

# YANG-BAXTER BASES OF 0-HECKE ALGEBRAS AND REPRESENTATION THEORY OF 0-ARIKI-KOIKE-SHOJI ALGEBRAS

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ABSTRACT. After reformulating the representation theory of 0-Hecke algebras in an appropriate family of Yang-Baxter bases, we investigate certain specializations of the Ariki-Koike algebras, obtained by setting  $q = 0$  in a suitably normalized version of Shoji's presentation. We classify the simple and projective modules, and describe restrictions, induction products, Cartan invariants and decomposition matrices. This allows us to identify the Grothendieck rings of the towers of algebras in terms of certain graded Hopf algebras known as the Mantaci-Reutenauer descent algebras, and Poirier quasi-symmetric functions. We also describe the Ext-quivers, and conclude with numerical tables.

## CONTENTS

|                                                                   |    |
|-------------------------------------------------------------------|----|
| 1. Introduction                                                   | 2  |
| 2. Notations and preliminaries                                    | 3  |
| 2.1. Words and permutations                                       | 3  |
| 2.2. Compositions                                                 | 4  |
| 2.3. Shuffle, shifted shuffle, and convolution                    | 5  |
| 2.4. Cycloribbons and anticycloribbons                            | 5  |
| 3. The Hecke algebra of type $A$ at $q = 0$ and Yang-Baxter bases | 7  |
| 3.1. Representation theory                                        | 7  |
| 3.2. Yang-Baxter bases of $H_n(0)$                                | 9  |
| 3.3. Yang-Baxter graphs of semi-combinatorial modules             | 10 |
| 3.4. Representations in Yang-Baxter bases                         | 13 |
| 4. The 0-Ariki-Koike-Shoji algebras                               | 15 |
| 4.1. The associative bilinear form                                | 18 |
| 5. Simple modules of $\mathcal{H}_{n,r}(0)$                       | 20 |
| 5.1. Induction of the simple 0-Hecke modules                      | 21 |
| 5.2. First Grothendieck ring                                      | 22 |
| 5.3. Restriction of simple modules                                | 25 |
| 5.4. The quasi-symmetric Mantaci-Reutenauer algebra               | 26 |
| 6. Projective modules                                             | 28 |
| 6.1. The Grothendieck ring                                        | 28 |
| 6.2. Cartan invariants and decomposition numbers                  | 31 |
| 7. Quivers                                                        | 34 |
| 8. Tables                                                         | 37 |
| References                                                        | 44 |

## 1. INTRODUCTION

Given an *inductive tower of algebras*, that is, a sequence of algebras

$$(1) \quad A_0 \hookrightarrow A_1 \hookrightarrow \dots \hookrightarrow A_n \hookrightarrow \dots,$$

with embeddings  $A_m \otimes A_n \hookrightarrow A_{m+n}$  satisfying appropriate compatibility conditions, one can introduce two *Grothendieck rings*

$$(2) \quad \mathcal{G} := \bigoplus_{n \geq 0} G_0(A_n), \quad \mathcal{K} := \bigoplus_{n \geq 0} K_0(A_n),$$

where  $G_0(A)$  and  $K_0(A)$  are the (complexified) Grothendieck groups of the categories of finite-dimensional  $A$ -modules and projective  $A$ -modules respectively, with multiplication of the classes of an  $A_m$ -module  $M$  and an  $A_n$ -module  $N$  defined by

$$(3) \quad [M] \cdot [N] = [M \hat{\otimes} N] = [M \otimes N \uparrow_{A_m \otimes A_n}^{A_{m+n}}].$$

On each of these Grothendieck rings, one can define a coproduct by means of restriction of representations. Under favorable circumstances, this turns these two rings into mutually dual Hopf algebras.

The basic example of this situation is the character ring of symmetric groups (over  $\mathbb{C}$ ), due to Frobenius. Here the  $A_n = \mathbb{C} \mathfrak{S}_n$  are semi-simple algebras, so that

$$(4) \quad G_0(A_n) = K_0(A_n) = R(A_n),$$

where  $R(A)$  denotes the vector space spanned by isomorphism classes of indecomposable modules which in this case are all simple and projective. The irreducible representations  $[\lambda]$  of  $A_n$  are parametrized by partitions  $\lambda$  of  $n$ , and the Grothendieck ring is isomorphic to the algebra  $Sym$  of symmetric functions under the correspondence  $[\lambda] \leftrightarrow s_\lambda$ , where  $s_\lambda$  denotes the Schur function associated with  $\lambda$ .

Other known examples with towers of group algebras over the complex numbers,  $A_n = \mathbb{C} G_n$ , include the cases of wreath products  $G_n = \Gamma \wr \mathfrak{S}_n$  (Specht), finite linear groups  $G_n = GL(n, \mathbf{F}_q)$  (Green), *etc.*, all related to symmetric functions (see [17, 27]).

Examples involving modular representations of finite groups and non-semisimple specializations of Hecke algebras have also been worked out (see [6, 14, 16]). For example, finite Hecke algebras of type  $A$  at roots of unity ( $A_n = H_n(\zeta)$ ,  $\zeta^k = 1$ ) yield quotients and subalgebras of  $Sym$  [14]

$$(5) \quad \mathcal{G} = Sym / (p_{km} = 0), \quad \mathcal{K} = \mathbb{C}[p_i \mid i \not\equiv 0 \pmod{k}],$$

where the  $p_i$  are the power sum symmetric functions, supporting level 1 irreducible representations of the affine Lie algebra  $\widehat{\mathfrak{sl}}_k$ , while Ariki-Koike algebras at roots of unity give rise to higher level representations of the same Lie algebras [1]. The 0-Hecke algebras  $A_n = H_n(0)$  correspond to the pair Quasi-symmetric functions - Noncommutative symmetric functions,  $\mathcal{G} = QSym$ ,  $\mathcal{K} = \mathbf{Sym}$  [12]. Affine Hecke algebras at roots of unity lead to  $U^+(\widehat{\mathfrak{sl}}_k)$  and  $U^+(\widehat{\mathfrak{sl}}_k)^*$  [1], and the cases of generic affine Hecke algebras can be reduced to a subcategory admitting as Grothendieck rings  $U^+(\widehat{\mathfrak{gl}}_\infty)$  and  $U^+(\widehat{\mathfrak{gl}}_\infty)^*$  [1].

A further interesting example is the tower of 0-Hecke-Clifford algebras [21, 5], giving rise to the peak algebras [13, 26].

Here, we shall show that appropriate versions at  $q = 0$  of the Ariki-Koike algebras derived from the presentation of Shoji [25, 24] admit as Grothendieck rings two known combinatorial Hopf algebras, the Mantaci-Reutenauer descent algebras (associated with the corresponding wreath products) [18], and their duals, a generalization of quasi-symmetric functions, introduced by Poirier in [22] and more recently considered in [20, 4]. This work can be understood as the solution of an *inverse problem* in representation theory: given the Grothendieck rings, reconstruct the algebras. The main point is that the standard presentations of Ariki-Koike algebras at  $q = 0$  do not give the required Grothendieck rings.

This article is structured as follows. After recalling some classical definitions, we introduce new combinatorial structures (cycloribbons and anticycloribbons) needed in the sequel (Section 2). Next, we study a limit case of the Yang-Baxter bases of Hecke algebras introduced in [15], which will be one of our main tools (Section 3). In Section 4, we finally introduce the fundamental objects of our study, the 0-Ariki-Koike-Shoji algebras  $\mathcal{H}_{n,r}(0)$ , and special bases, well suited for analyzing representations. Next, we obtain the classification of simple  $\mathcal{H}_{n,r}(0)$ -modules, which turn out to be all one-dimensional, labelled by the  $r(r+1)^{n-1}$  cycloribbons. We then describe induction products and restrictions of these simple modules, which allows us to identify the first Grothendieck ring  $\mathcal{G}$  with a Hopf subalgebra of Poirier's Quasi-symmetric functions, dual to the Mantaci-Reutenauer Hopf algebra (Section 5). Since  $\mathcal{H}_{n,r}(0)$  is self-injective, this duality gives immediately the Grothendieck ring  $\mathcal{K}$  associated with projective modules. An alternative labelling of the indecomposable projective modules leads then to a simple description of basic operations such as induction products, restriction to  $H_n(0)$ , or induction from a projective  $H_n(0)$ -module. Summarizing, we obtain an explicit description of the Cartan-Brauer triangle, in particular of the Cartan invariants and of the decomposition matrices (Section 6). We conclude with a description of the Ext-quiver of  $\mathcal{H}_{n,r}(0)$ . The last section contains tables of  $q$ -Cartan invariants (recording the radical filtrations of projective modules) and decomposition matrices.

## 2. NOTATIONS AND PRELIMINARIES

**2.1. Words and permutations.** Let  $w$  be a word on a totally ordered alphabet  $A$ . We denote by  $\bar{w}$  the *mirror image* (reading from right to left) of  $w$ . If the alphabet is  $\mathbb{N}^*$ , the *evaluation* of  $w$  is the sequence of number of occurrences of 1, 2, and so on until all letters of  $w$  have been seen. For example, the evaluation of 15423341511457 is (4, 1, 2, 3, 3, 0, 1).

We will generally use the operation of concatenation on words but we will also use the *shifted concatenation*  $u \bullet v$  of two words  $u$  and  $v$  over the positive integers consisting in concatenating  $u$  with  $v[k] := (v_1 + k, \dots, v_l + k)$ , where  $k = |u|$  is the length of  $u$ . For example  $1431 \bullet 232 = 1431676$ .

We shall also make use of another classical operation on words, known as *standardization*. The standardized word  $\text{Std}(w)$  of  $w$  is the permutation having the same

inversions as  $w$ , and its right standardized word is  $\text{RStd}(w) := \overline{\text{Std}(w)}$ . For example,  $\text{Std}(412461415) = 514692738$ ,  $\text{RStd}(412461415) = 734692518$ .

We will represent permutations as words of size  $n$  over the alphabet  $\{1, \dots, n\}$ . Denote by  $s_i$  the transposition exchanging  $i$  and  $i + 1$ , and by  $\omega_n$  the maximal permutation  $(n, n - 1, \dots, 1)$ . For  $\sigma = (\sigma_1, \dots, \sigma_n)$ , we set

$$(6) \quad \bar{\sigma} := (\sigma_n, \dots, \sigma_1) = \sigma\omega_n, \quad \sigma^\# := (n + 1 - \sigma_1, \dots, n + 1 - \sigma_n) = \omega_n\sigma.$$

**2.2. Compositions.** Recall that a *composition* of an integer  $n$  is any finite sequence of positive integers  $I = (i_1, \dots, i_k)$  of sum  $|I| := i_1 + \dots + i_k = n$ . It can be pictured as a ribbon diagram, that is, a set of rows composed of square cells of respective lengths  $i_j$ , the first cell of each row being attached under the last cell of the previous one.  $I$  is called the *shape* of the ribbon diagram. The *conjugate*  $I^\sim$  of a composition  $I$  is the composition built with the number of cells of the columns of its ribbon diagram read from *right to left*. Its *mirror image*  $\bar{I}$  is the reading of  $I$  from right to left. For example, if  $I = (2, 3, 1, 2)$ ,  $I^\sim$  is  $(1, 3, 1, 2, 1)$  and  $\bar{I}$  is  $(2, 1, 3, 2)$ .

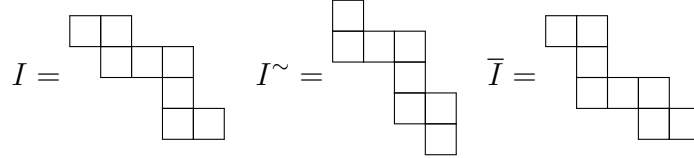


FIGURE 1. Ribbon diagrams of the compositions  $I = (2, 3, 1, 2)$ ,  $I^\sim = (1, 3, 1, 2, 1)$ , and  $\bar{I} = (2, 1, 3, 2)$ .

Given a filling of a ribbon diagram, we define its *row reading* as the reading of its rows from left to right and from top to bottom. Finally, the *descent set*  $D(I)$  of a composition  $I = (i_1, \dots, i_l)$  is the set  $\{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{l-1}\}$  composed of the *descents* of  $I$ . Recall also that the *descent set*  $D(\sigma)$  of a permutation  $\sigma$  is the set of  $i$  such that  $\sigma(i) > \sigma(i + 1)$  (the *descents* of  $\sigma$ ), and the *descent composition*  $C(\sigma)$  of  $\sigma$  is the unique composition  $I$  of  $n$  such that  $D(I) = D(\sigma)$ , that is, the shape of a filled ribbon diagram whose row reading is  $\sigma$  and whose rows are increasing and columns decreasing. For example, Figure 2 shows that the descent composition of  $(3, 5, 4, 1, 2, 7, 6)$  is  $I = (2, 1, 3, 1)$ .

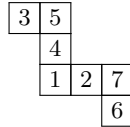


FIGURE 2. The ribbon diagram of the permutation 3541276.

Conversely, with a composition  $I$ , associate its *maximal permutation*  $\sigma = \omega(I)$  as the permutation with descent composition  $I$  and maximal inversion number. Similarly, the *minimal permutation*  $\alpha(I)$  is the permutation with descent composition  $I$  and minimal inversion number. For example, if  $I = (2, 1, 3, 1)$ ,  $\omega(I) = 6752341$  and  $\alpha(I) = 1432576$ .

**2.3. Shuffle, shifted shuffle, and convolution.** The *shuffle product*  $\sqcup$  of two words  $u$  and  $v$  over an alphabet  $A$  is inductively defined by:

$$(7) \quad \text{if } u = au' \text{ and } v = bv' \text{ with } a, b \in A, \text{ then } u \sqcup v = a(u' \sqcup v) + b(u \sqcup v'),$$

with the initial conditions  $u \sqcup \varepsilon = \varepsilon \sqcup u = u$ ,  $\varepsilon$  being the empty word.

The *shifted shuffle*  $\uplus$  of two permutations  $\sigma' \in \mathfrak{S}_k$  and  $\sigma'' \in \mathfrak{S}_l$  is the shuffle of  $\sigma'$  and  $\sigma''[k]$ . For example,

$$(8) \quad 21 \uplus 12 = 2134 + 2314 + 3214 + 2341 + 3241 + 3421.$$

The *convolution product*  $*$  of two permutations  $\sigma' \in \mathfrak{S}_k$  and  $\sigma'' \in \mathfrak{S}_l$  is the sum of permutations  $\sigma$  in  $\mathfrak{S}_{k+l}$  such that  $\text{Std}(\sigma_1 \cdots \sigma_k) = \sigma'$  and  $\text{Std}(\sigma_{k+1} \cdots \sigma_{k+l}) = \sigma''$ . Equivalently, it is the sum obtained by inverting the permutations occurring in the shifted shuffle of the inverses of  $\sigma'$  and  $\sigma''$ . It is well-known that the convolution of two permutations is an interval of the (left) weak order. For example, one has

$$(9) \quad 21 * 12 = 2134 + 3124 + 3214 + 4123 + 4213 + 4312 = [2134, 4312].$$

In all the paper, we make use of a set  $C = \{1, \dots, r\}$ , called the *color set*. The color words will be written in bold type.

The previous definitions can be extended to colored permutations as in [20], that is, pairs  $(\sigma, \mathbf{c}) \in \mathfrak{S}_n \times C^n$ . The *shifted shuffle* of two such elements  $(\sigma', \mathbf{c}') \uplus (\sigma'', \mathbf{c}'')$  is the sum of colored permutations obtained by shuffling with shift the two permutations, the colors remaining attached to letters. For example, representing colors as exponents,

$$(10) \quad \begin{aligned} 2^2 1^1 \uplus 1^1 2^2 &= 2^2 1^1 \sqcup 3^1 4^2 \\ &= 2^2 1^1 3^1 4^2 + 2^2 3^1 1^1 4^2 + 3^1 2^2 1^1 4^2 + 2^2 3^1 4^2 1^1 + 3^1 2^2 4^2 1^1 + 3^1 4^2 2^2 1^1. \end{aligned}$$

The *convolution*  $(\sigma', \mathbf{c}') * (\sigma'', \mathbf{c}'')$  of two colored permutations is the sum of colored permutations  $(\sigma, \mathbf{c}' \mathbf{c}'')$ , where  $\sigma$  runs over  $\sigma' * \sigma''$ . Equivalently, if one defines the *inverse* of a colored permutation  $(\sigma, \mathbf{c})$  as  $(\sigma^{-1}, \mathbf{c} \cdot \sigma^{-1})$ , the convolution of two colored permutations is the sum of the inverses of the elements occurring in the shifted shuffle of their inverses. For example,

$$(11) \quad 2^2 1^1 * 1^1 2^2 = 2^2 1^1 3^1 4^2 + 3^2 1^1 2^1 4^2 + 3^2 2^1 1^1 4^2 + 4^2 1^1 2^1 3^2 + 4^2 2^1 1^1 3^2 + 4^2 3^1 1^1 2^2.$$

The sums being multiplicity free, we can regard these as sets, and write, *e.g.*,  $2314 \in 21 \uplus 12$ , or  $3^2 1^1 2^1 4^2 \in 2^2 1^1 * 1^1 2^2$ .

**2.4. Cycloribbons and anticycloribbons.** Let  $I$  be a composition of  $n$  and  $\mathbf{c} \in C^n$  be a color word of length  $n$ . The pair  $[I, \mathbf{c}]$  is called a *colored ribbon*, and depicted as the filling of  $I$  whose row reading is  $\mathbf{c}$ . We say that this filling is a *cyclotomic ribbon* (*cycloribbon* for short) if it is *weakly increasing* in rows and *weakly decreasing* in columns. Notice that there are  $r(r+1)^{n-1}$  cycloribbons with at most  $r$  colors since when building the ribbon cell by cell, one has  $r$  possibilities for its first cell and then  $r+1$  possibilities for the next ones: 1 possibility for the  $r-1$  choices different from

the previous one and 2 possibilities for the same choice (right or down). Here are the five cycloribbons of shape  $(2, 1)$  with two colors:

$$(12) \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 2 & \\ \hline \end{array}$$

We say that a colored ribbon is an *anticyclotomic ribbon* (*anticycloribbon* for short) if it is *weakly decreasing* in rows and *weakly increasing* in columns. There are as many anticycloribbons as cycloribbons. The relevant bijection  $\phi$  from one set to the other is the restriction of an involution on all colored ribbons: read a ribbon  $R$  row-wise and build its image  $\phi(R)$  cell by cell as follows:

- if the  $(i + 1)$ -th cell has the same content as the  $i$ -th cell, glue it at the same position as in  $R$  (right or down),
- if the  $(i + 1)$ -th cell does not have the same content as the  $i$ -th cell, glue it at the other position (right or down).

For example, the colored ribbons of Figure 3 are exchanged by  $\phi$ :

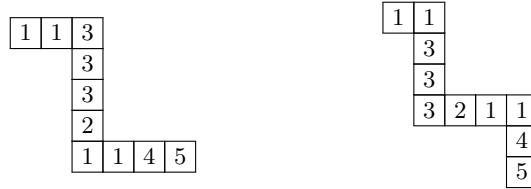


FIGURE 3. A cycloribbon  $R$  and the anticycloribbon  $\phi(R)$ .

Let us now extend to colored permutations and anticycloribbons the correspondances between permutations and ribbons: we say that  $i$  is an *anti-descent* of  $(\sigma, \mathbf{c})$  iff  $c_i < c_{i+1}$  or,  $c_i = c_{i+1}$  and  $\sigma_i < \sigma_{i+1}$ . Given a colored permutation  $(\sigma, \mathbf{c})$ , define its associated anticycloribbon as the pair  $[I, \mathbf{c}]$  where  $I$  has a descent at the  $i$ -th position iff  $i$  is an anti-descent of the colored permutation. Conversely, with a given anticycloribbon  $[I, \mathbf{c}]$ , associate its *maximal colored permutation*  $(\sigma, \mathbf{c})$  where  $\sigma = \omega(J)$ , where  $D(J) = \{i \mid c_i \neq c_{i+1}, \text{ or } c_i = c_{i+1} \text{ and } i \notin D(I)\}$ .

For example, the anti-descents of  $(81732564, 11132221)$  are  $\{2, 3, 5, 6\}$ . Its associated anticycloribbon is

$$(13) \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & \\ \hline 3 & 2 \\ \hline 2 & \\ \hline 2 & 1 \\ \hline \end{array}$$

and the maximal colored permutation having this associated anti-cycloribbon is  $\omega(1, 2, 1, 3, 1) = (86752341, 11132221)$ .

### 3. THE HECKE ALGEBRA OF TYPE $A$ AT $q = 0$ AND YANG-BAXTER BASES

In this section, we first recall some combinatorial aspects of the representation theory of  $H_n(0)$ , such as a realization of its simple and indecomposable projective modules in the left regular representation and the calculation of the composition factors of an induction product of two simple or indecomposable projective modules. We will show in the next Section how these constructions can be generalized to the 0-Ariki-Koike-Shoji algebra  $\mathcal{H}_{n,r}(0)$ . To achieve this, we will need to investigate in some detail a special case of the so-called Yang-Baxter bases of  $H_n(0)$ .

