HECKE GROUP ALGEBRAS AS QUOTIENTS OF AFFINE HECKE ALGEBRAS AT LEVEL 0

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ABSTRACT. The Hecke group algebra $\mathrm{H}\mathring{W}$ of a finite Coxeter group $\mathring{W}$, as introduced by the first and last authors, is obtained from $\mathring{W}$ by gluing appropriately its 0-Hecke algebra and its group algebra. In this paper, we give an equivalent alternative construction in the case when $\mathring{W}$ is the finite Weyl group associated to an affine Weyl group W. Namely, we prove that, for q not a root of unity of small order, $\mathrm{H}\mathring{W}$ is the natural quotient of the affine Hecke algebra $\mathrm{H}(W)(q)$ through its level 0 representation.

The proof relies on the following core combinatorial result: at level 0 the 0-Hecke algebra $\mathrm{H}(W)(0)$ acts transitively on $\mathring{W}$. Equivalently, in type A, a word written on a circle can be both sorted and antisorted by elementary bubble sort operators. We further show that the level 0 representation is a calibrated principal series representation $M(t)$ for a suitable choice of character t, so that the quotient factors (non-trivially) through the principal central specialization. This explains in particular the similarities between the representation theory of the 0-Hecke algebra $\mathrm{H}(\mathring{W})(0)$ and that of the affine Hecke algebra $\mathrm{H}(W)(q)$ at this specialization.

1. Introduction

The starting point of this research lies in the striking similarities between the representation theories of the degenerate (Iwahori)-Hecke algebras on one side and of the principal central specialization of the affine Hecke algebras on the other. For the sake of simplicity, we describe those similarities for type A in this introduction, but they carry over straightforwardly to any affine Weyl group W and its associated finite Weyl group $\mathring{W}$.

The representation theory of the degenerate Hecke algebras $\mathrm{H}_n(0)$ for general type has been worked out by Norton [Nor79] and special combinatorial features of type A have been described by Carter [Car86]. In particular, the projective modules P_I of the type A degenerate Hecke algebra $\mathrm{H}_n(0)$ are indexed by subsets I of $\{1,\dots,n-1\}$, and the basis of each P_I is indexed by those permutations of n whose descent set is I.

On the other hand, the classification of the irreducible finite-dimensional representations of the affine Hecke algebra $\widetilde{\mathrm{H}}_n(q)$ is due to Zelevinsky [Zel80]. They are indexed by simple combinatorial objects called multisegments. However, in this work, we are interested in a particular subcategory related to a central specialization for which the multisegments are also in bijection with subsets of $\{1,\dots,n-1\}$. This relation is as follows. It is well known from Bernstein and Zelevinsky [BZ77] and Lusztig [Lus83], that the center of the affine Hecke algebra is the ring of symmetric

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polynomials $\mathbb{C}[Y_1, \ldots, Y_n]^{\mathfrak{S}_n}$ in some particular elements $Y_1, \ldots, Y_n$ such that as vector space,

$$\widetilde{\mathrm{H}}_n(q) \simeq \mathrm{H}_n(q) \otimes \mathbb{C}[Y_1, \ldots, Y_n]\,. \tag{1}$$

As a center, it acts by scalar multiplication in all irreducible representations, and one way to select a particular class of representations is to specialize the center in the algebra itself. Thus any ring morphism from $\mathbb{C}[Y_1, \ldots, Y_n]^{\mathfrak{S}_n}$ to $\mathbb{C}$, or in other words any scalar alphabet, defines a quotient of the affine Hecke algebra of dimension

$$\dim\left(\mathrm{H}_n(q)\right)\, \dim\left(\mathbb{C}[Y_1, \ldots, Y_n]/\mathbb{C}[Y_1, \ldots, Y_n]^{\mathfrak{S}_n}\right) = n!^2\,. \tag{2}$$

Let us denote by $\mathcal{H}_n(q)$ the quotient of $\widetilde{\mathrm{H}}_n(q)$ obtained by the principal specialization of its center to the alphabet $\frac{1-q^n}{1-q} := \{1, q, \ldots, q^{n-1}\}$, that is

$$\mathcal{H}_n(q) := \widetilde{\mathrm{H}}_n(q) \;/\; \langle e_i(Y_1, \ldots, Y_n) - e_i(1, q, \ldots, q^{n-1}) \mid i = 1, \ldots, n \rangle\,, \tag{3}$$

where e_i denote the elementary symmetric polynomials. Then, in this particular case, the multisegments of Zelevinsky are in bijection with subsets I of $\{1, \ldots, n-1\}$ and the irreducible representations S_I of $\mathcal{H}_n(q)$ have their bases indexed by descent classes of permutations. Thus one expects a strong link between $\mathrm{H}_n(0)$ and $\mathcal{H}_n(q)$.

The goal of this paper is to explain this relation by means of the Hecke group algebra $\mathrm{H}\mathring{W}$ introduced by the first and the last authors [HT06, HT08]. Indeed, by definition, $\mathrm{H}\mathring{W}$ contains naturally the degenerated Hecke algebra $\mathrm{H}(\mathring{W})(0)$ and it was shown that the simple modules of $\mathrm{H}\mathring{W}$, when restricted to $\mathrm{H}(\mathring{W})(0)$ form a complete family of projective ones. The relation comes from the fact that there is a natural surjective morphism from the affine Hecke algebra $\mathrm{H}(W)(q)$ to $\mathrm{H}\mathring{W}$. As a consequence the simple modules of $\mathrm{H}\mathring{W}$ are also simple modules of $\mathrm{H}(W)(q)$ elucidating the similarities. This can be restated as follows:

Theorem 1.1. *For q not a root of unity, there is a particular finite-dimensional quotient $\mathrm{H}\mathring{W}$ of the affine Hecke algebra $\mathrm{H}(W)(q)$ which contains the 0-Hecke algebra $\mathrm{H}(\mathring{W})(0)$ and such that any simple $\mathrm{H}\mathring{W}$ module is projective when restricted to $\mathrm{H}(\mathring{W})(0)$.*

The remainder of this paper is structured as follows.

In Sections 2 and 3, we briefly review the required material on Coxeter groups, Hecke algebras, and Hecke group algebras, as well as on the central theme of this paper: the level 0 action of an affine Weyl group W on the associated finite Weyl group $\mathring{W}$ and the corresponding level 0 representation of the affine Hecke algebra on $\mathbb{C}\mathring{W}$.

In Section 4, we prove the core combinatorial property (Theorem 4.2) which states that, at level 0, the affine 0-Hecke algebra $\mathrm{H}(W)(0)$ acts transitively on the chambers of $\mathring{W}$ (or equivalently on the finite Weyl group). We first treat type A where Theorem 4.2 states that a word written on a circle can be both sorted and antisorted by elementary bubble sort operators (explicit (anti)sorting algorithms are also provided for types B, C, and D). We proceed with a type-free geometric proof of Theorem 4.2. The ideas used in the proof are inspired by private notes on finite-dimensional representations of quantized affine algebras by

Kashiwara [Kas08], albeit reexpressed in terms of alcove walks. We also mention connections with affine crystals.

In Section 5 we prove the main result of the paper, namely that for q not a root of unity of small order, the Hecke group algebra is the natural quotient of the (extended) affine Hecke algebra through its representation at level 0 (Theorem 5.1). The proof relies on the results from the subsequent sections, namely Corollary 6.2 for $q = 0$ and Theorem 7.7 for q non-zero and not a root of unity. Both yield a proof for generic q.

In Section 6, we derive new sets of generators for the Hecke group algebra of a finite Weyl which, together with the combinatorial results of Section 4 give Corollary 6.2.

Unlike for the affine Weyl group W, and interestingly enough, the torus Y does not degenerate trivially. In Section 7, we describe precisely this degeneracy, and show that, for a suitable choice of character on Y, the level 0 representation is a calibrated principal series representation (Theorem 7.1). This allows to us refine Theorem 5.1 to q not a root of unity.

Altogether, Theorems 5.1 and 7.1 can be interpreted as two new equivalent alternative constructions of the Hecke group algebra, while the latter provides a parametrized family of maximal decompositions of its identity into idempotents (Corollary 7.4).

2. Coxeter groups, Hecke algebras, and Hecke group algebras

In this and the next section, we briefly recall the notations and properties of Coxeter groups, (affine) Weyl groups, their Hecke and Hecke group algebras, as well as root systems and alcove walks that we need in the sequel. For further reading on those topics, we refer the reader to [Hum90, Kac90, Mac03, BB05, Ram06].

2.1. **Coxeter groups and their geometric representations.** Let W be a Coxeter group and I the index set of its Dynkin diagram. Denote by $(s_i)_{i\in I}$ its simple reflections and by w_0 its maximal element (when W is finite). A presentation of W is given by the generators s_i together with their quadratic and braid-like relations:

$$s_i^2 = 1 \qquad \text{and} \qquad \underbrace{s_i s_j \cdots}_{m(i,j)} = \underbrace{s_j s_i \cdots}_{m(i,j)} \quad \text{for } i \neq j, \tag{4}$$

where the $m(i,j)$'s are integers depending on W.

For $J \subset I$, write W_J for the parabolic subgroup generated by $(s_i)_{i\in J}$. The left and right descent sets of an element $w \in W$ are respectively

$$\mathrm{D}_L(w) := \{i \in I \mid s_i w < w\} \qquad \text{and} \qquad \mathrm{D}_R(w) := \{i \in I \mid w s_i < w\}\,.$$

The Coxeter group W can be realized geometrically as follows. Take the module $\mathfrak{h}^* := \mathfrak{h}^*_{\mathbb{K}} := \bigoplus_{i\in I} \mathbb{K}\alpha_i$ and its $\mathbb{K}$-dual $\mathfrak{h} := \mathfrak{h}_{\mathbb{K}} := \bigoplus_{i\in I} \mathbb{K}\Lambda_i^\vee$, with the natural pairing $\langle \Lambda_i^\vee, \alpha_j\rangle = \delta_{ij}$. The α_i are the *simple roots*, and the $\Lambda_i^\vee$ the *fundamental coweights*. The *simple coroots* are given by $\alpha_i^\vee := \sum_j a_{i,j}\Lambda_i^\vee$, where $M = (a_{i,j})_{i,j\in I}$ with $a_{i,j} = \langle \alpha_i^\vee, \alpha_i\rangle$ is the (generalized) Cartan matrix for W with coefficients in a ring $\mathbb{K} \subset \mathbb{R}$. The Coxeter group acts on $\mathfrak{h}$ by the number game:

$$s_i(x^\vee) := x^\vee - \langle x^\vee, \alpha_i\rangle \alpha_i^\vee \quad \text{for } x^\vee \in \mathfrak{h}, \tag{5}$$

and on $\mathfrak{h}^*$ by the dual number game:

$$s_i(x) := x - \langle \alpha_i^\vee, x\rangle \alpha_i \quad \text{for } x \in \mathfrak{h}^*. \tag{6}$$

Denote by $R := \{w(\alpha_i) \mid w \in W, i \in I\}$ the set of *roots*, and by $R^\vee := \{w(\alpha_i^\vee) \mid w \in W, i \in I\}$ the set of *coroots*. To each root α corresponds the reflection s_α across the associated coroot $\alpha^\vee$ and along the hyperplane H_α which splits $\mathfrak{h}$ into a positive H_α^+ and a negative half-space H_α^-:

$$\begin{aligned} H_\alpha &:= \{x^\vee \in \mathfrak{h} \mid \langle x^\vee, \alpha\rangle = 0\}\,, \\ H_\alpha^+ &:= \{x^\vee \in \mathfrak{h} \mid \langle x^\vee, \alpha\rangle > 0\}\,, \\ H_\alpha^- &:= \{x^\vee \in \mathfrak{h} \mid \langle x^\vee, \alpha\rangle < 0\}\,. \end{aligned} \tag{7}$$

Take now $\mathbb{K} = \mathbb{R}$. Define the *fundamental chamber* as the open simplicial cone $C := \{x^\vee \mid \langle x^\vee, \alpha_i\rangle > 0, \forall i \in I\}$. For each root α, the fundamental chamber C lies either entirely in H_α^+ or in H_α^-; R splits accordingly into the sets of *positive roots* $R^+ := \{\alpha \in R \mid C \subseteq H_\alpha^+\}$ and of *negative roots* $R^- := \{\alpha \in R \mid C \subseteq H_\alpha^-\} = -R^+$.

