is computed as follows. Let K' and K'' be the compositions such that $|K'| = |I|$ and either $K = K' \cdot K''$, or $K = K' \triangleright K''$. If K' is not coarser than I, then $D_{I,J}^K$ is 0. Otherwise,

$$D_{I,J}^K = \binom{l(K)}{l(I)}. \tag{34}$$

Proof – The proof follows essentially the same lines as the proof of Theorem 4.1, so we only sketch it. In fact, the details are much simpler than for Theorem 4.1. Looking at the definitions of the W-composition and of the convolution of packed words, it is clear that the multiset of the W-compositions of the words in the convolution of two packed words depends only on the W-compositions of the words. So the product is well-defined and $\mathbf{T}'$ is a quotient of **WQSym**.

Let us now see why the product $U_I U_J$ is given by Equations (33) and (34). Let us choose two words u and v such that $\mathrm{WC}(u) = I$, and $\mathrm{WC}(v) = J$. Since there is exactly one nondecreasing word having a given WC, we can assume that u and v are nondecreasing. Let $|v|$ be the size of v. Let us compute $\mathbf{M}_u \mathbf{M}_v$.

The idea is that a word $w \in u *_W v$ satisfies $\mathrm{WC}(w) = K$ iff the last $|v|$ letters of w have specific values, depending on K'. Indeed, by definition of WC, if K' is not coarser than I, or if $K'' \neq J$, then the coefficient of U_K is zero. Now, let us fix a composition $I = (i_1, \ldots, i_l)$ and a composition K satisfying the conditions of the theorem. Then any word of $u *_W v = w = u' \cdot v'$ satisfying $\mathrm{WC}(w) = K$ has also $|K|$ different letters. Now, for all $j < l$, if I_j and I_{j+1} come from the same part of K, the letter $u'_{i_1 + \cdots + i_j}$ has to appear in v', otherwise this letter does not appear in v'. Hence, given the letters appearing in u', the letters appearing in v' are also fixed, which completely determines v' too (since its packed word is given). The number of ways of choosing the letters appearing in u' obviously is the binomial coefficient $\binom{l(K)}{l(I)}$. ∎

Example 6.2. Let $I = (2, 2, 1)$, $J = (1, 3)$, and $K = (4, 1, 1, 3)$. We can choose $\sigma = 23212$ and $\tau = 2111$. Then $K' = (4, 1)$ and $K'' = (1, 3)$. The coefficient of K in $U_I U_J$ is $\binom{4}{2} = 6$ and, indeed, there are four packed words in the (modified) convolution $11223 *_W 1222$ with W-composition K:

$$112241333,\ 113341222,\ 112231444,\ 223341222. \tag{35}$$

Those four packed words are obtained as follows: since $(4,1,1,3)$ is obtained by gluing together the first two parts of I, this means that, if $\mathrm{WC}(w)=K$, the last four letters of w have to be the first letter of w or the one not used in its first five letters.

Note 6.3. This quotient is not a Hopf quotient, since again, $\mathcal{J}'$ is not a coideal. For example, $221 \sim 112$ but

$$\overline{\Delta}(\mathbf{M}_{221}) = \mathbf{M}_1 \otimes \mathbf{M}_{11}, \text{ and } \overline{\Delta}(\mathbf{M}_{112}) = \mathbf{M}_{11} \otimes \mathbf{M}_1. \tag{36}$$

6.3. **Change of bases in Sym.** As in the case of permutations, we have a map going from **Sym** to itself: start with the injection of **Sym** into **WQSym**, which reads

$$R_I := \sum_{\mathrm{D}(u)=I} \mathbf{M}_u, \tag{37}$$

and then go from **WQSym** to **Sym** by the W-quotient homomorphism.

Let ϕ' be the composition of those maps and let R'_I be the image of R_I by ϕ':

$$\begin{array}{rccc} \phi': & \mathbf{Sym} & \to & \mathbf{T}' \\ & R_I & \mapsto & R'_I. \end{array} \tag{38}$$

By definition of ϕ', we have

$$R'_I := \sum_{\mathrm{D}(u)=I} \overline{\mathbf{M}_u} = \sum_{\substack{\mathrm{D}(u)=I \\ \mathrm{WC}(u)=J}} \Psi_J, \tag{39}$$

where $\overline{\mathbf{M}}_u$ is the image of $\mathbf{M}_u$ by the W-quotient homomorphism. Then, since $\Psi_{1^n} = R_{1^n}$ and $R'_{1^n} = \overline{\mathbf{M}_{n\ldots 21}} = U_{1^n}$, we have $\phi'(\Psi_n) = U_n$ for all n, so that, thanks to the product formulas of Ψ_n and U_n, $\phi'(\Psi_I) = U_I$ for all compositions I.