**3.1. Representation theory.** Let us consider the Hecke algebra  $H_n(0)$  of type  $A$  at  $q = 0$  in the presentation of [19, 12], that is, the  $\mathbb{C}$ -algebra with generators  $T_i$ ,  $1 \leq i \leq n-1$  and relations:

$$(14) \quad T_i(1 + T_i) = 0 \quad (1 \leq i \leq n-1),$$

$$(15) \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i \leq n-2),$$

$$(16) \quad T_i T_j = T_j T_i \quad (|i - j| \geq 2).$$

Let  $\sigma =: \sigma_{i_1} \dots \sigma_{i_p}$  be a reduced word for a permutation  $\sigma \in \mathfrak{S}_n$ . The defining relations of  $H_n(0)$  ensure that the element  $T_\sigma := T_{i_1} \dots T_{i_p}$  is independent of the chosen reduced word for  $\sigma$ . Moreover, the well-defined family  $(T_\sigma)_{\sigma \in \mathfrak{S}_n}$  is a basis of the Hecke algebra, which is consequently of dimension  $n!$ .

**3.1.1. Simple modules and induction.** It is known that  $H_n(0)$  has  $2^{n-1}$  simple modules, all one-dimensional, naturally labelled by compositions  $I$  of  $n$  [19]: following the notation of [12], let  $\eta_I$  be the generator of the simple  $H_n(0)$ -module  $S_I$  associated with  $I$  in the left regular representation. It satisfies

$$(17) \quad \begin{cases} -T_i \eta_I = \eta_I & \text{if } i \in D(I), \\ (1 + T_i) \eta_I = \eta_I & \text{otherwise.} \end{cases}$$

The composition factors of the induction product  $S_I \hat{\otimes} S_J$  of two simple 0-Hecke modules are easily described in terms of permutations.

Let us say that a family  $(b)$  of elements of a  $H_n(0)$ -module  $M$  is a *combinatorial basis* if it is a basis of  $M$  such that  $T_i b$  is either  $-b$ , or 0 or another basis element  $b'$  for all  $i$ . Then  $M$  is completely encoded by the edge-labelled directed graph having as vertices the basis elements, with a loop labelled  $-T_i$  iff  $-T_i b = b$ , a loop labelled  $1 + T_i$  iff  $(1 + T_i) b = b$ , and an edge from  $b$  to  $b'$  labelled  $T_i$  iff  $T_i b_\tau$  is  $b_{s_i \tau}$  (if  $s_i \tau$  occurs in  $\sigma * \sigma'$ , and  $s_i \tau$  has one inversion more than  $\tau$ ) or  $-b_\tau$  or 0, depending on whether  $i$  is a descent of  $\tau^{-1}$  or not. The composition factors are then the simple modules indexed by the descent compositions of the inverses of the permutations occurring in the convolution [9, 12] (see Figure 4).

This can be interpreted as a computation in the algebra  $\mathbf{FQSym}^*$  [18, 8], where the product of two basis elements  $\mathbf{G}_\alpha \mathbf{G}_\beta$  is

$$(18) \quad \mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\gamma \in \alpha * \beta} \mathbf{G}_\gamma.$$

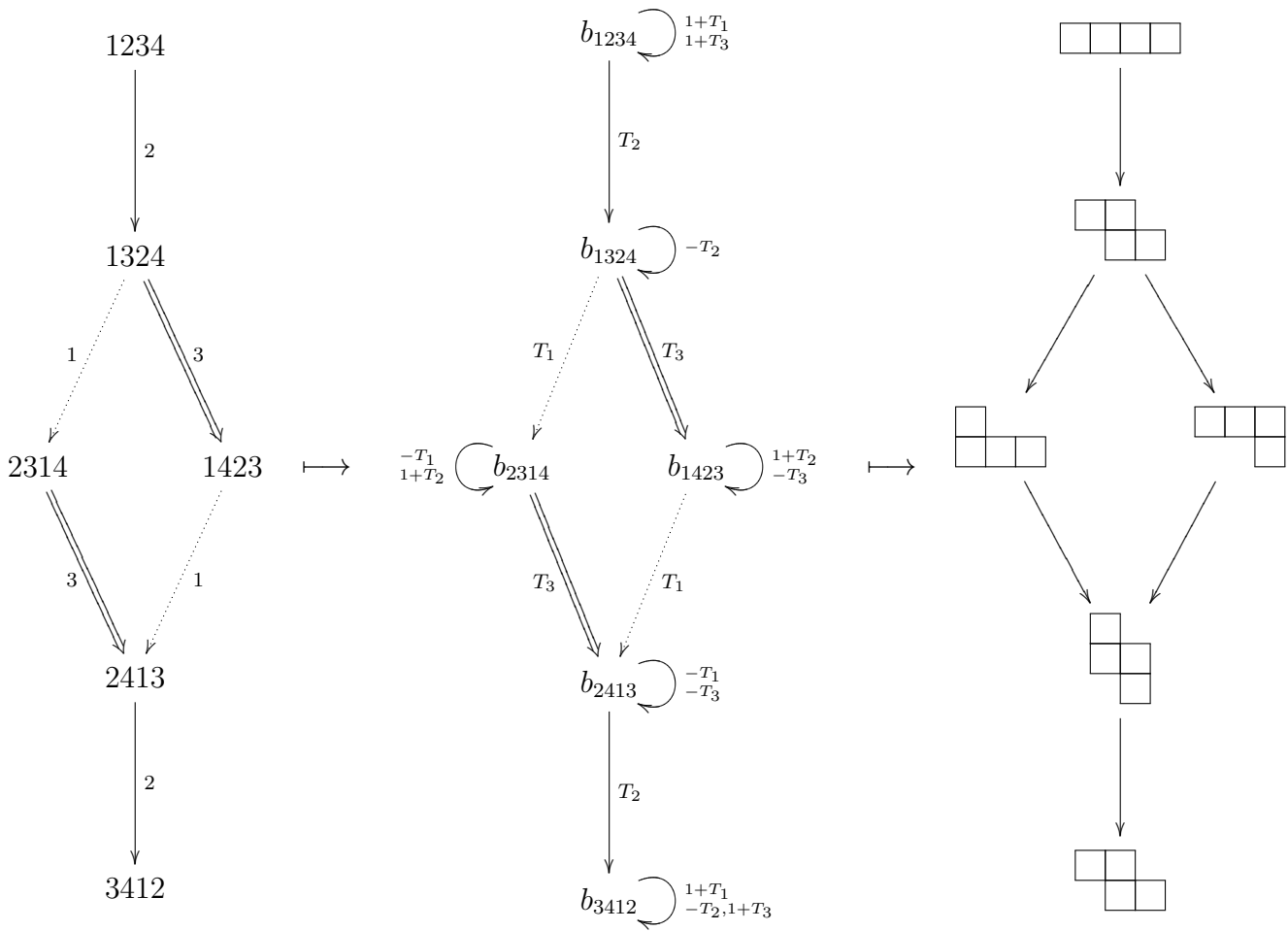


FIGURE 4. Induction product of two simple modules of  $H_2(0)$  to  $H_4(0)$ . The first graph is the restriction of the left weak order to the convolution of  $12*12$ . The second graph is its 0-Hecke graph and the third graph represents the composition factors, read from the descent positions of the inverse permutations.



Indeed, the composition factors of the induction product of two simple 0-Hecke modules can be obtained by computing the product  $\mathbf{G}_\sigma \mathbf{G}_{\sigma'}$  and taking the image of the result by the morphism sending  $\mathbf{G}_\tau$  to the quasi-symmetric function  $F_{C(\tau^{-1})} \in QSym$ . It is known that this amounts to take the commutative image of the realization of  $\mathbf{FQSym}^*$  by noncommutative polynomials [8]. Since this is an epimorphism, the Grothendieck ring  $\mathcal{G}$  of the tower of the 0-Hecke algebras is isomorphic to  $QSym$ . We shall later use a similar argument to identify the Grothendieck ring of the 0-Ariki-Koike-Shoji algebras.

**3.1.2. Indecomposable projective modules.** The indecomposable projective  $H_n(0)$ -module  $P_I$  labelled by a composition  $I$  of  $n$  is combinatorial in a certain basis ( $g_\sigma$ ). Its 0-Hecke graph coincides with the restriction of the left weak order to the interval  $[\alpha(I), \omega(I)]$ , that is the set of permutations with descent composition  $I$ , an edge  $\sigma \mapsto_i \tau$  meaning that  $T_i(g_\sigma) = g_\tau$ . The basis  $g_\sigma$  is defined in terms of Norton's generators:

$$(19) \quad g_{\alpha(I)} = \nu_I = T_{\alpha(I)} T'_{\alpha(\bar{I})},$$

where  $T'_i = 1 + T_i$  satisfies the braid relations, so that  $T'_\sigma$  is well defined for any permutation. Figure 5 shows the projective module  $P_{121}$  of  $H_4(0)$ .

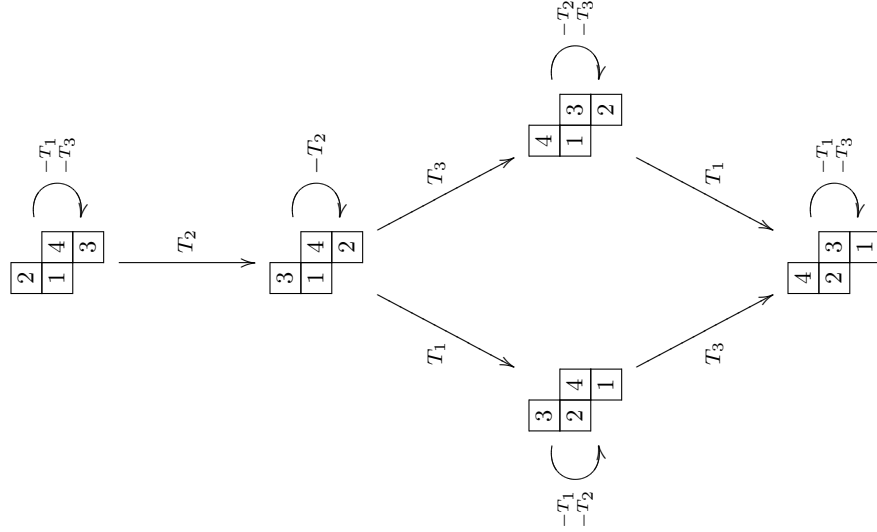


FIGURE 5. The projective module  $P_{121}$  of  $H_4(0)$ .

**3.2. Yang-Baxter bases of  $H_n(0)$ .** It is known that from a set of generators (depending on two parameters  $t$  and  $u$ )

$$(20) \quad Y_i(t, u) = a(t, u) + b(t, u)T_i \quad b \neq 0,$$

of  $H_n(0)$  satisfying the quantum Yang-Baxter equation

$$(21) \quad Y_i(t, u)Y_{i+1}(t, v)Y_i(u, v) = Y_{i+1}(u, v)Y_i(t, v)Y_{i+1}(t, u),$$

one can associate with any vector  $\mathbf{x} = (x_1, \dots, x_n)$  of “spectral parameters”, a Yang-Baxter basis  $(Y_\sigma(\mathbf{x}))_{\sigma \in \mathfrak{S}_n}$  of  $H_n(0)$ , inductively defined by  $Y_{id}(\mathbf{x}) = 1$  and

$$(22) \quad Y_{\sigma_j \tau}(\mathbf{x}) = Y_j(x_{\tau^{-1}(j)}, x_{\tau^{-1}(j+1)}) Y_\tau(\mathbf{x}) = Y_{\sigma_j}(\mathbf{x} \tau^{-1}) Y_\tau(\mathbf{x}),$$

if  $\sigma_j \tau$  has one inversion more than  $\tau$ . The special case  $Y_j(t, u) = 1 + (1 - u/t)T_j$  is studied in [14]. Here we need the solution

$$(23) \quad Y_j(t, u) = \begin{cases} T_j & \text{if } t > u, \\ 1 + T_j & \text{if } t \leq u, \end{cases}$$

Indeed, checking the six different possibilities of order among  $t, u, v$ , one has:

**Lemma 3.1.** *The  $Y_j(t, u)$  defined by Equation (23) satisfy Equation (21).* ■

The *Yang-Baxter graph* associated with a word  $\mathbf{x}$  is the graph whose vertices are  $Y_\sigma(\mathbf{x})$  and whose edges go from  $Y_\tau$  to  $Y_{\sigma_j \tau}$  with the label  $Y_j(x_{\tau(j)}, x_{\tau(j+1)})$  if  $\sigma_j \tau$  has one inversion more than  $\tau$ . For example, one can check on Figure 6 that  $Y_{3241}(2431) = (1 + T_2)T_1T_2T_3 = T_1T_2(1 + T_1)T_3 = T_1T_2T_3(1 + T_1)$ .

**Note 3.2.** The conditions of Equation (23) show that  $Y_\sigma(\mathbf{x})$  does not depend on  $\mathbf{x}$  but only on its standardized word  $\text{Std}(\mathbf{x})$ :

$$(24) \quad Y_\sigma(\mathbf{x}) = Y_\sigma(\text{Std}(\mathbf{x})).$$

We can therefore assume that  $\mathbf{x}$  is a color word with no repeated colors, hence a permutation.

**3.3. Yang-Baxter graphs of semi-combinatorial modules.** We shall now generalize the definition of a combinatorial module as follows: a *semi-combinatorial  $H_n(0)$ -module* is a  $H_n(0)$ -module with a basis  $(b)$  satisfying: either  $T_i b$ ,  $-T_i b$  or  $(1 + T_i)b$  is a basis element.

In particular, from any interval  $[Y_\sigma(\mathbf{x}), Y_\tau(\mathbf{x})]$  in a Yang-Baxter graph, one can build the graph of a semi-combinatorial module of  $H_n(0)$ , in the basis  $(Y_\rho)$ , by adding loops on each vertex  $Y_\rho$  as follows: if  $Y_{s_i \rho}$  does not belong to the interval, add a loop on  $Y_\rho$  labelled  $-T_i$  or  $1 + T_i$  depending on whether  $\mathbf{x}_{\rho^{-1}(i)} < \mathbf{x}_{\rho^{-1}(i+1)}$  or not. As in the case of combinatorial  $H_n(0)$ -modules, one can read on the graph  $G$  of a semi-combinatorial module as many composition series of the module as linear extensions of  $G$ . One can see an example of a restricted Yang-Baxter graph and the corresponding semi-combinatorial module on Figure 7.

Note that the interval  $[Y_\sigma(\mathbf{x}), Y_{\omega_n}(\mathbf{x})]$  corresponds to the left  $H_n(0)$ -module generated by  $Y_\sigma(\mathbf{x})$ . Consequently, any interval  $[Y_\sigma(\mathbf{x}), Y_\tau(\mathbf{x})]$  corresponds to a quotient of the previous  $H_n(0)$ -module, hence to an  $H_n(0)$ -module itself.

**Lemma 3.3.** *Let  $\mathbf{x}$  be a word,  $\sigma$  a permutation and consider the interval of the left weak order  $[Y_\sigma(\mathbf{x}), Y_{\omega_n}(\mathbf{x})]$  as a one-dimensional quotient of the  $H_n(0)$ -module generated by  $Y_\sigma(\mathbf{x})$ . It is isomorphic to the simple  $H_n(0)$ -module labelled by the shape of the anticycloribbon associated with  $(\sigma, \mathbf{x})^{-1}$ .*

*In particular,  $Y_{\omega_n}(\mathbf{x})$  is the simple  $H_n(0)$ -module labelled by the shape of the anticycloribbon associated with  $(\omega_n, \bar{\mathbf{x}})$ .*

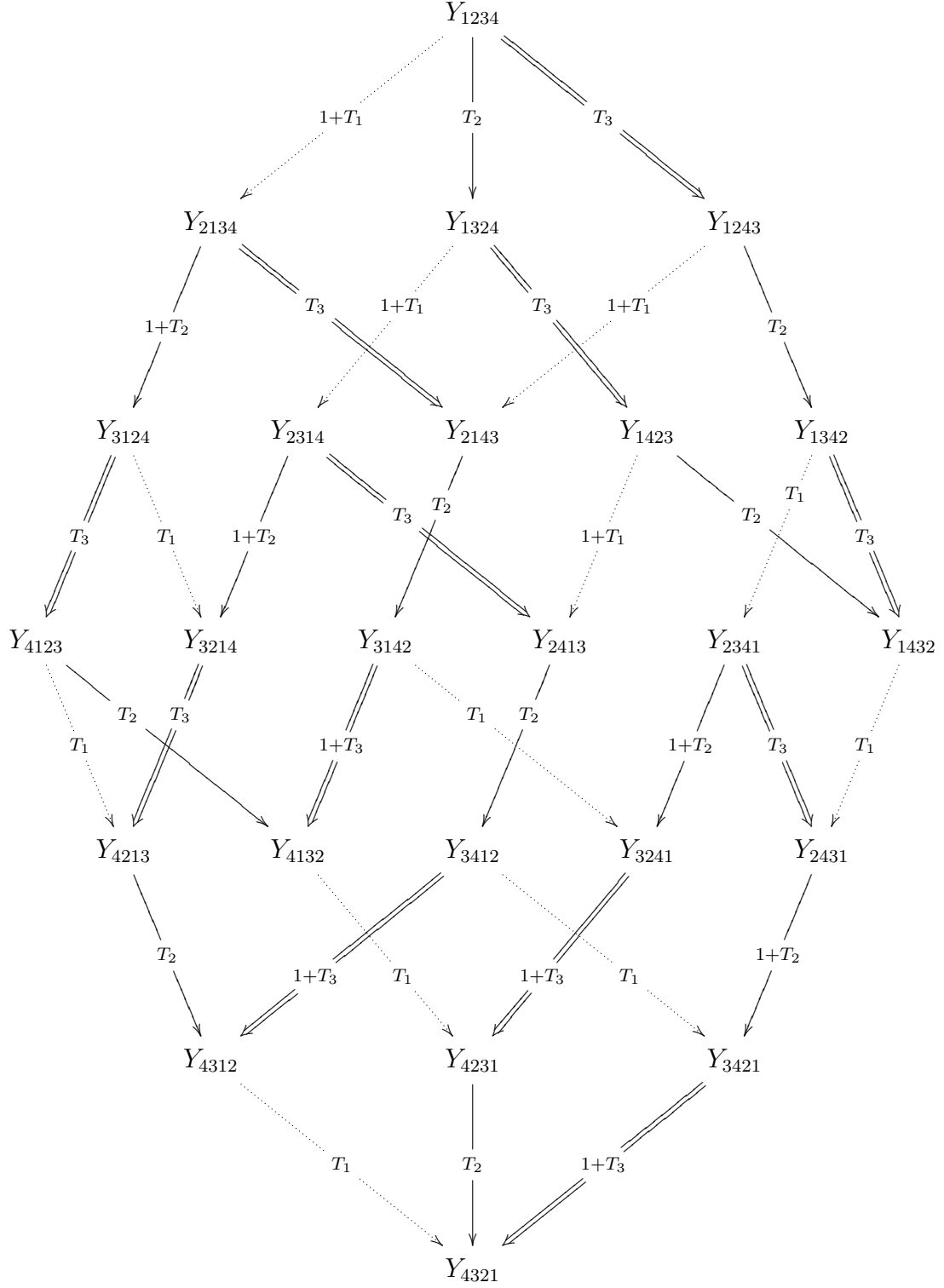


FIGURE 6. The Yang-Baxter graph associated with the color word 2431.