The closure $\overline{C}$ of C is a fundamental domain for the action of W on the *Tits cone* $U := \bigcup_{w\in W} w(\overline{C})$, and the elements w of W are in bijection with the *chambers* $w(C)$. This bijection induces both a left and a right actions of W on the chambers. The right action is particularly nice as the chambers $w(C)$ and $w(C).s_i = ws_i(C)$ share a common wall. Any sequence $i_1, \dots, i_r$ gives therefore rise to a sequence of adjacent chambers $C,\ s_{i_1}(C),\ s_{i_1}s_{i_2}(C),\ \dots,\ (s_{i_1}\cdots s_{i_r})(C)$ from C to $w(C)$ (where $w = s_{i_1}\cdots s_{i_r}$), called a *gallery*. For short, we often denote this gallery by just $i_1, \dots, i_r$.

2.2. **(Iwahori)-Hecke algebras.** Let W be a Coxeter group and q_1 and q_2 two complex numbers. When defined, set $q =: -\frac{q_1}{q_2}$. The (generic, Iwahori) (q_1, q_2)-Hecke algebra $\mathrm{H}(W)(q_1, q_2)$ of W is the $\mathbb{C}$-algebra generated by the operators T_i subject to the quadratic and braid-like relations:

$$(T_i - q_1)(T_i - q_2) = 0 \qquad \text{and} \qquad \underbrace{T_iT_j\cdots}_{m(i,j)} = \underbrace{T_jT_i\cdots}_{m(i,j)} \quad \text{for } i \neq j. \tag{8}$$

Its dimension is $|W|$, and a basis is given by the elements $T_w := T_{i_1}\cdots T_{i_r}$ where $w \in W$ and $i_1, \dots, i_r$ is a reduced word for w. The right regular representation of $\mathrm{H}(W)(q_1, q_2)$ is given by

$$T_wT_i = \begin{cases} (q_1+q_2)T_w - q_1q_2T_{ws_i} & \text{if } i \text{ descent of } w, \\ T_{ws_i} & \text{otherwise.} \end{cases} \tag{9}$$

Define the unique operators $\overline{T}_i$ such that $T_i + \overline{T}_i = q_1 + q_2$. They satisfy the same relations as the T_i, and further $T_i\overline{T}_i = \overline{T}_iT_i = q_1q_2$.

At $q_1 = 1, q_2 = -1$ (so $q = 1$), we recover the usual group algebra $\mathbb{C}[W]$ of W; in general, when $q_1 + q_2 = 0$ one still recovers $\mathbb{C}[W]$ up to a scaling of the generators: $s_i = \frac{1}{q_1}T_i$. Note that when q_1 and q_2 are non-zero and q is not a root of unity $\mathrm{H}(W)(q_1, q_2)$ is still isomorphic to $\mathbb{C}[W]$, but in a non-trivial way. On the opposite side, taking $q_1 = 0$ and $q_2 \neq 0$ (so $q = 0$) yields the 0-*Hecke algebra* $\mathrm{H}(W)(0)$; it is also a monoid algebra for the 0-*Hecke monoid* $\{\pi_w \mid w \in W\}$ generated by the idempotents $\pi_i := \frac{1}{q_2}T_i$. At $q_1 = q_2 = 0$, one obtains the nilCoxeter algebra. Traditionally, and depending on the application in mind, different authors choose different specializations of q_1 and q_2, typically $q_1 = q$ and $q_2 = -1$ (cf. [Wik08]), or $q_1 = t^{\frac{1}{2}}$ and $q_2 = t^{-\frac{1}{2}}$ (cf. for example [RY08]). For our needs, keeping the two eigenvalues generic yields more symmetrical formulas which are also easier to specialize to other conventions. There also exists a more general definition of the

Hecke algebra by allowing a different pair of parameters (q_1, q_2) for each conjugacy class of reflections in W. For the sake of simplicity, we did not try to extend the results presented in this paper to this larger setting, but would not expect specific difficulties either.

We may realize the 0-Hecke monoid geometrically on $\mathfrak{h}$ as follows. For each $i \in I$, define the (half-linear) idempotent π_i (resp. $\overline{\pi}_i$) which projects onto the negative (resp. positive) half space with respect to the root α_i:

$$(10)\quad \pi_i(x^\vee) := \begin{cases} s_i(x^\vee) & \text{if } \langle x^\vee, \alpha_i\rangle > 0, \\ x^\vee & \text{otherwise;} \end{cases} \qquad \overline{\pi}_i(x^\vee) := \begin{cases} s_i(x^\vee) & \text{if } \langle x^\vee, \alpha_i\rangle < 0, \\ x^\vee & \text{otherwise.} \end{cases}$$

As with the reflection s_i, these projections map chambers to chambers. None of the projections $\pi_1, \dots, \pi_n$ fix the fundamental chamber, and (when W is finite) all of them fix the negative chamber. The correspondence between chambers and Weyl group elements induces an action on the group W itself: this is the usual right regular actions of the 0-Hecke monoid, where π_i adds a left descent at position i if it is not readily there, and $\overline{\pi}_i$ does the converse. The action of the π_i's can be depicted by a graph on W, with an i-arrow from w to w' if $\pi_i(w) = w'$. Examples of such graphs are given in Figure 3 (ignoring the 0-arrows).

Let $\mathbb{C}W$ be the vector space of dimension $|W|$ spanned by W. Except for the nilCoxeter algebra ($q_1 = q_2 = 0$), the Hecke algebra $\mathrm{H}(W)(q_1, q_2)$ can be realized as acting on $\mathbb{C}W$ by interpolation, mapping T_i to $(q_1+q_2)\pi_i - q_1 s_i$. This amounts to identify $\mathbb{C}W$ with the right regular representation of $\mathrm{H}(W)(q_1, q_2)$ via $w \mapsto q_2^{-\ell(w)} T_w$, where $\ell(w)$ is the length of w. Through this mapping, $\overline{T}_i = (q_1+q_2)\overline{\pi}_i - q_2 s_i$.

2.3. **Hecke group algebras.** Let now W be a finite Coxeter group. As we have just seen, we may embed simultaneously the Hecke algebra $\mathrm{H}(W)(0)$ and the group algebra $\mathbb{C}[W]$ in $\mathrm{End}(\mathbb{C}W)$, via their right regular representations. The *Hecke group algebra* $\mathrm{H}W$ of W is the smallest subalgebra of $\mathrm{End}(\mathbb{C}W)$ containing them both (see [HT08]). It is therefore generated by $(\pi_i)_{i\in I}$ and $(s_i)_{i\in I}$, and by interpolation it contains all q_1, q_2-Hecke algebras where $(q_1, q_2) \neq (0, 0)$[1].

A basis for $\mathrm{H}W$ is given by $\{w\pi_{w'} \mid \mathrm{D}_R(w) \cap \mathrm{D}_L(w') = \emptyset\}$. A more conceptual characterization is as follows: call a vector v in $\mathbb{C}W$ *i-left antisymmetric* if $s_i v = -v$; then, $\mathrm{H}W$ is the subalgebra of $\mathrm{End}(\mathbb{C}W)$ of those operators which preserve all i-left antisymmetries [HT08].

3. Affine Weyl groups, Hecke algebras, and their level 0 actions

Now let W be an affine Weyl group, with index set $I := \{0, \dots, n\}$ and Cartan matrix M. We always assume that W is irreducible. We denote respectively by a_i and $a_i^\vee$ the coefficients of the canonical linear combination annihilating the columns and rows of M, respectively.

In the sequel, we stick to the number game / dual number game geometric setting of Section 2.1. (see also Figure 1) This differs slightly from the usual setting for affine or Kac-Moody Lie algebras [Kac90]; it turns out to be simpler yet sufficient for our purpose. Note first that $R := \{w(\alpha_i) \mid w \in W, i \in I\}$ is the set of *real*

[1]However, the nilCoxeter algebra does not embed naturally. More precisely, up to a scalar there is a single nilpotent element $d_i := 1 + s_i - 2\pi_i$ in the algebraic span of s_i and π_i. A direct calculation shows that, for example, d_1 and d_2 do not satisfy the braid relations.

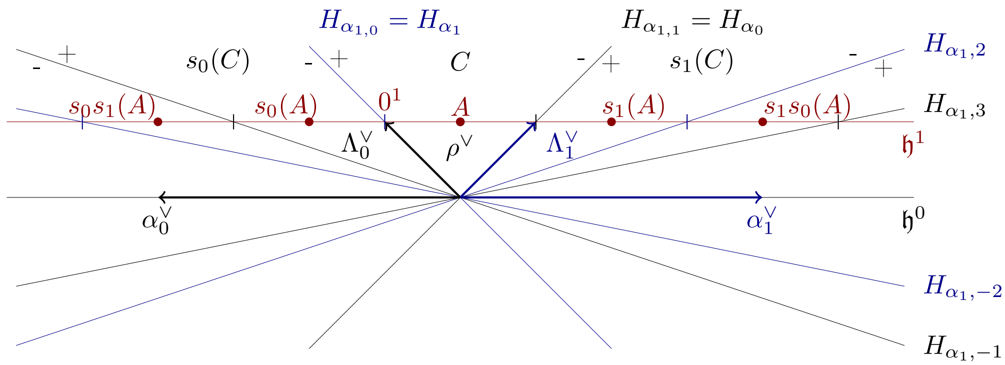

FIGURE 1. Realization of the alcove picture at the level 1 hyperplane $\mathfrak{h}^1$ of the coweight space $\mathfrak{h}$ in type $A_1^{(1)}$.

roots; by abuse, we call them roots, as the imaginary roots do not play a role for our purposes. The geometric representations $\mathfrak{h}^*_{\mathbb{Z}}$ and $\mathfrak{h}$ defined in Section 2.1 correspond to the root lattice and the coweight space, respectively; we do not use the central extension by $c := \sum_{i=0}^n a_i^\vee \alpha_i^\vee$. As a consequence, the coroot lattice $\bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$ does not embed faithfully in $\mathfrak{h}_{\mathbb{Z}}$ (since $c = 0$ in $\mathfrak{h}_{\mathbb{Z}}$). In particular, the set of coroots $R^\vee$ is finite, and (essentially) coincides with the set $\mathring{R}^\vee$ of coroots of $\mathring{W}$. We also keep separate the dual lattices, without embedding them in a single ambient space endowed with an inner product.

3.1. **Affine Weyl groups and alcove walks.** Let $\delta := \sum_{i\in I} a_i\alpha_i$ be the so-called *null root*[2]. The level of an element $x^\vee$ of $\mathfrak{h}$ is given by $\ell(x^\vee) = \langle x^\vee, \delta\rangle$; in particular, and by construction, all the coroots are of level 0. Since δ is fixed by W, the affine hyperplanes $\mathfrak{h}^\ell := \{x^\vee \mid \langle x^\vee, \delta\rangle = \ell\}$ are stabilized by W.