Since the U_n are algebraic generators of $\mathbf{T}'$, the algebra morphism ϕ' is an isomorphism of algebras, so that, applying ϕ'^{-1} to Equation (39), one gets

Theorem 6.4. *Let I be a composition of n. Then*

$$R_I = \sum_{J \vDash n} K_{IJ} \Psi_J, \tag{40}$$

where K_{IJ} is the number of permutations satisfying $\mathrm{D}(u)=I$ and $\mathrm{WC}(u)=J$. In particular, the K_{IJ} are nonnegative integers.

Combining this last result with Equation (4), one then gets back Corollary 5.2. Examples of the transition matrices are given in Section 7.2, together with the same matrices filled with the corresponding packed words.

7. Tables

7.1. **Coefficients G_{IJ}.** Here are the transition matrices from R to L (the matrices of the coefficients G_{IJ}) for $n=3$ and $n=4$, the compositions being in lexicographic order. To save space and for better readability, 0 has been represented by a dot.

$$M_3(R,L) = \begin{pmatrix} 1 & . & . & . \\ . & 2 & 1 & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix} \tag{41}$$

$$M_4(R,L) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ . & 3 & 2 & . & 1 & 1 & . & . \\ . & . & 2 & . & 1 & . & . & . \\ . & . & 1 & 3 & . & 2 & 1 & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 2 & 1 & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{pmatrix} \tag{42}$$

Here are the same matrices with the list of permutations having a given recoil composition (or descent composition of the inverse) and G-composition, instead of their number.

(43) $\mathfrak{M}_3 =$

GC \ Rec	3	21	12	111
3	123			
21		132 312	231	
12			213	
111				321

(44) $\mathfrak{M}_4 =$

GC \ Rec	4	31	22	211	13	121	112	1111
4	1234							
31		1243, 1423 4123	1342 3412		2341	2413		
22			1324 3124		2314			
211			3142	1432, 4132 4312		2431 4231	3241	
13					2134			
121						2143 4213	3421	
112							3214	
1111								4321

7.2. **Coefficients K_{IJ}.** Here are the transition matrices from R to Ψ (the matrices of the coefficients K_{IJ}) for $n=3$ and $n=4$, the compositions being in lexicographic order. To save space and for better readability, 0 has been represented by a dot.

$$M_3(R,\Psi) = \begin{pmatrix} 1 & . & . & . \\ 1 & 2 & 1 & . \\ 1 & . & 1 & . \\ 1 & 2 & 2 & 1 \end{pmatrix} \tag{45}$$

$$M_4(R,\Psi) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ 1 & 3 & 2 & . & 1 & 1 & . & . \\ 1 & . & 2 & . & 1 & . & . & . \\ 1 & 3 & 5 & 3 & 2 & 3 & 1 & . \\ 1 & . & . & . & 1 & . & . & . \\ 1 & 3 & 2 & . & 2 & 3 & 1 & . \\ 1 & . & 2 & . & 2 & . & 1 & . \\ 1 & 3 & 5 & 3 & 3 & 5 & 3 & 1 \end{pmatrix} \tag{46}$$

Here are the same matrices with the list of packed words having a given descent composition and W-composition, instead of their number.

(47) $\mathfrak{M}'_3 =$

WC \ D	3	21	12	111
3	111			
21	112	121 221	212	
12	122		211	
111	123	132 231	312 213	321

$\mathfrak{M}'_4 =$

WC \ D	4	31	22	211	13	121	112	1111
4	1111							
31	1112	1121, 1221 2221	2212 1212		2112	2121		
22	1122		1211 2211		2122			
211	1123	1132,1231 2231	1213,1312,2213 2312,3312	1321,2321 3321	2123 3123	2132,3132 3231	3213	
13	1222				2111			
121	1223	1232,1332 2331	1323 2313		2113 3112	2131,3121 3221	3212	
112	1233		1322 2311		2133 3122		3211	
1111	1234	1243,1342 2341	1324,1423,2314 2413,3412	1432,2431 3421	2134,3124 4123	2143,3142,3241 4132,4231	3214,4213 4312	4321

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(F. Hivert) LITIS, Université de Rouen ; Avenue de l'université ; 76801 Saint Étienne du Rouvray, France,

(J.-C Novelli, J.-Y. Thibon) Institut Gaspard Monge, Université Paris-Est Marne-la-Vallée, 5 Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France

(L. Tevlin) Physics Department, Yeshiva University, 500 West 185th Street, New York, N.Y. 10033, USA

E-mail address, F. Hivert: `hivert@univ-rouen.fr`
E-mail address, J.-C. Novelli: `novelli@univ-mlv.fr`
E-mail address, L. Tevlin: `tevlin@yu.edu`
E-mail address, J.-Y. Thibon: `jyt@univ-mlv.fr`