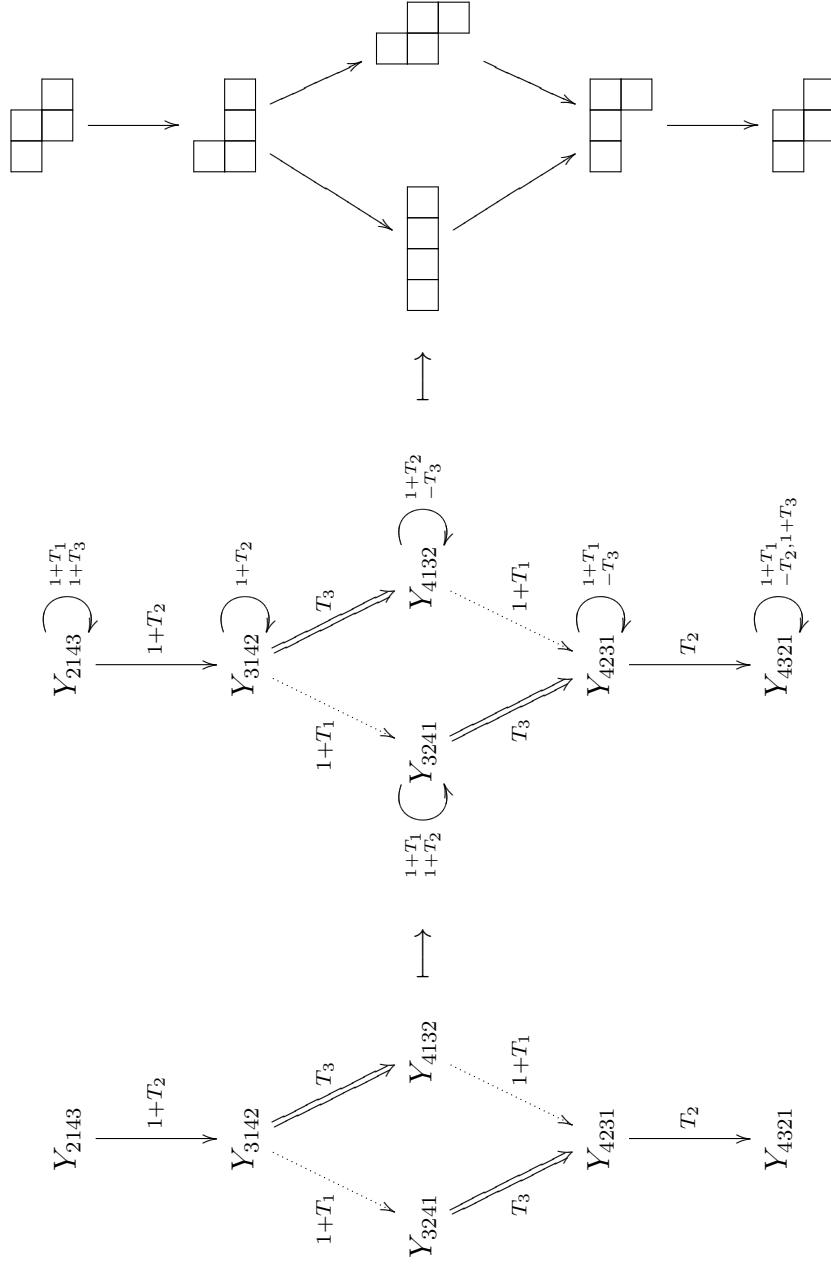


FIGURE 7. The first graph is the restriction of the Yang-Baxter graph associated with the color word 2314 to the convolution  $21*21$ . The second graph describes the corresponding semi-combinatorial  $H_n(0)$ -module. In the third graph, each basis element  $Y_\sigma$  has been replaced by the associated composition factor, given by the shape of the anticycloribbon associated with  $(\sigma, 2314)^{-1}$ .

*Proof.* The action of  $T_i$  on the quotient  $[Y_\sigma(\mathbf{x}), Y_\sigma(\mathbf{x})]$  is encoded by the Yang-Baxter graph associated with  $\mathbf{x}$ , and the lemma follows from the definitions. ■

For example, the one-dimensional quotient of the left  $H_n(0)$ -module generated by  $Y_{1342}(3213)$  has as corresponding anticycloribbon

$$(25) \quad \begin{array}{|c|c|c|} \hline 1^3 & & \\ \hline 4^3 & 2^2 & 3^1 \\ \hline \end{array}$$

so that it is the simple module  $S_{13}$ , as one can check on Figure 7 (the generator of the quotient module is annihilated by  $1 + T_1$ ,  $-T_2$  and  $-T_3$ ).

### 3.4. Representations in Yang-Baxter bases.

**3.4.1. Simple modules and indecomposable projective modules.** We can now rewrite the representation theory of  $H_n(0)$  in terms of Yang-Baxter bases. First, we identify the simple and indecomposable projective modules of  $H_n(0)$  as the semi-combinatorial modules associated with some intervals of Yang-Baxter graphs. Then, using Theorem 3.8, we will be able to describe in a very simple way the induction of those modules in terms of intervals in the Yang-Baxter graphs.

We first consider the case of simple  $H_n(0)$ -modules. The next lemma is a rewriting of a special case of Lemma 3.3. Recall that  $\eta_I$  is the generator of the simple module  $S_I$  (see Equation (17)).

**Lemma 3.4.** *Let  $I$  be a composition. Then for any permutation  $\tau$  such that  $\overline{C(\tau)} = I$ , one has*

$$(26) \quad \eta_I = Y_{\omega_n}(\tau).$$

Let  $\varphi$  be the anti-automorphism of  $H_n(0)$  such that  $\varphi(T_i) = T_i$  for all  $i$ . One easily proves by induction on the number of inversions of  $\sigma$  that

**Lemma 3.5.** *For all color words  $\tau$  and all permutations  $\sigma$ , one has*

$$(27) \quad \varphi(Y_\sigma(\tau)) = Y_{\sigma^{-1}}(\bar{\tau}^\# \cdot (\sigma^{-1})^\#).$$

These lemmas give in particular the description of  $\mathbb{C}\eta_I$  as a right  $H_n(0)$ -module:

**Proposition 3.6.** *For all color words  $\tau$ , one has*

$$(28) \quad \varphi(Y_{\omega_n}(\tau)) = Y_{\omega_n}(\bar{\tau}^\#),$$

so that, for all compositions  $I$ ,

$$(29) \quad \varphi(\eta_I) = \eta_{\bar{I}}.$$

Let us now consider the case of indecomposable projective  $H_n(0)$ -modules. For those modules, from the definition of Norton's generators (see Equation (19)), one obtains:

**Proposition 3.7.** *Let  $I$  be a composition. Then*

$$(30) \quad \nu_I = Y_{\alpha(I) \cdot \alpha(I^\sim)}(\omega(I)).$$

### 3.4.2. Induction product of modules.

**Theorem 3.8.** *Let  $M'$  and  $M''$  be the semi-combinatorial modules of  $H_k(0)$  and  $H_{n-k}(0)$  associated with the intervals  $[Y_{\alpha'}(\tau'), Y_{\beta'}(\tau')]$  and  $[Y_{\alpha''}(\tau''), Y_{\beta''}(\tau'')]$ . Then,  $M' \widehat{\otimes} M''$  is the semi-combinatorial module associated with the interval  $[Y_{\alpha}(\tau), Y_{\beta}(\tau)]$ , where  $\tau$  is any element of  $\tau' * \tau''$ ,  $\alpha = \alpha' \cdot \alpha''[k]$  and  $\beta = \beta''[k] \cdot \beta'$ .*

*Proof.* First, the semi-combinatorial module  $M$  corresponding to the restriction of the Yang-Baxter graph determined by  $\tau$  to the interval  $[\alpha, \beta]$  and  $M' \widehat{\otimes} M''$  have the same dimension and both are  $H_n(0)$ -modules. Moreover, the quotient of  $M$  by the submodule  $H_n(0)T_k M$  is isomorphic to  $M' \otimes M''$  as an  $H_k(0) \otimes H_{n-k}(0)$ -module. So  $M$  is isomorphic to  $M' \widehat{\otimes} M''$ .  $\blacksquare$

Since any simple or indecomposable projective module is an interval of a Yang-Baxter graph (seen as a semi-combinatorial module), the induction of such modules is also such a Yang-Baxter graph. The choice we have in color  $\tau$  in Theorem 3.8 will be useful when computing the induced module of two indecomposable projective modules in the 0-Ariki-Koike-Shoji algebras.

**Corollary 3.9.** *Let  $I_1$  and  $I_2$  be compositions of  $n_1$  and  $n_2$ , and  $\mathbf{x} = \mathbf{x}_1 \mathbf{x}_2$  a color word with  $\mathbf{x}_1 \in C^{n_1}$  and  $\mathbf{x}_2 \in C^{n_2}$ . Let  $\sigma_1 = \omega(I_1)$  and  $\sigma_2 = \omega(I_2)$ .*

*Then, the semi-combinatorial module corresponding to the restriction of the Yang-Baxter graph determined by  $\mathbf{x}$  to the convolution  $\sigma_1 * \sigma_2$  is isomorphic to the induced module  $S_{I'_1} \widehat{\otimes} S_{I'_2}$  where  $I'_1$  (resp.  $I'_2$ ) is the shape of the anticycloribbon associated with the colored permutation  $(\sigma_1, \mathbf{x}_1)^{-1}$  (resp.  $(\sigma_2, \mathbf{x}_2)^{-1}$ ).*

**Corollary 3.10.** *Let  $I'$  and  $I''$  be two compositions of integers  $n'$  and  $n''$ . Let  $\mathbf{x}'$  and  $\mathbf{x}''$  be two permutations such that  $I' = \overline{C(\mathbf{x}')}$  and  $I'' = \overline{C(\mathbf{x}'')}$ . Then the interval  $[Y_{\sigma}^{\mathbf{x}}, Y_{\omega_{n'+n''}}^{\mathbf{x}}]$  of the Yang-Baxter graph of color  $\mathbf{x}$  where  $\sigma = \omega_{n'} \bullet \omega_{n''}$  and  $\mathbf{x} = \mathbf{x}' \bullet \mathbf{x}''$  is isomorphic to the induced module  $S_{I'} \widehat{\otimes} S_{I''}$ .*

For example, the semi-combinatorial module represented on Figure 7 is isomorphic to  $S_2 \widehat{\otimes} S_2$ , as one can check on Figure 4. The edges are not labelled by the same elements but the composition factors are the same. Since the generators of both modules are the same, one can easily provide an explicit description of the basis of the semi-combinatorial module in the left regular representation: it is a product of factors  $T_i$  and  $1 + T_i$  multiplied on their right by  $\eta_{I'_1} \otimes \eta_{I'_2}$  seen as an element of  $H_{|I_1|+|I_2|}(0)$  through the morphism  $H_{|I_1|}(0) \otimes H_{|I_2|}(0) \rightarrow H_{|I_1|+|I_2|}(0)$ .

Another example is given Figure 8. The graphs are the semi-combinatorial modules corresponding to the restriction of the Yang-Baxter graph associated with the color words 3124 and 4231 to the same interval [2314, 4321]. Both are the graph of the induced module  $P_{(2,1)} \widehat{\otimes} P_{(1)}$  where  $P_I$  is the indecomposable projective module associated with  $I$ .

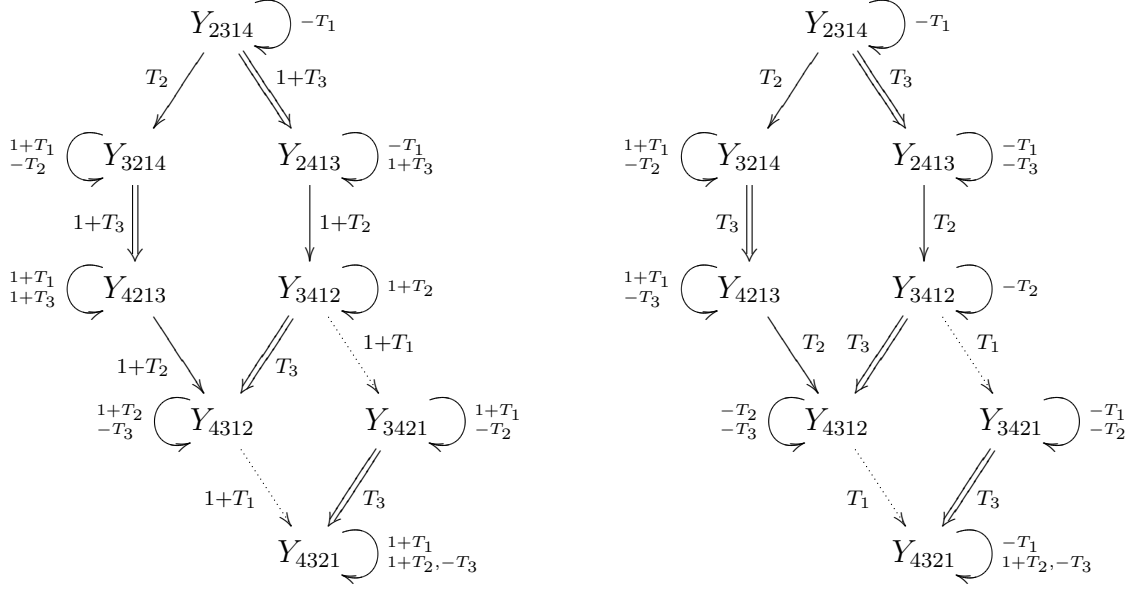


FIGURE 8. The graphs are the semi-combinatorial modules corresponding to the restriction of the Yang-Baxter graph associated with the color words 3124 and 4231 to the same interval  $[2314, 4321]$ . Both are the graph of the induced module  $P_{(2,1)} \hat{\otimes} P_{(1)}$

#### 4. THE 0-ARIKI-KOIKE-SHOJI ALGEBRAS

In [25], Shoji obtained a new presentation of the Ariki-Koike algebras defined in [2]. We shall first give a presentation very close to his, put  $q = 0$  in the relations and then prove some simple results about another basis of the resulting algebra. To get our presentation from Shoji's [25], one has to replace  $qa_i$  by  $T_{i-1}$  and  $q^2$  by  $q$ .

Let  $u_1, \dots, u_r$  be  $r$  distinct complex numbers. We shall denote by  $P_k(X)$  the Lagrange polynomial

$$(31) \quad P_k(X) := \prod_{1 \leq l \leq r, l \neq k} \frac{X - u_l}{u_k - u_l}.$$

The Ariki-Koike algebra  $\mathcal{H}_{n,r}(q)$  is the associative  $\mathbb{C}$ -algebra generated by elements  $T_1, \dots, T_{n-1}$  and  $\xi_1, \dots, \xi_n$  subject to the following relations:

$$\begin{aligned}
(32) \quad & (T_i - q)(T_i + 1) = 0 & (1 \leq i \leq n-1), \\
(33) \quad & T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & (1 \leq i \leq n-2), \\
(34) \quad & T_i T_j = T_j T_i & (|i - j| \geq 2), \\
(35) \quad & (\xi_j - u_1) \cdots (\xi_j - u_r) = 0 & (1 \leq j \leq n), \\
(36) \quad & \xi_i \xi_j = \xi_j \xi_i & (1 \leq i, j \leq n), \\
(37) \quad & T_i \xi_i = \xi_{i+1} T_i - (q-1) \sum_{c_1 < c_2} (u_{c_2} - u_{c_1}) P_{c_1}(\xi_i) P_{c_2}(\xi_{i+1}) & (1 \leq i \leq n-1), \\
(38) \quad & T_i(\xi_{i+1} + \xi_i) = (\xi_{i+1} + \xi_i) T_i & (1 \leq i \leq n-1), \\
(39) \quad & T_i \xi_j = \xi_j T_i & (j \neq i+1, i).
\end{aligned}$$

As noticed in [25], it is obvious from this presentation that a generating set is given by the  $\xi_1^{c_1} \cdots \xi_n^{c_n} \cdot T_\sigma$  with  $\sigma \in \mathfrak{S}_n$  and  $c_i$  such that  $0 \leq c_i \leq r-1$ . Shoji proves that this is indeed a basis of  $\mathcal{H}_{n,r}(q)$ . A simple adaptation of his proof would enable us to conclude that this property still holds at  $q = 0$ . We will prove it by introducing a new basis on which the product of generators has a simple expression (see Lemma 4.1.) This algebra  $\mathcal{H}_{n,r}(0)$ , which we call the 0-Ariki-Koike-Shoji algebra, will be our main concern in the sequel.

If  $\mathbf{c} = (c_1, \dots, c_n)$  is a word on  $C = \{1, \dots, r\}$ , we define

$$(40) \quad L_{\mathbf{c}} := P_{c_1}(\xi_1) \cdots P_{c_n}(\xi_n).$$

Since the Lagrange polynomials (31) associated with  $r$  distinct complex numbers form a basis of  $\mathbb{C}_{r-1}[X]$  (polynomials of degree at most  $r-1$ ), the set of elements of  $\mathcal{H}_{n,r}(q)$

$$(41) \quad B := \{B_{\mathbf{c},\sigma}(q) := L_{\mathbf{c}} T_\sigma\},$$

where  $\sigma$  runs over  $\mathfrak{S}_n$  and  $\mathbf{c} = (c_1, \dots, c_n)$  runs over the color words of size  $n$  span the same vector space as the  $\xi^{\mathbf{c}} T_\sigma$ .

Let us now describe the action of a generator on the left of  $B_{\mathbf{c},\sigma}$ : the generator  $\xi_i$  acts diagonally by multiplication by  $u_{c_i}$ , so that it only remains to explicit the product  $T_i L_{\mathbf{c}} T_\sigma$ . The relevant expression comes from the following apparently unnoticed relation in  $\mathcal{H}_{n,r}(q)$ :

**Lemma 4.1.** *The following relation holds in  $\mathcal{H}_{n,r}(q)$ :*

$$(42) \quad T_i L_{\mathbf{c}} = L_{\mathbf{c}s_i} T_i - (q-1) \begin{cases} -L_{\mathbf{c}} & \text{if } c_i < c_{i+1}, \\ 0 & \text{if } c_i = c_{i+1}, \\ L_{\mathbf{c}s_i} & \text{if } c_i > c_{i+1}. \end{cases}$$

where  $s_i$  acts on the right of  $\mathbf{c}$  by exchanging  $c_i$  and  $c_{i+1}$ .

*Proof.* It is enough to compute  $T_i P_a(\xi_i) P_b(\xi_{i+1})$  since all the other terms commute with  $T_i$ . Using Relations (36), (37), and (38), one proves

$$(43) \quad T_i \xi_i \xi_{i+1} = \xi_i \xi_{i+1} T_i,$$



so that  $T_i$  commutes with any symmetric function of  $\xi_i$  and  $\xi_{i+1}$ . The computation then reduces to evaluate the expression

$$(44) \quad \prod_{1 \leq l \leq r, l \neq a, l \neq b} \left( \frac{\xi_i - u_l}{u_a - u_l} \frac{\xi_{i+1} - u_l}{u_a - u_l} \right) T_i \frac{\xi_i - u_b}{u_a - u_b} \frac{\xi_{i+1} - u_a}{u_b - u_a}.$$

Now, writing  $(\xi_i - u_b)(\xi_{i+1} - u_a)$  as  $\xi_i \xi_{i+1} - u_b(\xi_i + \xi_{i+1}) + u_a u_b + (u_b - u_a)\xi_i$ , the first three terms commute with  $T_i$  and the last one, thanks to Equation (37), simplifies into a sum of two terms since all but one Lagrange polynomial vanish when multiplied with the left factor of Equation (44), the terms depending on the order relation between  $a$  and  $b$ .  $\blacksquare$

The last lemma implies that the set  $B$  defined in Equation (41) is a generating set of  $\mathcal{H}_{n,r}(q)$ . We now show that it is a basis of  $\mathcal{H}_{n,r}(q)$  for all values of  $q$  and in particular for  $q = 0$ . One can argue as in [25]. Let  $V = \bigoplus_{i=1}^r V_i$ , with  $V_i = \mathbb{C}^{m_i}$ ,  $m_i \geq n$ , and let  $W = \{v_1, \dots, v_m\}$  with  $m = m_1 + \dots + m_r$ , be a basis of  $V$  such that  $v_1, \dots, v_{m_1} \in V_1$ ,  $v_{m_1+1}, \dots, v_{m_1+m_2} \in V_2$ , and so on. Let  $w = v_{k_1} \otimes \dots \otimes v_{k_n} \in V^{\otimes n}$ . There is a classical right action of  $H_n(q)$  on  $V^{\otimes n}$  given by

$$(45) \quad \begin{cases} w \cdot T_i = & ws_i & \text{if } k_i < k_{i+1}, \\ w \cdot T_i = & qw & \text{if } k_i = k_{i+1}, \\ w \cdot T_i = & qws_i + (q-1)w & \text{if } k_i > k_{i+1}. \end{cases}$$

Let  $\xi$  be the linear map whose restriction to  $V_i$  is  $u_i \cdot id_{v_i}$ , and  $\xi_j = id^{\otimes(j-1)} \otimes \xi \otimes id^{\otimes(n-j)}$  acting on the right of  $V^{\otimes n}$ . Then, as observed by Shoji, the  $T_i$  and  $\xi_j$  generate a right action of  $\mathcal{H}_{n,r}(q)$  on  $V^{\otimes n}$ . It is known that the representation (45) of  $H_n(q)$  is faithful for all  $q$ , provided that  $m_i \geq n$ . This easily implies the next proposition:

**Proposition 4.2.** *The set  $B$  defined in Equation (41) is a basis of  $\mathcal{H}_{n,r}(q)$ , for all values of  $q$ .*

With exactly the same arguments, one would also prove:

**Proposition 4.3.** *The set  $B' = \{B'_{c,\sigma} := T_\sigma L_c\}$  is a basis of  $\mathcal{H}_{n,r}(q)$ , for all values of  $q$ .*

We now provide a new presentation of  $\mathcal{H}_{n,r}(0)$ . Since  $\mathcal{H}_{n,r}(0)$  satisfies the relations of the next proposition, a simple comparison of dimensions proves that:

**Proposition 4.4.** *The algebra with generators  $L_{\mathbf{c}}$ ,  $\mathbf{c}$  being any color word of size  $r$ , and  $T_i$  with  $1 \leq i \leq n-1$ , and relations*

$$(46) \quad T_i(1 + T_i) = 0 \quad (1 \leq i \leq n-1),$$

$$(47) \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i \leq n-2),$$

$$(48) \quad T_i T_j = T_j T_i \quad (|i - j| \geq 2),$$

$$(49) \quad L_{\mathbf{d}} L_{\mathbf{c}} = \delta_{\mathbf{d}, \mathbf{c}} L_{\mathbf{c}},$$

$$(50) \quad \begin{cases} \text{if } c_i < c_{i+1}, & (1 + T_i) L_{\mathbf{c}} = L_{\mathbf{c} s_i} T_i, \\ \text{if } c_i = c_{i+1}, & T_i L_{\mathbf{c}} = L_{\mathbf{c} s_i} T_i, \\ \text{if } c_i > c_{i+1}, & T_i L_{\mathbf{c}} = L_{\mathbf{c} s_i} (1 + T_i). \end{cases}$$

is isomorphic to  $\mathcal{H}_{n,r}(0)$ .