At level 0, the action cl of the affine Weyl group W on $\mathfrak{h}^0$ reduces to that of a finite Weyl group $\mathring{W} := \mathrm{cl}(W)$; in fact $\mathring{W} = \langle s_1, \dots, s_n\rangle$, assuming an appropriate labeling of the Dynkin diagram. This induces a right action of W on $\mathring{W}$: for w in $\mathring{W}$ and $s_i \in W$, $w.s_i := w\mathrm{cl}(s_i)$, where $\mathrm{cl} : W \to \mathring{W}$ denotes the canonical quotient map. We denote respectively by $\mathring{R} := \{w(\alpha_i) \mid w \in W, i = 1, \dots, n\}$ and $\mathring{R}^\vee := \{w(\alpha_i^\vee) \mid w \in W, i = 1, \dots, n\}$ the sets of roots and coroots of $\mathring{W}$. The coroot $\alpha_0^\vee$ is of the form $\alpha_0^\vee = \epsilon\alpha^\vee$ where $\alpha^\vee \in \mathring{R}^{\vee+}$ and $\epsilon < 0$. In the untwisted case, $\epsilon = -1$ so that $R^\vee = \mathring{R}^\vee$. In the other cases $R^\vee$ and $\mathring{R}^\vee$ may differ by the orbit of $\alpha_0^\vee$.

The reflections in W are given by

$$\{s_{\alpha,m} := s_{\alpha - m\delta} \mid \alpha \in \mathring{R}^+ \text{ and } m \in c_\alpha\mathbb{Z}\}\,. \tag{11}$$

Here $s_{\alpha,m}$ is the reflection across the hyperplane $H_{\alpha,m} := H_{\alpha-m\delta}$ along the coroot $\alpha^\vee$ of $\mathring{W}$, and $c_\alpha \in \mathbb{Q}$ ($c_\alpha = 1$ always in the untwisted case; for the twisted case see Kac [Kac90, Proposition 6.5]).

[2]Beware that this is not a root in the current setting!

At level ℓ, each positive root α of $\mathring{W}$ gives rise to a family $(H^\ell_{\alpha,m})_{m\in c_\alpha\mathbb{Z}}$ of parallel reflection hyperplanes (which all collapse to H^0_α at level 0):

$$H^\ell_{\alpha,m} := H_{\alpha-m\delta}\cap\mathfrak{h}^\ell = \{x^\vee\in\mathfrak{h}^\ell \mid \langle x^\vee,\alpha\rangle = \ell m\}\,. \tag{12}$$

The Tits cone is $\{x^\vee \mid \langle x^\vee,\delta\rangle > 0\}$, and slicing it at level $\ell > 0$ gives rise to the *alcove picture* (see Figure 1). The *fundamental alcove* $A := C\cap\mathfrak{h}^\ell$ is a simplex, and the *alcoves* $w(A)$ in its orbit form a tessellation of $\mathfrak{h}^\ell$. Each gallery $C, s_{i_1}(C),\dots,(s_{i_1}\cdots s_{i_r})(C)$ induces an *alcove walk* $A, s_{i_1}(A),\dots,(s_{i_1}\cdots s_{i_r})(A)$. As for galleries, we often denote this alcove walk by just $i_1,\dots,i_r$.

For a simple coroot $\alpha_i^\vee$, let $c_i = c_{\alpha_i}$ and define $t_{\alpha_i^\vee} = s_{\alpha_i,c_i}s_{\alpha_i,0}$; at level ℓ, $t_{\alpha_i^\vee}$ is the composition of two reflections along parallel hyperplanes, and acts as a translation by $c_i\ell\alpha_i^\vee$. For any $\lambda^\vee = \sum_{i=1}^n \lambda_i\alpha_i^\vee$ in the coroot lattice $\mathring{R}^\vee$ of $\mathring{W}$, set $c(\lambda^\vee) = \sum_{i=1}^n c_i\lambda_i\alpha_i^\vee$. Then, in general, $t_{\lambda^\vee}:\mathfrak{h}\to\mathfrak{h}$ defined by

$$t_{\lambda^\vee}(x^\vee) = x^\vee + \ell(x^\vee)c(\lambda^\vee) \tag{13}$$

belongs to W. More specifically, $t_{\lambda^\vee} = s_{i_1}\cdots s_{i_r}$, where $i_1,\dots,i_r$ is an alcove walk from A to the translated alcove $t_{\lambda^\vee}A$. By abuse, we call $t_{\lambda^\vee}$ a *translation* of W. This gives the usual semi-direct product decomposition $W = \mathring{W}\rtimes\mathring{R}^\vee$. In particular, $\mathrm{cl}: W\mapsto\mathring{W}$ is the group morphism which kills the translations $t_{\lambda^\vee}$, $\lambda^\vee\in\mathring{R}^\vee$.

The *fundamental chamber* for $\mathring{W}$ is the open simplicial cone

$$\{x^\vee\in\mathfrak{h}^\ell \mid \langle x^\vee,\alpha_i\rangle > 0, \forall i = 1,\dots,n\}\,.$$

We denote by 0^ℓ the intersection point of its walls $(H^\ell_{\alpha_i})_{i=1,\dots,n}$. The orientation of the alcove walls is the periodic orientation where only points infinitely deep inside the fundamental chamber for $\mathring{W}$ is on the positive side of all walls. Consider an *i-crossing* for $i\in\{0,\dots,n\}$ from an alcove $w(A)$ to the adjacent alcove $ws_i(A)$, and let $H_{\alpha,m}$ the crossed affine wall. The crossing is *positive* if $ws_i(A)$ is on the positive side of $H_{\alpha,m}$, and *negative* otherwise. For an alcove walk $i_1,\dots,i_k$, define $\epsilon_1,\dots,\epsilon_r$ by $\epsilon_k = 1$ if the kth crossing is positive and -1 otherwise.

The *height* of an alcove $w(A)$ is given by $\mathrm{ht}(w(A)) = \frac{1}{2}(\epsilon_1+\cdots+\epsilon_k)$, for any alcove walk $i_1,\dots,i_k$ from A to $w(A)$. This is well-defined, since $\epsilon_1+\cdots+\epsilon_k$ counts the number of hyperplanes $H_{\alpha,m}$ separating A from $w(A)$, where those with $w(A)$ on the positive side are counted positively, and the others negatively.

Remark 3.1. *The height of the alcove $t_{\lambda^\vee}(A)$ coincides with the* height *of the coroot $\lambda^\vee$ of $\mathring{W}$, $\mathrm{ht}(\lambda^\vee) := \langle\lambda^\vee,\mathring{\rho}\rangle$, where $\mathring{\rho} := \frac{1}{2}\sum_{\alpha\in\mathring{R}^+}\alpha$. In particular, a coroot is of height one if and only if it is a simple coroot ($\mathring{\rho}$ is also the sum of the fundamental weights of $\mathring{W}$).*

Proof. For each positive root α of $\mathring{W}$, the family of parallel hyperplanes $(H_{\alpha,m})_{m\in c_\alpha\mathbb{Z}}$ contributes to $\epsilon_1+\cdots+\epsilon_k$ the (relative) number of those separating $\frac{\ell}{n+1}\rho^\vee$ and $\frac{\ell}{n+1}\rho^\vee + \ell c(\lambda^\vee)$; this is given by $\langle\lambda^\vee,\alpha\rangle$. The result follows by summing up over all positive roots. □

3.2. **Affine Hecke algebras.** The affine Hecke algebra of W is $\mathrm{H}(W)(q_1,q_2)$. In particular, it is isomorphic to $\mathrm{H}(\mathring{W})(q_1,q_2)\otimes\mathbb{C}[Y]$, where

$$\mathbb{C}[Y] := \mathbb{C}.\{Y^{\lambda^\vee} \mid \lambda^\vee\in\mathring{R}^\vee\} \tag{14}$$

is the group algebra of the coroot lattice. The $Y^{\lambda^\vee}$'s have an expression in terms of the T_i's which generalizes that for translations $t_{\lambda^\vee}$ in the affine Weyl group [Mac03, Equation (3.2.10)]:

$$Y^{\lambda^\vee} := (\frac{1}{\sqrt{-q_1q_2}}T_{i_1})^{\epsilon_1}\cdots(\frac{1}{\sqrt{-q_1q_2}}T_{i_r})^{\epsilon_r} = (-q_1q_2)^{-\operatorname{ht}(\lambda^\vee)}T_{i_1}^{\epsilon_1}\cdots T_{i_r}^{\epsilon_r}\,, \tag{15}$$

where $i_1,\dots,i_r$ is an alcove walk from A to $t_{\lambda^\vee}(A)$. The center of $\mathrm{H}(W)(q_1,q_2)$ is the subring of invariants $Y^W := \{p\in Y \mid p.w = p\}$. In type A, this is the ring of symmetric functions.

As for W, the geometric realization at level 0 induces an action cl of the 0-Hecke monoid $\langle \pi_i \mid i\in I\rangle$ on the chambers of $\mathring{W}$, and therefore on $\mathring{W}$ itself:

$$w.\mathrm{cl}(\pi_i) := \begin{cases} ws_i & \text{if } \pi_i(w^{-1}(\mathring{\rho}^\vee)) = w^{-1}(\mathring{\rho}^\vee)\text{, that is } \langle w^{-1}(\mathring{\rho}^\vee),\alpha_i\rangle > 0,\\ w & \text{otherwise,}\end{cases} \tag{16}$$

where $\mathring{\rho}^\vee = \frac{1}{2}\sum_{\alpha^\vee\in\mathring{R}^\vee}\alpha^\vee$ is the canonical representative of the fundamental chamber of $\mathring{R}^\vee$. Geometrically, it can be interpreted as a quotient of the action at level ℓ by identifying a point in a chamber at level 0 with a point infinitely deep inside the corresponding chamber for $\mathring{W}$ at level ℓ. We recognize the usual action of $\pi_1,\dots,\pi_n$, where $w.\pi_i = w$ if i is a (right) descent of w and $w.\pi_i = ws_i$ otherwise. By extension 0 is called an *(affine) descent* if $w.\pi_0 = w$. Since there is no ambiguity, we write $w.\pi_i$ for $w.\mathrm{cl}(\pi_i)$. Let us relate affine descents and positivity of crossings.

Remark 3.2. *Consider an* i-crossing *for* $i\in\{0,\dots,n\}$ *from an alcove* $w(A)$ *to the adjacent alcove* $ws_i(A)$*. Let* $H_{\alpha,m}$ *be the wall separating* $w(A)$ *and* $ws_i(A)$*. Then* $w(\alpha_i)$ *can be written as* $w(\alpha_i) = \epsilon(\alpha - m\delta)$*, where* $\epsilon\in\mathbb{R}$ *(in fact* $\epsilon = \pm1$ *in the untwisted case). Furthermore, the following conditions are equivalent:*

(i) *The* i*-crossing is positive;*
(ii) i *is an (affine) descent of* $\mathrm{cl}(w)$*;*
(iii) $\epsilon < 0$.

Condition (iii) is to be interpreted as $\mathrm{cl}(w)$ *maps* α_i *(resp.* $\alpha_i^\vee$*) to a negative root (resp. coroot) for* $\mathring{W}$ *(possibly up to a positive scalar factor for* $i=0$ *in the twisted case).*

Proof. Note that $ws_i(A) = ws_iw^{-1}w(A) = s_{w(\alpha_i)}w(A)$, so $s_{w(\alpha_i)} = s_{\alpha,m}$. The form for $w(\alpha_i)$ follows. It remains to prove the equivalence between the three conditions.

(i) $\iff$ (ii): Let $\rho^\vee = \frac{\ell}{n+1}(\Lambda_0^\vee+\cdots+\Lambda_n^\vee)$ be the canonical representative of the fundamental alcove at level ℓ: for i in I, $\langle\rho^\vee,\alpha_i\rangle = \frac{\ell}{n+1} > 0$. We compute how the representative $w(\rho^\vee)$ of $w(A)$ is moved in the crossing:

$$\begin{aligned} ws_i(\rho^\vee) - w(\rho^\vee) &= s_{w(\alpha_i)}w(\rho^\vee) - w(\rho^\vee) = -\langle w(\rho^\vee), w(\alpha_i)\rangle w(\alpha_i^\vee)\\ &= -\langle\rho^\vee,\alpha_i\rangle w(\alpha_i^\vee) = -\frac{\ell}{n+1}w(\alpha_i^\vee)\,. \end{aligned} \tag{17}$$

The crossing is positive if $\langle ws_i(\rho^\vee) - w(\rho^\vee),\alpha\rangle > 0$, or equivalently

$$0 > \langle w(\alpha_i^\vee),\alpha\rangle = \langle w(\alpha_i^\vee), \frac{1}{\epsilon}w(\alpha_i) + m\delta\rangle = \frac{1}{\epsilon}\langle w(\alpha_i^\vee), w(\alpha_i)\rangle = \frac{2}{\epsilon}\,, \tag{18}$$

that is $\epsilon < 0$.