This description of the Ariki-Koike-Shoji algebra enables us to analyze the left regular representation of  $\mathcal{H}_{n,r}(0)$  in terms of our basis elements. As a first application, we shall obtain a classification of the simple  $\mathcal{H}_{n,r}(0)$ -modules.

**4.1. The associative bilinear form.** Recall that a bilinear form  $(\cdot, \cdot)$  on an algebra  $A$  is said to be associative if  $(ab, c) = (a, bc)$  for all  $a, b, c \in A$ , and that  $A$  is called a *Frobenius algebra* whenever it has a nondegenerate associative bilinear form. Such a form induces an isomorphism of left  $A$ -modules between  $A$  and the dual  $A^*$  of the right regular representation. Frobenius algebras are in particular self-injective, so that finitely generated projective and injective modules coincide (see [7]).

It is known that  $H_n(0)$  is a Frobenius algebra for the following bilinear form: for a basis  $(Y_\sigma)$ , we denote by  $(Y_\sigma^*)$  the dual basis. We set  $\chi = T_{\omega_n}^*$ . Then  $(f, g) = \chi(fg)$  is a non-degenerate associative bilinear form [8]. Moreover if we denote  $\eta_i = -T_i$  and  $\xi_i = 1 + T_i$  (both satisfy the braid relations) then the following properties hold:

$$(51) \quad \zeta_\sigma = (-1)^{\ell(\omega_\sigma^{-1})} \xi_{\omega_\sigma^{-1}}, \quad \xi_\alpha = \sum_{\beta \leq \alpha} T_\beta \quad \text{and} \quad \eta_\alpha = \sum_{\beta \leq \alpha} (-1)^{\ell(\beta)} \xi_\beta$$

shows that  $(\zeta_\sigma)$  and  $(\eta_\tau)$  are two adjoint bases of  $H_n(0)$ , that is

$$(52) \quad (\zeta_\sigma, \eta_\tau) = \delta_{\sigma, \tau}.$$

This bilinear form can be extended to  $\mathcal{H}_{n,r}(0)$  by setting

$$(53) \quad X := \sum_{\mathbf{c}} B_{\mathbf{c}, \omega}^*$$

where  $B$  is the basis defined by Equation (41).

**Proposition 4.5.** *The associative bilinear form defined by*

$$(54) \quad (f, g) := X(fg)$$

is non-degenerate on  $\mathcal{H}_{n,r}(0)$ . Therefore,  $\mathcal{H}_{n,r}(0)$  is a Frobenius algebra.

*Proof.* The form is obviously bilinear and associative. It remains to prove that it is non-degenerate. We have

$$(55) \quad L_{\mathbf{c}} T_\sigma = T_\sigma L_{\mathbf{c} \sigma^{-1}} + \text{smaller terms},$$

so that

$$(56) \quad (L_{\mathbf{c}}T_{\sigma}, T_{\tau}L_{\mathbf{d}}) = X(L_{\mathbf{c}}T_{\sigma}T_{\tau}L_{\mathbf{d}}) = \delta_{\mathbf{cd}} X(T_{\sigma}T_{\tau}).$$

This formula being linear on  $T_{\sigma}$  and  $T_{\tau}$ , we have as well,

$$(57) \quad X(L_{\mathbf{c}}\xi_{\alpha}\eta_{\beta}L_{\bar{\mathbf{d}}}) = \delta_{\mathbf{cd}}\delta_{\alpha\beta},$$

so that  $B'_{\mathbf{c},\alpha} := L_{\mathbf{c}}\xi_{\alpha}$  and  $B''_{\mathbf{d},\beta} := \eta_{\beta}L_{\bar{\mathbf{d}}}$  are two adjoint bases. ■

In particular, we have

**Corollary 4.6.**  $\mathcal{H}_{n,r}(0)$  is self-injective. ■

5. SIMPLE MODULES OF  $\mathcal{H}_{n,r}(0)$ 

Let  $[I, \mathbf{c}]$  be a cycloribbon. Let  $\eta_{[I, \mathbf{c}]}$  be the element of  $\mathcal{H}_{n,r}(0)$  defined as

$$(58) \quad \eta_{[I, \mathbf{c}]} := L_{\mathbf{c}} \eta_I,$$

where  $\eta_I$  is the generator of the simple  $H_n(0)$ -module associated with  $I$  as in Section 3.1. This element generates a simple  $\mathcal{H}_{n,r}(0)$ -module:

**Theorem 5.1.** *Let  $[I, \mathbf{c}]$  be a cycloribbon. Then*

$$(59) \quad S_{[I, \mathbf{c}]} := \mathcal{H}_{n,r}(0) \eta_{[I, \mathbf{c}]}$$

*is a simple module of  $\mathcal{H}_{n,r}(0)$ , realized as a minimal left ideal in its left regular representation. The eigenvalue of  $L_{\mathbf{d}}$  is 1 if  $\mathbf{d} = \mathbf{c}$  and 0 otherwise, and that of  $T_i$  is  $-1$  or 0 according to whether  $i$  is a descent of the shape of  $\phi(R)$  or not. All these simple modules are pairwise non isomorphic and of dimension 1. Moreover, all one-dimensional  $\mathcal{H}_{n,r}(0)$ -modules are isomorphic to some  $S_{[I, \mathbf{c}]}$ .*

*Proof.* We prove simultaneously that the  $S_{[I, \mathbf{c}]}$  are one-dimensional  $\mathcal{H}_{n,r}(0)$ -modules and that all one-dimensional modules are isomorphic to some  $S_{[I, \mathbf{c}]}$ . Let us first consider a simple module of dimension 1. Since  $\mathcal{H}_{n,r}(0)$  is self-injective (Corollary 4.6), it can be realized as a minimal left ideal in the left regular representation, that is, as  $\mathcal{H}_{n,r}(0)S$  with  $S = \sum_{\mathbf{c}, \sigma} C_{\mathbf{c}, \sigma} B_{\mathbf{c}, \sigma}$  with  $C_{\mathbf{c}, \sigma} \in \mathbb{C}$ . Since all  $B_{\mathbf{c}, \sigma}$  are eigenvectors of  $L_{\mathbf{d}}$  with eigenvalue 1 or 0 depending on whether  $\mathbf{d} = \mathbf{c}$  or not,  $S$  is an eigenvector for all  $L_{\mathbf{d}}$  iff the only non-zero coefficients correspond to the same  $\mathbf{c}$ . So  $S = L_{\mathbf{c}} S'$  with  $S' = \sum_{\sigma} C_{\sigma} T_{\sigma}$ . Since  $S$  is also an eigenvector for all  $T_i$ , we are left with the following possibilities:

- If  $c_i < c_{i+1}$ , then thanks to Formula (50), first case,  $S$  is an eigenvector of  $T_i$  iff  $T_i$  acts by 0 on  $S'$ .
- If  $c_i = c_{i+1}$ , then thanks to Formula (50), second case,  $S$  is an eigenvector of  $T_i$  iff  $T_i$  acts by 0 or  $-1$  on  $S'$ .
- If  $c_i > c_{i+1}$ , then thanks to Formula (50), third case,  $S$  is an eigenvector of  $T_i$  iff  $T_i$  acts by  $-1$  on  $S'$ .

So, in any case,  $S'$  is an eigenvector for all  $T_i$ , so it is equal to some  $\eta_I$ . Moreover,  $I$  has to be compatible with  $\mathbf{c}$  in the following sense: if  $c_i < c_{i+1}$  then  $i$  cannot be a descent of  $I$  and if  $c_i > c_{i+1}$  then  $i$  has to be a descent of  $I$ . This is precisely the definition of a cycloribbon. All those simple modules are non-isomorphic since the eigenvalues characterize them completely. ■

We will prove in Section 5.1 that all simple modules are of dimension one, thus isomorphic to some  $S_{[I, \mathbf{c}]}$  by following the same pattern as in [5], proving that all composition factors of the modules induced from simple modules of the 0-Hecke algebra are one-dimensional.

Notice that if  $[I, \mathbf{c}]$  is a cycloribbon, then  $\eta_{[I, \mathbf{c}]}$  is an eigenvector for all  $T_i$ , so that it can be written (since  $B'$  is a basis of  $\mathcal{H}_{n,r}(0)$ , see Proposition 4.3) as  $\eta_{I'} \sum_{\mathbf{c}'} \alpha_{\mathbf{c}'} L_{\mathbf{c}'}$ , where  $I'$  is the shape of the anticycloribbon  $\phi([I, \mathbf{c}])$  and  $\alpha_{\mathbf{c}'}$  are unknown coefficients. The result is actually simpler:

**Proposition 5.2.** *Let  $[I, \mathbf{c}]$  be a cycloribbon. Then,*

$$(60) \quad \eta_{[I, \mathbf{c}]} = \eta_{I'} L_{\bar{\mathbf{c}}},$$

where  $I'$  is the shape of the anticycloribbon  $\phi([I, \mathbf{c}])$ .

*Proof.* Let  $n := |I|$  and let  $\rho'$  be a color word involving  $n$  distinct colors such that

$$(61) \quad \begin{cases} \rho'_i < \rho'_j & \text{if } \bar{\mathbf{c}}_i < \bar{\mathbf{c}}_j, \\ \rho'_i > \rho'_j & \text{if } \bar{\mathbf{c}}_i > \bar{\mathbf{c}}_j. \end{cases}$$

Let us define another color word  $\rho$  as the permutation such that

$$(62) \quad \rho_i > \rho_j \text{ iff } \begin{cases} \rho'_i > \rho'_j & \text{and } \mathbf{c}_i = \mathbf{c}_j, \text{ or} \\ \rho'_i < \rho'_j & \text{and } \mathbf{c}_i \neq \mathbf{c}_j. \end{cases}$$

Thanks to Relation (50), one proves by induction that

$$(63) \quad Y_\sigma(\rho') L_{\bar{\mathbf{c}}} = L_{\bar{\mathbf{c}} \cdot \sigma^{-1}} Y_\sigma(\rho).$$

Thanks to Lemmas 3.5 and 3.4, one then gets

$$(64) \quad Y_{\omega_n}(\rho') = \eta_{\overline{C(\rho)}}.$$

Now, it is easy to build a permutation  $\rho'$  satisfying these properties, with descent composition equal to  $\bar{I}'$  iff  $I'$  is a composition such that  $[I', \mathbf{c}]$  is an anticycloribbon (the descent composition of  $\rho'$  is, by definition of  $\rho'$ , already included in this set of compositions). It then comes that

$$(65) \quad \eta_{I'} L_{\bar{\mathbf{c}}} = L_{\bar{\mathbf{c}} \cdot \omega_n} \eta_{\overline{C(\rho)}} = L_{\mathbf{c}} \eta_{\overline{C(\rho)}}.$$

By definition of  $\rho$ ,  $[\overline{C(\rho)}, \mathbf{c}]$  is the cycloribbon  $\phi([I', \mathbf{c}])$ . ■

This result was not unexpected: it is clear that the generator of  $S_{[I, \mathbf{c}]}$  has an expression as  $\eta_{I'} L_{\mathbf{c}'}$ , and from Lemma 3.4 and Equation (50), one can see that  $\mathbf{c}'$  has to be  $\bar{\mathbf{c}} = \mathbf{c} \omega_n$ .

**5.1. Induction of the simple 0-Hecke modules.** To describe the induction process, we need a partial order on colored ribbons. Let  $I$  be a composition and  $\mathbf{c} = (c_1, \dots, c_n)$ . The covering relation of the order  $\leq_I$  amounts to sort in increasing order any two adjacent elements in the rows of  $I$  or to sort in decreasing order any two adjacent elements in the columns of  $I$ . For example, the elements smaller than or equal to

$$(66) \quad T := \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 3 & 3 \\ \hline & 4 \\ \hline & 3 \\ \hline \end{array}$$

are

$$(67) \quad \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 3 & 3 \\ \hline & 4 \\ \hline & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 3 & 3 \\ \hline & 4 \\ \hline & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 3 & 4 \\ \hline & 3 \\ \hline & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 3 & 4 \\ \hline & 3 \\ \hline & 3 \\ \hline \end{array}$$

If  $I$  is a composition of  $n$ , let  $S_I := H_n(0)\eta_I$  be the corresponding simple module of  $H_n(0)$  realized as a minimal left ideal in its left regular representation (see [12]) and

$$(68) \quad M_I := S_I \uparrow_{H_n(0)}^{\mathcal{H}_{n,r}(0)}.$$

Clearly,  $M_I$  has dimension  $r^n$  and admits  $L_{\mathbf{c}}\eta_I$  as linear basis, when  $\mathbf{c}$  runs over color words. For  $\mathbf{c} \in C^n$ , let  $M_{\mathbf{c},I}$  be the  $\mathcal{H}_{n,r}(0)$ -submodule of  $M_I$  generated by  $L_{\mathbf{c}}\eta_I$ .

**Lemma 5.3.**

$$(69) \quad M_{\mathbf{c},I} \subseteq M_{\mathbf{c}',I} \iff \mathbf{c} \leq_I \mathbf{c}'.$$

*Proof.* Let  $i \in \{1, \dots, n-1\}$ .

- If  $c_i < c_{i+1}$ , and  $T_i$  acts by 0 on  $\eta_I$ , we get  $(1 + T_i)L_{\mathbf{c}}\eta_I = L_{\mathbf{c}\sigma_i}T_i\eta_I = 0$ .
- If  $c_i < c_{i+1}$ , and  $T_i$  acts by  $-1$  on  $\eta_I$ , we get  $(1 + T_i)L_{\mathbf{c}}\eta_I = L_{\mathbf{c}\sigma_i}T_i\eta_I = -L_{\mathbf{c}\sigma_i}\eta_I$ , and so  $-(1 + T_i)$  sorts in decreasing order  $c_i$  and  $c_{i+1}$ .
- If  $c_i = c_{i+1}$ , then  $T_iL_{\mathbf{c}}\eta_I = L_{\mathbf{c}}T_i\eta_I$  so the result is either 0 or  $-1$  times  $L_{\mathbf{c}}\eta_I$ .
- If  $c_i > c_{i+1}$ , and  $T_i$  acts by  $-1$  on  $\eta_I$ , we get  $T_iL_{\mathbf{c}}\eta_I = L_{\mathbf{c}\sigma_i}(1 + T_i)\eta_I = 0$ .
- If  $c_i > c_{i+1}$ , and  $T_i$  acts by 0 on  $\eta_I$ , we get  $T_iL_{\mathbf{c}}\eta_I = L_{\mathbf{c}\sigma_i}(1 + T_i)\eta_I = L_{\mathbf{c}\sigma_i}\eta_I$ , and so  $T_i$  sorts in increasing order  $c_i$  and  $c_{i+1}$ .

■

We can now complete Theorem 5.1 by proving that we have a complete set of simple  $\mathcal{H}_{n,r}(0)$ -modules:

**Theorem 5.4.** *The  $S_{[I,\mathbf{c}]}$  form a complete family of simple modules of  $\mathcal{H}_{n,r}(0)$ .*

*Proof.* It is sufficient to prove that any simple  $\mathcal{H}_{n,r}(0)$ -module  $M$  appears as a composition factor of some  $M_I$  since by the previous lemma,  $M_I$  admits a composition series involving only one-dimensional modules.

Now, if  $M$  is any simple  $\mathcal{H}_{n,r}(0)$ -module, let  $S_I$  be a simple  $H_n(0)$ -module occurring in the socle of  $M \downarrow H_n(0)$ . This means that we have a monomorphism  $i : S_i \hookrightarrow M$  of  $H_n(0)$ -modules. Inducing to  $\mathcal{H}_{n,r}(0)$ , we get a non-zero  $\mathcal{H}_{n,r}(0)$ -morphism  $f : M_I \rightarrow M$ , which has to be surjective since  $M$  is simple. So  $M$  is a composition factor of some  $M_I$  and all simple  $\mathcal{H}_{n,r}(0)$ -modules are one-dimensional. ■

**5.2. First Grothendieck ring.** The induction product of two simple modules of  $\mathcal{H}_{n,r}(0)$  can be worked out in a way very similar to the case of  $H_n(0)$ -modules such as recalled in Section 3.1. As in Proposition 5.2, we represent these modules by their anticycloribbons. In the context of  $\mathcal{H}_{n,r}(0)$ , one has to extend to colored permutations what was done with permutations in  $H_n(0)$ .

Let us consider two anticycloribbons  $[I, \mathbf{c}]$  and  $[I', \mathbf{c}']$ . The structure of the induction product  $S_{[I,\mathbf{c}]} \hat{\otimes} S_{[I',\mathbf{c}]}$  of the corresponding simple modules is completely encoded in the graph of the convolution of the associated colored permutations: as in  $H_n(0)$ , there exists a basis  $(b)$  of  $S_{[I,\mathbf{c}]} \hat{\otimes} S_{[I',\mathbf{c}]}$  naturally labelled by the elements occurring in the convolution, such that, either  $T_i$ ,  $-T_i$ , or  $1 + T_i$  sends a given basis element  $b$  to another one, and  $L_{\mathbf{c}}b$  is  $b$  or 0. The unique one-dimensional quotient of the module

generated by an element  $b$  is the one labelled by the anticycloribbon associated with its colored permutation. Let us describe more precisely this graph.

**Algorithm 5.5.**

Input: Two anticycloribbons  $[I_1, \mathbf{c}_1]$  and  $[I_2, \mathbf{c}_2]$ .

Output: A graph  $G$ .

Let  $(\sigma_1, \mathbf{c}_1)$  and  $(\sigma_2, \mathbf{c}_2)$  be the inverse of the corresponding maximal colored permutations and let  $\mathbf{c} = \mathbf{c}_1 \mathbf{c}_2$ .

Let  $G$  be the graph with vertices  $V_w$  labelled by the elements of the convolution  $\sigma_1 * \sigma_2$ . The edges of  $G$  are labelled by the generators of our presentation of  $\mathcal{H}_{n,r}(0)$  (see Proposition 4.4), that is, all Lagrange polynomials  $L_{\mathbf{d}}$  and, for all  $1 \leq i \leq n-1$ , the Hecke generator  $T_i$ ,  $-T_i$  or  $1 + T_i$ .

The edge labelled by  $L_{\mathbf{d}}$  on the vertex  $V_{\tau}$  is a loop on  $V_{\tau}$  iff  $\mathbf{d} = \mathbf{c} \cdot \tau^{-1}$ . Otherwise,  $V_{\tau}$  is annihilated and the edge is not represented. The edges labelled by the Hecke generators depend on the following cases:

- if the letters  $\tau_i$  and  $\tau_{i+1}$  come from the same factor of the convolution, there is a loop labelled by  $-T_i$  (resp.  $1 + T_i$ ) if  $i$  is (resp. not) an anti-descent of  $(\tau^{-1}, \mathbf{c} \cdot \tau^{-1})$ .
- if  $\tau_i$  comes from  $\sigma_1$  and  $\tau_{i+1}$  comes from  $\sigma_2$ , there is a vertex from  $V_{\tau}$  to  $V_{s_i \tau}$  labelled by  $1 + T_i$  (resp.  $T_i$ ) if  $i$  is (resp. not) an anti-descent of  $(\tau, \mathbf{c})^{-1}$ .
- if  $\tau_i$  comes from  $\sigma_2$  and  $\tau_{i+1}$  comes from  $\sigma_1$ , there is a loop labelled by  $-T_i$  (resp.  $1 + T_i$ ) if  $i$  is (resp. not) an anti-descent of  $(\tau, \mathbf{c})^{-1}$ .

For example, the second graph of Figure 9 is the graph associated with the anticycloribbons  $[(1, 1), 23]$  and  $[(1, 1), 13]$ .

**Theorem 5.6.** Let  $[I_1, \mathbf{c}_1]$  and  $[I_2, \mathbf{c}_2]$  be two anticycloribbons. Let  $\mathbb{C}V$  the vector space with basis  $V$  indexed by the vertices of the graph of Algorithm 5.5. Then the relations  $\rho V_i = V_j$  whenever  $V_i \xrightarrow{\rho} V_j$  is a presentation of the induced module  $S_{[I_1, \mathbf{c}_1]} \widehat{\otimes} S_{[I_2, \mathbf{c}_2]}$ . Its composition factors are given by the shapes of the anticycloribbons corresponding to the colored permutations  $(\tau, \mathbf{c})^{-1}$ , where  $\tau$  runs over the labellings of the vertices  $V$ .

For example, Figure 9 presents the induction product of the two simple modules  $S_{[(1,1),23]}$  and  $S_{[(1,1),13]}$  (written as anticycloribbons) of  $\mathcal{H}_{2,2}(0)$  to  $\mathcal{H}_{4,2}(0)$ . In particular, we get the following composition factors (written as anticycloribbons):

$$(70) \quad \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 3 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 3 & 2 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 3 & 3 & 2 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 3 \\ \hline 3 & 1 \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & 3 & 1 \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 3 & 2 \\ \hline \end{array}$$

Note that this construction gives an effective algorithm to compute the composition factors of the induction product of two simple modules. One recovers the same shapes as in Figure 4, that is, the induction of the simple modules (2) and (2) of  $H_2(0)$  in  $H_4(0)$ . Moreover, one recovers these shapes in the same order as in Figure 7. This property comes from Proposition 5.2, since any basis element  $V_{(\sigma, \mathbf{c})}$  can be written either in the form  $L_{\mathbf{d}} T$  or  $T' L_{\mathbf{d}'}$ , where  $T$  and  $T'$  are in  $H_n(0)$ .

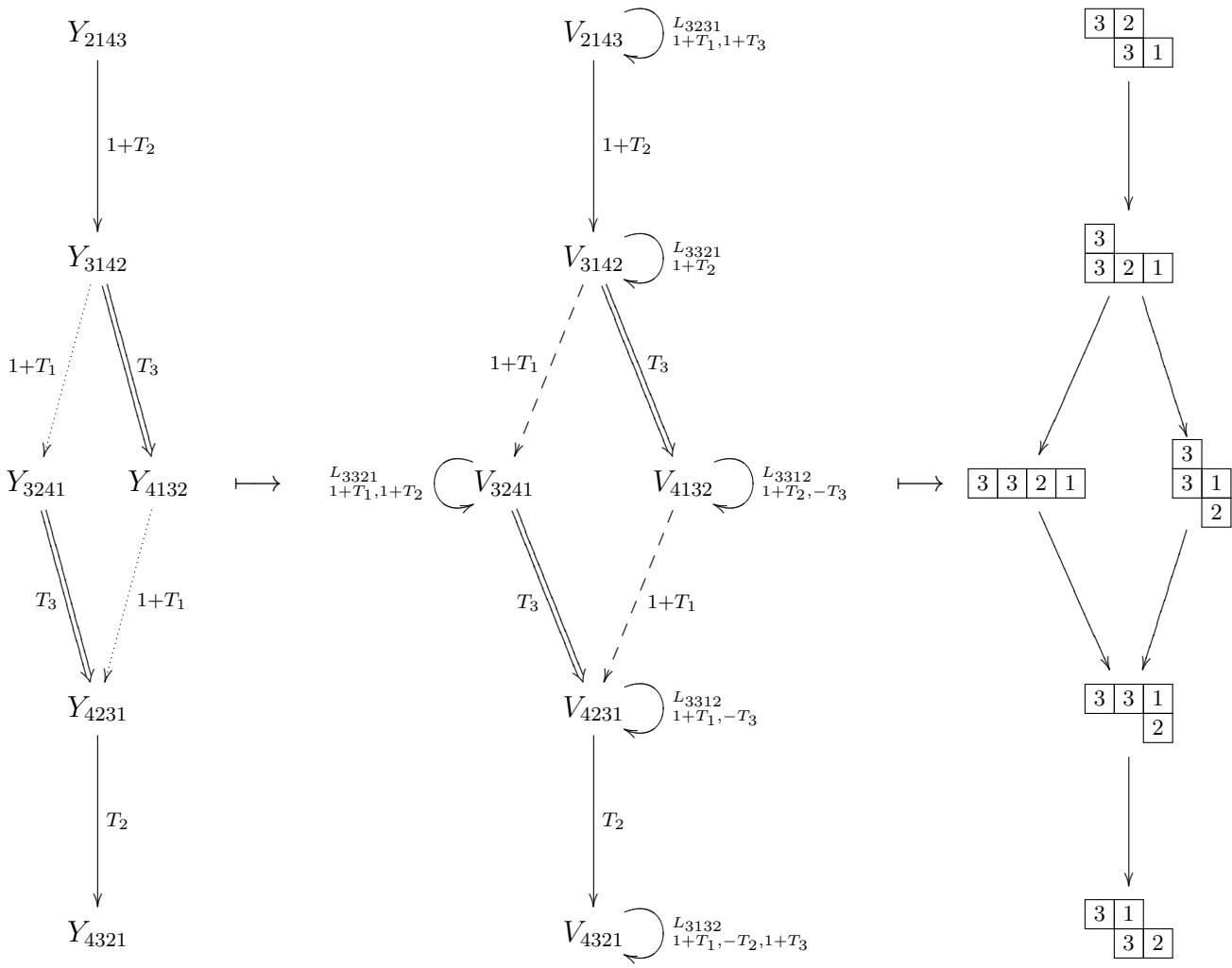


FIGURE 9. Induction product of the two simple modules  $S'_{[(1,1),(23)]}$  and  $S'_{[(1,1),(13)]}$  of  $\mathcal{H}_{2,2}(0)$  to  $\mathcal{H}_{4,2}(0)$ . The first graph is the restriction of the left weak order to the convolution  $21*21$ . In the second graph, we show the graph built by Algorithm 5.5, and the third graph represents the composition factors corresponding to each basis element.



*Proof.* Let us consider two simple modules  $M = S_{[I_1, \mathbf{c}_1]}$ ,  $N = S_{[I_2, \mathbf{c}_2]}$  of  $\mathcal{H}_{m,r}(0)$  and  $\mathcal{H}_{n,r}(0)$  and let  $\sigma_1$  and  $\sigma_2$  be the maximal colored permutations associated with their anticycloribbons. Let  $\mathbf{c} = \mathbf{c}_1 \mathbf{c}_2$  be the concatenation of the color words.

The induced module  $M \hat{\otimes} N$  of  $\mathcal{H}_{m+n,r}(0)$  is by definition

$$(71) \quad \mathcal{H}_{m+n,r}(0) \bigotimes_{\mathcal{H}_{m,r}(0) \otimes \mathcal{H}_{n,r}(0)} (M \otimes_{\mathbb{C}} N).$$

Since every  $L_{\mathbf{c}}$  belongs to  $\mathcal{H}_{m,r}(0) \otimes \mathcal{H}_{n,r}(0)$ , the induced module is isomorphic to  $H_{m+n}(0) \otimes_{H_m(0) \otimes H_n(0)} (M \otimes N)$  as a vector space and even as a  $H_{m+n}(0)$ -module. In particular, its dimension is  $\binom{m+n}{n}$ .

Now, as a  $H_{m+n}(0)$ -module, thanks to Lemma 3.1,  $M \hat{\otimes} N$  is described by the restriction  $G_1$  of the graph of  $\mathfrak{S}_{m+n}$  to  $\sigma_1 * \sigma_2$ , with the Yang-Baxter elements determined by the color word  $\mathbf{c}$ . Let  $G$  be the graph associated with  $M$  and  $N$  by Algorithm 5.5. The edges of  $G$  and  $G_1$  are the same since both are edges in  $\sigma_1 * \sigma_2$  with the same constraints. The loops of  $G$  and  $G_1$  labelled by the Hecke generators are the same thanks to Algorithm 5.5 and Corollary 3.9. So  $G$  satisfies the relations of the 0-Hecke algebra, relations (46), (47), and (48). It trivially satisfies Relations (49) and it satisfies Relations (50), since each vertex of the graph can be written as some Lagrange polynomial  $L_{\mathbf{c}}$  multiplied by an element of  $H_n(0)$ . So it satisfies all the relations of our presentation of the Ariki-Koike-Shoji algebra at  $q = 0$  (Proposition 4.4). Since it has the correct dimension and the correct generator, this graph encodes the structure of the induced module  $M \hat{\otimes} N$ .

Finally, the composition factors are the anticycloribbons associated with the colored permutations  $(\tau, \mathbf{c})^{-1}$ : the only non-zero  $L$  is the right one and the action of  $T_i$  is the one encoded by the shape of the anticycloribbon, thanks to Note 3.3. ■

### 5.3. Restriction of simple modules.

**Proposition 5.7.** *Let  $[I, \mathbf{c}]$  be a cycloribbon, regarded as a filling of the ribbon diagram of  $I$ . Let  $I'$  and  $I''$  be the compositions whose diagrams are formed respectively of the first  $k$  cells and the last  $n - k$  cells of  $I$  and  $\mathbf{c}' = (c_1, \dots, c_k)$ ,  $\mathbf{c}'' = (c_{k+1}, \dots, c_n)$ . Then  $[I', \mathbf{c}']$  and  $[I'', \mathbf{c}']$  are again cycloribbons and the restriction of  $S_{[I, \mathbf{c}]}$  to  $\mathcal{H}_{k,r}(0) \otimes \mathcal{H}_{n-k,r}(0)$  is  $S_{[I', \mathbf{c}']} \otimes S_{[I'', \mathbf{c}]}$ .*

For example, Figure 10 shows that the restriction of the first module  $S_{[31122, 122211323]}$  to  $\mathcal{H}_{5,r}(0) \otimes \mathcal{H}_{4,r}(0)$  is  $S_{[311, 12221]} \otimes S_{[22, 1323]}$ .

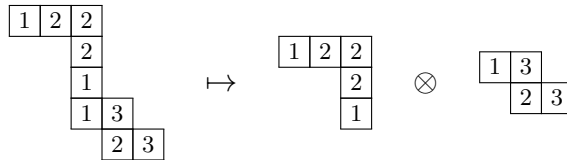


FIGURE 10. The restriction of the simple module represented on the left by a cycloribbon to  $\mathcal{H}_{5,r}(0) \otimes \mathcal{H}_{4,r}(0)$  gives the simple module on the right also represented by cycloribbons.

*Proof.* This follows from the obvious graphical description of the restriction of simple  $H_n(0)$ -modules, which consists in cutting the ribbon diagram, as was done before on cycloribbons, and from the commutation relations between  $L$  and  $T$ . ■

**5.4. The quasi-symmetric Mantaci-Reutenauer algebra.** A *colored composition* is a pair  $(I, u) = ((i_1, \dots, i_m), (u_1, \dots, u_m))$  formed of a composition  $I$  and a color word of the same length.

There is a bijection between colored compositions and anticycloribbons: starting with a colored composition, one rebuilds an anticycloribbon by separating adjacent blocks of the colored composition with different colors and gluing them back in the only possible way according to the criterion of being an anticycloribbon. Conversely, one separates two adjacent blocks of different colors and glue them one below the other. For example, Figure 11 shows a colored composition represented as a filling of a ribbon ( $u_i$  is written in row  $i$ ) and its corresponding anticycloribbon.



FIGURE 11. A colored composition and the corresponding anticycloribbon.

Let  $X^{(i)} = \{x_k^{(i)} | k \geq 1\}$  for  $1 \leq i \leq r$  be  $r$  infinite linearly ordered sets of commuting indeterminates. The *monochromatic monomial quasi-symmetric functions* labelled by a colored composition  $(I, u)$  is

$$(72) \quad M_{(I,u)}(X) = \sum_{j_1 < \dots < j_m} (x_{j_1}^{(u_1)})^{i_1} \dots (x_{j_m}^{(u_m)})^{i_m}.$$

It is known ([20] Proposition 5.2), that the  $M_{I,u}$  span a subalgebra  $QMR^{(r)}$  of  $\mathbb{C}[X^{(1)}, \dots, X^{(r)}]$ . This algebra is also a subalgebra of the algebra of level  $r$  quasi-symmetric functions defined in [22] (see also [20]).

We will prove that the Grothendieck ring of the tower of  $\mathcal{H}_{n,r}(0)$  algebras is isomorphic to  $QMR^{(r)}$ .

Define an order on colored compositions as follows: let

$$(I, u) = ((i_1, \dots, i_m), (u_1, \dots, u_m)) \quad \text{and} \quad (J, v) = ((j_1, \dots, j_p), (v_1, \dots, v_p))$$

two colored compositions. Then  $(I, u)$  is *finer* than  $(J, v)$  if there exists a sequence  $(l_0 = 0, l_1, \dots, l_p = m)$  such that for any integer  $k$ ,

$$(73) \quad j_k = i_{l_{k-1}+1} + \dots + i_{l_k} \quad \text{and} \quad v_k = u_{l_{k-1}+1} = \dots = u_{l_k}.$$

For example, the compositions fatter than  $((1, 1, 3, 2), (2, 1, 1, 2))$  are

$$(74) \quad ((1, 1, 3, 2), (2, 1, 1, 2)) \quad \text{and} \quad ((1, 4, 2), (2, 1, 2)).$$

This allows us to define the *monochromatic quasi-ribbon functions*  $F_{(I,u)}$  by

$$(75) \quad F_{(I,u)} = \sum_{(I',u') \leq (I,u)} M_{(I',u')}.$$

Notice that this last description of the order  $\leq$  is reminiscent of the order  $\leq'$  on descent sets used in the context of quasi-symmetric functions and non-commutative symmetric functions: more precisely, one gets the usual order when considering compositions colored with only one color.

**Lemma 5.8** ([4, 20]). *The  $F_{(I,u)}$ , where  $(I,u)$  runs over colored compositions, span  $QMR^{(r)}$ . This algebra is isomorphic to the dual of the Mantaci-Reutenauer algebra  $\mathbf{MR}^{(r)}$  defined in [18].* ■

Let

$$(76) \quad \mathcal{G}^{(r)} := \bigoplus_{n \geq 0} G_0(\mathcal{H}_{n,r}(0))$$

be the Grothendieck ring of the tower of  $\mathcal{H}_{n,r}(0)$  algebras. Define a characteristic map

$$(77) \quad \begin{aligned} ch : \quad \mathcal{G}^{(r)} &\rightarrow QMR^{(r)} \\ [S_{[I,\mathbf{c}]}] &\mapsto F_{[I,\mathbf{c}]} \end{aligned}$$

Comparing the descriptions of  $S_{[I',\mathbf{c}']} \hat{\otimes} S_{[I'',\mathbf{c}'']}$  and of  $S_{[I,\mathbf{c}]} \downarrow$  obtained in Sections 5.2 and 5.3 with the formulas for product and coproduct of the  $F$  basis in  $QMR^{(r)}$ , we obtain:

**Theorem 5.9.**  *$ch$  is an isomorphism of Hopf algebras.* ■

## 6. PROJECTIVE MODULES

**6.1. The Grothendieck ring.** The previous results imply, by duality, a description of the Grothendieck ring of the category of projective  $\mathcal{H}_{n,r}(0)$ -modules. Indeed, we already know that  $\mathcal{H}_{n,r}(0)$  is self injective. Moreover, it is easy to see that  $\mathcal{H}_{n,r}(0)$  is a free module and hence projective over any parabolic subalgebra

$$(78) \quad \mathcal{H}_{n_1,r}(0) \otimes \cdots \otimes \mathcal{H}_{n_k,r}(0).$$

These conditions are sufficient to ensure that, as a Hopf algebra,

$$(79) \quad \mathcal{K} = \bigoplus_{n \geq 0} K_0(\mathcal{H}_{n,r}(0)) \simeq \mathbf{MR}^{(r)}$$

is the graded dual of  $\mathcal{G}$ , and thus is isomorphic to the Mantaci-Reutenauer algebra. Under this isomorphism, the classes of indecomposable projective modules are mapped to the subfamily of the dual basis  $F_{[I,c]}$  of Poirier quasi-symmetric functions labelled by colored descent sets, or cycloribbons.

Recall that the Hopf algebra  $\mathbf{MR}^{(r)}$  can be defined (see [20]) as the free associative algebra over symbols  $(S_j^{(i)})_{j \geq 1; 1 \leq i \leq r}$ , graded by  $\deg S_j^{(i)} = j$ , and with coproduct

$$(80) \quad \Delta S_n^{(k)} = \sum_{i=0}^n S_i^{(k)} \otimes S_j^{(k)}.$$

For a colored composition  $(I, u)$  as above, one defines

$$(81) \quad S^{(I,u)} := S_{i_1}^{(u_1)} \cdots S_{i_p}^{(u_p)}.$$

Clearly, the  $S^{(I,u)}$  form a linear basis of  $\mathbf{MR}^{(r)}$ .

The *monochromatic colored ribbon basis*  $R_{(I,u)}$  of  $\mathbf{MR}^{(r)}$  can now be defined by the condition

$$(82) \quad S^{(I,u)} =: \sum_{(J,v) \leq (I,u)} R_{(J,v)}.$$

Let  $(I, u) = (i_1, \dots, i_p; u_1, \dots, u_p)$  and  $(J, v) = (j_1, \dots, j_q; v_1, \dots, v_q)$  be two colored compositions. We set

$$(83) \quad (I, u) (J, v) := (I J, u v),$$

where  $ab$  denotes concatenation.

Moreover, if  $u_p = v_1$ , we set

$$(84) \quad (I, u) \triangleright (J, v) := ((i_1, \dots, i_{p-1}, (i_p + j_1), j_2, \dots, j_q); (u_1, \dots, u_p, v_2, \dots, v_q)).$$

The colored ribbons satisfy the very simple multiplication rule:

$$(85) \quad R_{(I,u)} R_{(J,v)} = R_{(I,u) (J,v)} + \begin{cases} R_{(I,v) \triangleright (J,v)} & \text{if } u_p = v_1, \\ 0 & \text{if } u_p \neq v_1. \end{cases}$$

Let  $[K, c]$  be an anticycloribbon. Let  $P_{[K,c]}$  be the indecomposable projective module whose unique simple quotient is the simple module labelled by  $\phi([K, c])$  and let

$(I, u)$  be the corresponding colored composition. Summarizing this discussion, we have

**Theorem 6.1.** *The map*

$$(86) \quad \begin{aligned} \mathbf{ch} : \quad \mathcal{K} &\rightarrow \mathbf{MR}^{(r)} \\ [P_{[K, \mathbf{c}]}] &\mapsto R_{(I, u)} \end{aligned}$$

*is an isomorphism of Hopf algebras.* ■

The main interest of the labelling by colored compositions is that it allows immediate reading of some important information. For example, it follows from Theorem 6.1 that the products of complete functions  $S^{(I, u)}$  are the characteristics of the projective  $\mathcal{H}_{n,r}(0)$ -modules obtained as induction products of the one-dimensional projective  $\mathcal{H}_{m,r}(0)$ -modules on which all  $T_i$  act by 0 and all  $\xi_j$  by the same eigenvalue  $u_i$ .

We see that, as in the case of  $H_n(0)$ , each indecomposable projective  $\mathcal{H}_{n,r}(0)$ -module occurs as a direct summand of such an induced module, and that the direct sum decomposition is given by the anti-refinement order. For example, writing a bar over the parts of the composition corresponding to color 2 (and nothing over the parts corresponding to color 1), the identity

$$(87) \quad S^{(2\bar{1}\bar{2}13)} = R_{(2\bar{1}\bar{2}13)} + R_{(2\bar{3}13)} + R_{(2\bar{1}\bar{2}4)} + R_{(2\bar{3}4)}$$

indicates which indecomposable projective direct summands compose the projective  $\mathcal{H}_{9,2}(0)$ -module defined as the outer tensor product

$$(88) \quad S_2 \hat{\otimes} S_{\bar{1}} \hat{\otimes} S_{\bar{2}} \hat{\otimes} S_1 \hat{\otimes} S_3.$$

Since by multiplying a Lagrange element  $L_{\mathbf{c}}$  of  $\mathcal{H}_{n,r}(0)$  by any element of  $\mathcal{H}_{n,r}(0)$ , one can only obtain an element of the form  $L_{\mathbf{d}}T$  where  $\mathbf{d}$  has same evaluation as  $\mathbf{c}$  and  $T$  is any Hecke element, we have a direct sum decomposition of  $\mathcal{H}_{n,r}(0)$  into two-sided ideals:

$$(89) \quad \mathcal{H}_{n,r}(0) = \bigoplus_e H_{n,r}^e,$$

where  $e$  is any evaluation of  $C^n$  and  $H_{n,r}^e := \mathcal{H}_{n,r}(0) \sum_{\mathbf{c} \in e} L_{\mathbf{c}}$ . Actually all such sums are central idempotents of  $\mathcal{H}_{n,r}(0)$ . The ideals  $H_{n,r}^e$  where  $e = (i, i, \dots, i)$  are said to be *monochromatic* of color  $i$ . The monochromatic ideals are isomorphic to  $H_n(0)$  as algebras, and their indecomposable projective modules  $P_{[I, \mathbf{c}]}$  are obtained by defining the action of all  $L_{\mathbf{c}}$  by zero except  $L_{(i^n)}$  acting by one on the  $H_n(0)$ -modules  $P_I$ . Such modules are also called monochromatic. Note that one can realize  $P_{[I, i^k]}$  in the left regular representation as

$$(90) \quad P_{[I, i^k]} = \mathcal{H}_{n,r}(0) L_{i^n} \nu_I.$$

It follows from Theorem 6.1 and Formula (85) that

**Corollary 6.2.** *Every indecomposable projective module  $P_{[I, \mathbf{c}]}$  of  $\mathcal{H}_{n,r}(0)$  is obtained as an induction product of monochromatic modules*

$$(91) \quad P_{[I, \mathbf{c}]} = P_{[I_1, \mathbf{c}_1]} \hat{\otimes} \cdots \hat{\otimes} P_{[I_k, \mathbf{c}_k]},$$

where  $I = (I_1, \dots, I_k)$  and  $I_j$  is a composition,  $\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_k)$  and  $\mathbf{c}_j$  is a monochromatic word. ■

Corollary 6.2 and Theorem 3.8 implies that  $P_{[I, \mathbf{c}]}$  is a semi-combinatorial module. Indeed, it is possible to choose for the underlying  $H_n(0)$ -module  $P_{I_1} \hat{\otimes} \dots \hat{\otimes} P_{I_k}$  a Yang-Baxter graph such that all vertices are common eigenvectors of the Lagrange polynomials by making the right choice on the color vector  $\mathbf{d}$ : if  $c_i < c_j$  then  $d_i < d_j$  and if  $c_i > c_j$  then  $d_i > d_j$ .