(i) $\iff$ (iii): Using (16), i is a descent of $\mathrm{cl}(w)$ if and only if:

$$0 > \langle w^{-1}(\mathring{\rho}^\vee),\alpha_i\rangle = \langle\mathring{\rho}^\vee, w(\alpha_i)\rangle = \langle\mathring{\rho}^\vee,\epsilon(\alpha - m\delta)\rangle = \epsilon\langle\mathring{\rho}^\vee,\alpha\rangle\,, \tag{19}$$

or equivalently $\epsilon < 0$. □

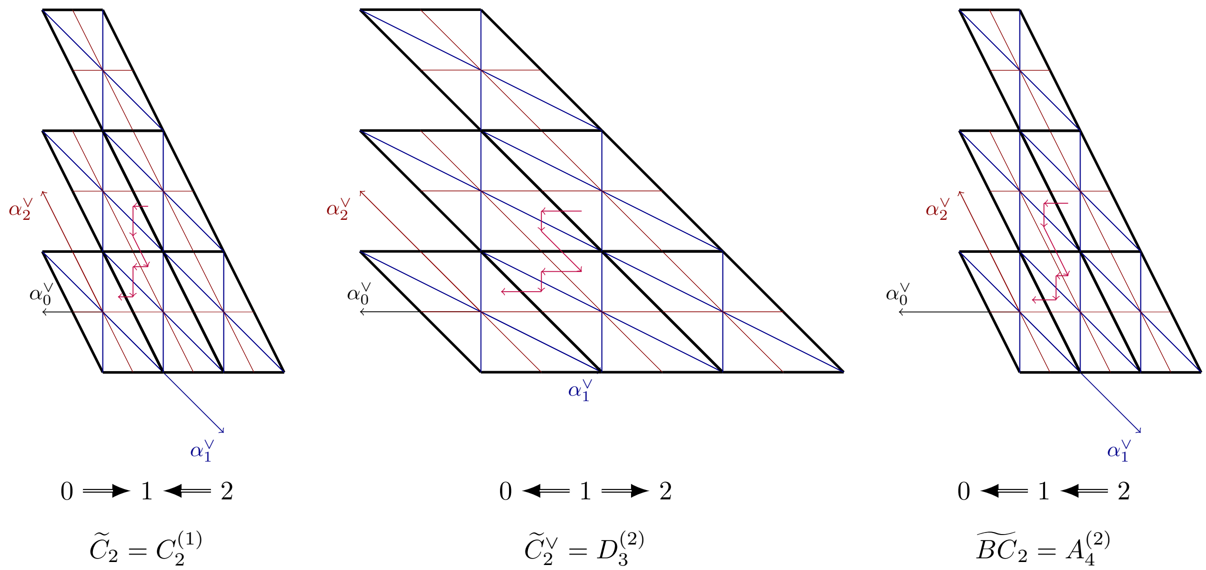

FIGURE 2. The alcove pictures and Dynkin diagrams for the three realizations of the Coxeter group $\widetilde{C}_2 = C_2^{(1)}$ as an affine Weyl group, drawn in the coweight lattice. The sample alcove walk is the same as in Figure 3. Notice that the pictures are identical up to a diagonal deformation.

By using the interpolation formula $T_i = (q_1+q_2)\pi_i - q_1 s_i$, the level 0 actions cl of the Weyl group W and of the 0-Hecke monoid $\langle \pi_i \mid i \in I \rangle$ on $\mathring{W}$ can be extended for any $(q_1, q_2) \neq (0,0)$ to a representation cl of the affine Hecke algebra $\mathrm{H}(W)(q_1,q_2)$ on $\mathbb{C}\mathring{W}$.

Interestingly enough, and this is the central topic of this paper, the algebra $\mathrm{cl}(\mathrm{H}(W)(q_1,q_2)) = \langle \mathrm{cl}(T_0), \dots, \mathrm{cl}(T_n) \rangle$ *turns out* not *to be the Hecke algebra* $\mathrm{H}(\mathring{W})(q_1,q_2)$, *except at* $q = 1$ *and certain roots of unity.*

3.3. **Cartan matrix independence.** In this subsection, we show that the geometric picture is independent of the chosen generalized Cartan matrix of W (see Figure 2). In other words, this paper is really about Coxeter groups which happen to have a realization as affine Weyl groups, and not about Weyl groups. In particular, one could always assume without loss of generality that the chosen geometric representation comes from a realization of W as an untwisted affine Weyl group.

Let W be any Coxeter group, and M and M' be two symmetrizable generalized Cartan matrices for W, and $D = (d_i)_{i \in I}$ be the diagonal matrix such that $M' = DMD^{-1}$. We denote by $\mathfrak{h}$ and $\mathfrak{h}'$ the corresponding geometric realizations of W, by $\mathfrak{h}^0$ and $\mathfrak{h}^{0\prime}$ the linear span of the coroots, etc. Consider the isomorphism $d : \mathfrak{h}^{*\prime} \to \mathfrak{h}^*$ determined by $d(\alpha_i') := \frac{1}{d_i}\alpha_i$. Further fix an isomorphism $d^\vee : \mathfrak{h}' \to \mathfrak{h}$ such that $d^\vee(\alpha_i^{\vee\prime}) := d_i \alpha_i^\vee$ ($d^\vee$ is a well-defined and unique isomorphism from $\mathfrak{h}^{0\prime}$ to $\mathfrak{h}^0$: given the relation between M' and M linear relations between the $\alpha_i^{\vee\prime}$'s are mapped to linear relations between the $\alpha_i^\vee$'s, and one can extend it to $\mathfrak{h}$).

Straightforward computations show that $\langle d^\vee(x^\vee), d(y)\rangle = \langle x^\vee, y\rangle$, $s_i(d^\vee(x^\vee)) = d^\vee(s_i(x^\vee))$ and $s_i(d(y)) = d(s_i(y))$, so that $d^\vee$ and d are W-morphisms. It follows that a root $\alpha' = w\alpha'_i$ of W in $\mathfrak{h}^{*\prime}$ is mapped by $d^\vee$ to a positive scalar multiple of $\alpha = w\alpha_i$ in $\mathfrak{h}^*$. So, $d^\vee$ preserves the hyperplane H_α and the half spaces H_α^+ and H_α^-. Therefore $d^\vee$ preserves chambers and in particular the fundamental one, the Tits cone, the bijection between chambers and elements of W; furthermore $d^\vee$ is a morphism for the action of the π_i's.

Assume now that W can be realized as an affine Weyl group. The action of W on the level 0-hyperplanes are isomorphic, and thus $\mathring{W}'$ and $\mathring{W}$ form the same quotient of W. Also, the level 0 action of W and of the 0-Hecke monoid on $\mathring{W}$, and therefore the representation of the q_1, q_2-affine Hecke algebra on $\mathbb{C}\mathring{W}$ match. The set of translations (elements of W acting trivially at level 0) are the same, and for $\lambda^\vee$ in the coroot lattice of $\mathring{W}$ we get identical expressions for $t_{\lambda^\vee}$ in terms of the s_i's, and for $Y^{\lambda^\vee}$ in terms of the T_i's.

Finally, $d^\vee$ can be chosen such as to further preserve the level and therefore the full alcove picture.

3.4. **Explicit (co)ambient space realizations for types A_n, B_n, C_n, D_n.** In the sequel, we use for types A_n, B_n, C_n, and D_n the following ambient space realizations of the finite coroot systems which realize $\mathring{W}$ as groups of signed permutations [BB05, EE98]. For type A_n, we take $\mathfrak{h} = \mathbb{Q}^{n+1}$ and for types B_n, C_n, and D_n $\mathfrak{h} = \mathbb{Q}^n$. Denoting by $(\varepsilon_i^\vee)_i$ the canonical basis of $\mathbb{Q}^{n+1}$ (resp. $\mathbb{Q}^n$) and identifying it with its dual basis $(\varepsilon_i)_i$, the simple roots are given by

$$\begin{aligned}
&\text{Type } A_n: && \alpha_i = \begin{cases} \varepsilon_{n+1} - \varepsilon_1 & \text{for } i = 0, \\ \varepsilon_i - \varepsilon_{i+1} & \text{for } 1 \le i \le n; \end{cases} \\
&\text{Type } B_n: && \alpha_i = \begin{cases} -\varepsilon_1 - \varepsilon_2 & \text{for } i = 0, \\ \varepsilon_i - \varepsilon_{i+1} & \text{for } 1 \le i < n, \\ \varepsilon_n & \text{for } i = n; \end{cases} \\
&\text{Type } C_n: && \alpha_i = \begin{cases} -2\varepsilon_1 & \text{for } i = 0, \\ \varepsilon_i - \varepsilon_{i+1} & \text{for } 1 \le i < n, \\ 2\varepsilon_n & \text{for } i = n; \end{cases} \\
&\text{Type } D_n: && \alpha_i = \begin{cases} -\varepsilon_1 - \varepsilon_2 & \text{for } i = 0, \\ \varepsilon_i - \varepsilon_{i+1} & \text{for } 1 \le i < n, \\ \varepsilon_{n-1} + \varepsilon_n & \text{for } i = n. \end{cases}
\end{aligned} \tag{20}$$

With this, the action (10) of π_i on $x^\vee = (x_1, x_2, \dots) \in \mathfrak{h}$ becomes

(21)

$$\text{Type } A_n\text{:} \qquad \pi_i(x^\vee) = \begin{cases} (x_{n+1}, x_2, \dots, x_n, x_1) & \text{if } i = 0 \text{ and } x_{n+1} > x_1, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_{n+1}) & \text{if } 1 \leq i \leq n \text{ and } x_i > x_{i+1}, \\ x & \text{otherwise;} \end{cases}$$

$$\text{Type } B_n\text{:} \qquad \pi_i(x^\vee) = \begin{cases} (-x_2, -x_1, x_3, \dots, x_n) & \text{if } i = 0 \text{ and } x_1 + x_2 < 0, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_n) & \text{if } 1 \leq i < n \text{ and } x_i > x_{i+1}, \\ (x_1, \dots, x_{n-1}, -x_n) & \text{if } i = n \text{ and } x_n > 0, \\ x & \text{otherwise;} \end{cases}$$

$$\text{Type } C_n\text{:} \qquad \pi_i(x^\vee) = \begin{cases} (-x_1, x_2, \dots, x_n) & \text{if } i = 0 \text{ and } x_1 < 0, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_n) & \text{if } 1 \leq i < n \text{ and } x_i > x_{i+1}, \\ (x_1, \dots, x_{n-1}, -x_n) & \text{if } i = n \text{ and } x_n > 0, \\ x & \text{otherwise;} \end{cases}$$

$$\text{Type } D_n\text{:} \qquad \pi_i(x^\vee) = \begin{cases} (-x_2, -x_1, x_3, \dots, x_n) & \text{if } i = 0 \text{ and } x_1 + x_2 < 0, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_n) & \text{if } 1 \leq i < n \text{ and } x_i > x_{i+1}, \\ (x_1, \dots, x_{n-2}, -x_n, -x_{n-1}) & \text{if } i = n \text{ and } x_{n-1} + x_n > 0, \\ x & \text{otherwise.} \end{cases}$$

We may pick $\rho^\vee := (d, d-1, \dots, 1)$ (where d is the dimension of $\mathfrak{h}$) as representative of the fundamental chamber for $\mathring{W}$: $\langle \rho^\vee, \alpha_i \rangle > 0$, for all $i = 1, \dots, n$. Instead of s_i and π_i acting on the coambient space, they can equivalently act on group elements themselves. The correspondence can be realized by evaluating $w(\rho^\vee)$. Whereas the action on the coambient space (21) is an action from the left, the action on the group itself is an action from the right.