Figure 12 shows the two examples of induction product  $P_{[(2,1),(1)]} \hat{\otimes} P_{[(1),(2)]}$  and  $P_{[(2,1),(2)]} \hat{\otimes} P_{[(1),(1)]}$ . On the first induced module, the color vector has to be in the convolution of 312 and 1 and have its last element greater than all the others, and so is 3124. On the second induced module, the color vector has to be in the convolution of 312 and 1 and have its last element smaller than all the others, and so is 4231. One can compare this figure with Figure 8.

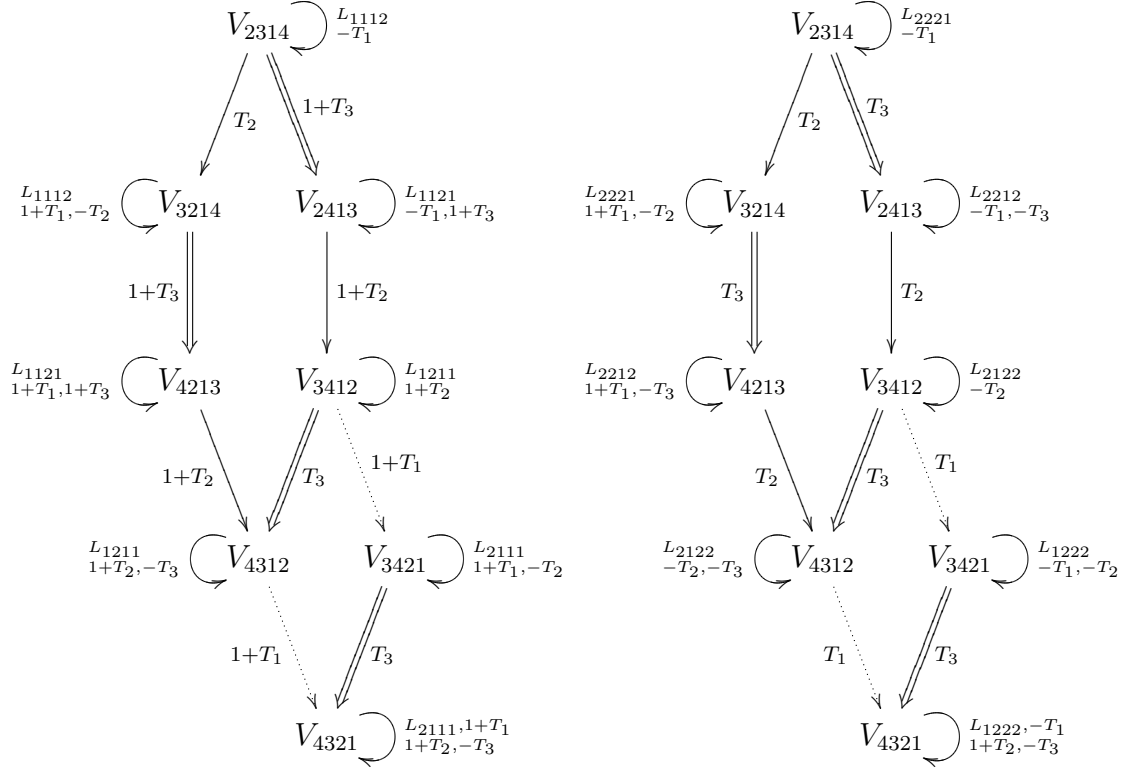


FIGURE 12. Induction product of the two indecomposable projective modules  $P_{[(2,1),(1)]} \hat{\otimes} P_{[(1),(2)]}$  (left module). Induction product of the two indecomposable projective modules  $P_{[(2,1),(2)]} \hat{\otimes} P_{[(1),(1)]}$  (right module).

Their restrictions to  $H_n(0)$  (and hence, their dimensions) can be computed by means of the following result:

**Corollary 6.3.** *The homomorphism of Hopf algebras*

$$(92) \quad \begin{aligned} \pi : \mathbf{MR}^{(r)} &\rightarrow \mathbf{Sym} \\ S_j^{(i)} &\mapsto S_j \end{aligned}$$

maps the class of a projective  $\mathcal{H}_{n,r}(0)$ -module to the class of its restriction to  $H_n(0)$ .

*Proof.* The previous considerations show in particular that the restriction of a monochromatic  $\mathcal{H}_{n,r}(0)$ -module  $P_{[I, i^n]}$  to  $H_n(0)$  is isomorphic to the indecomposable projective  $H_n(0)$ -module  $P_I$ . Hence, the restriction of any indecomposable projective module  $P_I$  given as in Formula (91) is

$$(93) \quad P_{I_1} \hat{\otimes} \cdots \hat{\otimes} P_{I_k},$$

so that the restriction map  $\pi$  is given by

$$(94) \quad \pi(R_{[I, c]}) = R_{I_1} \cdots R_{I_k},$$

which is equivalent to Formula (92). ■

Continuing the previous example, we see that the restriction of  $P_{2\bar{1}213}$  to  $H_9(0)$  is given by

$$(95) \quad \pi(R_{2\bar{1}213}) = \pi(R_2 R_{\bar{1}2} R_{13}) = R_2 R_{12} R_{13} = R_{21213} + R_{2133} + R_{3213} + R_{333}.$$

Dually, one can describe the induction of projective  $H_n(0)$ -modules to  $\mathcal{H}_{n,r}(0)$ . The next result is a consequence of Corollary 6.3.

**Corollary 6.4.** *Let  $I$  be a composition of  $n$  and let  $N_I$  be the  $\mathcal{H}_{n,r}(0)$ -module induced by the indecomposable projective  $H_n(0)$ -module  $P_I$ . Then*

$$(96) \quad N_I \simeq \bigoplus P_{[I, c]},$$

where the sum runs over all the anticycloribbons of shape  $I$ . ■

For example, let us complete the case  $I = (2, 1)$  with two colors. The following five anticycloribbons appear in the induction of  $P_I$ , with respective dimensions 3, 6, 3, 2, and 2:

$$(97) \quad \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 1 \\ \hline & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & 1 \\ \hline \end{array}$$

**6.2. Cartan invariants and decomposition numbers.** Finally, we can describe the Cartan invariants and the decomposition matrices by means of the maps

$$(98) \quad \begin{array}{ccc} \text{Sym}^{(r)} & \xrightarrow{d} & QMR^{(r)} \\ & \nwarrow e \quad \nearrow c & \\ & \mathbf{MR}^{(r)} & \end{array}$$

$$\begin{aligned} e : \mathbf{MR}^{(r)} &\rightarrow \text{Sym}^{(r)} = (\text{Sym})^{\otimes r} \simeq \text{Sym}(X_1, \dots, X_r), \\ S_j^{(i)} &\mapsto h_j(X_i) \end{aligned}$$

and

$$(99) \quad \begin{aligned} d : \text{Sym}^{(r)} &\hookrightarrow QMR^{(r)} \\ h_j(X_i) &\mapsto F_{[(j),i^j]}. \end{aligned}$$

Then the Cartan map is  $c = d \circ e$ , and the entry  $c_{[I,\mathbf{c}],[J,\mathbf{d}]}$  of the Cartan matrix giving the multiplicity of the simple module  $S_{[J,\mathbf{d}]}$  as a composition factor of the indecomposable projective module  $P_{[I,\mathbf{c}]}$  is equal to the coefficient of  $F_{[J,\mathbf{d}]}$  in  $c(R_{[I,\mathbf{c}]})$ .

For an  $r$ -partition  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ , let  $Z_\lambda(q)$  be the irreducible module of the generic algebra  $\mathcal{H}_{n,r}(q)$  as constructed in [23], Theorem 4.1. With our normalization, this module is defined for arbitrary values of  $q$ , including  $q = 0$ . For an  $r$ -colored composition  $(I, u)$  of  $n$ , let

$$(100) \quad d_{\lambda,(I,u)} := [S_{(I,u)} : Z_\lambda(0)]$$

be the decomposition numbers.

**Theorem 6.5.** *Let  $r_{(I,u)} = e(R_{(I,u)})$  be the commutative image of  $R_{(I,u)}$  in  $\text{Sym}^{(r)}$ . Let  $S_\lambda = s_{\lambda^{(1)}}(X_1) \cdots s_{\lambda^{(r)}}(X_r)$  as in [25]. Then the multiplicity of the simple module  $S_{(I,u)}$  as a composition factor of  $Z_\lambda(0)$  is equal to the coefficient of  $S_\lambda$  in  $r_{(I,u)}$ , that is*

$$(101) \quad d_{\lambda,(I,u)} := \langle S_\lambda, r_{(I,u)} \rangle$$

where  $\langle, \rangle$  is the scalar product on  $\text{Sym}^{(r)}$  for which the basis  $(S_\lambda)$  is orthonormal.

*Proof.* This follows from Shoji's character formula (Theorem 6.14 of [25]). We first need to reformulate it on another set of elements. Instead of the  $a_\mu$  of Shoji, we shall make use of elements  $b_{(J,v)}$  labelled by colored compositions, and defined by

$$(102) \quad b_{(J,v)} := L_{\mathbf{c}(J,v)} T_{w(J)}$$

where  $\mathbf{c}(J,v) = (v_1^{j_1} \cdots v_s^{j_s})$  and  $w(J) = \gamma_{j_1} \times \cdots \times \gamma_{j_s} \in \mathfrak{S}_{j_1} \times \cdots \times \mathfrak{S}_{j_s}$ , each  $\gamma_k = s_{k-1} \cdots s_1$  being a  $k$ -cycle. With each  $b_{(J,v)}$ , we associate a “non-commutative cycle index”

$$(103) \quad \mathbf{Z}(b_{(J,v)}) := (q-1)^{-l(J)} S^{(J,v)}((q-1)A) \in \mathbf{MR}_n^{(r)}.$$

Then, Shoji's formula (6.14.1) is equivalent to

$$(104) \quad \chi_q^\lambda(b_{(J,v)}) = \langle S_\lambda, \mathbf{Z}(b_{(J,v)}) \rangle$$

where  $S_\lambda$  is interpreted as an element of  $QMR^{(r)}$ .

Specializing this at  $q = 0$  gives

$$(105) \quad \chi_0^\lambda(b_{(J,v)}) = \langle S_\lambda, (-1)^{n-l(I)} \Lambda^{(J,v)}(A) \rangle,$$

where  $\Lambda^{(J,v)}(A) = \Lambda_{j_1}^{(v_1)} \cdots \Lambda_{j_s}^{(v_s)}$ , and the colored elementary functions  $\Lambda_j^{(i)}$  are

$$(106) \quad \sum_j (-1)^j \Lambda_j^{(i)} = \left( \sum_k S_k^{(i)} \right)^{-1}.$$

On another hand, using the two expressions

$$(107) \quad \eta_{(I,u)} = L_{\mathbf{c}'} \eta_{I'} = \eta_{I''} L_{\mathbf{c}''}$$



of the generator of  $S_{(I,u)}$ , one checks that the eigenvalue of  $b_{(J,v)}$  on  $\eta_{(I,u)}$  is

$$(108) \quad \langle F_{(I,u)}, \mathbf{Z}(b_{(J,v)}) \rangle.$$

Therefore,  $\chi_0^\lambda(b_{(J,v)})$  is also given by

$$(109) \quad \chi_0^\lambda(b_{(J,v)}) = \sum_{(I,u)} d_{\lambda,(I,u)} \langle F_{(I,u)}, \mathbf{Z}(b_{(J,v)}) \rangle.$$

The  $\mathbf{Z}(b_{(J,v)})$  being linearly independent, one can conclude that

$$(110) \quad \sum_{(I,u)} d_{\lambda,(I,u)} F_{(I,u)} = S_\lambda,$$

which is equivalent to Equation (100) since  $r_{(I,u)}$  is the image by  $e$  of  $R_{(I,u)}$  which is itself the dual basis of  $F_{(I,u)}$ . ■

## 7. QUIVERS

Being a finite dimensional elementary  $\mathbb{C}$ -algebra,  $\mathcal{H}_{n,r}(0)$  can be presented in the form  $\mathbb{C} Q_{n,r}/\mathcal{I}$ , where  $Q_{n,r}$  is a quiver,  $\mathbb{C} Q_{n,r}$  its path algebra, and  $\mathcal{I}$  an ideal contained in  $\mathcal{J}^2$  where  $\mathcal{J}$  is the ideal generated by the edges of  $Q_{n,r}$  [3]. The vertices of  $Q_{n,r}$  are naturally in bijection with the simple modules  $S_{[I,\mathbf{c}]}$ , and the number  $e_{[I,\mathbf{c}],[J,\mathbf{c}']}$  of edges  $S_{[I,\mathbf{c}]} \rightarrow S_{[J,\mathbf{c}]}$  is equal to

$$(111) \quad \dim \text{Ext}^1(S_{[I,\mathbf{c}]}, S_{[J,\mathbf{c}]}) = [S_{[J,\mathbf{c}]} : \text{rad } P_{[I,\mathbf{c}]} / \text{rad}^2 P_{[I,\mathbf{c}]}].$$

that is equal to  $[S_{[I,\mathbf{c}]} : \text{rad } P_{[J,\mathbf{c}]} / \text{rad}^2 P_{[J,\mathbf{c}]}]$  by auto-injectivity.

Therefore,  $e_{[I,\mathbf{c}],[J,\mathbf{c}']} = d_{[I,\mathbf{c}],[J,\mathbf{c}']}^{(1)}$ , where

$$(112) \quad d_{[I,\mathbf{c}],[J,\mathbf{c}']}^{(k)} := [S_{[J,\mathbf{c}']} : \text{rad}^k P_{[I,\mathbf{c}]} / \text{rad}^{k+1} P_{[I,\mathbf{c}]}]$$

are the coefficients of the  $q$ -Cartan invariants

$$(113) \quad d_{[I,\mathbf{c}],[J,\mathbf{c}']} (q) = \sum_{k \geq 0} d_{[I,\mathbf{c}],[J,\mathbf{c}']}^{(k)} q^k$$

associated with the radical series. Let  $D_n(q) = (d_{[I,\mathbf{c}],[J,\mathbf{c}']} (q))$  be the  $q$ -Cartan matrix. For  $n \leq 4$ , Section 8 provides examples of such matrices.

As one will see in the next paragraph, the quiver of  $\mathcal{H}_{n,r}(0)$  splits into connected components spanned by evaluation classes of the colors words of the anti-cycloribbons. In other words,  $\dim \text{Ext}^1(S_{[I,\mathbf{c}]}, S_{[J,\mathbf{c}]}) \neq 0$  implies that  $\mathbf{c}$  and  $\mathbf{c}'$  are permutations of one another.

The quivers corresponding to evaluations  $(1, 3)$  and  $(2, 2)$  of  $\mathcal{H}_{4,2}(0)$  are given on Figures 13 and 14.

The quiver  $Q_{n,r}$  can be described for all  $n$ : let  $[I, \mathbf{c}]$  and  $[J, \mathbf{c}']$  be cycloribbons. Since the simple modules are one-dimensional, isomorphism classes of non trivial extensions

$$(114) \quad 0 \rightarrow S_{[J,\mathbf{c}']} \rightarrow M \rightarrow S_{[I,\mathbf{c}]} \rightarrow 0$$

are in one-to-one correspondence with isomorphism classes of indecomposable two-dimensional (left) modules  $M$  of socle  $S_{[J,\mathbf{c}]}$  and such that  $M/\text{rad } M = S_{[I,\mathbf{c}]}$ .

Let  $M$  be such a module. Since the  $L_{\mathbf{c}}$  are orthogonal idempotents summing to 1, all but one or two of them act on  $M$  by zero.

If there only is one  $L_{\mathbf{c}}$  having a non-zero action on  $M$ , then it acts by 1 and the other  $L_{\mathbf{d}}$  act by 0 on  $M$ . So all Lagrange polynomials act as scalars on  $M$ , and  $\mathbf{c} = \mathbf{c}'$ . Since  $M$  is an indecomposable two-dimensional module of  $\mathcal{H}_{n,r}(0)$ ,  $M$  is also an indecomposable two-dimensional module of  $H_n(0)$ . Thanks to [8], there exist two elements  $b_1$  and  $b_2$  of  $M$  and an integer  $i$  such that  $T_i b_1 = b_2$  or  $(1 + T_i) b_1 = b_2$ , all  $T_j$  for  $j \in \{1, \dots, i-2\} \cap \{i+2, \dots, n-1\}$  act as scalars,  $T_{i-1}$  does not act as a scalar, and  $T_{i+1}$  can act as a scalar or not. Moreover, if  $T_i b_1 = b_2$ ,  $T_{i-1}$  acts by  $-1$  on  $b_1$  and by 0 on  $b_2$  and in the opposite way if  $(1 + T_i) b_1 = b_2$ . Finally, if  $T_{i+1}$  acts as a non-scalar, then it acts as  $T_{i-1}$ .

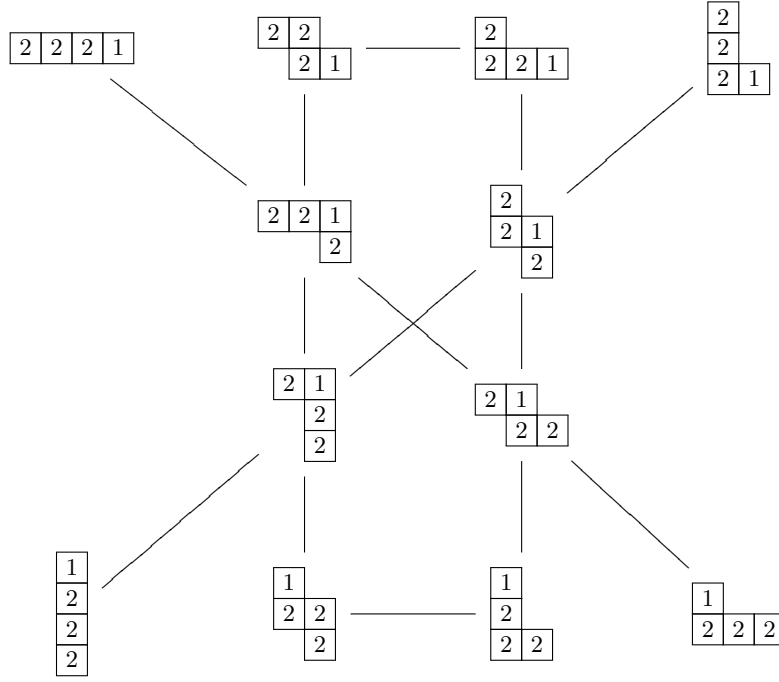
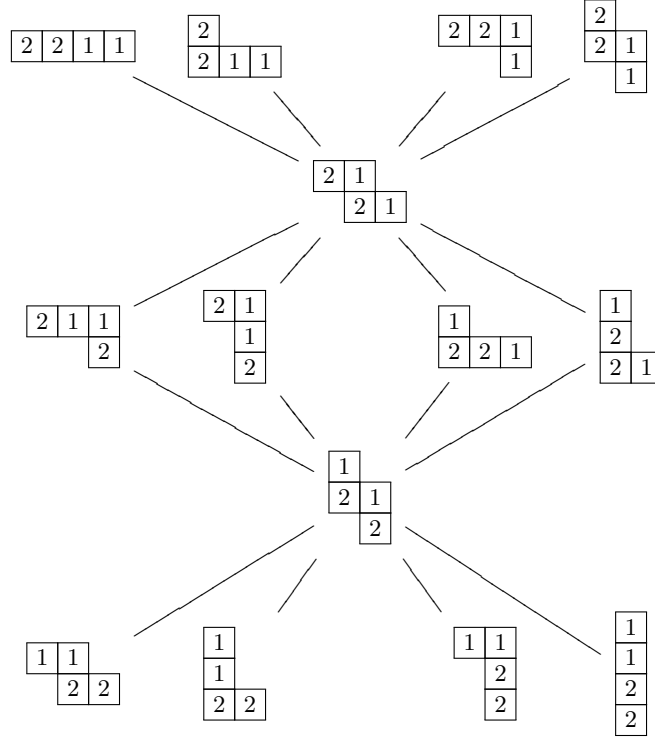


FIGURE 13. Quiver of  $\mathcal{H}_{4,2}(0)$  corresponding to anticycloribbons of evaluation  $(1, 3)$ .