4. Transitivity of the level 0 action of affine 0-Hecke algebras

In this section we state and prove the core combinatorial Theorem 4.2 of this paper about transitivity of the level 0 action of affine 0-Hecke algebras and mention some applications to crystal graphs.

4.1. Transitivity.

We start with type A_n to illustrate the results. Here, each π_i can be interpreted as a partial (anti)sort operator: it acts on a permutation (or word) $w := (w_1, \dots, w_{n+1})$ by exchanging w_i and w_{i+1} if $w_i < w_{i+1}$. By bubble sort, any permutation can be mapped via $\pi_1, \dots, \pi_n$ to the maximal permutation w_0, but not conversely. More precisely the (oriented) graph of the action is the usual right permutohedron, which is acyclic with 1 as minimal element and w_0 as maximal element.

Consider now w as written along a circle, and let π_0 act as above with i taken modulo $n+1$. As suggested by Figure 3 for $n = 2$, adding the 0 edges makes the graph of the action strongly connected.

Proposition 4.1. *$\pi_0, \dots, \pi_n$ act transitively on permutations of $\{1, \dots, n+1\}$.*

Proof. We start with any permutation w and identify it with $w(\rho^\vee) =: x^\vee = (x_1, \dots, x_{n+1})$. Then the π_i act as in (21).

Suppose that the letter $z = n+1$ is at position k in $x^\vee$. Then $\pi_0\pi_n\cdots\pi_{k+1}\pi_k(x^\vee)$ has letter z in position 1. The operator $\tilde{\pi}_0 = (\pi_0\pi_n\cdots\pi_1)^{n-1}(\pi_0\pi_n)(\pi_0)(\pi_{n-1}\cdots\pi_1)$ acts in the same way as π_0, except only on the last n positions:

$$\begin{array}{ll} z\ x_1\ x_2\ldots x_{n-1}\ x_n & \\ & \xrightarrow{\pi_{n-1}\cdots\pi_1} \\ x_1\ x_2\ldots x_{n-1}\ z\ x_n & \\ & \xrightarrow{\pi_0} \\ x_1'\ x_2\ldots x_{n-1}\ z\ x_n' & \\ & \xrightarrow{\pi_0\pi_n} \\ z\ x_2\ldots x_{n-1}x_n'\ x_1' & \\ & \xrightarrow{(\pi_0\pi_n\cdots\pi_1)^{n-1}} \\ z\ x_1'\ x_2\ldots x_{n-1}\ x_n' & \end{array} \tag{22}$$

where $x_1' = x_n$ and $x_n' = x_1$ if $x_n > x_1$ and $x_1' = x_1$ and $x_n' = x_n$ otherwise. In the last step we have used that the operator $\pi_0\pi_n\cdots\pi_1$ rotates the last n letters cyclically one step to the left, leaving the letter z in position 1 unchanged. The result follows by induction. □

Let now $\mathring{W}$ be any finite Weyl group, and $\mathrm{H}(\mathring{W})(0)$ its 0-Hecke algebra. Via $\pi_1,\ldots,\pi_n$ the identity of $\mathring{W}$ can be mapped to any $w\in\mathring{W}$, but not back (the graph of the action is just the Hasse diagram of the right weak Bruhat order). Now embed $\mathring{W}$ in an affine Weyl group W, and consider the extra generator π_0 of its 0-Hecke algebra acting on $\mathring{W}$. As the dominant chamber of $\mathring{W}$ is on the negative side of H_{α_0}, π_0 tends to map elements of $\mathring{W}$ back to the identity (see Figure 3).

Theorem 4.2. *Let W be an affine Weyl group, $\mathring{W}$ the associated finite Weyl group, and $\pi_0,\pi_1,\ldots,\pi_n$ the generators of the 0-Hecke algebra of W. Then, the level 0 action of $\pi_0,\pi_1,\ldots,\pi_n$ on $\mathring{W}$ (or equivalently on the chambers of $\mathring{W}$) is transitive.*

We prove Theorem 4.2 by a type free geometric argument using Lemma 4.3 below. Figure 3 illustrates the proof, and thanks to Section 3.3 covers all the rank 2 affine Weyl groups.

Lemma 4.3 (Cf. Remark 3.5 of [Ram06])**.** *Let $w(A)$ be an alcove in the dominant chamber of $\mathring{W}$, and consider a shortest alcove walk $i_1,\ldots,i_r$ from A to $w(A)$. Then, each crossing is positive. In particular, i_k is a descent of $cl(s_{i_1}\cdots s_{i_{k-1}})$.*

Proof. If $w(A)$ is the fundamental alcove A, the path is empty, and we are done. Otherwise, let $H_{\alpha,m}$ be the wall separating $w(A)$ from the previous alcove $s_{i_1}\cdots s_{i_{r-1}}(A)$. Assume that $w(A)$ is in $H^-_{\alpha,m}$. Taking some point $x^\vee$ in $w(A)$,

$$0 > \langle x^\vee, \alpha-\delta m\rangle = \langle x^\vee,\alpha\rangle - \ell m\,. \tag{23}$$

Then, using that $w(A)$ is in the fundamental chamber, $m > \frac{1}{\ell}\langle x^\vee,\alpha\rangle > 0$. On the other hand, since the alcove walk is shortest, $H_{\alpha,m}$ separates $w(A)$ and A, so $A\in H^+_{\alpha,m}$. Since 0^ℓ is in the closure of A, $0\le\langle 0^\ell,\alpha-\delta m\rangle = 0-\ell m$. It follows that $m\le 0$, a contradiction. □

Proof of Theorem 4.2. Take $w\in\mathring{W}$, and $w(A)$ the corresponding alcove. One can choose a long enough strictly dominant element $\lambda^\vee$ of the coroot lattice so that $t_{\lambda^\vee}(w(A))$ lies in the dominant chamber of $\mathring{W}$. Consider some shortest alcove walk $i_1,\ldots,i_r$ from $t_{\lambda^\vee}(w(A))$ back to the fundamental alcove A (see Figure 3). Then, in $\mathring{W}$, $w\mathrm{cl}(s_{i_1})\cdots\mathrm{cl}(s_{i_r}) = 1$. Furthermore, by Lemma 4.3, at each step i_k is not a descent of $w\mathrm{cl}(s_{i_1})\cdots\mathrm{cl}(s_{i_{k-1}})$. Therefore, $w.\pi_{i_1}\ldots\pi_{i_r} = w\mathrm{cl}(s_{i_1})\cdots\mathrm{cl}(s_{i_r}) = 1$, as desired. □

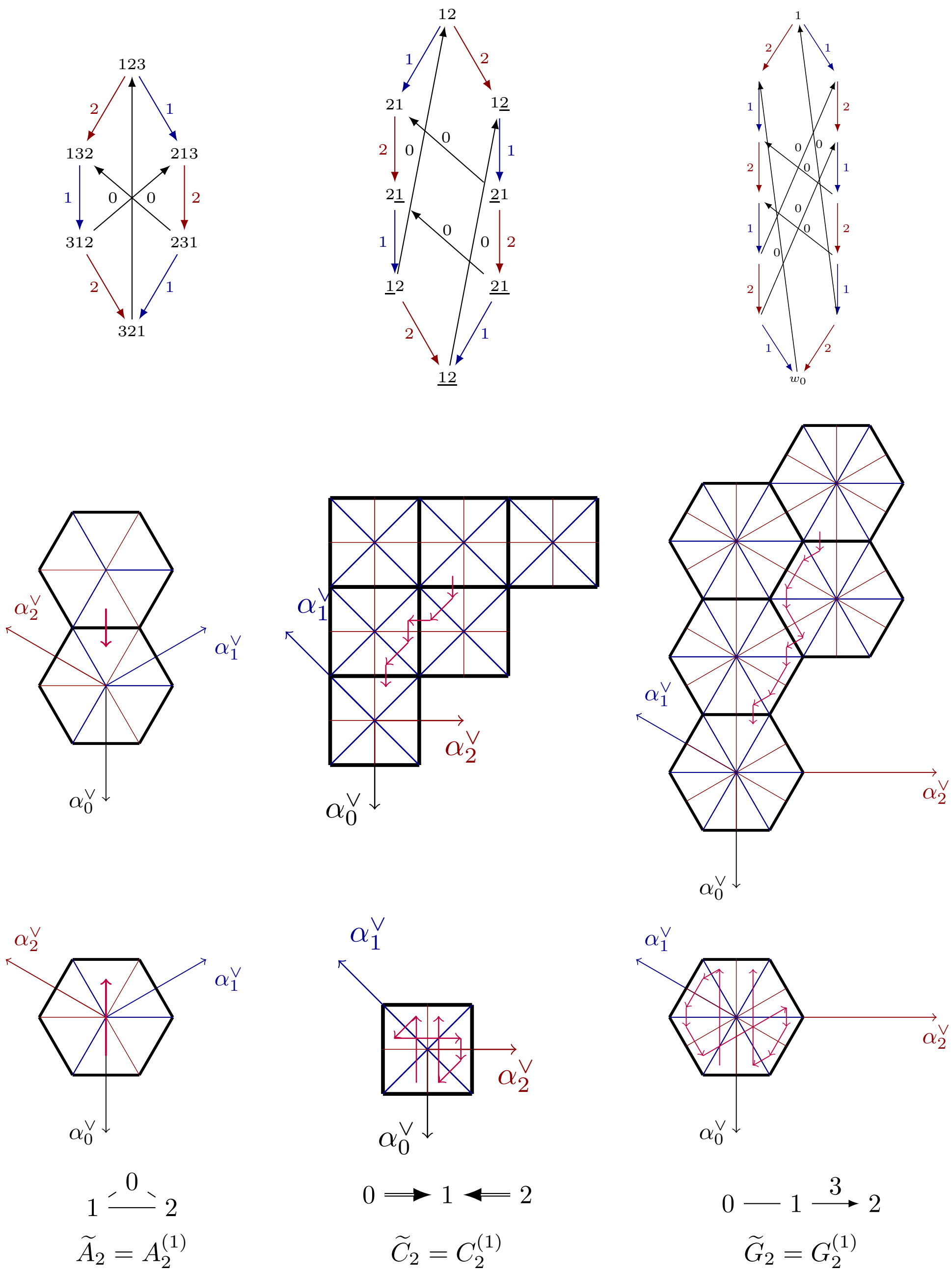

FIGURE 3. **Top:** Graph of the action of $\pi_0, \pi_1, \dots, \pi_n$ on the finite Weyl group $\mathring{W}$, using (signed) permutation notation.
Center: The alcove picture in the ambient space, with a shortest alcove walk from an alcove $w(A)$ in the dominant chamber such that $\mathrm{cl}(w) = w_0$ down to the fundamental alcove A. An i-crossing is negative if it goes down or straight to the left.
Bottom: The top graph can be realized geometrically in the *Steinberg torus*, quotient of the alcove picture by the translations, or equivalently by identification of the opposite edges of the fundamental polygon. An i-arrow in the graph corresponds to a negative i-crossing. The alcove walk of the center figure then becomes a path from the antifundamental chamber w_0(A) back the fundamental chamber A.

We now exhibit a recursive sorting algorithm for type B_n, where the operators π_i act on the coambient space as outlined in Section 3.4, similar to the recursive sorting algorithm for type A at the beginning of this section. This is an explicit algorithm which achieves the results of Theorem 4.2 (but not necessarily in the most efficient way). This sorting algorithm actually contains all the ingredients for type C_n and D_n, since the Dynkin diagram of type B_n contains both kinds of endings. We have also verified by computer that explicit recursive sorting algorithms exist for the exceptional types; the base cases B_2, B_3, C_2, and D_3 can be worked out explicitly. Details are available upon request.

Let w be a permutation of type B_n for $n \geq 4$. As before we identify w with $w(\rho^\vee) = x^\vee = (x_1, \ldots, x_n)$. We can bring the maximal letter $z = n$ to any position, as z or $-z$:

$$
\begin{array}{l}
x_1 \ldots x_{k-1}\ z\ x_k \ldots x_{n-1} \\
\quad \downarrow \pi_{n-1}\cdots\pi_k \\
x_1\ x_2 \ldots x_{n-1}\ z \\
\quad \downarrow \pi_n \\
x_1\ x_2 \ldots x_{n-1}\ -z \\
\quad \downarrow \pi_2\cdots\pi_{n-1} \\
x_1\ -z\ x_1 \ldots x_{n-1} \\
\quad \downarrow \pi_0 \\
z\ -x_1\ x_2 \ldots x_{n-1} \\
\quad \downarrow \pi_{k-1}\cdots\pi_1 \\
-x_1 \ldots x_{k-1}\ z\ x_k \ldots x_{n-1}
\end{array}
\tag{24}
$$

In particular, we can move z to the left of $y = n-1$ (or $-z$ to the right of $-y$). The pair zy (or $-y-z$) can move around in a circle to any position by similar arguments as above without disturbing any of the other letters, noting that if zy are in the last two positions of $x^\vee$, then $\pi_n\pi_{n-1}\pi_n(x^\vee)$ contains $-y-z$ in the last two positions, and if $-y-z$ is in the first two positions of $x^\vee$, then $\pi_0(x^\vee)$ contains zy in the first two positions.

Next suppose that zy occupy the first two positions of $x^\vee$. We construct $\tilde{\pi}_0$ on such $x^\vee$, which acts the same way as π_0, but on the last $n-2$ letters:

$$
\begin{array}{l}
z\ y\ x_1\ x_2 \cdots x_{n-2} \\
\quad \downarrow \pi_2\pi_1\pi_3\pi_2 \\
x_1\ x_2\ z\ y \cdots x_{n-2} \\
\quad \downarrow \pi_0 \\
x_1'\ x_2'\ z\ y \cdots x_{n-2}
\end{array}
\tag{25}
$$

followed by the above circling to move zy back to position 1 and 2.

Problem 4.4. *We had first proved a variant of Proposition 4.1 with the cycle $(1, \ldots, n)$ and $\pi_1, \ldots, \pi_n$ as operators. There, the sorting of a permutation σ involves decomposing it recursively in terms of the following strong generating set of $\mathfrak{S}_n$ (as a permutation group):*

$$
\big(((1, \ldots, i)^k)_{k=0,\ldots,i-1} \big)_{i=1,\ldots n}\,. \tag{26}
$$

The sequence $(k_n, \ldots, k_1)$ describing which power k_i of $(1, \ldots, i)$ is used for each base point i is (essentially) the flag code of σ, as defined in [AR01].

Similar flag codes have been defined for types B_n, C_n, D_n, and even for general reflection groups [ABR05, BC04, BB07]. *Do there exist related recursive sorting algorithms?*

4.2. **Strong connectivity of crystals.** Crystal bases are combinatorial bases of modules of quantum algebras $U_q(\mathfrak{g})$ as the parameter q tends to zero. They consist of a non-empty set B together with raising and lowering operators e_i and f_i for

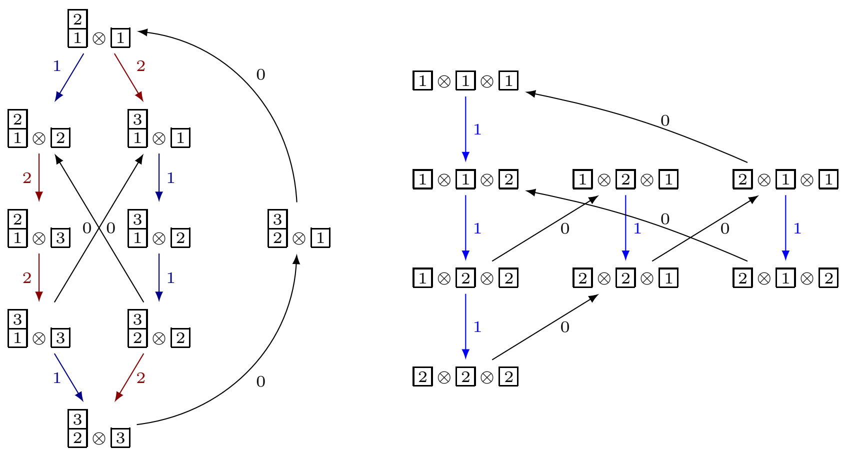

FIGURE 4. **Left**: Crystal $B^{2,1} \otimes B^{1,1}$ of type $A_2^{(1)}$. By contraction of all i-strings to a single edge i, one recovers the left most graph of Figure 3. **Right**: Crystal $(B^{1,1})^{\otimes 3}$ of type $A_1^{(1)}$.

$i \in I$ from B to $B \cup \{0\}$ and a weight function $\mathrm{wt} : B \to \mathcal{P}$. For more information on crystal theory see [HK02]. Of particular interest are crystals coming from finite-dimensional affine $U_q(\mathfrak{g})$-modules, where $\mathfrak{g}$ is an affine Kac-Moody algebra. These crystals are not highest weight. In this section we deduce from the transitivity of the level 0 action of the 0-Hecke algebra on $\mathring{W}$ of Theorem 4.2 that these finite-dimensional affine crystals are strongly connected; that is, any two elements $b, b' \in B$ can be connected via a sequence of operators f_i: $b' = f_{i_1} \cdots f_{i_r}(b)$ for $i_j \in I$.

There is an action of the Weyl group on any finite affine crystal B defined by

$$s_i(b) = \begin{cases} f_i^{\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle}(b) & \text{if } \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle > 0, \\ e_i^{-\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle}(b) & \text{if } \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle \leq 0, \end{cases} \tag{27}$$

where $b \in B$ and $i \in I$. This action is compatible with the weights, that is, $s_i(\mathrm{wt}(b)) = \mathrm{wt}(s_i(b))$. In particular we also have $\mathrm{wt}(\pi_i(b)) = \pi_i(\mathrm{wt}(b))$, where

$$\pi_i(b) := \begin{cases} f_i^{\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle}(b) & \text{if } \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle > 0, \\ b & \text{if } \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle \leq 0. \end{cases} \tag{28}$$

Remark 4.5. *Comparing* (10) *and* (28)*, it is clear that if a sequence $i_1, \ldots, i_r$ is such that at each step in $\pi_{i_r} \cdots \pi_{i_1}(\mathrm{wt}(b))$ the operator π_i acts as s_i, then the same holds in $\pi_{i_r} \cdots \pi_{i_1}(b)$.*

Theorem 4.6. *Let B be a finite connected affine crystal. Then B is strongly connected.*

Proof. It is sufficient to prove that if x and y in B are in the same i_0-string with $y = f_{i_0}^a(x)$ for some $i_0 \in I$ and $a > 0$, then there is an f-path from y to x. Using finiteness, we may further assume without loss of generality that $y = s_{i_0}(x) = \pi_{i_0}(x)$ (moving for example x and y to respectively to top and bottom of the string).

By Theorem 4.2, there exists a sequence $i_1, \ldots, i_r$ such that $\pi_{i_r} \cdots \pi_{i_1}(\mathrm{wt}(y)) = \mathrm{wt}(x)$. Choose such a sequence of minimal length, so that each π_{i_j} above acts as

s_{i_j}. Consider $p := \pi_{i_r} \cdots \pi_{i_1} \pi_{i_0}$, and $w := s_{i_r} \cdots s_{i_0}$. Then, $p(\mathrm{wt}(x)) = w(\mathrm{wt}(x)) = \mathrm{wt}(x)$. Now, $p(x)$ might not be x, but by Remark 4.5 we may apply p repeatedly and still have $p^k(x) = w^k(x)$. Since the crystal is finite, eventually we will have $p^k(x) = w^k(x) = x$. Since any application of π_i results from a sequence of applications of f_i, this proves the existence of an f-path from y back to x. □

Theorem 4.6 is equivalent to [Kas08, Theorem 3.37].

Remark 4.7. *As noted in the proof of Theorem 4.6, the action of the affine Weyl group on a crystal is* not *necessarily the level* 0 *action: only a power of* p *maps a given crystal element* x *to itself* $p^k(x) = x$. *Take for example* $x = \boxed{1} \otimes \boxed{1} \otimes \boxed{2}$ *in* $(B^{1,1})^{\otimes 3}$ *of type* $A_1^{(1)}$, *where* $B^{r,s}$ *denotes a Kirillov–Reshetikhin crystal. Then for* $p = s_0 s_1$ *we have* $p(\mathrm{wt}(x)) = \mathrm{wt}(x)$, *but only* $p^3(x) = x$ *as can be seen from Figure 4.*

Remark 4.8. *Interpreting the* π_i*'s as Demazure operators, Theorem 4.2 is related to properties of affine crystals. Let* $\mathfrak{g}$ *be an affine Kac–Moody algebra,* W *the corresponding affine Weyl group, and* $B^{r,s}$ *a Kirillov–Reshetikhin crystal of type* $\mathfrak{g}$ [HKO⁺02, OS08]. *Consider the affine crystal* $B := B^{n,1} \otimes B^{n-1,1} \otimes \cdots \otimes B^{1,1}$, *and define the Demazure operators on* $b \in B$ *as in* [Kas93]*:*

$$\Pi_i(b) = \begin{cases} \sum_{0 \le k \le \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle} f_i^k(b) & \text{if } \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle \ge 0\,, \\ -\sum_{1 \le k \le -\langle \alpha_i^\vee, \mathrm{wt}(b) \rangle} e_i^k(b) & \text{if } \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle < 0\,. \end{cases} \tag{29}$$

Let Λ_i *be the fundamental weights of* $\mathfrak{g}$, *and take for* u_i *the unique element in* $B^{i,1}$ *of weight* $\Lambda_i - \Lambda_0$. *Then, the transitivity of the action of* $\mathrm{H}(W)(0)$ *on* $\mathring{W}$ *is closely related to the strong connectivity of the graph generated by* $\Pi_0, \ldots, \Pi_n$ *acting on* $u_n \otimes \cdots \otimes u_1$ [Kas02, FSS07]*, see Figure 4.*

5. Hecke group algebras as quotients of affine Hecke algebras

We are now in the position to state the main theorem of this paper. Let W be an affine Weyl group and $\mathrm{H}(W)(q_1, q_2)$ its Hecke algebra. Let $\mathring{W}$ be the associated finite Weyl group, and $\mathrm{H}\mathring{W}$ its Hecke group algebra. Then the level 0-representation

$$\mathrm{cl} : \begin{cases} \mathrm{H}(W)(q_1, q_2) & \to \mathrm{End}(\mathbb{C}\mathring{W}) \\ T_i & \mapsto (q_1 + q_2)\pi_i - q_1 s_i \end{cases} \tag{30}$$

actually defines a morphism from $\mathrm{H}(W)(q_1, q_2)$ to $\mathrm{H}\mathring{W}$. (Note that π_α and in particular π_0 is indeed an element of $\mathrm{H}\mathring{W}$: it can be written as $\pi_\alpha = w \pi_i w^{-1}$ where w is an element of $\mathring{W}$ conjugating α to some simple root α_i.) When the Dynkin diagram has special automorphisms Ω, this morphism can be extended to the extended affine Hecke algebra by sending the special Dynkin diagram automorphisms to the corresponding element of the finite Weyl group $\mathring{W}$.

Theorem 5.1. *Let* W *be an affine Weyl group. Except when* $q_1 + q_2 = 0$ *(and possibly when* $q := -\frac{q_1}{q_2}$ *is a* k*th root of unity with* $k \le 2\,\mathrm{ht}(\theta^\vee)$*), the morphism* $\mathrm{cl} : \mathrm{H}(W)(q_1, q_2) \to \mathrm{H}\mathring{W}$ *is surjective and makes the Hecke group algebra* $\mathrm{H}\mathring{W}$ *into a quotient of the affine Hecke algebra* $\mathrm{H}(W)(q_1, q_2)$.