Since  $M$  is a module, we must have  $c_{i-1} = c_i = c_{i+1}$  and, it is also equal to  $c_{i+2}$  if  $T_{i+1}$  acts as  $T_{i-1}$ . Moreover, if  $c_k \neq c_{k+1}$  then  $T_k$  acts as a scalar, by  $-1$  if  $c_k < c_{k+1}$  and by  $0$  otherwise. One then checks that these relations are sufficient, so that there is an edge in  $Q_{n,r}$  between  $[I, \mathbf{c}]$  and  $[J, \mathbf{c}']$  iff there is an edge between  $I$  and  $J$  in the quiver of  $H_n(0)$ .

Let us now consider the case where two elements,  $L_{\mathbf{c}}$  and  $L_{\mathbf{c}'}$  act as non-zero elements on  $M$ . Since they commute, there exist a basis where both are diagonal. Given that their sum is  $1$  and that they are orthogonal, this proves that there exist two elements  $x$  and  $y$  of  $M$  such that  $x = L_{\mathbf{c}}x$  and  $y = L_{\mathbf{c}'}y$ . Since we assumed that  $M$  is indecomposable, this means that there exists an integer  $i$  such that, up to the exchange of  $x$  and  $y$ ,  $T_i x = \alpha x + \beta y$  with  $\beta \neq 0$ . Relations (37) then show that  $\mathbf{c}' = \mathbf{c}s_i$ , so that  $c_i \neq c_{i+1}$  and that  $\alpha = 0$  or  $-1$ , so that  $T_i x = y$ , or  $(1 + T_i)x = y$  depending on whether  $c_i > c_{i+1}$  or  $c_i < c_{i+1}$ . Then, as in [8], one proves that all  $T_j$  except  $T_i$  act diagonally on  $x$  and  $y$ , and that  $T_j$  acts as a scalar if  $j \notin \{i-1, i, i+1\}$ . Finally, since  $M$  is a module, the action of any  $T_j$  ( $j \neq i$ ) on  $x$  (resp.  $y$ ) is fixed if  $c_j \neq c_{j+1}$  (resp.  $c'_j \neq c'_{j+1}$ ). One then checks that these relations are sufficient, so that there is an edge in  $Q_{n,r}$  between  $[I, \mathbf{c}]$  and  $[J, \mathbf{c}']$  ( $\mathbf{c}' \neq \mathbf{c}$ ) iff there exist an integer  $i$  such that  $\mathbf{c}' = \mathbf{c}s_i$ , and any integer  $k \notin \{i-1, i, i+1\}$  is a descent of both or none of  $I$  and  $J$ .

So we can conclude:

FIGURE 14. Quiver of the block  $(2, 2)$  of  $\mathcal{H}_{4,2}(0)$ .

**Proposition 7.1.** *Let  $[I, \mathbf{c}]$  and  $[J, \mathbf{c}']$  be two cycloribbons. There is an edge in  $Q_{n,r}$  between  $[I, \mathbf{c}]$  and  $[J, \mathbf{c}']$  iff*

- $\mathbf{c} = \mathbf{c}'$  and there is an edge in the quiver of  $H_n(0)$ ,
- $\mathbf{c} \neq \mathbf{c}'$  and there exists an integer  $i$  such that  $\mathbf{c}' = \mathbf{c}s_i$ , and if  $k$  is a descent of both or none of  $I, J$  for  $k \notin \{i-1, i, i+1\}$ .

Note that there cannot be any edge between two cycloribbons if  $\mathbf{c}$  and  $\mathbf{c}'$  are not permutations of one another. So  $Q_{n,r}$  splits into components generated by rearrangement classes of color vectors. These rearrangement classes will be called *evaluations* in the sequel.

Since the product of two basis elements  $B_{\mathbf{c},\sigma}$  and  $B_{\mathbf{c}',\sigma'}$  is non-zero only if  $\mathbf{c}$  and  $\mathbf{c}'$  are permutations of one another, the simple modules  $S_{[I,\mathbf{c}]}$  occurring in any indecomposable module must be in the same evaluation class, so that the Cartan matrix of any  $\mathcal{H}_{n,r}(0)$  is a block matrix, each block being indexed by anticycloribbons with the same evaluation. The monochromatic blocks coincide with the Cartan matrices of  $H_n(0)$ . The other one correspond to blocks (in the sense of indecomposable two-sided ideals) of  $\mathcal{H}_{n,r}(0)$ .

## 8. TABLES

The first tables present the  $q$ -Cartan matrices of the blocks (2) and (1, 1) of  $\mathcal{H}_{2,2}(0)$  and the blocks (3), (2, 1), (1, 2), and (1, 1, 1) of  $\mathcal{H}_{3,3}(0)$ . Note that the block (1, 2) can be deduced from the block (2, 1) by changing any anticycloribbon  $(I, \mathbf{c})$  into  $(\bar{I}, \mathbf{c}')$  where  $\mathbf{c}'$  is obtained from  $\mathbf{c}$  by exchanging its smallest letter with its greatest, its second smallest with its second greatest, and so on. Note also that since each permutation is the color word of exactly one anticycloribbon, one can see a standard anticycloribbon as its color word. Then the matrix of the block (1, 1, 1) is the matrix whose entry  $(\sigma, \tau)$  is  $q^{l(\sigma^{-1}\tau)}$ .

Next, we present the blocks (4), (3, 1), (2, 2), (2, 1, 1), and (1, 2, 1) of  $\mathcal{H}_{4,4}(0)$ . The other blocks can be deduced from the previous ones: (1, 1, 2) comes from (2, 1, 1) and (1, 1, 1, 1) is again  $q^{l(\sigma^{-1}\tau)}$ .

These  $q$ -Cartan matrices are symmetric along both diagonals since the algebra is auto-injective so that the quiver is an unoriented graph (first main diagonal) and the morphism sending  $L_{\mathbf{c}}$  to  $L_{\bar{\mathbf{c}}}$  and  $-T_i$  to  $1 + T_{n-i}$  is an automorphism of  $\mathcal{H}_{n,r}(0)$  (second main diagonal). Moreover, for  $q = 1$ , the Cartan matrix is the product of the decomposition matrix with its transpose. Finally, we give three examples of decomposition matrices.

$$\begin{array}{ccc}
\begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \\
\begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} & 1 & . \\
\begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & . & 1
\end{array}
\quad
\begin{array}{ccc}
\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 1 \\ \hline \end{array} & \\
\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} & 1 & q \\
\begin{array}{|c|c|} \hline 2 & 1 \\ \hline \end{array} & q & 1
\end{array}$$

FIGURE 15. The two blocks of the  $q$ -Cartan matrix of  $\mathcal{H}_{2,2}(0)$ .

$$\begin{array}{cccccc}
& & \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array} \\
\begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array} & 1 & . & . & . & . \\
\begin{array}{|c|} \hline 1 \\ \hline 1 & 1 \\ \hline \end{array} & . & 1 & q & . & . \\
\begin{array}{|c|} \hline 1 & 1 \\ \hline 2 \\ \hline \end{array} & . & q & 1 & . & . \\
\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array} & . & . & . & 1 & .
\end{array}
\quad
\begin{array}{cccccc}
& & \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline 2 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline \end{array} \\
\begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} & 1 & . & q & q^2 & . \\
\begin{array}{|c|} \hline 1 & 1 \\ \hline 2 \\ \hline \end{array} & . & 1 & q & . & q^2 \\
\begin{array}{|c|} \hline 1 \\ \hline 2 & 1 \\ \hline \end{array} & q & q & q^2 + 1 & q & q \\
\begin{array}{|c|} \hline 2 & 1 \\ \hline 1 \\ \hline \end{array} & q^2 & . & q & 1 & . \\
\begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline \end{array} & . & q^2 & q & . & 1
\end{array}$$
  

$$\begin{array}{cccccc}
& & \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & \\ \hline 2 & 1 \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 2 & 2 & 1 \\ \hline \end{array} \\
\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 2 \\ \hline \end{array} & 1 & . & q & q^2 & . \\
\begin{array}{|c|} \hline 1 \\ \hline 2 & 2 \\ \hline \end{array} & . & 1 & q & . & q^2 \\
\begin{array}{|c|} \hline 2 & 1 \\ \hline 2 \\ \hline \end{array} & q & q & q^2 + 1 & q & q \\
\begin{array}{|c|} \hline 2 \\ \hline 2 & 1 \\ \hline \end{array} & q^2 & . & q & 1 & . \\
\begin{array}{|c|c|c|} \hline 2 & 2 & 1 \\ \hline \end{array} & . & q^2 & q & . & 1
\end{array}
\quad
\begin{array}{cccccc}
& & \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & \\ \hline 3 & 1 \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline \end{array} \\
\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} & 1 & q & q & q^2 & q^2 & q^3 \\
\begin{array}{|c|} \hline 1 \\ \hline 3 & 2 \\ \hline \end{array} & q & 1 & q^2 & q & q^3 & q^2 \\
\begin{array}{|c|} \hline 2 & 1 \\ \hline 3 \\ \hline \end{array} & q & q^2 & 1 & q^3 & q & q^2 \\
\begin{array}{|c|} \hline 3 & 1 \\ \hline 2 \\ \hline \end{array} & q^2 & q & q^3 & 1 & q^2 & q \\
\begin{array}{|c|} \hline 2 \\ \hline 3 & 1 \\ \hline \end{array} & q^2 & q^3 & q & q^2 & 1 & q \\
\begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline \end{array} & q^3 & q^2 & q^2 & q & q & 1
\end{array}$$

FIGURE 16. The four blocks of the  $q$ -Cartan matrix of  $\mathcal{H}_{3,3}(0)$ .

|                                                          | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ |
|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | 1                                                        | .                                                        | .                                                        | .                                                        | .                                                        | .                                                        | .                                                        | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | .                                                        | 1                                                        | $q$                                                      | $q^2$                                                    | .                                                        | .                                                        | .                                                        | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | .                                                        | $q$                                                      | $q^2 + 1$                                                | $q$                                                      | .                                                        | $q$                                                      | .                                                        | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | .                                                        | $q^2$                                                    | $q$                                                      | 1                                                        | .                                                        | .                                                        | .                                                        | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | .                                                        | .                                                        | .                                                        | .                                                        | 1                                                        | $q$                                                      | $q^2$                                                    | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | .                                                        | .                                                        | $q$                                                      | .                                                        | $q$                                                      | $q^2 + 1$                                                | $q$                                                      | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | .                                                        | .                                                        | .                                                        | .                                                        | $q^2$                                                    | $q$                                                      | 1                                                        | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ | .                                                        | .                                                        | .                                                        | .                                                        | .                                                        | .                                                        | .                                                        | .                                                        | 1                                                        |

 FIGURE 17. The block (4) of the  $q$ -Cartan matrix of  $\mathcal{H}_{4,4}(0)$ .

|                                                          | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ |
|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|----------------------------------------------------------|
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | 1                                                        | .                                                        | .                                                        | .                                                        | $q$                                                      | .                                                        | $q^2$                                                    | .                                                        | $q^3$                                                    | .                                                        | .                                                        | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | .                                                        | 1                                                        | $q$                                                      | .                                                        | $q$                                                      | $q^2$                                                    | $q^3$                                                    | $q^2$                                                    | .                                                        | $q^4$                                                    | $q^3$                                                    | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | .                                                        | $q$                                                      | 1                                                        | .                                                        | $q^2$                                                    | $q$                                                      | $q^2$                                                    | $q^3$                                                    | .                                                        | $q^3$                                                    | $q^4$                                                    | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | .                                                        | .                                                        | .                                                        | 1                                                        | .                                                        | $q$                                                      | .                                                        | $q^2$                                                    | .                                                        | .                                                        | .                                                        | $q^3$                                                    | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $q$                                                      | $q$                                                      | $q^2$                                                    | .                                                        | $q^2 + 1$                                                | $q^3$                                                    | $q^4 + q$                                                | $q$                                                      | $q^2$                                                    | $q^3$                                                    | $q^2$                                                    | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | .                                                        | $q^2$                                                    | $q$                                                      | $q$                                                      | $q^3$                                                    | $q^2 + 1$                                                | $q$                                                      | $q^4 + q$                                                | .                                                        | $q^2$                                                    | $q^3$                                                    | $q^2$                                                    | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $q^2$                                                    | $q^3$                                                    | $q^2$                                                    | .                                                        | $q^4 + q$                                                | $q$                                                      | $q^2 + 1$                                                | $q^3$                                                    | $q$                                                      | $q$                                                      | $q^2$                                                    | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | .                                                        | $q^2$                                                    | $q^3$                                                    | $q^2$                                                    | $q$                                                      | $q^4 + q$                                                | $q^3$                                                    | $q^2 + 1$                                                | .                                                        | $q^2$                                                    | $q$                                                      | $q$                                                      | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | $q^3$                                                    | .                                                        | .                                                        | .                                                        | $q^2$                                                    | .                                                        | $q$                                                      | .                                                        | 1                                                        | .                                                        | .                                                        | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | .                                                        | $q^4$                                                    | $q^3$                                                    | .                                                        | $q^3$                                                    | $q^2$                                                    | $q$                                                      | $q^2$                                                    | .                                                        | 1                                                        | $q$                                                      | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | .                                                        | $q^3$                                                    | $q^4$                                                    | .                                                        | $q^2$                                                    | $q^3$                                                    | $q^2$                                                    | $q$                                                      | .                                                        | $q$                                                      | 1                                                        | .                                                        | .                                                        |
| $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$ | .                                                        | .                                                        | .                                                        | $q^3$                                                    | .                                                        | $q^2$                                                    | .                                                        | $q$                                                      | .                                                        | .                                                        | .                                                        | .                                                        | 1                                                        |

 FIGURE 18. The block (3, 1) of the  $q$ -Cartan matrix of  $\mathcal{H}_{4,4}(0)$ .

|                                                     |       |       |       |       |              |           |           |           |           |              |       |       |       |       |
|-----------------------------------------------------|-------|-------|-------|-------|--------------|-----------|-----------|-----------|-----------|--------------|-------|-------|-------|-------|
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | 1     | .     | .     | .     | $q$          | $q^2$     | $q^2$     | .         | .         | $q^3$        | $q^4$ | .     | .     | .     |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | .     | 1     | .     | .     | $q$          | .         | $q^2$     | .         | $q^2$     | $q^3$        | .     | $q^4$ | .     | .     |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | .     | .     | 1     | .     | $q$          | $q^2$     | .         | $q^2$     | .         | $q^3$        | .     | .     | $q^4$ | .     |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | .     | .     | .     | 1     | $q$          | .         | .         | $q^2$     | $q^2$     | $q^3$        | .     | .     | .     | $q^4$ |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | $q$   | $q$   | $q$   | $q$   | $3q^2 + 1$   | $q^3 + q$ | $q^3 + q$ | $q^3 + q$ | $q^3 + q$ | $q^4 + 3q^2$ | $q^3$ | $q^3$ | $q^3$ | $q^3$ |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | $q^2$ | .     | $q^2$ | .     | $q^3 + q$    | $q^4 + 1$ | $q^2$     | $q^2$     | .         | $q^3 + q$    | $q^2$ | .     | $q^2$ | .     |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | $q^2$ | $q^2$ | .     | .     | $q^3 + q$    | $q^2$     | $q^4 + 1$ | .         | $q^2$     | $q^3 + q$    | $q^2$ | $q^2$ | .     | .     |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | .     | .     | $q^2$ | $q^2$ | $q^3 + q$    | $q^2$     | .         | $q^4 + 1$ | $q^2$     | $q^3 + q$    | .     | .     | $q^2$ | $q^2$ |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | .     | $q^2$ | .     | $q^2$ | $q^3 + q$    | .         | $q^2$     | $q^2$     | $q^4 + 1$ | $q^3 + q$    | .     | $q^2$ | .     | $q^2$ |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | $q^3$ | $q^3$ | $q^3$ | $q^3$ | $q^4 + 3q^2$ | $q^3 + q$ | $q^3 + q$ | $q^3 + q$ | $q^3 + q$ | $3q^2 + 1$   | $q$   | $q$   | $q$   | $q$   |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | $q^4$ | .     | .     | .     | $q^3$        | $q^2$     | $q^2$     | .         | .         | $q$          | 1     | .     | .     | .     |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | .     | $q^4$ | .     | .     | $q^3$        | .         | $q^2$     | .         | $q^2$     | $q$          | .     | 1     | .     | .     |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | .     | .     | $q^4$ | .     | $q^3$        | $q^2$     | .         | $q^2$     | .         | $q$          | .     | .     | 1     | .     |
| $\begin{array}{ c } \hline 1 \\ \hline \end{array}$ | .     | .     | .     | $q^4$ | $q^3$        | .         | .         | $q^2$     | $q^2$     | $q$          | .     | .     | .     | 1     |

FIGURE 19. The block  $(2, 2)$  of the  $q$ -Cartan matrix of  $\mathcal{H}_{4,4}(0)$ .