Proof. Here we outline the proof which relies on material in the next two sections.

When $q_1 + q_2 = 0$, the image of cl is obviously $\mathbb{C}[\mathring{W}]$ (or just $\{0\}$ if $q_1 = q_2 = 0$); so the morphism is not surjective.

If $q_1 = 0$ and $q_2 \neq 0$, this is exactly Corollary 6.2 below. If $q_2 = 0$ and $q_1 \neq 0$, then $\mathrm{cl}(\overline{T}_i) = q_1\overline{\pi}_i$, and by symmetry, we can also use Corollary 6.2. The theorem follows right away for all values of $q = -q_1/q_2$ but a finite number using a standard specialization argument: take q formal, and consider the family B_q obtained from B by replacing each π_i by $(1-q)\pi_i + qs_i$. This family has polynomial coefficients when expressed in terms of the basis $\{w\pi_{w'} \mid \mathrm{D}_R(w) \cap \mathrm{D}_L(w') = \emptyset\}$ of $\mathrm{H}\mathring{W}$. Its determinant is a polynomial in q with a non-zero constant since B_0 is a basis. Thus it vanishes for at most a finite number of values of q.

Theorem 7.7 below allows to further reduce the possible inappropriate values of q to kth roots of unity with k small. Note however that Theorem 7.7 does not apply at $q_1 = 0$ or $q_2 = 0$. □

Theorem 5.1 raises immediately the following problem, currently under investigation together with Nicolas Borie.

Problem 5.2. *Determine for which roots of unity q the morphism* cl *is not surjective.*

6. Alternative generators for Hecke group algebras

In this section we show that the Hecke group algebra can be entirely generated by $\pi_0, \pi_1, \ldots, \pi_n$.

Proposition 6.1. *Let $\mathring{W}$ be a finite Coxeter group, and $\mathcal{S}$ be a set of roots of $\mathring{W}$ such that the associated projections $\{\pi_\alpha \mid \alpha \in \mathcal{S}\}$ act transitively on $\mathring{W}$. Then, the Hecke group algebra $\mathrm{H}\mathring{W}$ is generated as an algebra by $\{\pi_\alpha \mid \alpha \in \mathcal{S}\}$.*

Proof. First note that π_α is indeed an element of $\mathrm{H}\mathring{W}$: it can be written as $\pi_\alpha = w\pi_i w^{-1}$ where w is an element of $\mathring{W}$ conjugating α to some simple root α_i. In Proposition 6.4 below, we exhibit a sufficiently large family of operators which are linearly independent, because they display the same triangularity property as the basis $\{w\pi_{w'} \mid \mathrm{D}_R(w) \cap \mathrm{D}_L(w') = \emptyset\}$ of $\mathrm{H}\mathring{W}$ (see Lemma 3.8 of [HT08]). □

Corollary 6.2. *Let W be an affine Weyl group, $\mathring{W}$ be the associated finite Weyl group, and $\pi_0, \ldots, \pi_n$ be the projections associated to the roots $\mathrm{cl}(\alpha_0), \ldots, \mathrm{cl}(\alpha_n)$ of the finite Weyl group. Then, the Hecke group algebra $\mathrm{H}\mathring{W}$ is generated as an algebra by $\pi_0, \ldots, \pi_n$.*

Alternatively, π_0 may be replaced by any $\Omega \in W$ mapping α_0 to some simple root, typically one induced by some special Dynkin diagram automorphism.

Let $w \in \mathring{W}$. An *$\mathcal{S}$-reduced word* for w is a word $i_1, \ldots, i_r$ of minimal length such that $i_j \in \mathcal{S}$ and $w^{-1}.\pi_{i_1} \ldots \pi_{i_r} = 1$. Since the $\{\pi_\alpha \mid \alpha \in \mathcal{S}\}$ acts transitively on $\mathring{W}$, there always exists such an $\mathcal{S}$-reduced word, and we choose once for all one of them for each w. More generally, for a right coset $w\mathring{W}_J$, we choose an $\mathcal{S}$-reduced word $i_1, \ldots, i_r$ of minimal length such that there exists $\nu \in \mathring{W}_J w^{-1}$ and $\mu \in \mathring{W}_J$ with $\nu.\pi_{i_1} \ldots \pi_{i_r} = \mu$.

Example 6.3. *In type C_2, the word $0,1,2,0,1,0$ is $\mathcal{S}$-reduced for $w_0 = {w_0}^{-1} = (\underline{1}, \underline{2})$, where we write $\underline{1}$ and $\underline{2}$ for -1 and -2 (see Figure 3).*

In type A_3 the word $1,0$ is $\mathcal{S}$-reduced for $4123\mathring{W}_{\{1,3\}}$. Here $w = 4123$, $\nu = w^{-1} = 2341$, and $\mu = 1243$. Looking at $\mathring{W}_J$ left-cosets is the Coxeter equivalent to looking at words with repetitions: we may think of left $\mathring{W}_{\{1,3\}}$-cosets as identifying *the values* $1,2$ *and* $3,4$*, and represent* $\mathring{W}_{\{1,3\}}w^{-1} = 2341$ *by the word* 1331*; this word gets sorted by* $\pi_1\pi_0$ *to* 1133 *which represents* $\mathring{W}_{\{1,3\}}$*.*

Setting $\nabla_i := (\pi_i - 1)$, define the operator $\nabla_{w\mathring{W}_J} := \nabla_{i_1}\cdots\nabla_{i_r}$ where $i_1,\dots,i_r$ is the chosen $\mathcal{S}$-reduced word. The operator may actually depend on the choice of the $\mathcal{S}$-reduced word, but this is irrelevant for our purpose.

Proposition 6.4. *The following family forms a basis for* $\mathrm{H}\mathring{W}$*:*

$$B := \{\nabla_{w\mathring{W}_{\mathrm{D}_L(w')}}\pi_{w'} \mid \mathrm{D}_L(w)\cap\mathrm{D}_R(w') = \emptyset\}\,. \tag{31}$$

Proof. The number of elements of B is the same as the dimension of the Hecke group algebra by Section 2.3. Corollary 6.7 shows that the elements in B are linearly independent. This proves the claim. □

Lemma 6.5. *Let* $w\mathring{W}_J$ *be a right coset in* $\mathring{W}$*, and* $i_1,\dots,i_r$ *be the corresponding* $\mathcal{S}$*-reduced word. Set* $w' = s_{i_1}\cdots s_{i_r}$*. Then,* $\pi_{i_1}\cdots\pi_{i_r}$ *restricted to* $\mathring{W}_J w^{-1}$ *acts by right multiplication by* w'*. In particular, it induces a bijection from* $\mathring{W}_J w^{-1}$ *to* $\mathring{W}_J$*.*

Proof. Take ν in $\mathring{W}_J w^{-1}$ such that $\nu.\pi_{i_1}\dots\pi_{i_r}\in\mathring{W}_J$. By minimality of the $\mathcal{S}$-reduced word, no π_i acts trivially, so $\nu.\pi_{i_1}\dots\pi_{i_r} = \nu w'$. Furthermore, $\pi_{i_1}\cdots\pi_{i_r}$ is in $\mathrm{H}\mathring{W}$ and thus preserves left-antisymmetries. Taking $i\in I$, this implies that $(s_i\nu).\pi_{i_1}\dots\pi_{i_r}$ is either $s_i\nu w'$ or $\nu w'$. By minimality of the $\mathcal{S}$-reduced word, the latter case is impossible: indeed if any of the π_{i_j} acts trivially we get a strictly shorter $\mathcal{S}$-reduced word from $s_i\nu\in\mathring{W}_J w^{-1}$ to $\nu w'\in\mathring{W}_J$. Applying transitivity, we get that $\pi_{i_1}\cdots\pi_{i_r}$ acts by multiplication by w' on $\mathring{W}_J w^{-1}$. □

Let $<$ be any linear extension of the right Bruhat order on $\mathring{W}$. Given an endomorphism f of $\mathbb{C}\mathring{W}$, we order the rows and columns of its matrix $M_f := [f_{\mu\nu} := f(\nu)_{|\mu}]$ according to $<$ (beware that, the action being on the right, $M_{fg} = M_g M_f$). Denote by $\mathrm{init}(f) := \min\{\mu \mid \exists\nu, f_{\mu\nu}\neq 0\}$ the index of the first non-zero row of M_f.

Lemma 6.6. *Let* $f := \nabla_{w\mathring{W}_J}$*. Then, for any* $\mu\in\mathring{W}$*, there exists a unique* $\nu\in\mathring{W}$ *such that the coefficient* $f_{\mu\nu}$ *is non-zero; this coefficient is either* 1 *or* -1 *(in other words,* f *is the transpose of a signed-monoidal application).*

In particular, if $\mu\in\mathring{W}_J$ *then* ν *belongs to* $\mathring{W}_J w^{-1}$*, and* $f_{\mu\nu} = 1$*.*

Proof. This is clear if $f = \nabla_J$; here is for example the matrix of ∇_1 in type A_1:

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}. \tag{32}$$

By products, this extends to any f.

Take now $\mu\in\mathring{W}_J$. Using Lemma 6.5, let ν be the unique element in $\mathring{W}_J w^{-1}$ such that $\nu.\pi_{i_1}\dots\pi_{i_r} = \mu$. By minimality of the $\mathcal{S}$-reduced word, μ cannot occur in any other term of the expansion of

$$\nu.\nabla_{i_1}\dots\nabla_{i_r} = \nu.(\pi_{i_1}-1)\dots(\pi_{i_r}-1)\,. \tag{33}$$

Therefore, $f_{\mu\nu} = 1$, and $f_{\mu\nu'} = 0$ for $\nu'\neq\nu$. □

We get as a corollary that the basis B is triangular.

Corollary 6.7. *Let* $f := \nabla_{w\mathring{W}_{\mathrm{D}_L(w')}}\pi_{w'}$ *in* B. *Then,* $\mathrm{init}(f) = w'$, *and*

$$f_{w'\nu} = \begin{cases} 1 & \text{if } \nu \in \mathring{W}_{\mathrm{D}_L(w')}w^{-1}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

7. Hecke group algebras and principal series representations of affine Hecke algebras

Let $t : Y \to \mathbb{C}^*$ be a character of the multiplicative group Y (or equivalently of the additive group $\mathfrak{h}^*_{\mathbb{Z}}$). It induces a representation $M(t) := t \uparrow_{\mathbb{C}[Y]}^{\mathrm{H}(W)(q_1,q_2)}$ called *principal series representation* of the affine Hecke algebra $\mathrm{H}(W)(q_1, q_2)$. Since $\mathrm{H}(W)(q_1, q_2) = \mathrm{H}(\mathring{W})(q_1, q_2)\otimes\mathbb{C}[Y]$, this representation is of dimension $|\mathring{W}|$. When t is regular, the representation is *calibrated*: it admits a basis $(E_w)_{w\in\mathring{W}}$ which diagonalizes the action of Y with a distinct character wt on each E_w. This basis can be constructed explicitly by means of intertwining operators τ_i which skew commute with the elements of Y. We refer to [Ram03, Section 2.5] for details. Note also that the construction of the τ_i operators by deformation of the T_i is reminiscent of Yang-Baxter graphs [Las03, § 10.7], in which t corresponds to a choice of *spectral parameters*.

The main result of this section is that for $q_1, q_2 \neq 0$ and q not a root of unity, there exists a suitable character t, such that the level 0 representation of the affine Hecke algebra is isomorphic to the principal series representation $M(t)$ (Theorem 7.1), and to deduce that the morphism $\mathrm{cl} : \mathrm{H}(W)(q_1, q_2) \mapsto \mathrm{H}\mathring{W}$ is surjective (Theorem 7.7).

Theorem 7.1. *Assume* q_1, q_2 *are such that* $q_1, q_2 \neq 0$ *and* $q := -\frac{q_1}{q_2}$ *is not a* k*th root of unity with* $k \leq 2\,\mathrm{ht}(\theta^\vee)$. *Then, the level* 0 *representation of the affine Hecke algebra* $\mathrm{H}(W)(q_1, q_2)$ *is isomorphic to the principal series representation* $M(t)$ *for the character* $t : Y^{\lambda^\vee} \mapsto q^{-\mathrm{ht}(\lambda^\vee)}$.

Note that $t(Y^{\alpha_i^\vee}) = q^{-1}$ for any simple coroot. By a result of Kato [Kat81, Theorem 2.2] (see also [Ram03, Theorem 2.12 (c)]) one sees right away that $M(t)$ is not irreducible. Note also that this is, up to inversion, the same character as for the action of $\mathbb{C}[Y]$ on the constant Macdonald polynomial $\mathbf{1}$ [RY08, Equation (3.4)].

Proof. In the upcoming Lemma 7.5, we prove that w_0 is an eigenvector for the character t, and check that t is regular (that is the orbit $\mathring{W}t$ of t is of size $|\mathring{W}|$). We then mimic [Ram03] and use the intertwining operators to explicitly diagonalize the action of Y on $\mathbb{C}\mathring{W}$ in Proposition 7.3. Although this is more than strictly necessary to prove the desired isomorphism, the results will be useful for the subsequent Theorem 7.7. □

Lemma 7.2. *Let* $i_1, \ldots, i_r$ *be an alcove walk from the fundamental alcove, and* $\epsilon_1, \ldots, \epsilon_r$ *as defined in Section 3.1. Then,*

$$w_0.T_{i_1}^{\epsilon_1}\cdots T_{i_r}^{\epsilon_r} = q_2^{\epsilon_1+\cdots+\epsilon_r} w_0 s_{i_1}\cdots s_{i_r}\,. \tag{35}$$

Proof. Take $w \in \mathring{W}$, and $i \in \{0, \ldots, n\}$. If i is not a descent of w, then, using (30):

$$w.T_i = w.\mathrm{cl}(T_i) = w.\left((q_1 + q_2)\pi_i - q_1 s_i\right) = w((q_1 + q_2)s_i - q_1 s_i) = q_2 w s_i\,. \tag{36}$$

Inverting this equation yields that, when i is a descent of w, $w.T_i^{-1} = q_2^{-1} w s_i$.

We conclude by induction since $\epsilon_k = 1$ if and only if i_k is a descent of $w_{k-1} = s_{i_1} \cdots s_{i_{k-1}}$ (cf. Remark 3.2), that is not a descent of $w_0 s_{i_1} \cdots s_{i_{k-1}}$. □

Proposition 7.3. *Assuming the same conditions as in Theorem 7.1, there exists a basis* $(E_w)_{w\in\mathring{W}}$ *of* $\mathbb{C}\mathring{W}$ *which diagonalizes simultaneously all* $Y^{\lambda^\vee}$*:*

$$E_w.Y^{\lambda^\vee} = (wt)(Y^{\lambda^\vee})E_w\,, \tag{37}$$

where $(wt)(Y^{\lambda^\vee}) := q^{-\operatorname{ht}(w(\lambda^\vee))}$. *In particular, the eigenvalue for* $Y^{\lambda^\vee}$ *on* E_w *is* q^{-1} *if and only if* $w(\lambda^\vee)$ *is a simple coroot.*

Note that acting with $\overline{T}_i$'s instead, or equivalently defining T_i's in term of the operators $\overline{\pi}_i$'s would allow to revert the picture and use 1 as initial eigenvector instead of w_0. We also get the following side result on the Hecke group algebra.

Proof. First note that t is regular; indeed, $\mathring{\rho}$ is regular, and q is not a kth root of unity with k too small, so one can use

$$(wt)(Y^{\alpha_i^\vee}) = q^{-\operatorname{ht}(w(\alpha_i^\vee))} = q^{-\langle \alpha_i^\vee, w^{-1}(\mathring{\rho})\rangle} \tag{38}$$

to recover the coordinates of $w(\mathring{\rho})$ on each ith fundamental weight. For the same reason, $(wt)(Y^{\alpha_i^\vee})$ is never 1.

We first prove in Lemma 7.5 that $E_1 = w_0$ is an appropriate eigenvector, and then define intertwining operators τ_i to construct the other E_w (Lemma 7.6). □

Corollary 7.4. *Each choice of* q_1 *and* q_2 *as in Theorem 7.1 determines in* $\mathrm{H}\mathring{W}$ *a maximal decomposition of the identity into idempotents, namely,* $1 = \sum_{w\in W} p_w$*, where* p_w *is the projection onto* E_w*, orthogonal to all* $E_{w'}$*,* $w' \neq w$*.*

Proof. Since t is regular, one can construct each p_w from $\operatorname{cl}(Y^{\alpha_1^\vee}), \dots, \operatorname{cl}(Y^{\alpha_n^\vee}) \in \mathrm{H}\mathring{W}$ by multivariate Lagrange interpolation. Therefore p_w belongs to $\mathrm{H}\mathring{W}$. □

Lemma 7.5. *Let* w_0 *be the maximal element of* $\mathring{W}$ *in* $\mathbb{C}\mathring{W}$*, and* $\lambda^\vee$ *an element of the (finite) coroot lattice. Then* w_0 *is an eigenvector for* $Y^{\lambda^\vee}$ *with eigenvalue* $q^{-\operatorname{ht}(\lambda^\vee)}$*.*

Proof. Let $i_1, \dots, i_r$ be an alcove walk for the translation $t_{\lambda^\vee}$. Then, $s_{i_1} \cdots s_{i_r}$ acts trivially on the finite Weyl group: $w_0 s_{i_1} \cdots s_{i_r} = w_0$. Therefore,

$$\begin{aligned} w_0.Y^{\lambda^\vee} &= w_0.(-q_1q_2)^{-\operatorname{ht}(\lambda^\vee)} T_{i_1}^{\epsilon_1} \cdots T_{i_r}^{\epsilon_r} \\ &= (-q_1q_2)^{-\frac{1}{2}(\epsilon_1+\cdots+\epsilon_r)} q_2^{(\epsilon_1+\cdots+\epsilon_r)} w_0 s_{i_1} \dots s_{i_r} \\ &= \left(-\frac{q_1}{q_2}\right)^{-\frac{1}{2}(\epsilon_1+\cdots+\epsilon_r)} w_0 = q^{-\operatorname{ht}(\lambda^\vee)} w_0\,, \end{aligned} \tag{39}$$

using Equation (15), Lemma 7.2, and Remark 3.1. □

As in [RY08], define $\tau_i := T_i - \frac{q_1+q_2}{1-Y^{-\alpha_i^\vee}} \in \operatorname{End}(\mathbb{C}\mathring{W})$ for $i = 1, \dots, n$. Note that this operator is a priori only defined for eigenvectors of $Y^{-\alpha_i^\vee}$ for an eigenvalue $\neq 1$. Whenever they are well-defined, they satisfy the braid relations, as well as the following skew-commutation relation: $\tau_i Y^{\lambda^\vee} = Y^{s_i(\lambda^\vee)}\tau_i$. Therefore, τ_i sends an Y-weight space for the character wt to an Y-weight space for the character ws_it.

For $w \in \mathring{W}$, define $E_w := w_0.\tau_{i_1} \cdots \tau_{i_r}$ where $i_1, \dots, i_r$ is any reduced word for w.

Lemma 7.6. *The* $(E_w)_{w\in\mathring{W}}$ *are well-defined, and triangular with respect to the canonical basis of* $\mathbb{C}\mathring{W}$*:*

$$E_w = (-q_1)^{\ell(w)} w_0 w + \sum_{w' > w_0 w} c_{w,w'} w' , \tag{40}$$

for some coefficients $c_{w,w'} \in \mathbb{C}$*. In particular, the* E_w *are all non-zero.*

Proof. The definition of E_w does not depend on the choice of the reduced word thanks to the braid relations. Furthermore, at each step the application of τ_i on E_{ws_i} is well-defined because $(ws_i t)Y^{-\alpha_i^\vee} \neq 1$.

The triangularity is easily proved by induction: when i is not a descent of w:

$$E_{ws_i} = E_w.\tau_i = E_w.\left((q_1+q_2)\pi_i - q_1 s_i - \frac{q_1+q_2}{1-Y^{-\alpha_i^\vee}}\right) , \tag{41}$$

and only the second term can contribute to the coefficient of $w_0 w$. □

Theorem 7.7. *The morphism* cl *from the affine Hecke algebra* $\mathrm{H}(W)(q_1, q_2)$ *to the Hecke group algebra* $\mathrm{H}\mathring{W}$ *is surjective for* q_1, q_2 *as in Theorem 7.1.*

Proof. Consider the decomposition $1 = \sum_{w\in W} p_w$ of the identity of $\mathrm{H}\mathring{W}$ given in Corollary 7.4.

Writing $(1-Y^{-\alpha_i^\vee})^{-1} = \sum_{w\in\mathring{W}} p_w (1-Y^{-\alpha_i^\vee})^{-1} = \sum_{w\in\mathring{W}} p_w (1-(wt)(Y^{-\alpha_i^\vee}))^{-1}$ shows that $1-Y^{-\alpha_i^\vee}$ is invertible not only in $\mathrm{End}(\mathbb{C}\mathring{W})$ but even inside $\mathrm{cl}(\mathrm{H}(W)(q_1,q_2))$. Therefore $\tau_i = T_i - \frac{q_1+q_2}{1-Y^{-\alpha_i^\vee}}$ also belongs to $\mathrm{cl}(\mathrm{H}(W)(q_1,q_2))$.

Consider the operator $p_w\tau_i$ which kills all eigenspaces $\mathbb{C}.E_{w'}, w \neq w'$, and sends the eigenspace $\mathbb{C}.E_w$ to $\mathbb{C}.E_{ws_i}$.

The *calibration graph* is the graph on W with an arrow from w to ws_i if $p_w\tau_i \neq 0$, or equivalently if $E_w.\tau_i \neq 0$. We claim that this is the case if and only if $\mathrm{D}_L(w) \subset \mathrm{D}_L(ws_i)$. Take indeed $w \in W$ with a non-descent at position i. Then, $\mathrm{D}_L(w) \subset \mathrm{D}_L(ws_i)$ and by Lemma 7.6, $E_w.\tau_i = E_{ws_i} \neq 0$. Next, there is no arrow back from ws_i to w if and only if $E_w.(\tau_i)^2 = 0$. Using the quadratic relation satisfied by τ_i,

$$\tau_i^2 = \frac{(q_1+q_2Y^{\alpha_i^\vee})(q_1+q_2Y^{-\alpha_i^\vee})}{(1-Y^{\alpha_i^\vee})(1-Y^{-\alpha_i^\vee})} , \tag{42}$$

this is the case if $\mathrm{ht}(w(\alpha_i^\vee)) = \pm 1$. Since i is not a descent of w, this is equivalent to $w(\alpha_i^\vee) = -\alpha_j^\vee$ for some simple coroot $\alpha_j^\vee$, that is $ws_i = s_j w$. In turn, this is equivalent to $\mathrm{D}_L(ws_i) = \mathrm{D}_L(s_j w) \supsetneq \mathrm{D}_L(w)$, which concludes the claim.

For each w and w' with $\mathrm{D}_L(w) \subset \mathrm{D}_L(w')$ there exists a path $i_1, \dots, i_r$ from w to w' in the calibration graph; choose one, and set $\tau_{w,w'} = \tau_{i_1}\cdots\tau_{i_r}$. The following family

$$\{p_w\tau_{w,w'} \mid \mathrm{D}_L(w) \subset \mathrm{D}_L(w')\} \tag{43}$$

is linearly independent, and by dimension comparison with $\mathrm{H}\mathring{W}$ forms a basis $\mathrm{cl}(\mathrm{H}(W)(q_1,q_2))$. Therefore, $\mathrm{cl}(\mathrm{H}(W)(q_1,q_2)) = \mathrm{H}\mathring{W}$. □

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