|                                                          | $\begin{smallmatrix} 1 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 1 \\ 3 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 1 \\ 3 & 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 2 & 1 \\ 3 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 2 \\ 3 & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 3 & 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 2 & 1 \\ 1 \\ 3 \end{smallmatrix}$ | $\begin{smallmatrix} 2 & 1 & 1 \\ 3 \end{smallmatrix}$ | $\begin{smallmatrix} 3 & 1 \\ 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 3 & 1 & 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 2 & 1 \\ 3 & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 3 & 2 & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 2 \\ 3 & 1 \\ 1 \end{smallmatrix}$ | $\begin{smallmatrix} 2 \\ 3 & 1 & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 3 & 2 & 1 \\ 1 \end{smallmatrix}$ | $\begin{smallmatrix} 3 & 2 & 1 & 1 \end{smallmatrix}$ |
|----------------------------------------------------------|----------------------------------------------------------|--------------------------------------------------------|----------------------------------------------------------|--------------------------------------------------------|---------------------------------------------------------|---------------------------------------------------------|---------------------------------------------------------|---------------------------------------------------------|--------------------------------------------------------|---------------------------------------------------------|--------------------------------------------------------|--------------------------------------------------------|--------------------------------------------------------|--------------------------------------------------------|---------------------------------------------------------|--------------------------------------------------------|--------------------------------------------------------|-------------------------------------------------------|
| $\begin{smallmatrix} 1 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$ | 1                                                        | .                                                      | $q$                                                      | .                                                      | $q$                                                     | $q^2$                                                   | $q^2$                                                   | $q^2$                                                   | .                                                      | $q^3$                                                   | .                                                      | $q^3$                                                  | $q^3$                                                  | $q^4$                                                  | $q^4$                                                   | .                                                      | $q^5$                                                  | .                                                     |
| $\begin{smallmatrix} 1 & 1 \\ 2 \\ 3 \end{smallmatrix}$  | .                                                        | 1                                                      | .                                                        | $q$                                                    | $q$                                                     | $q^2$                                                   | $q^2$                                                   | .                                                       | $q^2$                                                  | .                                                       | $q^3$                                                  | $q^3$                                                  | $q^3$                                                  | $q^4$                                                  | .                                                       | $q^4$                                                  | .                                                      | $q^5$                                                 |
| $\begin{smallmatrix} 1 \\ 1 \\ 3 & 2 \end{smallmatrix}$  | $q$                                                      | .                                                      | 1                                                        | .                                                      | $q^2$                                                   | $q^3$                                                   | $q$                                                     | $q^3$                                                   | .                                                      | $q^2$                                                   | .                                                      | $q^4$                                                  | $q^2$                                                  | $q^3$                                                  | $q^5$                                                   | .                                                      | $q^4$                                                  | .                                                     |
| $\begin{smallmatrix} 1 & 1 \\ 3 & 2 \end{smallmatrix}$   | .                                                        | $q$                                                    | .                                                        | 1                                                      | $q^2$                                                   | $q^3$                                                   | $q$                                                     | .                                                       | $q^3$                                                  | .                                                       | $q^2$                                                  | $q^4$                                                  | $q^2$                                                  | $q^3$                                                  | .                                                       | $q^5$                                                  | .                                                      | $q^4$                                                 |
| $\begin{smallmatrix} 1 \\ 2 & 1 \\ 3 \end{smallmatrix}$  | $q$                                                      | $q$                                                    | $q^2$                                                    | $q^2$                                                  | $q^2 + 1$                                               | $q^3 + q$                                               | $2q^3$                                                  | $q$                                                     | $q$                                                    | $q^4$                                                   | $q^4$                                                  | $2q^2$                                                 | $q^4 + q^2$                                            | $q^5 + q^3$                                            | $q^3$                                                   | $q^3$                                                  | $q^4$                                                  | $q^4$                                                 |
| $\begin{smallmatrix} 1 \\ 2 \\ 3 & 1 \end{smallmatrix}$  | $q^2$                                                    | $q^2$                                                  | $q^3$                                                    | $q^3$                                                  | $q^3 + q$                                               | $q^4 + 1$                                               | $q^4 + q^2$                                             | $q^2$                                                   | $q^2$                                                  | $q^3$                                                   | $q^3$                                                  | $q^3 + q$                                              | $q^5 + q$                                              | $q^4 + q^2$                                            | $q^2$                                                   | $q^2$                                                  | $q^3$                                                  | $q^3$                                                 |
| $\begin{smallmatrix} 1 \\ 3 & 1 \\ 2 \end{smallmatrix}$  | $q^2$                                                    | $q^2$                                                  | $q$                                                      | $q$                                                    | $2q^3$                                                  | $q^4 + q^2$                                             | $q^2 + 1$                                               | $q^4$                                                   | $q^4$                                                  | $q$                                                     | $q$                                                    | $q^5 + q^3$                                            | $q^3 + q$                                              | $2q^2$                                                 | $q^4$                                                   | $q^4$                                                  | $q^3$                                                  | $q^3$                                                 |
| $\begin{smallmatrix} 2 & 1 \\ 1 \\ 3 \end{smallmatrix}$  | $q^2$                                                    | .                                                      | $q^3$                                                    | .                                                      | $q$                                                     | $q^2$                                                   | $q^4$                                                   | 1                                                       | .                                                      | $q^5$                                                   | .                                                      | $q$                                                    | $q^3$                                                  | $q^4$                                                  | $q^2$                                                   | .                                                      | $q^3$                                                  | .                                                     |
| $\begin{smallmatrix} 2 & 1 & 1 \\ 3 \end{smallmatrix}$   | .                                                        | $q^2$                                                  | .                                                        | $q^3$                                                  | $q$                                                     | $q^2$                                                   | $q^4$                                                   | .                                                       | 1                                                      | .                                                       | $q^5$                                                  | $q$                                                    | $q^3$                                                  | $q^4$                                                  | .                                                       | $q^2$                                                  | .                                                      | $q^3$                                                 |
| $\begin{smallmatrix} 3 & 1 \\ 1 \\ 2 \end{smallmatrix}$  | $q^3$                                                    | .                                                      | $q^2$                                                    | .                                                      | $q^4$                                                   | $q^3$                                                   | $q$                                                     | $q^5$                                                   | .                                                      | 1                                                       | .                                                      | $q^4$                                                  | $q^2$                                                  | $q$                                                    | $q^3$                                                   | .                                                      | $q^2$                                                  | .                                                     |
| $\begin{smallmatrix} 3 & 1 & 1 \\ 2 \end{smallmatrix}$   | .                                                        | $q^3$                                                  | .                                                        | $q^2$                                                  | $q^4$                                                   | $q^3$                                                   | $q$                                                     | .                                                       | $q^5$                                                  | .                                                       | 1                                                      | $q^4$                                                  | $q^2$                                                  | $q$                                                    | .                                                       | $q^3$                                                  | .                                                      | $q^2$                                                 |
| $\begin{smallmatrix} 2 & 1 \\ 3 & 1 \end{smallmatrix}$   | $q^3$                                                    | $q^3$                                                  | $q^4$                                                    | $q^4$                                                  | $2q^2$                                                  | $q^3 + q$                                               | $q^5 + q^3$                                             | $q$                                                     | $q$                                                    | $q^4$                                                   | $q^4$                                                  | $q^2 + 1$                                              | $q^4 + q^2$                                            | $2q^3$                                                 | $q$                                                     | $q$                                                    | $q^2$                                                  | $q^2$                                                 |
| $\begin{smallmatrix} 1 \\ 3 & 2 & 1 \end{smallmatrix}$   | $q^3$                                                    | $q^3$                                                  | $q^2$                                                    | $q^2$                                                  | $q^4 + q^2$                                             | $q^5 + q$                                               | $q^3 + q$                                               | $q^3$                                                   | $q^3$                                                  | $q^2$                                                   | $q^2$                                                  | $q^4 + q^2$                                            | $q^4 + 1$                                              | $q^3 + q$                                              | $q^3$                                                   | $q^3$                                                  | $q^2$                                                  | $q^2$                                                 |
| $\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}$   | $q^4$                                                    | $q^4$                                                  | $q^3$                                                    | $q^3$                                                  | $q^5 + q^3$                                             | $q^4 + q^2$                                             | $2q^2$                                                  | $q^4$                                                   | $q^4$                                                  | $q$                                                     | $q$                                                    | $2q^3$                                                 | $q^3 + q$                                              | $q^2 + 1$                                              | $q^2$                                                   | $q^2$                                                  | $q$                                                    | $q$                                                   |
| $\begin{smallmatrix} 2 \\ 3 & 1 \\ 1 \end{smallmatrix}$  | $q^4$                                                    | .                                                      | $q^5$                                                    | .                                                      | $q^3$                                                   | $q^2$                                                   | $q^4$                                                   | $q^2$                                                   | .                                                      | $q^3$                                                   | .                                                      | $q$                                                    | $q^3$                                                  | $q^2$                                                  | 1                                                       | .                                                      | $q$                                                    | .                                                     |
| $\begin{smallmatrix} 2 \\ 3 & 1 & 1 \end{smallmatrix}$   | .                                                        | $q^4$                                                  | .                                                        | $q^5$                                                  | $q^3$                                                   | $q^2$                                                   | $q^4$                                                   | .                                                       | $q^2$                                                  | .                                                       | $q^3$                                                  | $q$                                                    | $q^3$                                                  | $q^2$                                                  | .                                                       | 1                                                      | .                                                      | $q$                                                   |
| $\begin{smallmatrix} 3 & 2 & 1 \\ 1 \end{smallmatrix}$   | $q^5$                                                    | .                                                      | $q^4$                                                    | .                                                      | $q^4$                                                   | $q^3$                                                   | $q^3$                                                   | $q^3$                                                   | .                                                      | $q^2$                                                   | .                                                      | $q^2$                                                  | $q^2$                                                  | $q$                                                    | $q$                                                     | .                                                      | 1                                                      | .                                                     |
| $\begin{smallmatrix} 3 & 2 & 1 & 1 \end{smallmatrix}$    | .                                                        | $q^5$                                                  | .                                                        | $q^4$                                                  | $q^4$                                                   | $q^3$                                                   | $q^3$                                                   | .                                                       | $q^3$                                                  | .                                                       | $q^2$                                                  | $q^2$                                                  | $q^2$                                                  | $q$                                                    | .                                                       | $q$                                                    | .                                                      | 1                                                     |

 FIGURE 20. The block  $(2, 1, 1)$  of the  $q$ -Cartan matrix of  $\mathcal{H}_{4,4}(0)$ .

|                                                          | $\begin{smallmatrix} 1 \\ 2 \\ 2 \\ 3 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 2 & 2 \\ 3 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 2 \\ 3 & 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 \\ 3 & 2 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & \\ 3 & 2 & 2 \end{smallmatrix}$ | $\begin{smallmatrix} 2 & 1 \\ 2 \\ 3 \end{smallmatrix}$ | $\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}$ | $\begin{smallmatrix} 2 \\ 2 & 1 \\ 3 \end{smallmatrix}$ | $\begin{smallmatrix} 2 & 2 & 1 \\ 3 \end{smallmatrix}$ | $\begin{smallmatrix} 3 & 1 \\ 2 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 3 & 1 \\ 2 & 2 \end{smallmatrix}$ | $\begin{smallmatrix} 2 \\ 3 & 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 3 & 2 & 1 \\ 2 \end{smallmatrix}$ | $\begin{smallmatrix} 2 \\ 2 \\ 3 & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 2 & 2 \\ 3 & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 2 \\ 3 & 2 & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 3 & 2 \\ 2 & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 3 & 2 & 2 & 1 \end{smallmatrix}$ |
|----------------------------------------------------------|----------------------------------------------------------|---------------------------------------------------------|---------------------------------------------------------|---------------------------------------------------------|----------------------------------------------------------|---------------------------------------------------------|--------------------------------------------------------|---------------------------------------------------------|--------------------------------------------------------|---------------------------------------------------------|--------------------------------------------------------|---------------------------------------------------------|--------------------------------------------------------|---------------------------------------------------------|--------------------------------------------------------|--------------------------------------------------------|--------------------------------------------------------|-------------------------------------------------------|
| $\begin{smallmatrix} 1 \\ 2 \\ 2 \\ 3 \end{smallmatrix}$ | 1                                                        | .                                                       | $q$                                                     | $q^2$                                                   | .                                                        | $q$                                                     | $q^2$                                                  | $q^2$                                                   | .                                                      | $q^3$                                                   | .                                                      | $q^3$                                                   | $q^4$                                                  | $q^3$                                                   | .                                                      | $q^4$                                                  | $q^5$                                                  | .                                                     |
| $\begin{smallmatrix} 1 \\ 2 & 2 \\ 3 \end{smallmatrix}$  | .                                                        | 1                                                       | $q$                                                     | .                                                       | $q^2$                                                    | $q$                                                     | $q^2$                                                  | .                                                       | $q^2$                                                  | .                                                       | $q^3$                                                  | $q^3$                                                   | $q^4$                                                  | .                                                       | $q^3$                                                  | $q^4$                                                  | .                                                      | $q^5$                                                 |
| $\begin{smallmatrix} 1 \\ 2 \\ 3 & 2 \end{smallmatrix}$  | $q$                                                      | $q$                                                     | $q^2 + 1$                                               | $q$                                                     | $q$                                                      | $2q^2$                                                  | $q^3 + q$                                              | $q^3$                                                   | $q^3$                                                  | $q^2$                                                   | $q^2$                                                  | $q^4 + q^2$                                             | $2q^3$                                                 | $q^4$                                                   | $q^4$                                                  | $q^5 + q^3$                                            | $q^4$                                                  | $q^4$                                                 |
| $\begin{smallmatrix} 1 \\ 3 & 2 \\ 2 \end{smallmatrix}$  | $q^2$                                                    | .                                                       | $q$                                                     | 1                                                       | .                                                        | $q^3$                                                   | $q^2$                                                  | $q^4$                                                   | .                                                      | $q$                                                     | .                                                      | $q^3$                                                   | $q^2$                                                  | $q^5$                                                   | .                                                      | $q^4$                                                  | $q^3$                                                  | .                                                     |
| $\begin{smallmatrix} 1 \\ 3 & 2 & 2 \end{smallmatrix}$   | .                                                        | $q^2$                                                   | $q$                                                     | .                                                       | 1                                                        | $q^3$                                                   | $q^2$                                                  | .                                                       | $q^4$                                                  | .                                                       | $q$                                                    | $q^3$                                                   | $q^2$                                                  | .                                                       | $q^5$                                                  | $q^4$                                                  | .                                                      | $q^3$                                                 |
| $\begin{smallmatrix} 2 & 1 \\ 2 \\ 3 \end{smallmatrix}$  | $q$                                                      | $q$                                                     | $2q^2$                                                  | $q^3$                                                   | $q^3$                                                    | $q^2 + 1$                                               | $q^3 + q$                                              | $q$                                                     | $q$                                                    | $q^4$                                                   | $q^4$                                                  | $q^4 + q^2$                                             | $q^5 + q^3$                                            | $q^2$                                                   | $q^2$                                                  | $2q^3$                                                 | $q^4$                                                  | $q^4$                                                 |
| $\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}$   | $q^2$                                                    | $q^2$                                                   | $q^3 + q$                                               | $q^2$                                                   | $q^2$                                                    | $q^3 + q$                                               | $q^4 + 1$                                              | $q^2$                                                   | $q^2$                                                  | $q^3$                                                   | $q^3$                                                  | $q^5 + q$                                               | $q^4 + q^2$                                            | $q^3$                                                   | $q^3$                                                  | $q^4 + q^2$                                            | $q^3$                                                  | $q^3$                                                 |
| $\begin{smallmatrix} 2 \\ 2 & 1 \\ 3 \end{smallmatrix}$  | $q^2$                                                    | .                                                       | $q^3$                                                   | $q^4$                                                   | .                                                        | $q$                                                     | $q^2$                                                  | 1                                                       | .                                                      | $q^5$                                                   | .                                                      | $q^3$                                                   | $q^4$                                                  | $q$                                                     | .                                                      | $q^2$                                                  | $q^3$                                                  | .                                                     |
| $\begin{smallmatrix} 2 & 2 & 1 \\ 3 \end{smallmatrix}$   | .                                                        | $q^2$                                                   | $q^3$                                                   | .                                                       | $q^4$                                                    | $q$                                                     | $q^2$                                                  | .                                                       | 1                                                      | .                                                       | $q^5$                                                  | $q^3$                                                   | $q^4$                                                  | .                                                       | $q$                                                    | $q^2$                                                  | .                                                      | $q^3$                                                 |
| $\begin{smallmatrix} 3 & 1 \\ 2 \\ 2 \end{smallmatrix}$  | $q^3$                                                    | .                                                       | $q^2$                                                   | $q$                                                     | .                                                        | $q^4$                                                   | $q^3$                                                  | $q^5$                                                   | .                                                      | 1                                                       | .                                                      | $q^2$                                                   | $q$                                                    | $q^4$                                                   | .                                                      | $q^3$                                                  | $q^2$                                                  | .                                                     |
| $\begin{smallmatrix} 3 & 1 \\ 2 & 2 \end{smallmatrix}$   | .                                                        | $q^3$                                                   | $q^2$                                                   | .                                                       | $q$                                                      | $q^4$                                                   | $q^3$                                                  | .                                                       | $q^5$                                                  | .                                                       | 1                                                      | $q^2$                                                   | $q$                                                    | .                                                       | $q^4$                                                  | $q^3$                                                  | .                                                      | $q^2$                                                 |
| $\begin{smallmatrix} 2 \\ 3 & 1 \\ 2 \end{smallmatrix}$  | $q^3$                                                    | $q^3$                                                   | $q^4 + q^2$                                             | $q^3$                                                   | $q^3$                                                    | $q^4 + q^2$                                             | $q^5 + q$                                              | $q^3$                                                   | $q^3$                                                  | $q^2$                                                   | $q^2$                                                  | $q^4 + 1$                                               | $q^3 + q$                                              | $q^2$                                                   | $q^2$                                                  | $q^3 + q$                                              | $q^2$                                                  | $q^2$                                                 |
| $\begin{smallmatrix} 3 & 2 & 1 \\ 2 \end{smallmatrix}$   | $q^4$                                                    | $q^4$                                                   | $2q^3$                                                  | $q^2$                                                   | $q^2$                                                    | $q^5 + q^3$                                             | $q^4 + q^2$                                            | $q^4$                                                   | $q^4$                                                  | $q$                                                     | $q$                                                    | $q^3 + q$                                               | $q^2 + 1$                                              | $q^3$                                                   | $q^3$                                                  | $2q^2$                                                 | $q$                                                    | $q$                                                   |
| $\begin{smallmatrix} 2 \\ 2 \\ 3 & 1 \end{smallmatrix}$  | $q^3$                                                    | .                                                       | $q^4$                                                   | $q^5$                                                   | .                                                        | $q^2$                                                   | $q^3$                                                  | $q$                                                     | .                                                      | $q^4$                                                   | .                                                      | $q^2$                                                   | $q^3$                                                  | 1                                                       | .                                                      | $q$                                                    | $q^2$                                                  | .                                                     |
| $\begin{smallmatrix} 2 & 2 \\ 3 & 1 \end{smallmatrix}$   | .                                                        | $q^3$                                                   | $q^4$                                                   | .                                                       | $q^5$                                                    | $q^2$                                                   | $q^3$                                                  | .                                                       | $q$                                                    | .                                                       | $q^4$                                                  | $q^2$                                                   | $q^3$                                                  | .                                                       | 1                                                      | $q$                                                    | .                                                      | $q^2$                                                 |
| $\begin{smallmatrix} 2 \\ 3 & 2 & 1 \end{smallmatrix}$   | $q^4$                                                    | $q^4$                                                   | $q^5 + q^3$                                             | $q^4$                                                   | $q^4$                                                    | $2q^3$                                                  | $q^4 + q^2$                                            | $q^2$                                                   | $q^2$                                                  | $q^3$                                                   | $q^3$                                                  | $q^3 + q$                                               | $2q^2$                                                 | $q$                                                     | $q$                                                    | $q^2 + 1$                                              | $q$                                                    | $q$                                                   |
| $\begin{smallmatrix} 3 & 2 \\ 2 & 1 \end{smallmatrix}$   | $q^5$                                                    | .                                                       | $q^4$                                                   | $q^3$                                                   | .                                                        | $q^4$                                                   | $q^3$                                                  | $q^3$                                                   | .                                                      | $q^2$                                                   | .                                                      | $q^2$                                                   | $q$                                                    | $q^2$                                                   | .                                                      | $q$                                                    | 1                                                      | .                                                     |
| $\begin{smallmatrix} 3 & 2 & 2 & 1 \end{smallmatrix}$    | .                                                        | $q^5$                                                   | $q^4$                                                   | .                                                       | $q^3$                                                    | $q^4$                                                   | $q^3$                                                  | .                                                       | $q^3$                                                  | .                                                       | $q^2$                                                  | $q^2$                                                   | $q$                                                    | .                                                       | $q^2$                                                  | $q$                                                    | .                                                      | 1                                                     |

FIGURE 21. The block  $(1, 2, 1)$  of the  $q$ -Cartan matrix of  $\mathcal{H}_{4,4}(0)$ .

$$\begin{array}{cc}
 \begin{array}{c} \boxed{1} \\ \boxed{1} \end{array} & \boxed{111} \\
 (2) & \cdot \quad 1 \\
 (11) & 1 \quad \cdot
 \end{array}
 \qquad
 \begin{array}{cc}
 \begin{array}{c} \boxed{1} \\ \boxed{2} \end{array} & \boxed{211} \\
 (1,1) & 1 \quad 1
 \end{array}$$

 FIGURE 22. The two blocks (2), (1, 1) of the decomposition matrices of  $\mathcal{H}_{2,2}(0)$ .

$$\begin{array}{ccccc}
 \begin{array}{c} \boxed{1} \\ \boxed{1} \\ \boxed{1} \end{array} & \begin{array}{c} \boxed{1} \\ \boxed{11} \end{array} & \begin{array}{c} \boxed{11} \\ \boxed{1} \end{array} & \boxed{1111} & \\
 (3) & \cdot & \cdot & \cdot & 1 \\
 (21) & \cdot & 1 & 1 & \cdot \\
 (111) & 1 & \cdot & \cdot & \cdot
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \begin{array}{c} \boxed{1} \\ \boxed{1} \\ \boxed{2} \end{array} & \begin{array}{c} \boxed{11} \\ \boxed{2} \end{array} & \begin{array}{c} \boxed{1} \\ \boxed{21} \end{array} & \begin{array}{c} \boxed{21} \\ \boxed{1} \end{array} & \boxed{2111} \\
 (2,1) & \cdot & 1 & 1 & \cdot & 1 \\
 (11,1) & 1 & \cdot & 1 & 1 & \cdot
 \end{array}$$
  

$$\begin{array}{ccccc}
 \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{2} \end{array} & \begin{array}{c} \boxed{1} \\ \boxed{22} \end{array} & \begin{array}{c} \boxed{21} \\ \boxed{2} \end{array} & \begin{array}{c} \boxed{2} \\ \boxed{21} \end{array} & \boxed{2211} \\
 (1,2) & \cdot & 1 & 1 & \cdot & 1 \\
 (1,11) & 1 & \cdot & 1 & 1 & \cdot
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array} & \begin{array}{c} \boxed{1} \\ \boxed{32} \end{array} & \begin{array}{c} \boxed{21} \\ \boxed{3} \end{array} & \begin{array}{c} \boxed{31} \\ \boxed{2} \end{array} & \begin{array}{c} \boxed{2} \\ \boxed{31} \end{array} & \boxed{3211} \\
 (1,1,1) & 1 & 1 & 1 & 1 & 1 & 1
 \end{array}$$

 FIGURE 23. The blocks (3), (2, 1), (1, 2), and (1, 1, 1) of the decomposition matrix of  $\mathcal{H}_{3,3}(0)$ .



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